Two-sector stochastic growth models[1]

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Abstract. This paper develops the study of two-sector growth models of the form introduced by Arrow and Kurz (1970). We extend their deterministic model by allowing the population process to become random and by allowing the population to choose their level of effort. We find that under suitable conditions the government is able to tax and borrow in such as way as to induce the private sector to invest and consume along the path which the government considers optimal. Moreover, we also find that in some important cases the model can be solved explicitly in closed form, to the extent that we can write down expressions for tax rates and interest rates. This leads to new one-factor interest rate models, with related taxation policies; numerical examples show very reasonable behaviour.

Key words: Two-sector growth model, stochastic, fiscal policy, debt policy.

JEL classification system: O41, E43, E63.

1 Introduction

The history of growth models is long and illustrious, stretching back at least to Ramsey (1928). Throughout this development, much attention has been devoted to single-sector models, where there is just one type of capital or good, which is produced at a rate depending on current capital levels, labour force and technology levels, and is then either consumed or reinvested into capital. One analogy is a farm producing corn which can either be eaten or used to produce more corn. There are two basic types of continuous time single-sector growth model appearing in the economic literature. Firstly the Solow model as developed by Solow (1956) and Swan (1956). This is a growth model with an exogenously given savings rate which determines the proportion of capital reinvested (and hence also the proportion consumed). Denison (1961) showed that this model was able to explain trends in empirical growth data for the United States. Secondly there is the Ramsey model.

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This was originally conceived by Ramsey (1928) but the term is now used to refer to the version as refined[4] by Cass (1965) and Koopmans (1965). This is a growth model with consumer optimization - households choose their rates of consumption over time to maximise a utility functional. See, for example, the books of Romer (2001) and Barro and Sala-I-Martin (1995) for a more complete description of these models and their variants.

The first two-sector model was developed by Uzawa (1961), (1963) who considered an economy with two produced goods, a consumer good and an investment good, produced by investment capital and labour. Again using the farm analogy, this is using labour and tractors to make corn and tractors. Uzawa (1965) then refined this model to one where the two goods are physical capital and human capital, both of which are required for production of further physical capital (by manufacturing) and human capital (by education). Arrow and Kurz (1970) chose public capital rather than human capital and our work in this paper develops this model.

Arrow & Kurz proposed a deterministic model where there were two types of capital, government capital and private capital, which were both needed in the production of the single consumption good. They first set about solving the government's optimization problem, where the government's objective was to maximise the integrated discounted felicity from *per capita* consumption, where the felicity also depends on the *per capita* level of government capital. This feature of the model recognises that the felicity of the population is improved if the provision of education, healthcare, transportation, *etc.* is improved, and that such infrastructure is provided largely (if not exclusively) by government capital. Since Arrow & Kurz assume that private and government capital can be freely switched at any time, it is clear that the state of the optimally-controlled system at any time is completely described by the total amount of capital, the split between government and private sectors being dictated by optimality.

The problem gets more interesting when it comes to the behaviour of the private sector, which is viewed as very large collection of identical non-collaborating small households, each individually optimising its common objective, which is again an integrated discounted felicity of *per capita* consumption and government capital, but not of course the same as the government's objective. History and fashion have overwhelmed the centrally-planned economy, so we suppose that the government's control of the economy is exercised only through levying various proportional taxes, and issuing and retiring riskless debt from time to time. The central question studied by Arrow & Kurz is: *can the government manipulate taxes and debt in such as way as to induce the private sector to follow the government's optimal policy?*

The analysis of Arrow & Kurz is quite involved, but they are able to conclude that, under certain conditions, various combinations of taxes and debt can steer the

[4]Although Ramsey's original model was actually more subtle than Cass/Koopmans in some respects, for example it included a disutility function to reflect the amount of labour supplied (i.e. the longer the hours worked the less the utility). We will adopt a similar approach.

economy along the government's desired trajectory. However, the solutions they find are in terms of deterministic trajectories for the various tax rates for all future times, and this leaves undecided the interpretation of the solution: *is this open-loop or closed-loop control?* That is, do we think of the solution for the income tax rate (which will be an explicit function of time) as something that the government commits to at time 0, or do we rather think of the income tax rate as being a function of the underlying state variable (the total amount of capital)? The former interpretation seems viable only if we assume that the world really is deterministic, and that the government can predict with perfect foresight for all time. Casual observation suggests that this is very unlikely to be the case, so we would prefer to have a solution where tax rates would be expressed in terms of the current state of the economy, rather than being set according to a centuries-old plan[5]. In the deterministic model of Arrow & Kurz, these two cannot be distinguished.

Another feature of Arrow & Kurz's solution is that we have little insight into the properties of the tax regimes the government would need to follow: in particular, are the tax rates always between 0 and 1? If not, are the suggested values actually credible?

To address these issues, we plan in this paper to take the model of Arrow & Kurz, and modify it in two respects:

(i) introduce random fluctuations in output and population size;

(ii) allow the population to choose their level of effort.

The first modification allows us to distinguish between solutions which are functions of the underlying state of the economy, and solutions which are pre-determined processes. Without the second modification, we find that the effects of income tax are unrealistic. Once again, it turns out that the optimal solution of the government's problem can be expressed in terms of a single underlying state variable, the technology-adjusted *per capita* capital in the economy, which now follows a stochastic differential equation, and is thus a diffusion. We are then able to solve the private sector's problem, and deduce relations which must be satisfied by the various tax rates and by government debt in order to induce the private sector to follow the trajectory desired by government. In particular, we look for (and find) solutions for the tax rates which are functions only of the state process.

As yet, these expressions for tax rates are still quite opaque, so we are no better placed to decide whether they will always be between 0 and 1, for example. Our response to that has been to find explicit examples which can be *solved in closed form*, and where it is possible to find the range of any of the tax rates, as these are expressed now as explicit functions of the state variable. A collection of such

[5]See Christiaans (2001) for further discussion on this point. He concludes that open-loop solutions of dynamic optimization problems are unstable and therefore provide no reasonable basis for a positive theory of economic growth.

examples helps us to build up a library of possible behaviours, may lead us to other interesting questions, and allows us to check further analytical and numerical work. The approach we use is simply to take the inverse problem; write down the solution we would like, and then see whether we can find a model to which that is the solution! So we obtain a simple solution to a possibly slightly complicated model, rather than no solution to a simple model. This approach applies even to the basic one-sector model, and we show in an appendix some of the solutions which can be obtained for that. Our consideration of explicit examples is similar to the so called "inverse optimal" problem first studied by Kurz (1968) of constructing the class of objective functions that could give rise to given specified consumption-investment functions. Chang (1988) solves a similar stochastic inverse optimal problem.

Shortly after the work of Arrow & Kurz growth theory fell out of favour, not making a return until the mid-1980s. Lucas (1988) extended the work of Denison (1961) by showing that a two-sector model can explain not only the trends in growth data, but also diversity between countries in the data. Consequently much of the recent growth literature deals with economies with two capital goods, the first usually being physical capital and examples of the second including human capital, public capital, financial capital, quality of products and embodied and disembodied knowledge (Mulligan and Sala-I-Martin 1993).

Models considering directly the effects of public investment come in two formulations. The first considers how the *rate* of government expenditure on public services effects the productivity of the economy; see Aschauer (1988) for a discrete example or Barro (1990) for a continuous time model. The second type of formulation considers the total *stock* of public capital, invested in such things as roads and hospitals, as the key input to the production rate. This was the problem first studied by Arrow & Kurz, with the stock of government capital appearing in the utility function as well as the production function. This second approach is arguably more realistic but has not been widely adopted, although Futagami, Morita, and Shibata (1993) have extended the model of Barro (1990) to include government capital, and Fisher and Turnovsky (1998) have adopted a Ramsey style framework, although in both these models the public capital only appears in the production function and not also the utility[6]. Baxter and King (1993) considered a discrete time model very similar to that of Arrow and Kurz.

Use of continuous time stochastic calculus in economic growth models first appeared in the papers of Bourguignon (1974), Merton (1975) and Bismut (1975). These extend the Solow growth model to a random setting by addition of a Brownian element to the labour supply (Bourguignon, Merton) or to the production process (Bourguignon, Bismut). Merton also considers a stochastic version of the Ramsey problem, again with Brownian motion appearing in the dynamics of the labour supply. Chapter 3 of Malliaris and Brock (1982) contains a good overview of these and similar models. More recent contributions building on Merton's 'Stochastic

[6]This is true for other two-sector models too. Usually the utility function is restricted to being a function of consumption and not of levels of capital or rates of investment.

Ramsey Model' include Foldes (1978), (2001) who adds Brownian motions to further parameters of the model, and Amilon and Bermin (2001) who allow the government to control the expected growth rate of the labour supply. We have been unable to find any continuous time stochastic two-sector (private sector and government capital) models anywhere in the literature.

One of the possible uses of a stochastic growth model is to study interest rate dynamics. Merton (1975) does this for the stochastic Solow model using a Cobb-Douglas production function and a constant savings ratio. Amilon and Bermin (2001) use a stochastic Ramsey model and generate a variety of interest rate processes by considering different production and utility functions. Cox, Ingersoll, and Ross (1985a), (1985b) develop a simple stochastic model of capital growth which they use to determine the behaviour of asset prices including the term structure of interest rates. Sundaresan (1984) builds on this work and that of Merton by considering multiple consumption goods with a Cobb-Douglas production function.

The layout of the remainder of the paper is as follows. In Part I, we develop the theory of the model, firstly (in Section 2) characterising the solution to the government's problem in terms of the Lagrangian shadow price function, and then in Section 3 we introduce taxation and a private sector independently optimising its own utility functional subject to taxation constraints. We find conditions that the tax rates must satisfy in order to induce the private sector to follow the *government's* optimal choices. In Part II, we try to find examples of the model studied in Part I which can be solved in closed form. As mentioned above, we do this by solving the *inverse* problem, where we postulate a particular solution to the government's problem, in terms of their utility, the proportion of capital held by the government as a fraction of the total capital in the economy, and the value function, and then we seek a production function which would give rise to this solution. One issue that needs to be addressed (which is a methodological innovation of this paper) is that if we have selected the government's solution then we can only hope to know the production function and its derivatives *along the optimal path*; can the production function then be extended off this path so as to remain concave and homogeneous of degree 1? We are able to show that under mild conditions this *is* possible, even though the exact form of the production function may be rather indirect. We go on to present some quite concrete examples, and look at the kinds of tax regimes that would be applied by the government. We also find quite explicit and novel one-factor models for interest rates. We conclude in Section 6, which is followed by four appendices. Appendix A has proofs of statements made earlier in the paper. Appendix B is a (technical) discussion of the behaviour of the level of government debt. Appendix C shows how our results simplify to the one-sector Ramsey model. Finally Appendix D contains a useful summary of the notation used in the paper.

PART I

2 The government's problem

The dynamics of the total capital K_t in the economy at time t evolves according to the equation[7]

$$\dot{K}(t) = F(K_p(t), K_g(t), T(t)L(t)\pi(t)) - \delta K_t - C_t, \tag{2.1}$$

where L_t is the size of the population at time t, $\pi(t) \in [0,1]$ is the proportion of the population's effort devoted to production, and $K_p(t)$ is the amount of private capital in existence at time t, $K_g(t) \equiv K(t) - K_p(t)$ the amount of government capital at time t. The parameter δ is the rate of depreciation, a positive constant, the process C is the aggregate consumption rate, and the process T is the labour-augmenting effect of improvements in technology. We shall assume always that K, K_g, K_p and C are non-negative. Following Arrow & Kurz (1970), we shall suppose that capital can be freely switched between government and private sectors; the implications of this assumption are discussed in detail by Arrow & Kurz, and we refer the reader there for more detail. Suffice it to say that the problem is hard enough already with this simplifying assummption. Concerning the production function F, we shall make the usual assumption of homogeneity of degree 1, which is to say that

$$F(\lambda K_p, \lambda K_g, \lambda L) = \lambda F(K_p, K_g, L) \tag{2.2}$$

for any $\lambda > 0$. The dynamics (2.1) are the same as the dynamics of Arrow & Kurz (1971), but where our account begins to differ is in the assumptions we make concerning population growth. While Arrow & Kurz took this to be deterministic, we shall suppose (perhaps more realistically) that

$$\begin{aligned} dL_t &= L_t(\sigma dW_t + \mu_L dt), && (2.3)\\ dT_t &= \mu_T T_t dt, \qquad T_0 = 1, && (2.4) \end{aligned}$$

where μ_L and $\mu_T \geq 0$ are constants and where W is standard Brownian motion.

The objective of the government is to maximise

$$E\int_0^\infty e^{-\rho_g t} L_t\, U\biggl(\frac{C_t}{L_t}, \frac{K_g(t)}{L_t},\, \pi_t\biggr)\, dt, \tag{2.5}$$

where U is strictly concave, and increasing in the first two arguments, decreasing in the last. The objective (2.5) depends on *per capita* consumption and per capita government capital, and the felicity is weighted according to the current population size. In order to have the prospect of a time-homogeneous solution, we require that

[7] As a notational convenience, we use subscript and argument notations $K_t \equiv K(t)$ interchangeably throughout, and will omit appearance of the time argument where there is no risk of confusion.

U is also homogeneous of degree $(1-R_g)$ for some $R_g > 0$[8]; this means that U can be represented as

$$U(C, K_g, \pi) = K_g^{1-R_g} h(\xi, \pi), \qquad \xi \equiv C/K_g \tag{2.6}$$

for some C^2 function h strictly concave and increasing in its first argument and decreasing in its second[9].

As a consequence of the assumptions so far, it turns out to be advantageous to work with *per capita* technology-adjusted variables, rather than their aggregated equivalents. So if we define

$$\eta_t \equiv L_t T_t = L_0 \exp\left\{ \sigma W_t + (\mu_L + \mu_T - \frac{1}{2}\sigma^2)t \right\}, \tag{2.7}$$

and then define

$$k_t \equiv K_t/\eta_t, \quad k_g(t) \equiv K_g(t)/\eta_t, \quad k_p(t) \equiv K_p(t)/\eta_t, \quad c_t \equiv C_t/\eta_t, \tag{2.8}$$

and so forth, we find that the dynamics of k follow from the dynamics (2.1) of K:

$$dk_t = -k_t \sigma dW_t + \left[F(k_p(t), k_g(t), \pi_t) - \gamma k_t - c_t \right] dt, \tag{2.9}$$

where

$$\gamma \equiv \delta + \mu_L + \mu_T - \sigma^2.$$

It is now necessary to re-express the government objective (2.5) in terms of *per capita* technology-adjusted variables, and here the assumption that U is homogeneous of degree $(1-R_g)$ enters in an essential way. We find that the objective of the government can be expressed as

$$\begin{aligned} E\int_0^\infty e^{-\rho_g t} L_t\, U\left(\frac{C_t}{L_t}, \frac{K_g(t)}{L_t}, \pi_t \right) dt &= E\int_0^\infty e^{-\rho_g t} L_t\, U(c_t T_t, k_g(t) T_t, \pi_t)\, dt \\ &= E\int_0^\infty e^{-\rho_g t} L_t T_t^{1-R_g} U(c_t, k_g(t), \pi_t)\, dt \\ &= L_0 E_g \int_0^\infty e^{-\lambda_g t} U(c_t, k_g(t), \pi_t)\, dt \end{aligned} \tag{2.10}$$

where

$$\lambda_g \equiv \rho_g - (1-R_g)\mu_T - \mu_L,$$

[8]We also assume that $R_g \neq 1$, not because the case of logarithmic utility is in any way difficult, but rather because some of the expressions to be developed have a different appearance in this special case.

[9]In fact for U to have the required properties we will also need that $(1-R_g)h > \xi h_\xi$, $\xi^2 h_{\xi\xi} + 2R_g \xi h_\xi - R_g(1-R_g)h < 0$ and $R_g h_\xi^2 < -(1-R_g) h h_{\xi\xi}$.

and the final expectation is with respect to the measure P_g which is absolutely continuous with respect to P on every $\mathcal{F}_t$[10], and has density

$$\left.\frac{dP_g}{dP}\right|_{\mathcal{F}_t} = \exp(\sigma W_t - \frac{t}{2}\sigma^2).$$

The effect of changing measure from P to P_g is to introduce additional drift[11] into the Brownian motion σW; precisely, we have

$$W_t = w_t + \sigma t,$$

where w is a P_g-Brownian motion. This therefore transforms the dynamics (2.9) into

$$dk_t = -k_t \sigma dw_t + \left[F(k_p(t), k_g(t), \pi_t) - \gamma_g k_t - c_t \right] dt, \tag{2.11}$$

where the constant γ_g is given by

$$\gamma_g = \gamma + \sigma^2 = \delta + \mu_L + \mu_T.$$

In order to maximise (2.10) with the dynamics (2.11), we can proceed to find the Hamilton-Jacobi-Bellman equation for the value function

$$V(k) \equiv \sup E_g \left[\int_0^\infty e^{-\lambda_g t} U(c_t, k_g(t), \pi_t)\, dt \;\middle|\; k_0 = k \right]. \tag{2.12}$$

The HJB equation satisfied by V is

$$\sup_{c, k_g, 0 \leq \pi \leq 1} U(c, k_g, \pi) - \lambda_g V(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + [F(k - k_g, k_g, \pi) - \gamma_g k - c] V'(k) = 0. \tag{2.13}$$

From this, we deduce the necessary first-order conditions for optimality:

$$U_c(c, k_g, \pi) = V'(k) \tag{2.14}$$
$$U_g(c, k_g, \pi) = V'(k)(F_p(k_p, k_g, \pi) - F_g(k_p, k_g, \pi)), \tag{2.15}$$
$$U_\pi(c, k_g, \pi) = -V'(k) F_L(k_p, k_g, \pi) \tag{2.16}$$

where we use subscripts to denote differentiation, as in the abbreviations:

$$U_c \equiv \frac{\partial U}{\partial c}, \quad U_g \equiv \frac{\partial U}{\partial k_g}, \quad f_p \equiv \frac{\partial f}{\partial k_p}, \quad f_g \equiv \frac{\partial f}{\partial k_g}.$$

The conditions (2.14), (2.15) and (2.16) arise from considering the optimization problem

$$\sup_{c, k_g, 0 \leq \pi \leq 1} U(c, k_g, \pi) + p[\, F(k - k_g, k_g, \pi) - c\,]; \tag{2.17}$$

[10]The filtration $(\mathcal{F}_t)_{t \geq 0}$ denotes the working filtration, with respect to which all processes are adapted.

[11]This is the famous Cameron-Martin-Girsanov Theorem; see, for example, Rogers and Williams (2000) for an account

implicit in the statements (2.14) and (2.15) is the following assumption:

> *For every $p, k > 0$, the problem* (2.17) *has an interior solution which depends in a C^1 fashion on (p, k)* (2.18)

(In fact, the assumed strict concavity of U makes an interior solution unique.) This assumption does not always hold, but we shall make it for the sake of the simplifications in the statements and proofs of results; no doubt similar conclusions can be reached without it, but we leave that as an issue for further research.

The observation that the optimising values (c, k_g, π) are uniquely determined as functions of (p, k) reduces the HJB equation (2.13) to a non-linear differential equation for V; once the solution is found, we are able to express the optimal values of (c, k_g, π) as functions of $(V(k), k)$, or, more simply put, functions of k. We shall henceforth use the notation c^*, k_g^* and π^* for these optimal functions[12] of the underlying state variable k, and also we shall introduce the notation

$$\Phi(k) = F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma_g k - c^*(k) \tag{2.19}$$

for the drift in the dynamics (2.11), which therefore are more compactly expressed as

$$dk_t = -\sigma k_t dw_t + \Phi(k_t)dt. \tag{2.20}$$

Under the original measure P the dynamics (2.9) can be written as

$$dk_t = -\sigma k_t dW_t + \tilde{\Phi}(k_t)dt, \tag{2.21}$$

with the identification

$$\tilde{\Phi}(k) \equiv \Phi(k) + \sigma^2 k.$$

Under mild conditions[13] on Φ, (2.20) has a pathwise-unique strong solution, and the value function V will satisfy the equation

$$U(c^*(k), k_g^*(k), \pi^*(k)) - \lambda_g V(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + \Phi(k)V'(k) = 0. \tag{2.22}$$

Although there may be some issues concerning smoothness of the (c, k_g) optimizing in (2.18), the following result is the starting point of our investigations.

Theorem 1 *(i) Assuming that the value function* (2.12) *is finite valued and C^2, and that assumption* (2.18) *holds then there exist differentiable functions Φ, c^*, k_g^*, π^* and twice-differentiable $\Psi \equiv V'$ such that the equalities*

$$0 = U - \lambda_g V + \frac{1}{2}\sigma^2 k^2 V'' + \Phi V' \tag{G1}$$

$$U_c = V' \tag{G2}$$

$$U_g = (F_p - F_g)V' \tag{G3}$$

$$U_\pi = -F_L V' \tag{G4}$$

[12] The notation k_p^* will also be used, with the obvious interpretation $k_p^*(k) = k - k_g^*(k)$.

[13] Global Lipschitz will certainly be enough: Rogers & Williams (2000) again, Theorem V.11.2.

hold along the path given by $(c^*(k), k_g^*(k), \pi^*(k))$[14]*, where*

$$\Phi = F - \gamma_g k - c. \tag{G5}$$

(ii) Conversely suppose that there exist differentiable functions Φ*,* c^**,* k_g^**,* π^* *and* C^3 *function*[15] V *such that the equalities* (G1)–(G5) *hold along the path given by* $(c^*(k), k_g^*(k), \pi^*(k))$*. If* k^* *is the solution to the SDE* (2.20) *then provided the transversality condition*

$$\sup_t e^{-\lambda_g t} k_t^* V'(k_t^*) \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_g t} k_t^* V'(k_t^*) = 0 \tag{GT}$$

holds, the policy given by (c^*, k_g^*, π^*) *is optimal for the government, the optimally-controlled economy follows the dynamics* (2.20) *and* V *is the value function.*

Proof. (i) follows from the discussion above. (ii) - see Appendix A

Theorem 1 characterises the optimal solution to the government's problem, but what can we do with it? Are there examples where the solution can be expressed in closed form? In view of the complicated way in which the optimising values c^*, k_g^* and π^* were defined, it appears at first sight unlikely, but we shall later see that it *is* possible to exhibit explicit solutions.

Though nothing in the analysis so far (or for some time to come) requires it, we have it in mind that we are looking for a solution where k *is an ergodic diffusion;* this is the stochastic equivalent of a *balanced growth path* in a deterministic model. A balanced growth path is a path where k is (or tends to) a fixed point of the dynamics, and in some sense describes an economy where everything grows in step with the technology-adjusted population. It makes most sense economically to consider such situations; models where this does not happen either offer the population something for nothing, or nothing for something.

3 Government borrowing and taxation

The government's optimal policy has been determined, but the issue now is how to implement that policy when the government cannot directly control the private sector, but can only shape its choices through taxation and the issuing of government debt. Since the optimal policy of the previous Section was Markovian, in the sense that the total technology-adjusted *per capita* capital k was a Markov process (even a diffusion), we shall now seek Markovian taxation policies, which are defined by the property that the rates of tax are functions only of k.

[14]This means, for example, that $U_g(c^*(k), k_g^*(k), \pi^*(k)) = (F_p - F_g)(k_p^*(k), k_g^*(k), \pi^*(k))\Psi(k)$ in the case of (G3).

[15]We require 3 continuous derivatives because the proof uses the Lagrangian process $e^{-\lambda_g t}V'(k_t)$, and to apply Itô's formula to this we request two continuous derivatives for V'.

Before we can understand the effects of government fiscal policy, we have to understand the behaviour of the private sector on which it acts, and we turn to that now. We think of the private sector as made up of a very large number L_0 of identical households; if one of these households receives a flow $(\Delta C_t)_{t\geq 0}$ of the consumption good, then it values this flow as

$$E\int_0^\infty e^{-\rho_p t}u\left(\frac{L_0\Delta C_t}{L_t}, \frac{K_g(t)}{L_t}, \pi_t\right) dt, \tag{3.1}$$

and it wishes to maximise this. Here, u is strictly concave, increasing in its first two arguments, and decreasing in the third, and $\rho_p > 0$ is constant. The felicity u depends on the *per capita* level of government capital, and on the *per capita* rate of consumption for the household, which is assumed to be subject to the same size fluctuations as the entire population; it also varies inversely with the proportion of effort devoted to production. As with the government objective, we assume that u is homogeneous, of degree $(1 - R_p)$, where $R_p > 0$ is different from 1, and typically different from R_g.

We suppose that the objectives of the government and private sector are different, and that the government aims to set taxes and to borrow in such a way as to induce the private sector to follow the government's desired path. We need now to decompose the dynamics (2.1) of the economy so as to understand the effects of the taxes. Homogeneity of order 1 of F implies[16] that we may express the output as the sum of three terms,

$$F(K_p, K_g, L) = K_pF_p(K_p, K_g, L) + K_gF_g(K_p, K_g, L) + LF_L(K_p, K_g, L), \tag{3.2}$$

which are interpreted as the return on private capital, the return on government capital, and the return on labour, respectively.

We shall suppose that the government is able to appropriate some fixed proportion $1 - \theta_p - \theta_L$ of the returns to its capital by direct charging for services such as toll roads, university tuition fees, subsidised rail fares, and some healthcare costs, but it is in the nature of government expenditure that much of the return on government capital cannot be directly appropriated, so in practice this proportion may be near to zero. A proportion θ_p of the returns to government capital are included in the returns to private capital, and the remaining proportion θ_L is included in returns to labour, so that from an accounting point of view we suppose that the returns on private capital and labour are (respectively)

$$K_pF_p + \theta_pK_gF_g, \quad \theta_LK_gF_g + \pi LTF_L, \tag{3.3}$$

with the remaining $(1 - \theta_p - \theta_L)K_gF_g$ going directly to government.

The evolution of the levels of private and government capital are determined by the equations

$$\begin{aligned} dK_p &= dI_p - \delta K_p dt & (3.4)\\ dK_g &= dI_g - \delta K_g dt, & (3.5) \end{aligned}$$

[16]Differentiate the identity (2.2) with respect to λ.

where $I_p(t)$ is the cumulative amount invested in private capital by time t.

The government will issue debt and levy taxes; returns on private capital will be taxed at rate $1-\beta_k$, income at rate $1-\beta_w$, consumption at rate $1-\beta_c$, and interest on government debt at rate $1-\beta_r$, so that the private sector's aggregate budget equation[17] is therefore

$$dI_p+dD+\beta_c^{-1}Cdt = \beta_k\big[\,K_pF_p+\theta_pK_gF_g\,\big]dt+r\beta_rDdt+\beta_w\big[\,F-K_pF_p-(1-\theta_L)K_gF_g\,\big]dt \tag{3.6}$$

where D_t denotes the amount of government debt at time t. The interpretation of the left-hand side is that this is the total outgoings of the private sector: the investment in private capital, the investment in government debt, and the cost of consumption. The right-hand side (3.6) is the after-tax income of the private sector: return on private capital plus interest on government debt plus wage income.

The relation (3.4) can be used to eliminate dI_p and rewrite the private-sector budget equation as

$$dK_p+dD = K_p(\beta_kF_p-\delta)dt+r\beta_rDdt-\beta_c^{-1}Cdt+\beta_w\pi\eta F_Ldt+(\beta_k\theta_p+\beta_w\theta_L)K_gF_gdt \tag{3.7}$$

which bears the simple interpretation that the change in private-sector wealth is accounted for by the return on private capital (adjusted for depreciation) plus the return on government debt, less consumption, plus the wage income, plus unappropriated return on government capital.

Recall that we seek tax rates as functions of k which will cause the private sector to follow the government's optimal trajectory. So we shall suppose that such tax rates have been set, the economy as a whole is following the government's optimal policy as discussed in Section 2, and shall consider the optimisation problem faced by a single household. *If any deviation from the government's optimal path is suboptimal for the individual household, then we have an equilibrium in which all households follow the government's optimal path;* we shall suppose that this is what happens, and deduce the implications for the tax rates and borrowing policy. These are summarised in the following result.

Theorem 2 *Suppose that the government sets proportional taxes $1-\beta_c$ on consumption, $1-\beta_w$ on income, $1-\beta_k$ on returns on private capital, and $1-\beta_r$ on returns on government debt, all functions only of the total technology-adjusted* per capita *capital k in the economy at the time. If there exists a C^2 function ψ, and a*

[17] Arrow & Kurz have also a tax on savings, which alters the term dI_p+dD in (3.6) to $\beta_s^{-1}(dI_p+dD)$. Since this could be absorbed into our formulation simply by reinterpreting the other $\beta_\cdot$, we lose no generality by studying the equations as given.

function r such that the equations

$$0 = \psi(\beta_k F_p - \gamma - \lambda_p) + \psi'(\tilde{\Phi} + \sigma^2 k) + \frac{1}{2}\sigma^2 k^2 \psi'' \tag{PS1}$$

$$u_c = \beta_c^{-1}\psi \tag{PS2}$$

$$u_\pi = -\beta_w F_L \psi \tag{PS3}$$

$$\beta_k F_p = r\beta_r + \delta \tag{PS4}$$

all hold along the government's optimal path[18]*, where $\lambda_p = \rho_p - (1 - R_p)\mu_T$, then the private sector faced with these tax rates will choose to follow the government's optimal path, provided the transversality condition*

$$\sup_t e^{-\lambda_p t}|x_t|\psi(k_t^*) \in L^1, \qquad \lim_{t\to\infty} e^{-\lambda_p t} x_t \psi(k_t^*) = 0 \tag{PST}$$

is satisfied, where $x \equiv k_p + \Delta_p$ is the total technology-adjusted per capita wealth of the private sector, split between private capital k_p and government debt Δ_p.

Proof. See Appendix A.

Remarks. (i) Of course, the way we plan to use Theorem 2 is to enable us to *find* the tax regimes which will persuade the private sector to follow the government optimal path. So if we suppose that the government's optimal path has been determined, as in Section 2, we want now to know whether it is possible to have the conditions (PS1), (PS2), (PS3) and (PS4) all holding at the same time. But this is in fact quite easy: for example, if we *choose* the functional form of β_c and β_r, then (PS2) determines the function ψ and then β_k, β_w and r are determined from (PS1), (PS3) and (PS4) respectively.

(ii) Note the similarities between conditions (PS1), (PS2) and (PS3) and the corresponding conditions (G1), (G2) and (G4) of Theorem 1. If we set the tax rates to zero (so $\beta_k = 1$ etc.) then these conditions of Theorem 2 are identical in form to those of Theorem 1; however they depend on the private sector parameters λ_p and γ and on the private sector utility function u rather than the corresponding government quantities. Only if the private sector and government share identical values $\lambda_p = \lambda_g$, $\gamma = \gamma_p$ and $u \equiv U$ will the private sector follow the government's optimal path under a no-tax regime.

(iii) We do not claim (nor is it true in general) that the solution is Markovian in the sense defined above, because the process Δ_p may fail to be a function only of k^*. However, it is possible[19] to express the private sector's wealth $x \equiv k_p + \Delta_p$ as

$$x_t = k_t \left[e^{-\int_s^t G_0(k_u)du} \frac{x_s}{k_s} + \int_s^t e^{-\int_v^t G_0(k_u)du} \frac{\tilde{B}(k_v)}{k_v} dv \right], \tag{PSW}$$

[18]For example, in full (PS3) says $u_\pi(c^*(k), k_g^*(k), \pi^*(k)) = -\beta_w(k) F_L(k_p^*(k), k_g^*(k), \pi^*(k))\psi(k)$.

[19]See (A.5).

where

$$\begin{aligned} G_0(k) &= (F-c^*)/k - \beta_k F_p, \qquad (3.8) \\ \tilde{B}(k) &= k_g^*(\beta_k\theta_p + \beta_w\theta_L)F_g + \beta_w\pi^* F_L - \beta_c^{-1}c^*, \qquad (3.9) \end{aligned}$$

with the understanding that F and its derivatives are evaluated along the optimal path; see Appendix B for the details.

In general, the expression (PSW) for the private-sector wealth will be path dependent; the effects of earlier borrowing persist. In this sense, the solution is not Markovian, in that private-sector wealth depends not just on k, but on the history of k. A special situation obtains when

$$\tilde{B}(k) = qkG_0(k) > 0; \qquad (3.10)$$

the integral in (PSW) is exact, and under mild conditions (ergodicity of k will certainly be enough) we can let $s \to -\infty$ in (PSW) to learn that

$$x_t = qk_t; \qquad (3.11)$$

the private sector's wealth is a fixed proportion of the total capital in the economy.

(iv) Note the interpretation of (PS4): the net return on private capital $\beta_k F_p$ is equal to the net return on debt $r\beta_r$ plus depreciation δ.

PART II

4 Explicit solutions: the government's problem.

Given the government's felicity function U, impatience parameter λ_g, and the production function F, it will in general not be easy to find the value function V. Our approach here is to solve the *inverse* problem: given the government's felicity function U, impatience parameter λ_g *and* V, try to find a *production function* F for which V is the value.

The homogeneity of degree $1-R_g$ assumed for U gives the expression

$$U(c, k_g, \pi) = k_g^{1-R_g} h(\xi, \pi) \qquad (4.1)$$

where $h(x,\pi) \equiv U(x,1,\pi)$, and $\xi \equiv c/k_g$. Differentiation gives

$$\begin{aligned} U_c(c,k_g,\pi) &= k_g^{-R_g} h_\xi(\xi,\pi), \qquad (4.2) \\ U_g(c,k_g,\pi) &= k_g^{-R_g}\left[\,(1-R_g)h(\xi,\pi) - \xi h_\xi(\xi,\pi)\,\right], \qquad (4.3) \\ U_\pi(c,k_g,\pi) &= k_g^{1-R_g} h_\pi(\xi,\pi). \qquad (4.4) \end{aligned}$$

Accordingly, the conditions (G1)–(G5) of Theorem 1 take the form

$$0 = U - \lambda_g V + \frac{1}{2}\sigma^2 k^2 V'' + \Phi V' \tag{g1}$$

$$U_c = k_g^{-R_g} h_\xi = V' \tag{g2}$$

$$U_g = k_g^{-R_g} \left[\, (1-R_g)h - \xi h_\xi \,\right] = (f_p - f_g)V' \tag{g3}$$

$$U_\pi = k_g^{1-R_g} h_\pi = -f_L V' \tag{g4}$$

$$\Phi = f - \gamma_g k - c \tag{g5}$$

The reason for the notational switch from F to f is that in Theorem 1 the production function F was known, with certain assumed properties, such as concavity, monotonicity and homogeneity. Here however, we shall be starting with assumed forms for h, V, and ξ, π, and will use (g1)–(g5) to try to find F. Of course, we will use (g1)–(g5) to try to define f, f_p, f_g, f_L, but there is *a priori* no reason to suppose that there is any (concave, increasing, homogeneous) function F relating them, and the use of a different notation is to emphasise that no such relations should be assumed. For F, the homogeneity property (3.2) holds[20], giving us

$$F = k_p F_p + k_g F_g + \pi F_L, \tag{4.5}$$

and differentiating $F = F(k) \equiv F(k_p^*(k), k_g^*(k), \pi^*(k))$ gives us

$$F' = k_p' F_p + k_g' F_g + \pi' F_L \tag{4.6}$$

along the path, but there is no reason to suppose that the corresponding properties

$$f = k_p f_p + k_g f_g + \pi f_L, \tag{g6}$$

and

$$f' = k_p' f_p + k_g' f_g + \pi' f_L \tag{g7}$$

should hold for f, f_p, f_g, f_L obtained from (g1)–(g5). Certainly properties (g6) and (g7) are necessary for f, f_p, f_g, f_L to be related through a production function F; the remarkable thing is that (g6) and (g7) are effectively *sufficient* for such a relation, as the following result establishes.

Theorem 3 (**Extension Theorem.**) *Suppose given monotone*[21] *C^1 functions $z : \mathbb{R}_+ \to \mathbb{R}_+^d$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+^d$, where ψ and z are co-monotone*[22]*, and C^1 $\phi : R_+ \to \mathbb{R}_+$. Then the following are equivalent.*

(i) There exists some concave $F : \mathbb{R}_+^d \to \mathbb{R}_+$ which is increasing in each argument, and homogeneous of degree 1, such that for all $t \geq 0$

$$\phi(t) = F(z(t)) \tag{4.7}$$

[20] For this Section, we will omit the superscript stars when not essential.

[21] By this we mean that each component of z is monontone; some components may be increasing while others are decreasing.

[22] That is, ψ increases in the components where z decreases, and *vice versa*.

and $\psi(t)$ is a supergradient[23] *to F at $z(t)$.*

(ii)

$$\phi(t) = z(t) \cdot \psi(t) \tag{4.8}$$
$$\phi'(t) = z'(t) \cdot \psi(t) \tag{4.9}$$

Remarks. The amazing thing about this result is that knowledge of ϕ and ψ only tells us about the function F and its gradient *at points on the path z*; nevertheless, co-monotonicity and (4.8)-(4.9) are together sufficient to extend F off the path z so as to be *globally* concave, homogeneous and increasing.

Proof.

$(i) \Rightarrow (ii)$ is immediate in view of the previous discussion.

$(ii) \Rightarrow (i)$. Define for each $x \in \mathbb{R}^+_d$ and each $t \geq 0$

$$\begin{aligned} \Lambda(x;t) &\equiv \phi(t) + (x - z(t)) \cdot \psi(t) \\ &= x \cdot \psi(t) \end{aligned}$$

in view of (4.8). Notice that $\Lambda(\cdot;t)$ is concave, increasing, and homogeneous of degree 1. If there were to be a concave F with the properties we seek, then $\Lambda(\cdot;t)$ would have to be a supporting hyperplane to F at $z(t)$. Consequently, we define F by

$$F(x) \equiv \inf_{t \geq 0} \Lambda(x;t) \tag{4.10}$$

and observe that $F(z(t)) \leq \phi(t)$. We also observe that $\psi(t)$ is a supergradient to F at $z(t)$, so all that now remains is to establish (4.7), for which we must check is that for all $t, w \geq 0$,

$$\phi(t) \leq \Lambda(z(t);w) = \phi(w) + (z(t) - z(w)) \cdot \psi(w).$$

However, for $0 \leq t < w$,

$$\begin{aligned} \phi(w) - \phi(t) &= \int_t^w \phi'(s)\, ds \\ &= \int_t^w z'(s) \cdot \psi(s)\, ds \\ &\geq \int_t^w z'(s) \cdot \psi(w)\, ds \\ &= (z(w) - z(t)) \cdot \psi(w), \end{aligned}$$

using (4.9) going from the first line to the second, and using the co-monotonicity going from the second line to the third. The case $w < t$ follows *mutatis mutandis.*

[23] That is, $F(y) \leq F(z(t)) + \psi(t) \cdot (y - z(t))$ for all y.

■

The strategy should now be clear. We assume given U, V, λ_g, γ_g, and shall propose $\pi(\cdot)$, $\xi(\cdot)$. We now use (g2) to determine $k_g(\cdot)$, which will tell us what $U(k) \equiv U(c(k), k_g(k), \pi(k))$ should be, and then we use (g3) to give us $f_p - f_g$, (g4) to give us f_L. As yet, we do not know f_p, but from (g6) we have

$$f = kf_p - (f_p - f_g)k_g + \pi f_L,$$

so substituting this into (g5), (g1), gives us f_p in terms of known functions. All that remains is to check (g7) and the co-monotonicity of (k_p, k_g, π) and (f_p, f_g, f_L), and the Extension Theorem finishes the job for us, constructing a production function F. The check of (g7) will impose an equation to be satisfied by the proposed $\pi(\cdot)$, $\xi(\cdot)$, so we are not able to choose both of these freely.

Carrying out this programme in more detail, we have firstly from (g2) that

$$k_g = (h_\xi / V')^{1/R_g}, \tag{4.11}$$

then dividing (g3) by (g2) we learn that

$$f_p - f_g = (1 - R_g)\frac{h}{h_\xi} - \xi; \tag{4.12}$$

dividing (g4) by (g2) gives us

$$f_L = -\frac{k_g h_\pi}{h_\xi}. \tag{4.13}$$

Using (g5), (g6) and (g1) gives two alternative expressions for Φ:

$$\begin{aligned}
\Phi &= k(f_p - \gamma_g) - k_g(f_p - f_g) + \pi f_L - c \\
&= k(f_p - \gamma_g) - k_g(f_p - f_g) + \pi f_L - \xi k_g \\
&= \frac{\lambda_g V - \frac{1}{2}\sigma^2 k^2 V''}{V'} - \frac{U}{V'} \\
&= \frac{\lambda_g V - \frac{1}{2}\sigma^2 k^2 V''}{V'} - \frac{k_g h}{h_\xi}.
\end{aligned}$$

Rearrangement gives an expression for f_p:

$$f_p - \gamma_g = \frac{\lambda_g V - \frac{1}{2}\sigma^2 k^2 V''}{kV'} - \frac{k_g h}{k h_\xi} + \frac{k_g(f_p - f_g)}{k} - \frac{\pi f_L}{k} + \frac{\xi k_g}{k}. \tag{4.14}$$

Now (g6) holds by construction, so (g7) will hold if and only if

$$0 = kf_p' - k_g(f_p' - f_g') + \pi f_L',$$

a condition equivalent (in view of (4.14) to

$$0 = \left(\frac{\lambda_g V - \frac{1}{2}\sigma^2 k^2 V''}{kV'}\right)' - \left(\frac{k_g h}{k h_\xi}\right)' + (f_p - f_g)\left(\frac{k_g}{k}\right)' - f_L\left(\frac{\pi}{k}\right)' + \left(\frac{\xi k_g}{k}\right)'. \tag{4.15}$$

5 Particular choices.

5.1 Special forms for V and U.

The equation (4.15) is as far as we can expect to get without more specific assumptions. In this Section, we shall make some very specific assumptions to begin the exploration of this model. We shall see that these assumptions are in a sense too special, but they give us a place to begin. We shall take the value function to be CRRA:

$$V(k) = \frac{Ak^{1-S}}{1-S} \tag{5.16}$$

for some $S \in (1, R_g)$, and positive A. Notice that this eliminates the first term in (4.15). Inspection of this equation suggests that we should introduce the function

$$\varphi(k) \equiv k_g(k)/k. \tag{5.17}$$

Our next assumption concerns h, which we shall suppose is of Cobb-Douglas form,

$$h(\xi, \pi) = -\xi^{-\nu}(1-\pi)^{-\kappa}, \tag{5.18}$$

for positive ν, κ, where we assume that

$$\omega \equiv R_g - 1 - \nu > 0. \tag{5.19}$$

Returning to (g2), (g3), (g4) gives us respectively

$$\begin{aligned} \xi &= \left(k^{S-R_g}\varphi^{-R_g}(1-\pi)^{-\kappa}/A \right)^{1/(1+\nu)}, && (5.20) \\ f_p - f_g &= \frac{\omega\xi}{\nu}, && (5.21) \\ f_L &= k\varphi\frac{\kappa\xi}{\nu(1-\pi)}. && (5.22) \end{aligned}$$

This now allows us to write ξ as a function of π, and, assuming φ has been chosen, to treat (4.15)

$$0 = \frac{\nu+1}{\nu}(\xi\varphi)' + \frac{\omega\varphi'\xi}{\nu} - \frac{\kappa\xi\varphi k}{\nu(1-\pi)}\left(\frac{\pi}{k}\right)'$$

as an equation to determine π. The derivatives of π vanish from this equation, and π is determined simply as

$$\pi(k) = \frac{R_g - S}{R_g - S + \kappa}, \tag{5.23}$$

a *constant*[24] in $(0,1)$! Abbreviating $(1-\pi)^{-\kappa/(1+\nu)}A^{-1/(1+\nu)}$ to a_g, we are able to simplify various expressions:

$$\begin{aligned} c &= a_g k^{(S-\omega)/(1+\nu)}\varphi^{-\omega/(1+\nu)} && (5.24)\\ f_L &= \frac{\kappa}{\nu(1-\pi)}\, c && (5.25)\\ f_p - f_g &= \frac{a\omega}{\nu}\, k^{(S-R_g)/(1+\nu)}\varphi^{-R_g/(1+\nu)} = \frac{\omega c}{\nu k\varphi} && (5.26)\\ f_p &= \gamma_g + Q + \frac{S}{\nu}\frac{c}{k} && (5.27)\\ f &= k(\gamma_g+Q) + \frac{(1+\nu)c}{\nu} && (5.28)\\ \Phi &= Qk + \nu^{-1}c && (5.29)\\ f_g &= \gamma_g + Q + \frac{c}{\nu k\varphi}\big(S\varphi - \omega\big) && (5.30) \end{aligned}$$

where

$$Q = \frac{\lambda_g + \frac{1}{2}\sigma^2 S(1-S)}{1-S}.$$

5.2 A special form for φ.

This is as far as we can get without specific choices for φ. Let us then assume that there exist positive constants a_0, a_1, b, and α such that

$$\varphi(k) = a_0 + a_1(1+bk)^{-\alpha}. \tag{5.31}$$

Notice that φ is decreasing; the interpretation of this is that the government's share of capital should decrease as the total stock of capital grows. There may be interest in other types of behaviour, but this seems a natural enough property for any realistic economy. Note also that we shall require that

$$a_0 + a_1 \leq 1 \tag{5.32}$$

in order that $0 \leq \varphi \leq 1$ everywhere. We propose also to restrict attention to situations where both k_g and k_p are increasing, just to fix ideas. This will not happen for all possible parameter choices, but by insisting that

$$\alpha \leq 1 \tag{5.33}$$

we guarantee that k_g is increasing (k_p is automatically increasing). We now have to ensure the co-monotonicity, which is to say that f_p and f_g are both decreasing (the monotonicity of f_L is irrelevant in view of the fact that π is constant.) For the decrease of f_p (equivalently, of c/k), it is sufficient that

$$\alpha \leq \frac{(R_g - S)(1+\nu)}{\omega}, \tag{5.34}$$

[24] In fact, it can be shown that π is constant whenever h is separable.

a condition that we shall assume. Finally, we have to guarantee that f_g decreases, and that it is non-negative. For non-negativity, it is sufficient to suppose that

$$a_0 > \omega/S \tag{5.35}$$

which we shall also assume[25]. Now f_g is decreasing as a function of k if and only if it is increasing as a function of $z \equiv (1+bk)^{-\alpha}$; a few calculations now lead to the conclusion that the condition (5.34) will also ensure that f_g is decreasing.

We therefore have the required co-monotonicity to apply the Extension Theorem, and deduce that there does exist a production function consistent with the solution constructed here. It remains for us to check the transversality condition (GT), which concerns the growth of k_t^*. Since k^* solves the SDE

$$dk = \sigma k dw + (Qk + \nu^{-1}c)dt,$$

by the Yamada-Watanabe stochastic comparison theorem (see, for example, Rogers & Williams (2000), V.43), since $c \geq 0$ we can say that k is pathwise everywhere above the solution y to the SDE

$$dy = \sigma y dw + Qy dt.$$

This then bounds

$$\begin{aligned} e^{-\lambda_g t} k_t^* V'(k_t^*) &\leq e^{-\lambda_g t} y_t^{1-R_g} \\ &= \exp\{(1-R_g)\sigma w_t + (1-R_g)(Q - \frac{1}{2}\sigma^2)t - \lambda_g t\} \\ &= \exp\left\{ (1-R_g)\sigma w_t + \left(\frac{\lambda_g(R_g - S)}{S-1} - \frac{1}{2}\sigma^2(S-1)(R_g-1) \right) t \right\} \end{aligned}$$

using the particular expression for Q. In view of this, provided the inequality

$$\frac{\lambda_g(R_g - S)}{S-1} < \frac{1}{2}\sigma^2(S-1)(R_g-1) \tag{5.36}$$

is satisfied (which we shall assume), the transversality condition (GT) is satisfied.

5.3 Taxation and the private sector

The government's choice of taxes will depend on the private sector's preferences, which we here will assume are of the form

$$u(c, k_g, \pi) = -k_g{}^{-\omega_p} c^{-\nu_p} (1-\pi)^{-\kappa_p}, \tag{5.37}$$

where $\nu_p > 0$, $\omega_p \equiv R_p - 1 - \nu_p > 0$ and $\kappa_p > 0$. We modify the notation of the previous subsections by writing ω_g in place of ω, ν_g in place of ν and so on,

[25]Note that this imposes the condition $\omega \equiv R_g - 1 - \nu < S$.

to emphasise the distinction between government and private-sector parameters in what is an otherwise similar specification. With the private sector's felicity function specified as above conditions (PS2) and (PS3) from Theorem 2 combined with the very similar conditions (G2) and (G4) from Theorem 1 tell us that

$$\beta_c \beta_w = \frac{u_\pi U_c}{u_c U_\pi} = \frac{\kappa_p \nu_g}{\kappa_g \nu_p} \equiv K^{-1}, \tag{5.38}$$

say, a constant.

The equations (PS1)–(PS4) do not determine the tax rates uniquely; we could, for example, pick any nice enough ψ and then invert those four equations to find the β's. We therefore propose to study two possible approaches, applied to a few numerical examples.

We will consider the following parameter regimes for government:

Cautious government: $R_g = 4$, $S_g = 3$, $\lambda_g = 0.02$, $\sigma = 0.02$, $\delta = 0.1$, $\mu_L = 0.02$, $\mu_T = 0.03$, $\nu_g = 1.2$, $a_0 = 0.1$, $a_1 = 0.8$, $b = 1$, $\kappa_g = 0.2$, $\alpha = 0.9$;

Adventurous government: $R_g = 1.5$, $S_g = 1.4$, $\lambda_g = 0.2$, $\sigma = 0.1$, $\delta = 0.2$, $\mu_L = 0.05$, $\mu_T = 0.05$, $\nu_g = 0.2$, $a_0 = 0.3$, $a_1 = 0.65$, $b = 1$, $\kappa_g = 0.2$, $\alpha = 0.5$;

and the following parameter regimes for the private sector:

Cautious private sector: $R_p = 4$, $\nu_p = 2.5$, $\kappa_p = 0.06$, $\lambda_p = 0.04$;

Adventurous private sector: $R_p = 2$, $\nu_p = 0.5$, $\kappa_p = 0.06$, $\lambda_p = 0.14$;

The cautious government has a relatively high coefficient of relative risk aversion $R_g = 4$, and has quite a long mean look-ahead time in its objective (50 years). The volatility and mean of population growth are both quite small, as would be expected. The technology is also growing at a modest rate of 3%. When $k = 0$, the government holds 90% of all the capital, but as $k \to \infty$, this falls to 10%.

The adventurous government (perhaps in a developing nation) has quite small coefficient of relative risk aversion, and a short mean look-ahead time, 5 years (next election?!). The population is growing faster, with more volatility, and technological progress is also faster.

The cautious private sector is more risk-averse, and also has a longer mean look-ahead (25 years) than the adventurous private sector.

Approach 1: constant β_c, β_w. This approach is driven by the fact that the product $\beta_c\beta_w$ is constant. We use the relation (PS2) to determine ψ, and then the ODE (PS1) gives us β_k. We shall make the not unreasonable assumption that $\beta_k = \beta_r$ as the way to determine the short rate r. We present the plots of the invariant density (scaled to have height 1), β_k, and the short rate r for the four possible combinations of government and private sector.

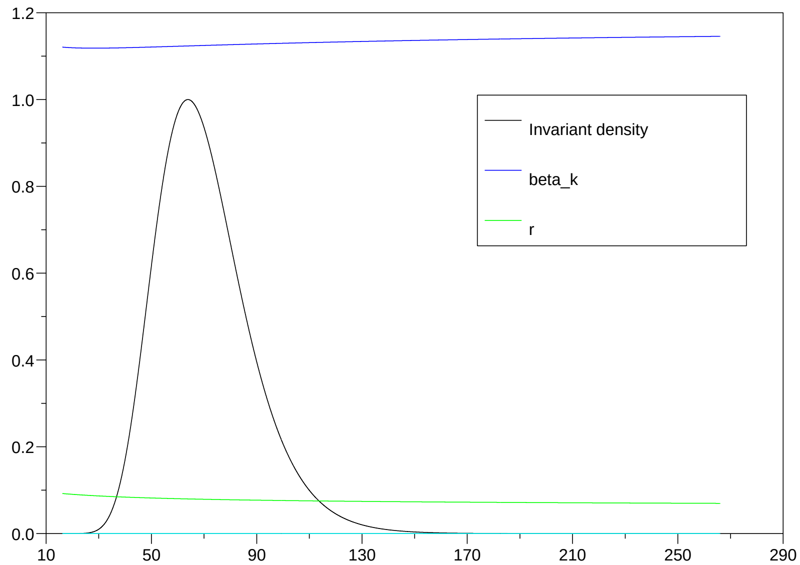

Figure 1: Invariant density, β_k, and r, cautious government, and cautious private sector (β_k above r, $\beta_c = \beta_w = 0.38$).

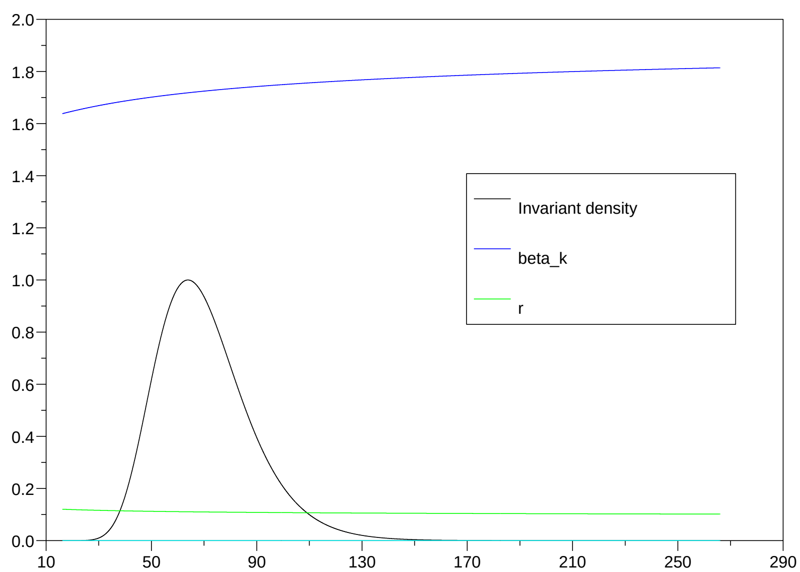

Figure 2: Invariant density, β_k, and r, cautious government, and adventurous private sector (β_k above r, $\beta_c = \beta_w = 0.849$).

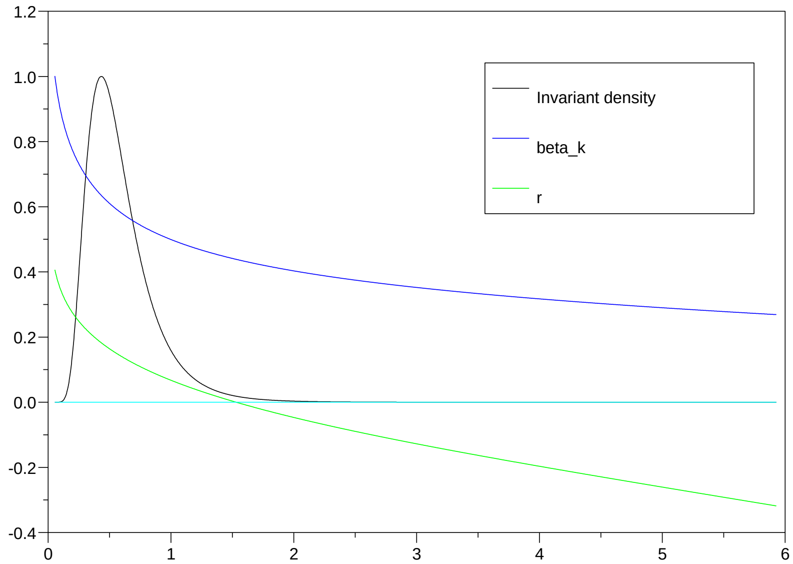

Figure 3: Invariant density, β_k, and r, adventurous government, and cautious private sector (β_k above r, $\beta_c = \beta_w = 0.155$).

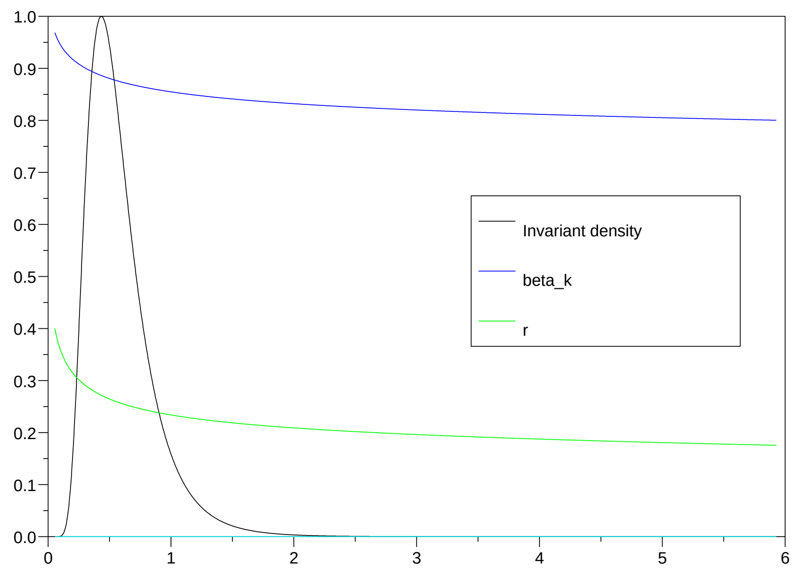

Figure 4: Invariant density, β_k, and r, adventurous government, and adventurous private sector (β_k above r, $\beta_c = \beta_w = 0.346$).

We see some very reasonable values for the tax rates and interest rates, reflecting the nature of the two participants in the economy. For example, the cautious government has to offer the cautious private sector a tax break on returns to capital in order to get the private sector to invest. The adventurous government must actually penalise the cautious private sector for investing in government debt if k is too high - there is too much capital around for the government's liking, and the private sector is to be induced to consume. In all four examples, the riskless rate falls as capital rises, which is intuitively reasonable; high rates of interest are needed at low k to get the private sector to invest so as to raise capital levels, but the need for this diminishes as k rises.

Approach 2: $\psi(k) \propto k^{-S_p}$. Looking at (5.24), we see that for small values of k we have $c(k) \sim k^{(S_g-\omega_g)/(1+\nu_g)}$, so a few calculations show that

$$u_c(k) = \beta_c^{-1}\psi(k) \sim k^{-S_p}$$

using (PS2). Here, the constant S_p satisfies

$$\frac{S_p - \omega_p}{1+\nu_p} = \frac{S_g - \omega_g}{1+\nu_g}.$$

We therefore propose to take $\psi(k) = a_\psi k^{-S_p}$ for all $k > 0$, and derive the forms of the βs from that.

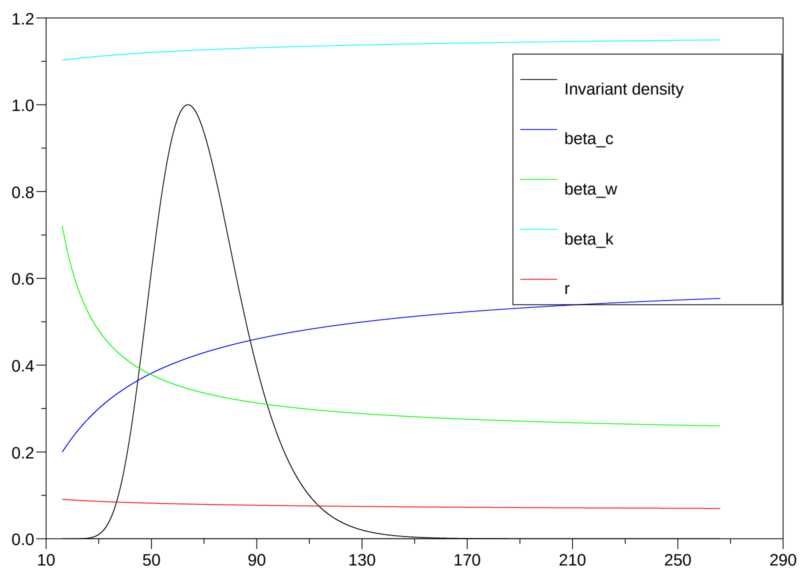

Figure 5: Invariant density, β_c, β_w, β_k, and r: cautious government, and cautious private sector. ($\beta_k > \beta_c > \beta_w > r$ on the right of the diagram)

Once again, we see tax rates that make a lot of sense: the cautious government facing the cautious private sector gives a subsidy on returns to capital, but gives a

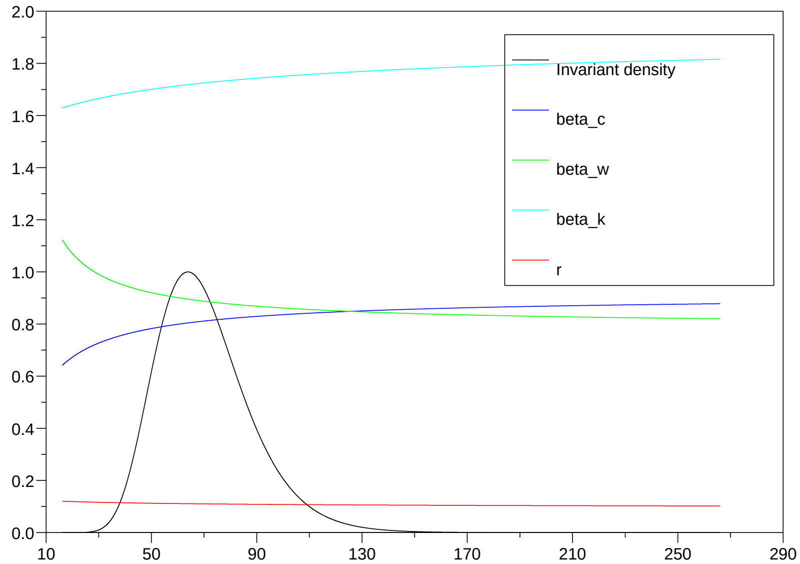

Figure 6: Invariant density, β_c, and β_w, β_k, and r: cautious government, and adventurous private sector. ($\beta_k > \beta_c > \beta_w > r$ on the right of the diagram)

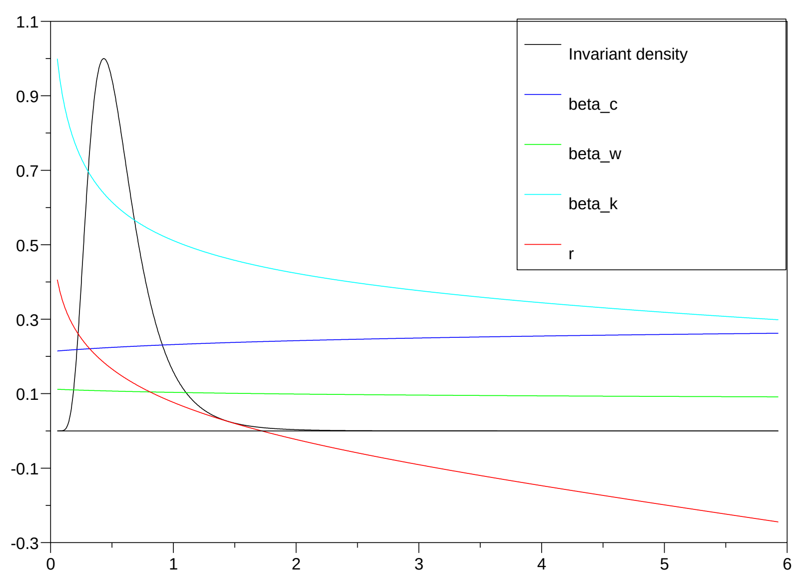

Figure 7: Invariant density, β_c, and β_w, β_k, and r: adventurous government, and cautious private sector. ($\beta_k > \beta_c > \beta_w > r$ on the right of the diagram)

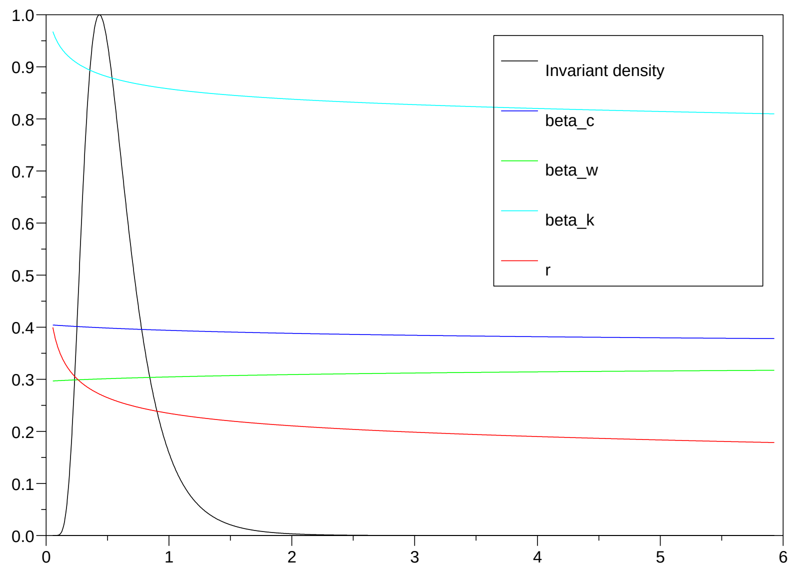

Figure 8: Invariant density, β_c, and β_w, β_k, and r: adventurous government, and adventurous private sector. ($\beta_k > \beta_c > \beta_w > r$ on the right of the diagram)

low rate of interest on riskless investment. The cautious government encouages the adventurous private sector by tax breaks on returns to invested capital, and lowered tax on consumption as k rises, but takes more of labour income in tax as k rises. The adventurous government faced with a cautious private sector gives negative rates of interest when k is high, but quite generous positive rates when k is low so as to encourage investment. In contrast, the rates of tax on consumption and on labour income remain almost constant; as a result, as would be expect then, Figures 3 and **??**-**??** look very similar. Similar comments apply to the case of adventurous government and adventurous private sector.

Let us briefly explore the implications of this model for the riskless rate. Under the assumption that $\beta_r = \beta_k$, we have that

$$r = F_p - \delta/\beta_k,$$

which in this case gives us explicitly that

$$r = F_p \frac{\gamma + \lambda_p - \delta + S_p(\sigma^2 + \tilde{\Phi}(k)/k) - \frac{1}{2}\sigma^2 S_p(1 + S_p)}{\gamma + \lambda_p + S_p(\sigma^2 + \tilde{\Phi}(k)/k) - \frac{1}{2}\sigma^2 S_p(1 + S_p)}. \tag{5.39}$$

The interest rate is thus expressed as a function of the underlying diffusion k, which itself solves the SDE

$$dk = -\sigma k dW + ((Q + \sigma^2)k + \nu_g^{-1}c(k)). \tag{5.40}$$

If we take φ to be constant, then c is proportional to k^B, where $B \equiv (S_g - \omega_g)/(1+\nu_g)$; the SDE can now be reduced to *linear* form by considering instead the variable $\zeta \equiv k^{1-B}$ which solves

$$d\zeta = \sigma(B-1)\zeta dW + (1-B)\big\{(Q + \sigma^2 - \frac{1}{2}\sigma^2 B)\zeta + a\nu_g^{-1}\big\}dt$$

Merton (1975) finds structurally similar interest rate processes in a study of a single-sector growth model, and Kloeden and Platen (1992) present this under the name of the stochastic Verhulst equation.

6 Conclusions.

We have introduced stochastic population fluctuations into the model of Arrow and Kurz (1970), which we have further modified by allowing the population to choose the proportion of its time to devote to working, as in the original model of Ramsey (1928). With these modifications we have then solved the government's central-planning problem. Under the assumption that tax rates are chosen so that the private sector, optimising its own utility functional, follows the optimal path of the government we have found tax and interest rates as functions of per-capita technology-adjusted capital - in other words, closed-loop control.

While it is not obvious whether the original problem can be solved in closed form if we assume that the production function is given, in a methodological innovation we have shown that the inverse problem, of finding a production function which gives rise to a particular (simple explicit) solution, may be solved in considerable generality. This opens the way to a host of explicit solutions, some of which we have begun to explore. The dependence of tax rates and interest rates on the state variable in the few examples we have studied takes on credible forms; there is scope for fitting the model to data, but such a study must wait til later.

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Appendix A Proofs.

Proof of Theorem 1. Suppose that the process k_t has dynamics given by (2.11) for some consumption process c_t and some choice $\theta_t = k_g(t)/k(t)$ of the proportion of capital held by the government. We define a P_g-Brownian motion w by $-z^L \equiv \sigma w$ and introduce a (Lagrangian) semimartingale $e^{-\lambda_g t}\Psi_t \equiv e^{-\lambda_g t}\Psi(k_t^*)$ where k^* is the conjectured optimal process, satisfying (2.20), and where

$$d\Psi_t \equiv \Psi_t(a_t dw + b_t dt).$$

We have for any stopping time τ that (omitting explicit appearance of t in most places)

$$\begin{aligned}\int_0^\tau e^{-\lambda_g t}U(c,k_g,\pi)dt &= \int_0^\tau e^{-\lambda_g t}\Big[U(c,k_g,\pi) + \Psi(F(k_p,k_g,\pi) - \gamma_g k - c) + \\ &\qquad k\Psi(b-\lambda_g) + \sigma a k \Psi\Big]\,dt + k_0\Psi_0 - e^{-\lambda_g \tau}k_\tau\Psi_\tau + M_\tau\end{aligned}$$

for some P_g-local martingale M; this is just obtained by integrating the process $e^{-\lambda_g t}\Psi_t k_t$ by parts. Taking a stopping time τ which reduces M strongly, we can now take expectations to obtain

$$\begin{aligned}E_g\int_0^\tau e^{-\lambda_g t}U(c,k_g,\pi)dt = E_g\int_0^\tau e^{-\lambda_g t}\Big[&U(c,k_g,\pi) + \Psi(F(k_p,k_g,\pi) - \gamma_g k - c) + \\ &k\Psi(b-\lambda_g) + \sigma a k\Psi\Big]\,dt + k_0\Psi_0 - E_g e^{-\lambda_g\tau}k_\tau\Psi_\tau.\end{aligned} \tag{A.1}$$

We now consider the maximisation over k, c, k_g and π of the integrand on the right-hand side of (A.1): the first-order conditions we obtain will be

$$\begin{aligned}\Psi(k^*)(F_p(k_p,k_g,\pi) - \gamma_g) &= (\lambda_g - b - a\sigma)\Psi(k^*) \\ U_c(c,k_g,\pi) &= \Psi(k^*) \\ U_g(c,k_g,\pi) &= \Psi(k^*)(F_p - F_g) \\ U_\pi(c,k_g,\pi) &= -\Psi(k^*)F_L.\end{aligned}$$

The last three of these are satisfied at $c = c^*(k^*)$, $k_g = k_g^*(k^*)$, $\pi = \pi^*(k^*)$ in view of (G2), (G4) and (G3). The first is satisfied due to (G1), since from the Itô expansion of $\Psi(k^*)$ we must have that

$$\begin{aligned} a &= \frac{\sigma k^* \Psi'(k^*)}{\Psi(k^*)} \\ b &= \frac{\Phi(k^*)\Psi'(k^*) + \frac{1}{2}\sigma^* k^{*2}\Psi''(k^*)}{\Psi(k^*)}. \end{aligned}$$

To summarise then: the integrand on the right-hand side of (A.1) is maximised at $c = c^*(k^*)$, $k_p = k_p^*(k^*)$, $k_g = k_g^*(k^*)$, $\pi = \pi^*(k^*)$. Reversing the integration-by-parts argument by which we arrived at (A.1), we conclude that[26]

$$\begin{aligned} E_g \int_0^\tau e^{-\lambda_g t} U(c, k_g, \pi)\, dt &\leq E_g \int_0^\tau e^{-\lambda_g t} U(c^*(k_t^*), k_g^*(k_t^*), \pi^*(k_t^*))\, dt \\ &\quad + E_g\big[\, e^{-\lambda_g \tau}(k_\tau^* - k_\tau)\Psi(k_\tau^*)\,\big] \\ &\leq E_g \int_0^\tau e^{-\lambda_g t} U(c^*(k_t^*), k_g^*(k_t^*), \pi^*(k_t^*))\, dt \\ &\quad + E_g\big[\, e^{-\lambda_g \tau} k_\tau^* \Psi(k_\tau^*)\,\big]. \end{aligned}$$

Now it only remains to let the reducing time τ tend to infinity, and appeal to the transversality condition (GT), together with the fact that U does not change sign to give us the required optimality result.

Finally, suppose that we take $V(k)$ given by

$$V(k) \equiv \int_1^k \Psi(y)\, dy + V_1$$

where

$$V_1 \equiv \frac{1}{\lambda_g}\left[\Psi(1)\Phi(1) + \frac{1}{2}\sigma^2\Psi'(1) + U(c^*(1), k_g^*(1), \pi^*(1))\right].$$

If we differentiate

$$-\lambda_g V(k) + V'(k)\Phi(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + U(c(k), k_g(k), \pi(k)) \qquad \text{(A.2)}$$

with respect to k, using the fact that $V'(k) = \Psi(k)$ we obtain

$$\begin{aligned} &\Psi(-\lambda_g + (1 - k_g')F_p + k_g' F_g + \pi' F_L - \gamma_g - c') \\ &\quad + \Psi'(\Phi + \sigma^2 k) + \frac{1}{2}\sigma^2 k^2 \Psi'' + c' U_c + k_g' U_g + \pi' U_\pi = 0 \end{aligned}$$

[26]There is a detail here: the stopping time τ which reduced M strongly may not reduce the corresponding local martingale for the optimal process. We can nevertheless replace τ by a stopping time which is no larger and which reduces both local martingales. Since we are interested in letting the reducing time tend to infinity, this little change affects nothing in the end.

by (G1)–(G3). Hence expression (A.2) is constant and this constant is 0 by the construction of V_1. □

PROOF OF THEOREM 2. The strategy is firstly to discover the dynamics faced by a single household optimising in an economy which is following the government's optimal path. Next we rework the private household's objective, expressing it in intensive variables. We then use the Lagrangian method to characterise the private household's optimal path,

So suppose we consider what happens if we add one more household to the (large) economy which is following the government's optimal path. The total labour available has increased by L_t/L_0, an $O(1)$ quantity, and the total amounts of both types of capital and of government debt will also have changed by an $O(1)$ quantity. If ΔC, ΔK_p, ΔD denote the changes in the corresponding aggregate quantities, and $\tilde{\pi}$ denote the proportion of effort which the new household devotes to production, then the perturbation of (3.7) to leading order is **not**

$$\begin{aligned} d\Delta K_p + d\Delta D &= \Delta K_p(\beta_k F_p - \delta)dt + r\beta_r \Delta D dt - \beta_c^{-1}\Delta C dt \\ &\quad + \beta_w \tilde{\pi}_t \frac{\eta_t}{L_0} F_L dt + \frac{\eta_t}{L_0} k_g(\beta_k \theta_p + \beta_w \theta_L) F_g dt. \end{aligned} \tag{A.3}$$

This is because if we consider the change in (3.7) when the new household joins, not only do the total amounts of capital, labour, consumption and debt change by the $O(1)$ amounts indicated in (A.3), *but the coefficients β. and the derivatives f. also get changed*, by amounts which are $O(1/L_0)$. Since these changes then get multiplied by quantities which are $O(L_0)$, the net impact on the budget equation of these changes is still $O(1)$. Nevertheless, *we argue that equation (A.3) is the correct equation for the evolution of the new household's wealth*, where the tax rates and all derivatives of f are evaluated along the *original* (government-optimal) path. This is because the quantities on the right-hand side of (A.3) are items directly visible to the new household: the return on *its* private capital, the wages for *its* labour, etc.. The other $O(1)$ changes in the budget equation, such as the changes in total wages due to the $O(1/L_0)$ shift in wage rates, get distributed among the population as a whole, and so have only an $O(1/L_0)$ effect on any one household.

This agreed, the problem facing the typical private sector household is to optimise the objective (3.1) with the dynamics given by (A.3), where the tax rates, the β., the f., r and f are all evaluated along the government's optimal path. As with the government's problem, we first reduce to technology-adjusted *per capita* variables, expressing the objective as

$$\begin{aligned} E\int_0^\infty e^{-\rho_p t} u\left(\frac{L_0 \Delta C_t}{L_t}, \frac{K_g(t)}{L_t}, \tilde{\pi}_t\right) dt, &= E\int_0^\infty e^{-\rho_p t} T_t^{1-R_p} u(c_t, k_g^*(t), \tilde{\pi}_t)\, dt \\ &= E\int_0^\infty e^{-\lambda_p t} u(c_t, k_g^*(t), \tilde{\pi}_t)\, dt, \end{aligned} \tag{A.4}$$

where we are reserving starred variables (k_p^*, k_g^*) for the government's optimal values, and are using the notations[27]

$$k_p \equiv \Delta K_p L_0/\eta, \quad \Delta_p \equiv \Delta D L_0/\eta, \quad c_t \equiv \Delta C_t L_0/\eta_t, \quad \lambda_p \equiv \rho_p - (1-R_p)\mu_T.$$

The dynamics (A.3) implies the following dynamics for the (technology-adjusted *per capita*) private-sector wealth process $x \equiv k_p + \Delta_p$:

$$\begin{aligned} dx &= k_p\big[\,\sigma dW + (\beta_k F_p - \gamma)dt\,\big] + \beta_w \tilde{\pi} F_L dt \\ &\qquad + \Delta_p\big[\,\sigma dW + (\delta + r\beta_r - \gamma)dt\,\big] - \beta_c^{-1} c dt + B dt, \\ &= x(\sigma dW + (\delta + r\beta_r - \gamma)dt) + (B + \beta_w \tilde{\pi} F_L - \beta_c^{-1} c)dt \\ &\equiv x(\sigma dW + (\delta + r\beta_r - \gamma)dt) + \tilde{B} dt \end{aligned} \tag{A.5}$$

where we have used (4.9) and the abbreviations $B = k_g^*(\beta_k\theta_p + \beta_w\theta_L)F_g$, $\tilde{B} = B + \beta_w\tilde{\pi}F_L - \beta_c^{-1}c$.

Let us now combine the objective (A.4) with the dynamics (A.5) using a Lagrangian process $e^{-\lambda_p t}\psi_t^* \equiv e^{-\lambda_p t}\psi(k_t^*)$, where by Itô's formula

$$d\psi^* = \psi^*\big[\,-a^* dZ + b^* dt\,\big], \tag{A.6}$$

using the notation $a_t^* = a(k_t^*)$, $b_t^* = b(k_t^*)$, and where

$$a(k) \;=\; k\psi'(k)/\psi(k) \tag{A.7}$$

$$b(k) \;=\; \frac{\frac{1}{2}\sigma^2 k^2 \psi''(k) + \psi'(k)\tilde{\Phi}(k)}{\psi(k)}. \tag{A.8}$$

Again omitting superfluous appearances of the time variable, integrating $xe^{-\lambda_p t}\psi^*$ by parts gives us for the Lagrangian

$$\begin{aligned} &\int_0^\tau e^{-\lambda_p t}\Big[u(c, k_g^*, \tilde{\pi}) + \psi^*\big\{x(\beta_k F_p - \gamma) + \tilde{B}\big\} + a^*\psi^* x\sigma^2 + x\psi^*(b^* - \lambda_p)\Big]\, dt \\ &\qquad\qquad + x_0\psi_0^* - x_\tau e^{-\lambda_p \tau}\psi_\tau^* + M_\tau \\ &= \int_0^\tau e^{-\lambda_p t}\Big[u(c, k_g^*, \tilde{\pi}) + \psi^* x\big\{\beta_k F_p - \gamma - \lambda_p + b^* + a^*\sigma^2\big\} + \psi^*\tilde{B}\Big]\, dt \\ &\qquad\qquad + x_0\psi_0^* - x_\tau e^{-\lambda_p \tau}\psi_\tau^* + M_\tau \\ &= \int_0^\tau e^{-\lambda_p t}\Big[u(c, k_g^*, \tilde{\pi}) + x\big\{\psi^*(\beta_k F_p - \gamma - \lambda_p) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) + \psi'(k^*)(\tilde{\Phi} + \sigma^2 k) \\ &\qquad\qquad + \psi^*\tilde{B}\Big]\, dt + x_0\psi_0^* - x_\tau e^{-\lambda_p \tau}\psi_\tau^* + M_\tau \end{aligned}$$

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[27] This notation conflicts slightly with the earlier use of c, k_p for the technology-adjusted *per capita* consumption C/η and private capital K_p/η. For the remainder of this proof, we shall treat c and k_p as local variables, distinct from those discussed earlier, and to be freely chosen by the private sector household. It will turn out in the end that the private sector will choose $c_t = c^*(k_t^*)$, $k_p(t) = k_p^*(k_t^*)$, of course.

where M is some continuous local martingale. Now because we are assuming that the conditions

$$\begin{aligned} 0 &= \psi^*(\beta_k F_p - \gamma - \lambda_p) + \psi'(k^*)(\tilde{\Phi} + \sigma^2 k) + \frac{1}{2}\sigma^2 k^2 \psi''(k^*) \\ u_c(c^*, k_g^*, \pi^*) &= \beta_c^{-1}\psi^* \\ u_\pi(c^*, k_g^*, \pi^*) &= -\beta_w F_L \psi^* \\ \beta_k F_p &= r\beta_r + \delta \end{aligned}$$

of Theorem 2 hold, and using the identity $\sigma^2 = \gamma_g - \gamma$, we deduce that

$$\begin{aligned} \int_0^\tau e^{-\lambda_p t} u(c, k_g^* \tilde{\pi})\, dt &\leq \int_0^\tau e^{-\lambda_p t}\big[u(c^*, k_g^*, \pi^*) + \psi^* \tilde{B}\big] dt + x_0 \psi_0^* - x_\tau e^{-\lambda_p \tau}\psi_\tau^* + M_\tau \\ &= \int_0^\tau e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + (x_\tau^* - x_\tau) e^{-\lambda_p \tau} \psi_\tau^* + \tilde{M}_\tau \\ &\leq \int_0^\tau e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + x_\tau^* e^{-\lambda_p \tau} \psi_\tau^* + \tilde{M}_\tau \end{aligned}$$

for some other local martingale $\tilde{M}$. Here, we obtained the second line by reversing the integration-by-parts used on the Lagrangian form. Taking expectations gives us that

$$E\int_0^\tau e^{-\lambda_p t} u(c, k_g^*, \tilde{\pi})\, dt \leq E\int_0^\tau e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt + E x_\tau^* e^{-\lambda_p \tau} \psi_\tau^*,$$

and the transversality condition (PST) allows us to let $\tau \to \infty$ to conclude that

$$E\int_0^\infty e^{-\lambda_p t} u(c, k_g^*, \tilde{\pi})\, dt \leq E\int_0^\infty e^{-\lambda_p t} u(c^*, k_g^*, \pi^*)\, dt$$

as required. □

Appendix B The wealth process x.

The dynamics

$$dk = \sigma k dW + \tilde{\Phi}(k) dt. \tag{B.1}$$

of k and the dynamics (A.5) of x

$$dx = x(\sigma dW + (\delta + r\beta_r - \gamma)dt) + \tilde{B} dt$$

allow us to develop x/k:

$$\begin{aligned} d\left(\frac{x}{k}\right) &= \frac{x}{k}\left\{\delta + r\beta_r - \gamma - \frac{\tilde{\Phi}(k)}{k}\right\} dt + \frac{\tilde{B}}{k}\, dt \\ &\equiv -\frac{x}{k}\, G_0(k)\, dt + \frac{\tilde{B}}{k}\, dt, \end{aligned} \tag{B.2}$$

where

$$\begin{aligned} G_0(k) &= k^{-1}\tilde{\Phi}(k) + \gamma - \beta_k F_p \\ &= k^{-1}\Phi(k) + \gamma_g - \beta_k F_p \\ &= (F(k_p^*, k_g^*, \pi^*) - c^*)/k - \beta_k F_p. \end{aligned}$$

The SDE (B.2) can be solved reasonably explicitly: for $s < t$,

$$x_t = k_t \left[e^{-\int_s^t G_0(k_u)du} \frac{x_s}{k_s} + \int_s^t e^{-\int_v^t G_0(k_u)du} \frac{\tilde{B}(k_v)}{k_v} \, dv \right] \tag{B.3}$$

Appendix C The one-sector problem

In the one-sector problem there is no distinction between public and private capital, and we can follow a similar development; or we may alternatively deduce the one-sector results as special cases of the two-sector results above. Either way, we will assume that the private sector works all the hours available to them ($\pi = 1$ in the previous notation) so that the rate of production is given simply by $F(K, LT) \equiv LTf(k)$ and the objective of the government is to maximise

$$E \int_0^\infty e^{-\rho_g t} L_t U\left(\frac{C_t}{L_t}\right) dt = L_0 E_g \int_0^\infty e^{-\lambda_g t} U(c_t) dt$$

where we use exactly the same notation as in the two-sector problem, and again assume that U is homogeneous of order $1 - R_g$. The optimality equations corresponding to those of Theorem 1 are

$$0 = U(c) - \lambda_g V(k) + \frac{1}{2}\sigma^2 k^2 V''(k) + \Phi(k)V'(k) \tag{C.1}$$

$$\Phi(k) = f(k) - \gamma_g k - c \tag{C.2}$$

$$U'(c) = V'(k). \tag{C.3}$$

We have assumed that U is homogeneous of order $1 - R_g$ so it must have the Constant Relative Risk Aversion (CRRA) form

$$U(c) = \frac{c^{1-R_g}}{1 - R_g},$$

with $R_g > 0$ and $R_g \neq 1$. Again it is possible to construct an explicit solution to the government's problem; choosing V, we find the optimal c from (C.3), then deduce Φ from (C.1), and then deduce f from (C.2). It remains only to check that the f so obtained is concave, increasing and non-negative.

As a simple example, if we pick a value function that is also CRRA

$$V(k) = \frac{A_V^{-R_g} k^{1-S}}{1 - S},$$

with $A_V > 0$, $S > 0$ and $S \neq 1$ then (C.3) gives us

$$c(k) = A_V k^{S/R_g}$$

and then (C.1) yields

$$\begin{aligned}\Phi(k) &= \left(\frac{\lambda_g}{1-S} + \frac{1}{2}\sigma^2 S\right) k - \frac{A_V k^{S/R_g}}{1-R_g} \\ &\equiv Qk - \frac{A_V k^{S/R_g}}{1-R_g}.\end{aligned}$$

Finally (C.2) gives

$$\begin{aligned}f(k) &= (\gamma_g + Q)k + \left(1 - \frac{1}{1-R_g}\right) c \\ &= (\gamma_g + Q)k - \frac{R_g A_V k^{S/R_g}}{1-R_g}.\end{aligned}$$

For these last two equations to make economic sense we require that

$$Q + \gamma_g \geq 0, \quad R_g > S > 1.$$

Appendix D Summary of notation

A t argument/subscript denotes a quantity at time t. Other subscripts are used to denote partial differentiation in the case of functions of two or more variables (e.g $f_g \equiv \partial f / \partial k_g$). Notation unique to the section on explicit solutions (Section 4) is not covered in this appendix.

C	Consumption rate
D	Level of government debt
I_g	Amount invested in government capital
I_p	Amount invested in private capital
K	Total capital
K_g	Government capital
K_p	Private sector capital
L	Labour force / population size
T	Technology level
X	Total private sector wealth $K_p + D$
c, k, k_g, k_p, x	$\equiv C/LT, K/LT, K_g/LT, K_p/LT, X/LT$

$c^*(k), k_g^*(k), \pi^*(k)$	Optimal values of c, k_g, π for a given k
$1 - \beta_c$	Tax rate on consumption
$1 - \beta_k$	Tax rate on returns on private capital
$1 - \beta_r$	Tax rate on returns on government debt
$1 - \beta_w$	Tax rate on wages
Δ_p	$\equiv D/\eta$
$\Delta C, \Delta D, \Delta K_p$	Per household rate of consumption, holding in government debt and amount of private capital
η	$\equiv LT$
ξ	$\equiv C/K_g \equiv c/k_g$
π	Proportion of time devoted to production
$F(K_p, K_g, \pi LT)$	Production (rate) function
$U(c, k_g, \pi)$	Government felicity function
$u(c, k_g, \pi)$	Private sector felicity function
$V(k)$	Government value function
$\Phi(k)$	$\equiv F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma_g k - c^*(k)$. The drift in k along the optimal path under P_g
$\tilde{\Phi}(k)$	$\equiv F(k_p^*(k), k_g^*(k), \pi^*(k)) - \gamma k - c^*(k)$. The drift in k along the optimal path under P
ψ	The Lagrange multiplier process corresponding to the private sector's optimization problem.
E, E_g	Expectation taken under P, P_g respectively
P	Real world probability measure
P_g	Government's valuation measure
R_g	U is homogeneous of order $1 - R_g$ in c,k_g
R_p	u is homogeneous of order $1 - R_p$ in c,k_g
δ	Rate of depreciation of capital
γ	$\equiv \delta + \mu_L + \mu_T + -\sigma^2$
γ_g	$\equiv \gamma + \sigma^2$
θ_p, θ_L	Proportion of return on government's capital included in returns to private sector capital and labour respectively
λ_g	$\equiv \rho_g - (1 - R_g)\mu_T - \mu_L$
λ_p	$\equiv \rho_p - (1 - R_p)\mu_T$
μ_L	Exponential drift term of labour
μ_T	Exponential growth rate of technology level
ρ_g, ρ_p	Government and private sector utility time-discount factors