The Maximum Maximum of a Martingale Constrained by an Intermediate Law.

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Abstract

Let (M_t) be any martingale with $M_0 \equiv 0$, an intermediate law $M_1 \sim \mu_1$, and terminal law $M_2 \sim \mu_2$, and let $\bar{M}_2 \equiv \sup_{0 \le t \le 2} M_t$. In this paper we prove that there exists an upper bound, with respect to stochastic ordering of probability measures, on the law of $\bar{M}_2$. We construct, using excursion theory, a martingale which attains this maximum. Finally we apply this result to the robust hedging of a lookback option.

1 Introduction

Suppose that $(M_t)_{t \ge 0}$ is a martingale and $\bar{M}_t = \sup_{0 \le s \le t} M_s$, its supremum process. Let μ_1 denote the law of M_1 and ν_1 the law of the maximum at time 1. The relationship between the law of a martingale and its maximum have been studied by a number of authors. Given only a law, μ_1, and no fixed initial law, Blackwell and Dubins [3] and Dubins and Gilat [5] showed that $\mu_1 \preceq \nu_1 \preceq \mu_1^*$, where $\preceq$ denotes stochastic ordering ($\rho \preceq \pi$ if and only if $F_\rho(x) \ge F_\pi(x)$ $\forall x$ with the obvious notational convention) and μ_1^* is the Hardy transform of μ_1. The converse was shown by Kertz and Rösler [10]; that if ρ is a probability measure such that $\mu_1 \preceq \rho \preceq \mu_1^*$

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then there exists a martingale, M, with time 1 distribution given by μ_1 and time one maximum given by ρ. A classification of all possible joint laws of $(M_1, \bar{M}_1)$ was given by Rogers [13], along with a method of constructing such martingales using excursion methods.

If we fix the initial law, μ_0, as well as the time one law, μ_1 then provided they have the same finite mean, a necessary condition for the existence of a martingale with those distributions is

$$\int_x^\infty (y-x)\, d\mu_0(y) \le \int_x^\infty (y-x)\, d\mu_1(y) \quad \forall x \tag{1}$$

which can be seen by Jensen's inequality. This is also a sufficient condition, see Strassen [16], or Meyer [11] for a proof.

Given initial and terminal laws μ_0, μ_1 Hobson [8] gave an explicit description of the stochastic upper bound of the law of the maximum ν_1, and a method of constructing a martingale which achieves this maximum. This paper deals with an extension of these results. We now consider the case where we specify a martingale starting at the origin, with a given terminal law and also some fixed law at an intermediate time. For simplicity we shall assume that we are given the laws at times zero, one and two. Suppose μ_1, μ_2 are zero mean probability measures. We denote the family of martingales, $(M_t)_{0\le t\le 2}$, with $M_0 \equiv 0$, the law at time one given by μ_1 and terminal law μ_2, by $\mathcal{M}(\mu_1, \mu_2)$. This set will be non-empty if and only if

$$0 \le \int_x^\infty (y-x)\, d\mu_1(y) \le \int_x^\infty (y-x)\, d\mu_2(y) \ \ \forall x\,.$$

Let ν_2 denote the law of the terminal maximum $\bar{M}_2$, and $\mathcal{P}(\mu_1, \mu_2) := \{\nu_2 | M \in \mathcal{M}(\mu_1, \mu_2)\}$. We construct an element, $M^* \in \mathcal{M}(\mu_1, \mu_2)$ using excursion theory whose maximal law ν_2^* stochastically dominates any $\nu \in \mathcal{P}$. This construction is an extension of the Azema-Yor Skorohod embedding. In some cases the imposition of an intermediate law will have no effect, the maximum maximum is the same as for a martingale with the same initial and terminal laws, however this will not always be the case.

2 A proof that the maximum maximum is attained by the Azema-Yor construction.

Let $\mathcal{M}(\mu_1)$ denote the family of martingales $(M_t)_{0\le t\le 1}$ for which $M_0 = 0$ and the terminal law is μ_1, where μ_1 is centred e.g. $\int |x|\, d\mu_1(x) < \infty, \int x\, d\mu_1(x) = 0$. We will give an outline of the Azema-Yor construction and a new (but long) proof that this construction gives a martingale whose law of the maximum dominates the law of the maximum for any other element of $\mathcal{M}(\mu_1)$. The purpose of this section is to introduce in this simple setting the main ideas and some of the results which will be used in proving our main theorem.

2.1 The Azema-Yor proof of the Skorohod embedding.

First we define the barycentre by

$$b(x) = \left(\int_{[x,\infty)} z \, d\mu_1(z) \right) / \left(\int_{[x,\infty)} d\mu_1(z) \right) \tag{2}$$

for all x such that $\mu_1([x,\infty)\,) > 0$, else $b(x) = x$. Then b is positive, non-decreasing, and if μ_1 has no atoms it is continuous. Define ξ to be the right inverse of the barycentre.

Let B be a Brownian motion with supremum process $\bar{B}$. Define a stopping time by $\tau = \inf\{t \mid B_t \le \xi(\bar{B}_t)\}$; then $\tau < \infty$ almost surely. We can imagine plotting the path of $(B_t, \bar{B}_t)$,

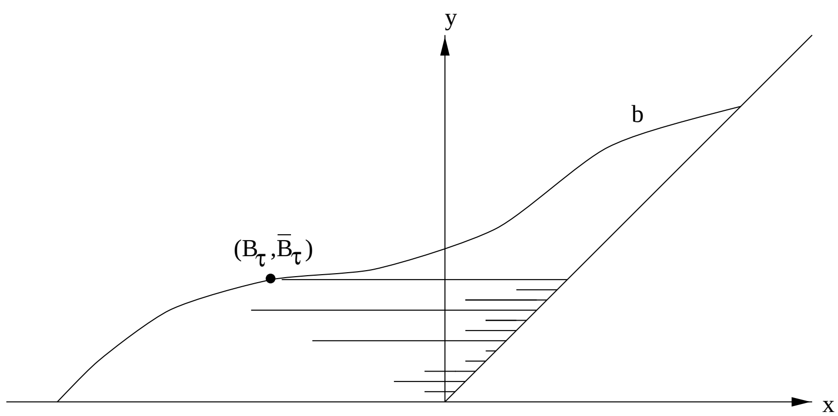

Figure 1: Stop once the excursions cross b

running B_t along the x-axis, at a height on the y-axis corresponding to $\bar{B}_t$. We stop the first time an excursion away from the maximum crosses ξ, as illustrated in Figure 1. Recall the excursion theory result that if the Brownian motion has reached a new maximum at s then the probability that it will achieve a maximum greater than y is equal to $\exp(-\int_{(s,y]} 1/(r - \xi_r)\, dr)$, see for example [14]. It can be shown from this that B_τ has law μ_1 and so we can define M^* as a time change of $B_{t\wedge\tau}$ by

$$M^*_t = B_{(\frac{t}{1-t})\wedge\tau} \qquad t \le 1\,.$$

The assumption that μ_1 is centred is necessary to prove that M^* is a true martingale and not just a local martingale. Then $M^* \in \mathcal{M}(\mu_1)$ and note that we have the following important relationship between maximum and position when μ_1 does not have an atom at ξ_y

$$\bar{M}_1^* \geq y \text{ if and only if } M_1^* \geq \xi(y) \tag{3}$$

and for general μ_1 the sets $\{\bar{M}_1^* \geq y\}$ and $\{M_1^* \geq \xi(y), \bar{M}_1^* \geq y\}$ are equal. With the possible exception of the supremum of the support of μ_1 (if this is finite) the law of $\bar{M}_1$ contains no atoms even if μ_1 does. This is a brief outline of the Azema-Yor construction; for a thorough treatment see [2] or for an excursion theoretic approach [12].

2.2 A martingale inequality.

For any $t > 0$ we define

$$c_t(\xi) = I\!E((M_t - \xi)^+)$$

where $(M_t - \xi)^+$ denotes the positive part of $M_t - \xi$. For fixed $t > 0$, c_t is positive, decreasing and convex. We now introduce a useful martingale inequality, originally given in a different form in [3], relating the laws of the position and maximum.

Lemma 2.1 *If $(M)_{0\leq t\leq\infty}$ is a right-continuous martingale then for all $t > 0$*

$$I\!P(\bar{M}_t \geq y) \leq \inf_{\zeta<y} \frac{c_t(\zeta)}{y - \zeta}\,. \tag{4}$$

Proof: Fix some $\zeta < y$ and we define the random variable Y as follows

$$Y = \frac{(M_t - \zeta)^+}{y - \zeta} + \frac{y - M_t}{y - \zeta} 1_{(\bar{M}_t \geq y)} = \frac{(M_t - \zeta)^+}{y - \zeta} + \left(1 - \frac{M_t - \zeta}{y - \zeta}\right) 1_{(\bar{M}_t \geq y)}\,.$$

Since M is right continuous we can apply Doob's inequality, hence $I\!E((y - M_t); \bar{M}_t \geq y) \leq 0$, and so $I\!E(Y) \leq c_t(\zeta)/(y - \zeta)$. Also it is straightforward to see that $Y \geq 1_{(\bar{M}_t \geq y)}$, and by taking expectations

$$I\!P(\bar{M}_t \geq y) \leq \frac{c_t(\zeta)}{y - \zeta}.$$

Since this must hold for all $\zeta < y$ the result follows. **Q.E.D.**

2.3 The maximum maximum is attained.

We wish to show that the law of the maximum of the martingale constructed by Azema and Yor, M^*, dominates the law of the maximum of any $M \in \mathcal{M}(\mu_1)$.

Lemma 2.2 *Let $M \in \mathcal{M}(\mu_1)$, then*

$$I\!P(\bar{M}_1^* \geq y) \geq I\!P(\bar{M}_1 \geq y) \quad \forall y > 0\,. \tag{5}$$

Proof: We will prove this by showing that the inequality (4), when $t = 1$, is attained by M^* for all $y > 0$. First assume that μ_1 has no atoms. Fix $y > 0$ and denote $\xi(y)$ by ξ. Doob's inequality becomes exact for continuous martingales and so from (3)

$$\int_\xi^\infty (x - y)\, d\mu_1(x) = E(M_1^* - y; M_1^* \geq \xi) = E(M_1^* - y; \bar{M}_1^* \geq y) = 0 \ ;$$

thus

$$\begin{aligned} P(\bar{M}_1^* \geq y) = P(M_1^* \geq \xi) &= \int_\xi^\infty d\mu_1(x) \\ &= \int_\xi^\infty \frac{y-\xi}{y-\xi}\, d\mu_1(x) + \int_\xi^\infty \frac{x-y}{y-\xi}\, d\mu_1(x) = \frac{c_1(\xi)}{y-\xi} \ . \end{aligned}$$

Hence $P(\bar{M}_1^* \geq y) = c_1(\xi)/(y-\xi)$, which must be the infimum else we contradict (4). This holds for all y and (5) follows if M is right continuous. However since the martingale which attains the maximum maximum must be continuous (5) will still hold for general M.

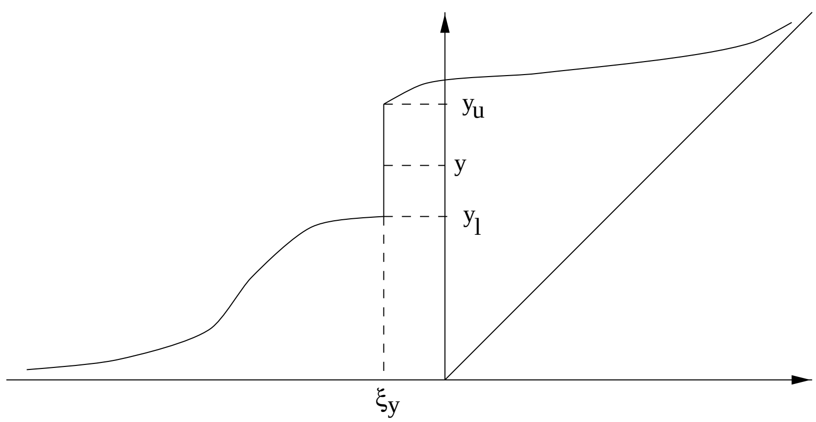

Figure 2: A barycentre with an atom at ξ_y

If μ_1 has atoms then the barycentre contains jumps and so ξ_y is constant over certain intervals, see Figure 2. Suppose that ξ is constant over the interval (y_l, y_u). By the previous argument, since $\{\bar{M}_1^* \geq y_l\} = \{M_1^* \geq \xi_y\}$, it is clear that $P(\bar{M}_1^* \geq y_l) = c(\xi)/(y_l - \xi)$ and so

$$\begin{aligned} P(\bar{M}_1^* \geq y) &= P(\bar{M}_1^* \geq y_l).P_{y_l}(H_y < H_\xi) \\ &= \frac{c(\xi)}{(y_l - \xi)}.\frac{(y_l - \xi)}{y - \xi} \quad = \frac{c(\xi)}{(y-\xi)} \ . \end{aligned}$$

Here $P_{y_l}(H_y < H_\xi)$ denotes the probability that a Brownian motion started at y_l will hit y before it hits ξ. So again the inequality (4) is attained and so (5) follows. **Q.E.D.**

Remark 2.3 We have shown that if ξ is the (left) inverse of the barycentre then for every y, ξ_y minimises the function $c_1(\xi)/(y-\xi)$.

Remark 2.4 This proof requires only the characterisation given earlier that, for all $y > 0$ if ξ_y is not an atom of μ_1, then $(\bar{M}_1^* \geq y) \equiv (M_1^* \geq \xi_y)$.

Remark 2.5 Instead of starting with the barycentre we could start our analysis by defining ξ_y to minimise $c_1(\xi)/(y-\xi)$. Since c_1 is convex it follows that ξ_y must be the point where the supporting tangent to c_1 which meets the x-axis at y, touches c_1 (see Figure 3). While ξ_y

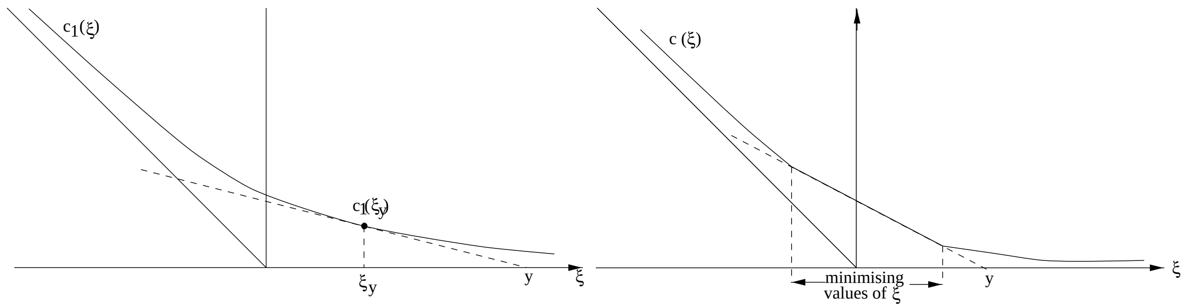

Figure 3: Defining ξ_y to minimise $c_1(\xi)/(y-\xi)$

may not be unique, if there is more than one value where the minimisation occurs it is clear from the convexity of c_1 that there must be a closed interval of minimising values. Similarly if $y_1 < y_2$ any value where $c_1(\xi)/(y_1-\xi)$ is minimised must be less than any possible minimising value of $c_1(\xi)/(y_2-\xi)$. Thus if we insist upon some additional regularity e.g. left continuity we can define ξ as follows: ξ_y is the unique left continuous function such that for each $y \geq 0$, ξ_y minimises $c_1(\xi)/(y-\xi)$ over all $\xi < y$. We now summarise this in a lemma.

Lemma 2.6 *If we define for $y > 0$*

$$k_y \equiv \inf_{x<y} \frac{c_1(x)}{y-x}$$

then this infimum is attained when $x = \xi_1(y)$ and

$$k_y = I\!P(\bar{M}_1^* \geq y) = \exp\left(-\int_0^y \frac{ds}{s-\xi_1(s)}\right). \tag{6}$$

From (6) it is clear that k_y has well defined left and right derivatives everywhere, and is almost surely differentiable. Where it exists $k'_y = -k_y/(y-\xi_1(y))$.

2.4 An example.

We now discuss a simple example to show how the introduction of an intermediate law can affect the law of the maximum maximum.

Let $\mu_0 = \delta_0$, and the distribution of M_1 be uniform on $[-1, +1]$. Let μ_2 have atoms at $-3/2, 0, 3/2$ with weights α, β, α respectively. For μ_2 to be an admissible measure for M_2 we must have $\alpha \geq 1/6$. Given this restriction there are two distinct cases, first $1/2 \geq \alpha \geq 1/4$ in which case the time one and time two barycentres do not overlap, and second $1/4 > \alpha \geq 1/6$ where an overlap exists.

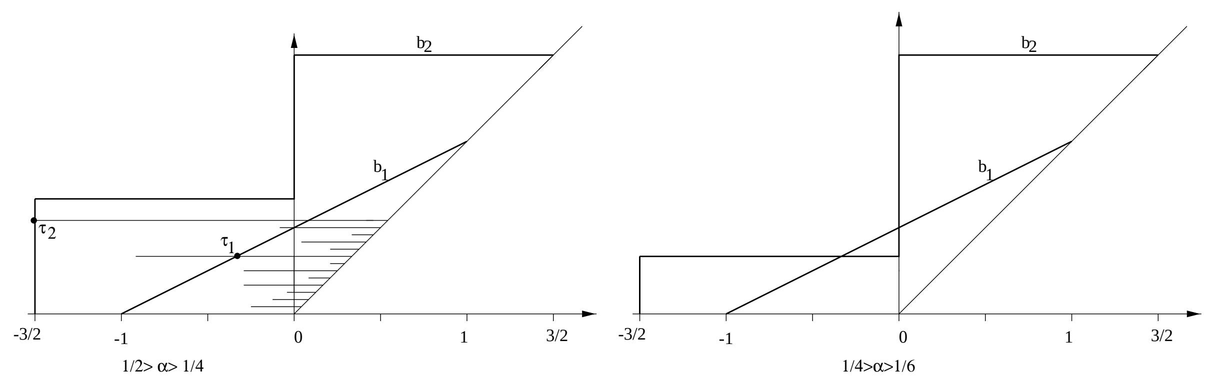

Figure 4: Barycentres b_1, b_2 with and without an overlap.

In the case where the barycentres do not overlap the constraint of the time one law has no effect. We can achieve the maximum maximum for a martingale started at time zero at the origin and at time two with law μ_2. We can simply adapt the Azema-Yor construction as follows. Let B be a Brownian motion and $\bar{B}_t$ its maximum, then we define the stopping times

$$\tau_1 = \inf\{t \mid B_t \leq b_1^{-1}(\bar{B}_t)\}\ , \qquad \tau_2 = \inf\{t \mid B_t \leq b_2^{-1}(\bar{B}_t)\}\,.$$

Plotting B_t along the x-axis and its current maximum $\bar{B}_t$ on the y-axis, we stop first when we cross b_1 and secondly when we cross b_2. Clearly $\tau_1 \leq \tau_2$ and from the Azema-Yor embedding we know that B_{τ_1} has law μ_1, and B_{τ_2} has law μ_2. A time change leads to a construction of a martingale which attains the maximum maximum.

In general, if the barycentres do not overlap we can always use the above construction. This simple example however demonstrates that it is possible for barycentres to overlap, in which case a more sophisticated approach is required.

3 An intermediate law

We again assume that we have two centred probability measures, μ_1, μ_2 such that the space of martingales, $\mathcal{M}(\mu_1, \mu_2)$ is non-empty. We shall introduce a martingale inequality which generalises that in Lemma 2.1 by relating the laws of position of a martingale at times one and two, to the law of its maximum at time two. We will use this to define two functions akin to the barycentre which will be used in an extension of the Azema-Yor Skorohod embedding. We then show that the martingale inequality becomes exact for this martingale and finally we show that the constructed martingale has the given laws at times one and two. Thus we will have shown by construction the existence of an element of $\mathcal{M}(\mu_1,\mu_2)$ whose maximum at time two has a stochastically maximal law.

3.1 A Martingale Inequality.

Lemma 3.1 *If $M\in\mathcal{M}(\mu_1,\mu_2)$ and is right continuous then for any $y>0$*

$$\mathbb{P}(\bar{M}_2\geq y)\leq \inf_{\zeta_1<y,\zeta_2<y}\left(\frac{c_2(\zeta_2)}{y-\zeta_2}-1_{\{\zeta_2>\zeta_1\}}(\frac{c_1(\zeta_2)}{y-\zeta_2}-\frac{c_1(\zeta_1)}{y-\zeta_1})\right). \tag{7}$$

Proof: First fix $y>0$, $\zeta_1<y$, and $\zeta_2<y$. If $\zeta_1\geq\zeta_2$ then we wish to show that

$$\mathbb{P}(\bar{M}_2\geq y)\leq\frac{c_2(\zeta_2)}{y-\zeta_2}$$

which follows from Lemma 2.1. Now consider $\zeta_1<\zeta_2$, then we claim that the following inequality holds:

$$\begin{aligned}1_{\{\bar{M}_2\geq y\}}\leq&\frac{(M_2-\zeta_2)^+}{y-\zeta_2}+\frac{(M_1-\zeta_1)^+}{y-\zeta_1}-\frac{(M_1-\zeta_2)^+}{y-\zeta_2}\\&+1_{\{\bar{M}_1\geq y\}}\frac{y-M_1}{y-\zeta_1}+1_{\{\bar{M}_1\geq y,M_1\geq\zeta_2\}}\frac{M_1-M_2}{y-\zeta_2}+1_{\{\bar{M}_1<y,\bar{M}_2\geq y\}}\frac{y-M_2}{y-\zeta_2}\end{aligned} \tag{8}$$

this is easy to check on a case by case basis. Note that by Doobs inequality

$$\mathbb{E}\left(\frac{y-M_1}{y-\zeta_1};\bar{M}_1\geq y\right)=0\quad,\quad\mathbb{E}\left(\frac{y-M_2}{y-\zeta_2};\bar{M}_1<y,\bar{M}_2\geq y\right)=0$$

and since M is a martingale by taking expectations in (8) we obtain

$$\mathbb{P}(\bar{M}_2\geq y)\leq\frac{c_2(\zeta_2)}{y-\zeta_2}-\frac{c_1(\zeta_2)}{y-\zeta_2}+\frac{c_1(\zeta_1)}{y-\zeta_1}.$$

Thus since y,ζ_1, and ζ_2 are arbitrary the result follows. **Q.E.D.**

3.2 Two Functions

We now use the above inequality to define two functions which will play a role akin to the barycentre in the Azema-Yor embedding. We wish to find two functions $\xi_1(y),\xi_2(y)$ such that for any $y>0$ choosing $\zeta_1=\xi_1(y)$ and $\zeta_2=\xi_2(y)$ will minimise (7). First we show that ξ_1 can be taken to equal the inverse of the barycentre, b_1, of μ_1.

Consider $y > 0$ fixed. Suppose $\xi_1(y), \xi_2(y)$ are values of ζ_1, ζ_2 where (7) is minimised. Then if $\xi_2(y) > b_1^{-1}(y)$ we may take $\xi_1(y) = b_1^{-1}(y)$ since $\zeta \to c_1(\zeta)/(y-\zeta)$ is decreasing to the left of $b_1^{-1}(y)$ and increasing to the right, by the convexity of c_1. If $\xi_2(y) \le b_1^{-1}(y)$ then we may choose any $\xi_1(y) > \xi_2(y)$ and so we choose $\xi_1(y) = b_1^{-1}(y)$. We consider the simpler, but equivalent problem; to choose $\xi_2(y)$ as the value of ζ_2 which minimises

$$\frac{c_2(\zeta_2)}{y-\zeta_2} - 1_{\{\zeta_2 > \xi_1(y)\}} \left[\frac{c_1(\zeta_2)}{y-\zeta_2} - \frac{c_1(\xi_1(y))}{y-\xi_1(y)} \right]. \tag{9}$$

This will not necessarily be unique and more will be said about this later, but for now choose any $\xi_2(y)$ to be a value of $\zeta_2 < y$ for which (9) is minimised.

Lemma 3.2 *ξ_2 is increasing in y.*

Proof: Fix $y > 0$ and define $l_0(x)$ to be the tangent to c_1 at $\xi_1(y)$ passing through y. So

$$l_0(x) \equiv \frac{(y-x)}{y-\xi_1(y)} c_1(\xi_1(y)).$$

Thus

$$c_1(x,y) = (c_1(x) - l_0(x))\, 1_{\{x > \xi_1(y)\}}$$

is non-negative, increasing and convex (Figure 5). Recall the definition of k_y from Lemma 2.6, then

$$c_1(x,y) = (c_1(x) - (y-x)k_y)\, 1_{\{x > \xi_1(y)\}}.$$

We define $c(x,y) = c_2(x) - c_1(x,y)$ so that the minimisation of (9) is the minimisation of

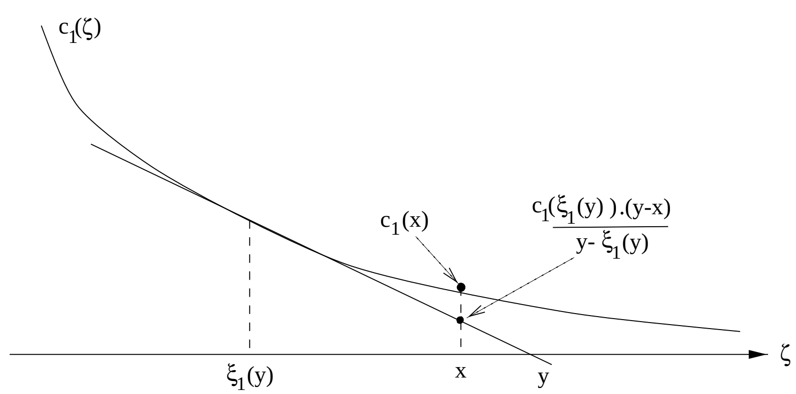

Figure 5: If $x > \xi_1(y)$ then $c_1(x,y)$ is equal to the difference between the two lines.

$c(\zeta,y)/(y-\zeta)$ over ζ. The supporting tangent to $c(\cdot,y)$ which passes through the x-axis at y must meet the curve at $\xi_2(y)$. Fix $\delta > 0$. First we consider the case when $\xi_1(y) > \xi_2(y)$ then

$c(x, y+\delta) = c(x, y)$ for all $x < \xi_2(y)$ and so clearly $\xi_2(y+\delta)$ cannot be smaller than $\xi_2(y)$. We now consider the remaining cases: $\xi_1(y) \le \xi_2(y)$.

Let $l_1(x)$ be a supporting tangent to $c(\cdot, y)$ at $\xi_2(y)$,

$$l_1(x) = c(\xi_2(y), y) + [x - \xi_2(y)](K_1 - k_y)$$

where K_1 lies between the left and right derivatives of $c_2(\cdot) - c_1(\cdot)$ at $\xi_2(y)$. Since l_1 passes through the x axis at y we can rewrite l_1 as

$$l_1(x) = \frac{y - x}{y - \xi_2(y)} c(\xi_2(y), y)$$

which implies that

$$K_1 - k_y = -\frac{c(\xi_2(y), y)}{y - \xi_2(y)} = -\frac{c_2(\xi_2(y)) - c_1(\xi_2(y))}{y - \xi_2(y)} - k_y\,;$$

thus $K_1 \le 0$. Since $l_1(y) = 0$, it is clear that $l_1(y+\delta) = \delta(K_1 - k_y)$.

Now consider $c(\cdot, y+\delta) - c(\cdot, y) = c_1(\cdot, y) - c_1(\cdot, y+\delta)$, which is non-decreasing, non-negative and convex. In fact it is zero for $x < \xi_1$, convex between $\xi_1(y)$ and $\xi_1(y+\delta)$, and a straight line for $x > \xi_1(y+\delta)$ (Figure 6). Let l_2 denote a supporting tangent to $c(\cdot, y+\delta) - c(\cdot, y)$ at $\xi_2(y)$. Since $c(\cdot, y+\delta) - c(\cdot, y)$ is convex, $l_1 + l_2$ must be a supporting tangent to $c(\cdot, y+\delta)$ at $\xi_2(y)$. We shall show that $(l_1 + l_2)(y+\delta) \le 0$ which will imply that $\xi_2(y+\delta) \ge \xi_2(y)$.

Suppose now that $\xi_1(y) \le \xi_1(y+\delta) \le \xi_2(y)$. Consider

$$l_2(x) = c(\xi_2(y), y+\delta) - c(\xi_2(y), y) + [x - \xi_2(y)](k_y - k_{y+\delta})$$

then $l_2(x)$ is a supporting tangent to $c(\cdot, y+\delta) - c(\cdot, y)$ at $\xi_2(y)$. Now

$$l_1(y+\delta) + l_2(y+\delta) = \delta[K_1 - k_y] + c_1(\xi_2(y), y) - c_1(\xi_2(y), y+\delta) + [y+\delta-\xi_2(y)](k_y - k_{y+\delta}) \quad (10)$$

and since $\xi_2(y) > \xi_1(y+\delta)$

$$c_1(\xi_2(y), y) - c_1(\xi_2(y), y+\delta) = -[y - \xi_2(y)](k_y - k_{y+\delta}) + \delta k_{y+\delta}\,,$$

and so from (10) we see $l_1(y+\delta) + l_2(y+\delta) = \delta K_1 \le 0$.

Conversely suppose $\xi_1(y) \le \xi_2(y) < \xi_1(y+\delta)$ then since ξ_1 is increasing it follows that $y \le \alpha := b_1(\xi_2(y)) \le y + \delta$, where $b_1 = \xi_1^{-1}$ is the barycentre of μ_1. We define

$$l_2(x) = c_1(\xi_2(y)) - [y - \xi_2(y)]k_y + [x - \xi_2(y)](k_y - k_\alpha) \quad (11)$$

then l_2 is a supporting tangent to $c(\cdot, y+\delta) - c(\cdot, y)$ at $\xi_2(y)$. Hence

$$\begin{aligned}(l_1+l_2)(y+\delta) &= \delta[K_1 - k_y] + c_1(\xi_2(y)) - [y - \xi_2(y)]k_y + [y+\delta-\xi_2(y)](k_y - k_\alpha) \\ &= \delta K_1 + c_1(\xi_2(y)) - [\alpha - \xi_2(y)]k_\alpha - [y+\delta-\alpha]k_\alpha .\end{aligned}$$

By definition $k_\alpha = c_1(\xi_2(y))/[\alpha - \xi_2(y)]$ and so again $(l_1+l_2)(y+\delta) \le 0$. **Q.E.D.**

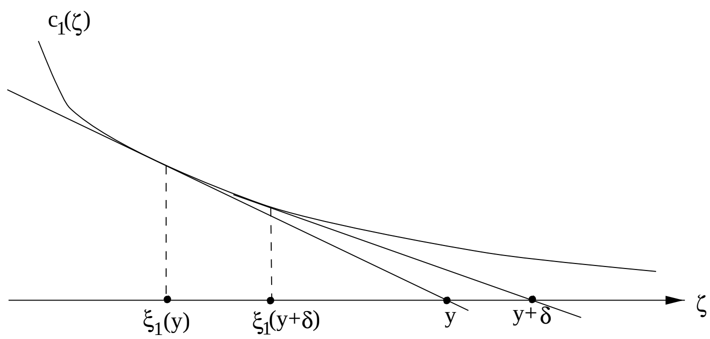

Figure 6: $c_1(x,y) - c_1(x, y+\delta)$ is equal to $c_1(x,y)$ for $x < \xi_1(y+\delta)$ and the difference between the two straight lines otherwise.

Definition For all $y > 0$ we define $\xi_1(y)$ to be the inverse of the barycentre of μ_1 and $\xi_2(y)$ to be the largest value of ζ_2 such that (9) is minimised. Then ξ_1, ξ_2 are increasing functions of y.

3.3 An extension of the Azema-Yor Skorohod Embedding.

We shall denote the martingale constructed here by M^* throughout the rest of this paper. Remember that $\xi_1(y) = b_1^{-1}(y)$, is the inverse of the barycentre for μ_1. Letting this guide our intuition we first use the Azema-Yor construction to embed the law μ_1 at time one. Between time one and time two there are three different ways to construct the martingale depending upon the starting point $(M_1^*, \bar{M}_1^*)$.

First if $M_1^* > \xi_2(\bar{M}_1^*)$ (i.e. starting below $\eta = \xi_2^{-1}$, for example the point (x_1, y_1) in Figure 8). In this case we use excursion methods. Similarly to the Azema-Yor construction we imagine running a Brownian motion from M_1, plotting its position along the x-axis, and its maximum in the y-axis, we stop the first time the excursion hits η (Figure 7).

If $M_1^* > \xi_2(\bar{M}_1)$ we define a new martingale, $(N_t)_{1\le t\le\infty}$ as follows. To begin with $(N_1, \bar{N}_1)$ has the law of $(M_1^*, \bar{M}_1^*)$. If B be a Brownian motion, then for $t > 1$ we set

$$N_t := M_1 + B_{t-1} \ ,$$

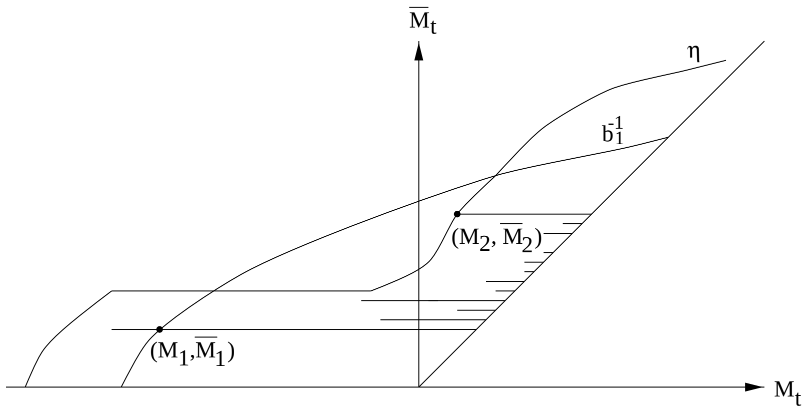

Figure 7: Excursions from $(M_1, \bar{M}_1)$

and the maximum process

$$\bar{N}_t := \bar{M}_1 \vee \sup_{1 \le s \le t} N_t \ .$$

We now introduce the stopping time $\tau := \inf\{t \ge 1 | N_t \le \xi_2(\bar{N}_t)\}$, and define M^* by rescaling the time of $N_{t \wedge \tau}$

$$M_t^* = N_{1/(2-t)) \wedge \tau} \quad t \in (1, 2] \ .$$

Finally we have to show that we obtain a true martingale, not just a local martingale. This follows by a straightforward extension to Lemma 2.3 in Rogers [13] and an appeal to Theorem 1 in Azéma, Gundy and Yor [1].

Second If $M_1^* < \xi_2(\bar{M}_1^*)$ and $\xi_2(y-) < M_1^* < \xi_2(y+)$ for some $y < \bar{M}_1^*$, then we run from the point $(M_1^*, \bar{M}_1^*)$ until some stopping time τ' at which $M_{\tau'} \sim \mathcal{L}\{M_2 \mid \xi_2(y-) \le M_2 \le \xi_2(y+)\}$. Although it is not obvious, it is possible to do this, as we shall prove shortly, and then $\bar{M}_{\tau'} = \bar{M}_1^*$. The situation is illustrated by the point (x_2, y_2) in Figure 8.

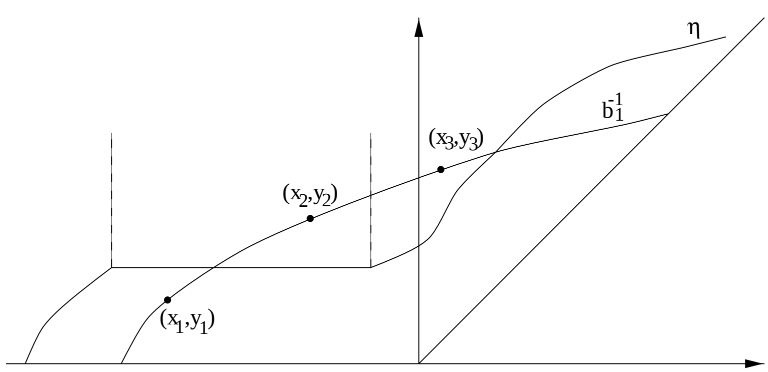

Figure 8: Three different starting points at time 1.

Third for any other starting point, e.g. point (x_3, y_3) in Figure 8, stop immediately. i.e.

$M_s^* = M_1^*$, $s \in (1,2]$.

3.4 The Main Result.

We are now in a position to state precisely the main result of this paper. If we denote by M^* the martingale constructed above and ν^* the law of $\bar{M}_2^*$ then we have the following result.

Theorem 3.3 *$M^* \in \mathcal{M}(\mu_1, \mu_2)$ and ν^* stochastically dominates any $\nu \in \mathcal{P}(\mu_1, \mu_2)$.*

The proof of Theorem 3.3 occupies the remainder of Section 3. First we show that the martingale embedding used in the second case of the construction is possible, next that

$$\forall y > 0 \ , \quad I\!\!P(\bar{M}_2^* \geq y) = \frac{c_2(\xi_2(y))}{y - \xi_2(y)} - 1_{\{\xi_1(y) < \xi_2(y)\}} \left(\frac{c_1(\xi_2(y))}{y - \xi_2(y)} - \frac{c_1(\xi_1(y))}{y - \xi_1(y)} \right)$$

and finally that $M^* \in \mathcal{M}(\mu_1, \mu_2)$. This means that the upper bound on the probability that $\bar{M}_2 \geq y$ for $M \in \mathcal{M}(\mu_1, \mu_2)$ is attained for all y, by M^*.

3.5 Non-unique values of ξ_2

We first show that the embedding when there is a jump in the value of ξ_2, the second step in the above construction, is possible. Fix $y > 0$ at a value where there is a jump in ξ_2, we will write $\alpha \equiv \xi_2(y-)$ and $\beta \equiv \xi_2(y+)$ for short during this whole section.

We need to distinguish two cases, first $\alpha \equiv \xi_2(y-) \geq \xi_1(y)$. Recall that the supporting tangent to $c(\cdot, y)$ passing through y touches the curve at α and β, and possibly in between these two points as well. We set m equal to the slope of this tangent

$$m = \frac{c(\alpha, y) - c(\beta, y)}{\alpha - \beta}$$

and let

$$\varepsilon_- = -c'(\alpha-, y) - m \quad , \quad \varepsilon_+ = m + c'(\beta+, y).$$

Where $c'(x\pm, y)$ represents the left/right derivative of $c(\cdot, y)$ at x. Now we define the measures

$$\tilde{\mu}_2 = \mu_2|_{[\alpha,\beta]} - \varepsilon_- \delta_\alpha - \varepsilon_+ \delta_\beta$$

$$\tilde{\mu}_1 \equiv \mu_1|_{[\alpha,\beta]}$$

We wish to show that it is possible to embed $\tilde{\mu}_2$ as a martingale starting from $\tilde{\mu}_1$. First we compare the total masses and then the means. The total mass of $\tilde{\mu}_2$ is $c_2'(\beta+) - c_2'(\alpha-) - \varepsilon_- - \varepsilon_+$

and the total mass of $\tilde{\mu}_1$ is $c_1'(\beta+) - c_1'(\alpha-)$, so that the difference is $c'(\beta+, y) - c'(\alpha-, y) - \varepsilon_+ - \varepsilon_- = 0$.

The mean of $\tilde{\mu}_2$ is

$$\begin{aligned}
\int_{[\alpha,\beta]} x\,\mu_2(dx) - \varepsilon_-\alpha - \varepsilon_+\beta &= \int_{[\alpha,\beta]} (x-\alpha)\mu_2(dx) - \varepsilon_-\alpha - \varepsilon_+\beta + \alpha\mu_2[\alpha,\beta] \\
&= \int_{[\alpha,\infty)} (x-\alpha)\,\mu_2(dx) - \int_{[\beta,\infty)} (x-\alpha)\,\mu_2(dx) \\
&\qquad -\varepsilon_-\alpha - \varepsilon_+\beta + \alpha\mu_2[\alpha,\beta] \\
&= c_2(\alpha) - c_2(\beta) + (\beta-\alpha)\mathbb{P}(M_2 > \beta) - \varepsilon_-\alpha - \varepsilon_+\beta + \alpha\mu_2[\alpha,\beta] \\
&= c_2(\alpha) - c_2(\beta) + (\beta-\alpha)c_2'(\beta+) - \varepsilon_-\alpha - \varepsilon_+\beta + \alpha\mu_2[\alpha,\beta]\,.
\end{aligned}$$

The mean of $\tilde{\mu}_1$ is

$$\int_{[\alpha,\beta]} x\,\mu_1(dx) = c_1(\alpha) - c_1(\beta) + (\beta-\alpha)c_1'(\beta+) + \alpha\mu_1([\alpha,\beta])\,.$$

The difference between the two means is

$$\begin{aligned}
&c(\alpha,y) - c(\beta,y) + (\beta-\alpha)c'(\beta+,y) - \varepsilon_-\alpha - \varepsilon_+\beta + \alpha\mu_2([\alpha,\beta]) - \alpha\mu_1([\alpha,\beta]) \\
&\quad = (\beta-\alpha)(m + c'(\beta+,y)\,) - \varepsilon_-\alpha - \varepsilon_+\beta + \alpha\mu_2([\alpha,\beta]) - \alpha\mu_1([\alpha,\beta]) \\
&\quad = (\beta-\alpha)\varepsilon_+ - \varepsilon_-\alpha - \varepsilon_+\beta + \alpha\mu_2([\alpha,\beta]) - \alpha\mu_1([\alpha,\beta]) \\
&\quad = -\varepsilon_+\alpha - \varepsilon_-\alpha + \alpha\mu_2([\alpha,\beta]) - \alpha\mu_1([\alpha,\beta]) = 0
\end{aligned}$$

Since the total mass and means match we need only show that

$$\int (x-z)^+\tilde{\mu}_2(dx) \geq \int (x-z)^+\tilde{\mu}_1(dx) \qquad \forall z \in [\alpha,\beta]\,.$$

Now for $z \in [\alpha,\beta]$

$$\begin{aligned}
\int (x-z)^+\tilde{\mu}_2(dx) &= \int (x-z)^+\mu_2(dx) - \varepsilon_+(\beta-z) - \int_{(\beta,\infty)} (x-z)\mu_2(dz) \\
&= c_2(z) - \varepsilon_+(\beta-z) - c_2(\beta) - (\beta-z)\mu_2(\beta,\infty)\,.
\end{aligned}$$

So

$$\begin{aligned}
\int (x-z)^+\tilde{\mu}_2(dx) - \int (x-z)^+\tilde{\mu}_1(dx) &= c_2(z) - c_1(z) - \varepsilon_+(\beta-z) - c_2(\beta) + c_1(\beta) \\
&\qquad -(\beta-z)\mu_2(\beta,\infty) + (\beta-z)\mu_1(\beta,\infty) \\
&= (\beta-z)\left(\mu_1(\beta,\infty) - \mu_2(\beta,\infty) - \varepsilon_+ + k_y\right) \\
&\qquad +c(z,y) - c(\beta,y) \\
&\geq (\beta-z)\left(m + \mu_1(\beta,\infty) - \mu_2(\beta,\infty) + k_y - \varepsilon_+\right) \\
&= (\beta-z)\left(m + c'(\beta+,y) - \varepsilon_+\right) = 0\,.
\end{aligned}$$

Thus we can perform the required embedding.

The second case is when $\alpha \equiv \xi_2(y-) < \xi_1(y)$. In this case we take $\tilde{\mu}_2$ to be defined as in the last case and $\tilde{\mu}_1$ to be defined by

$$\tilde{\mu}_1 = \mu_1|_{(\xi_1(y),\beta]} + \theta\delta_{\xi_1(y)}$$

where

$$\theta \equiv \frac{c_1(\xi_1(y))}{y-\xi_1(y)} + c_1'(\xi_1(y)+) \equiv m + c_1'(\xi_1(y)+)\,.$$

In a similar fashion to the first case the embedding can be proved. **Q.E.D.**

3.6 Proof that the extended Azema-Yor construction gives correct law.

The rest of the proof of Theorem 3.3 splits into two parts. In Lemma 3.1 we proved an upper bound for the law of the maximum of any element of $\mathcal{M}(\mu_1,\mu_2)$. We shall first show in Lemma 3.4 that our constructed martingale M^* achieves this maximum. Once this is proved all that remains is to show that M^* is itself an element of $\mathcal{M}(\mu_1,\mu_2)$. We already know that M_1^* has law μ_1 so we will need only prove that M_2^* has the law μ_2.

Lemma 3.4 *If we define for* $y>0$

$$K_y \equiv \inf_{x<y} \frac{c(x,y)}{y-x} \quad \textit{then} \quad K_y = \mathbb{P}(\bar{M}_2^* \geq y)\,.$$

We need some more results before we can prove this lemma.

Lemma 3.5 *Recall the definition of* k_y *from Lemma 2.6, then*

$$c_1(x,y) = \int_{-\infty}^{x} dt \int_y^\infty \frac{k_s}{s-\xi_1(s)} 1_{\{k_s > -c_1'(t)\}} ds\,.$$

Proof: We have

$$c_1(x,y) = \int_{-\infty}^{x} (c_1'(t)+k_y)^+ dt$$

and as $y \uparrow \infty$, $c_1(x,y) \downarrow 0$. Hence

$$\begin{aligned} c_1(x,y) &= -\int_{-\infty}^{x} dt \int_y^\infty k_s' 1_{\{c_1'(t)+k_s>0\}} ds \\ &= \int_{-\infty}^{x} dt \int_y^\infty \frac{k_s}{s-\xi_1(s)} 1_{\{c_1'(t)+k_s>0\}} ds \end{aligned}$$

as required. **Q.E.D.**

Lemma 3.6 *The function* $y \mapsto K_y$ *is absolutely continuous.*

Proof: We shall prove that K is locally Lipschitz. Firstly let's notice that $0 \le c_1(x,y) - c_1(x, y+\delta) \le (x - \xi_1(y))^+(k_y - k_{y+\delta})$ so that for $\delta > 0$

$$\begin{aligned} K_{y+\delta} = \inf_{x<y+\delta} \frac{c(x, y+\delta)}{y+\delta-x} &\le \inf_{x<y+\delta} \left(\frac{c(x,y)}{y+\delta-x} + \frac{(x-\xi_1(y))^+(k_y - k_{y+\delta})}{y+\delta-x} \right) \\ &\le \frac{c(\xi_2(y), y)}{y+\delta-\xi_2(y)} + \frac{(\xi_2(y)-\xi_1(y))^+(k_y-k_{y+\delta})}{y+\delta-\xi_2(y)} \\ &\le \left(1 + \frac{\delta}{y-\xi_2(y)}\right) K_y + \frac{(\xi_2(y)-\xi_1(y))^+}{y-\xi_2(y)}(k_y - k_{y+\delta}) \,. \end{aligned}$$

Thus

$$K_{y+\delta} - K_y \le \delta \left(\frac{K_y}{y-\xi_2(y)} + \frac{(\xi_2(y)-\xi_1(y))^+}{y-\xi_2(y)} \frac{k_y - k_{y+\delta}}{\delta} \right) .$$

Since k always has well defined left and right derivatives we obtain the required inequality. For the other inequality we suppose that $\delta > 0$ is small enough that $\xi_2(y+\delta) < y$. Then

$$\begin{aligned} K_y \le \frac{c(\xi_2(y+\delta), y)}{y - \xi_2(y+\delta)} &\le \frac{c(\xi_2(y+\delta), y+\delta)}{y-\xi_2(y+\delta)} \\ &= K_{y+\delta}\left(1 + \frac{\delta}{y-\xi_2(y+\delta)}\right) . \end{aligned}$$

Q.E.D.

Proof of Lemma 3.4: We know that $K_y \le \inf_{x<y} c_2(x)/(y-x)$, which tends to zero as $y \to \infty$, so from Lemma 3.6 we may write

$$K_y = -\int_y^\infty K'_t \, dt \ .$$

Now since K is absolutely continuous, for almost every y, both K and ξ_2 are differentiable at y, and ξ_1 is continuous at y. Assuming that y has all of these properties,

$$\begin{aligned} K'_y &\equiv \lim_{h\to 0} \frac{1}{h} \left(\frac{c(\xi_2(y+h), y+h)}{y+h-\xi_2(y+h)} - \frac{c(\xi_2(y), y)}{y-\xi_2(y)} \right) \\ &= \lim_{h\to 0} \frac{1}{h} \left(\left\{ \frac{1}{y+h-\xi_2(y+h)} - \frac{1}{y-\xi_2(y)} \right\} c(\xi_2(y+h), y+h) \right. \\ &\qquad\qquad \left. + \frac{c(\xi_2(y+h, y+h) - c(\xi_2(y), y)}{y - \xi_2(y)} \right) \\ &= \frac{\xi'_2(y) - 1}{y-\xi_2(y)} K_y + \frac{1}{y-\xi_2(y)} \left(\lim_{h\to 0} \frac{1}{h} \left\{ c(\xi_2(y+h), y+h) - c(\xi_2(y), y) \right\} \right) . \end{aligned} \tag{12}$$

Let's now assume that $c(\cdot, y)$ is differentiable at $\xi_2(y)$, the other case is dealt with later. Then

$$\begin{aligned} K'_y &= \frac{\xi'_2(y) - 1}{y - \xi_2(y)} K_y + \frac{1}{y-\xi_2(y)} \left(c'(\xi_2(y), y)\xi'_2(y) + \frac{\partial c}{\partial y}(\xi_2(y), y) \right) \\ &= -\frac{K_y}{y-\xi_2(y)} + \frac{1}{y-\xi_2(y)} \frac{\partial c}{\partial y}(\xi_2(y), y) \end{aligned}$$

$$\begin{aligned} &= -\frac{K_y}{y-\xi_2(y)} + \frac{1}{y-\xi_2(y)} \int_{-\infty}^{\xi_2(y)} dt \, \frac{k_y}{y-\xi_1(y)} 1_{\{c_1'(t)+k_y>0\}} \\ &= -\frac{K_y}{y-\xi_2(y)} + \frac{k_y}{(y-\xi_2(y))(y-\xi_1(y))}(\xi_2(y)-\xi_1(y))^+ . \end{aligned}$$

The third line follows from Lemma 3.5. Recall that $-K_y$ is the slope of the supporting tangent to $c(\cdot, y)$ at $\xi_2(y)$. From this it is clear that if $c(\cdot, y)$ were not differentiable at $\xi_2(y)$, it has to be that the right derivative of $c(\cdot, y)$ is greater or equal to $-K_y$, the left derivative less than or equal to $-K_y$, and the two are different. If we rule out the (null) cases where y is exactly at the end of some flat interval of ξ_2, we shall have that $\xi_2(y+h) = \xi_2(y)$ for all small enough h and so $\xi_2'(y) = 0$ and from (12)

$$\begin{aligned} K_y' &= -\frac{K_y}{y-\xi_2(y)} + \frac{1}{y-\xi_2(y)} \frac{\partial c}{\partial y}(\xi_2(y), y) \\ &= -\frac{K_y}{y-\xi_2(y)} + \frac{k_y}{(y-\xi_2(y))(y-\xi_1(y))}(\xi_2(y)-\xi_1(y))^+ \end{aligned} \tag{13}$$

exactly as before.

The only way M^* can achieve a new maximum between times one and two is if we are in the first case discussed in Section 3.3, and we run Brownian excursions from $(M_1^*, \bar{M}_1^*)$ until we first hit ξ_2^{-1}. So from excursion theoretical results, briefly discussed in Section 2.1, we see that

$$I\!P(\bar{M}_2^* \geq y, \bar{M}_1^* < y) = \int_0^y \frac{k_s \, ds}{s-\xi_1(s)} \frac{(\xi_1(s)-\xi_2(s))^+}{s-\xi_2(s)} \exp\left(-\int_s^y \frac{dr}{r-\xi_2(r)}\right). \tag{14}$$

What do we know of $I\!P(\bar{M}_2^* \geq y)$? Certainly $I\!P(\bar{M}_2^* \geq y) \to 0$ as $y \to \infty$ and

$$\tilde{K}_y \equiv I\!P(\bar{M}_2^* \geq y) = I\!P(\bar{M}_1^* \geq y) + \int_0^y \frac{k_s \, ds}{s-\xi_1(s)} \frac{(\xi_1(s)-\xi_2(s))^+}{s-\xi_2(s)} \exp\left(-\int_s^y \frac{dr}{r-\xi_2(r)}\right)$$

so that if we cross multiply by $\exp(\int_0^y dr/(r-\xi_2(r)))$ and differentiate we get

$$\begin{aligned} \frac{\tilde{K}_y}{y-\xi_2(y)} + \tilde{K}_y' &= k_y' + \frac{k_y}{y-\xi_2(y)} + \frac{k_y}{y-\xi_1(y)} \frac{(\xi_1(y)-\xi_2(y))^+}{y-\xi_2(y)} \\ &= k_y \left(-\frac{1}{y-\xi_1(y)} + \frac{1}{y-\xi_2(y)} + \frac{(\xi_1(y)-\xi_2(y))^+}{(y-\xi_1(y))(y-\xi_2(y))}\right) \\ &= k_y \frac{(\xi_2(y)-\xi_1(y))^+}{(y-\xi_1(y))(y-\xi_2(y))} \end{aligned}$$

and by comparing the above with equation (13) we are done. **Q.E.D.**

We are now in a position to finish the proof of Theorem 3.3, all that remains is to show that $M^* \in \mathcal{M}(\mu_1, \mu_2)$, and to do this we need only show that $M_2^* \sim \mu_2$.

Lemma 3.7 $M_2^* \sim \mu_2$

Proof: From Lemma 3.6, since $\lim_{y\to\infty} K_y = 0$, we must have

$$K_y = -\int_y^\infty K_t' dt = -\int_y^\infty 1_B(t) K_t'\, dt \ ,$$

where B is a set whose compliment is null. Take

$$B = \{t \mid \xi_2 \text{ is differentiable at } t,\ \ \xi_1(t-) = \xi_1(t+)\}\,.$$

For almost all y such that $\xi_1(y) < \xi_2(y)$ if $|\,\delta\,|$ is sufficiently small then $\xi_1(y+\delta) < \xi_2(y)$. Fix some $y > 0$ with this property, and also such that ξ_2 is differentiable at y, and $\xi_2(y)$ is not an atom of μ_2. It can be seen from a direct calculation of

$$\frac{d}{dy} K_y = \lim_{\delta\to 0} \frac{1}{\delta}\left[\frac{c(\xi_2(y+\delta), y+\delta)}{y+\delta-\xi_2(y+\delta)} - \frac{c(\xi_2(y), y)}{y-\xi_2(y)}\right]$$

that c_1 must be differentiable at $\xi_2(y)$. We define $\bar{F}_2(x) = \mu_2([x,\infty)\,)$. Consider the following two cases:

Case 1; $\xi_1(y) < \xi_2(y)$, and all of the above conditions hold. Here, since c_1 is differentiable at $\xi_2(y)$, we get

$$\begin{aligned} K_y &= -c_2'(\xi_2(y)) + c_1'(\xi_2(y)) + k_y \\ &= \bar{F}_2(\xi_2(y)) + \int_y^\infty \frac{k_s}{s-\xi_1(s)} 1_{\{k_s > -c_1'(\xi_2(y))\}} ds \\ &= \bar{F}_2(\xi_2(y)) + \int_y^\infty \frac{k_s}{s-\xi_1(s)} 1_{\{\xi_1(s)<\xi_2(y)\}} ds \\ &= \bar{F}_2(\xi_2(y)) + I\!P(y \le \bar{M}_1^* < b_1(\xi_2(y))\,) \end{aligned}$$

and since

$$I\!P(\bar{M}_2^* \ge y) = I\!P(M_2^* \ge \xi_2(y)) + I\!P(y \le \bar{M}_1^* < b_1(\xi_2(y))\,) = K_y$$

it follows that $I\!P(M_2^* \ge \xi_2(y)) = \bar{F}_2(\xi_2(y))$.

Case 2; $\xi_1(y) \ge \xi_2(y)$, and ξ_2 is differentiable in y and $\xi_2(y)$ is not an atom of μ_2. Here $K_y = -c_2'(\xi_2(y)) = \bar{F}_2(\xi_2(y)) = I\!P(\bar{M}_2^* \ge y) = I\!P(M_2^* \ge \xi_2(y)\,)$.

So we conclude that for all y with the stated properties

$$I\!P(M_2^* \ge \xi_2(y)\,) = \bar{F}_2(\xi_2(y)\,)\ \ .$$

So we've matched the distribution of M_2^* to μ_2 at almost every point outside of jumps intervals of ξ_2. However the jump intervals are taken care of by the earlier construction. Thus $M_2^* \in \mathcal{M}(\mu_1, \mu_2)$ and Lemma 3.7 is proved, which completes the proof of Theorem 3.3. **Q.E.D.**

4 Financial interpretation

We can reinterpret much of the earlier work in financial terms. Let M_t denote the price process of a risky asset and suppose interest rates are zero. Then standard arguments from the theory of complete markets show that when pricing contingent claims or derivative securities it is natural to treat M as if it were a martingale. The simplest and most liquidly traded contingent claims are European call options, which give the buyer the right to buy a unit of the asset at a fixed time, T (the maturity), for a fixed price k (the strike) regardless of its current price. Clearly this has a payoff equal to $(M_T - k)^+$, and the fair price is the $\mathbb{E}[(M_T - k)^+]$ where the expectation is taken with respect to the martingale measure. From knowledge of the price of the call option at all possible strikes for a given maturity it is possible to infer the law of M_T.

The standard approach to pricing derivatives is to assume a model for the dynamics of the price process of the underlying asset and use this to derive prices. However this approach has been reversed by Ross [15], Breeden and Litzenberger [4], and Dupire [6],[7], where no model for the underlying is assumed. Instead we start from the traded call prices at a given maturity to infer a law for the underlying. Options can then be priced using this implied law. If the value of the contingent claim depends only upon the underlying at the times when the law is known then we obtain a unique price. However if we only know the law at discrete times then we do not uniquely identify the behaviour of the underlying.

A digital is an option which has unit payoff if the underlying ever crosses some barrier in a given time period, and a look-back option has a payoff equal to the maximum achieved by the underlying in a given time period. In this case the best we can hope to achieve is the lowest upper bound to the set of possible prices. This has been studied by Hobson [9] in the case where the initial and terminal laws were known. Here we shall extend these results by introducing an intermediate law.

Our analysis of the maximal law of the maximum given a martingale defined only by its distribution at given points, leads, via the martingale inequality used in the proofs, to a super replicating strategy for digital and look-back options. These are the cheapest possible hedging strategies which guarantee to provide at least the required payoff.

4.1 A super-replicating strategy for a digital option.

We assume that a continuum of calls at all possible strikes can be bought and sold at arbitrary quantities. Further transactions in the forward market of the underlying are possible, there are no interest rates or transaction costs and trading can be executed instantaneously.

Suppose we are given the call prices of an asset at times one and two, then given the implied laws μ_1, μ_2 we wish to hedge a payoff $1_{\{\bar{M}_2 \geq y\}}$. The maximum possible price of a digital option must be the supremum of $\mathbb{P}(\bar{M}_2 \geq y)$ over all $M \in \mathcal{M}(\mu_1, \mu_2)$. Recall Lemma 2.1,

$$\mathbb{P}(\bar{M}_2 \geq y) \leq \inf_{\zeta_1, \zeta_2 < y} \left(\frac{c_2(\zeta_2)}{y - \zeta_2} - 1_{\zeta_2 > \zeta_1} \left(\frac{c_1(\zeta_2)}{y - \zeta_2} - \frac{c_1(\zeta_1)}{y - \zeta_1} \right) \right)$$

and note that $c_T(k) := \mathbb{E}((M_T - k)^+)$ is the price of a call option strike k, maturity T. The proof of Lemma 3.1 contains the following inequalities. First if $\zeta_2 \leq \zeta_1$

$$1_{\{\bar{M}_2 \geq y\}} \leq \frac{(M_2 - \zeta_2)^+}{y - \zeta_2} + 1_{\{\bar{M}_2 \geq y\}} \frac{y - M_2}{y - \zeta_2} . \tag{15}$$

We can interpret terms in the above inequality as a hedging strategy. First buy $1/(y - \zeta_2)$ calls strike ζ_2 with maturity 2. This has payoff $(M_2 - \zeta_2)^+/(y - \zeta_2)$. Next when the price first reaches y sell forward $1/(y - \zeta_2)$ units of the underlying to time 2. This has payoff $1_{\{\bar{M}_2 \geq y\}}(y - M_2)/(y - \zeta_2)$. Clearly from (15) this strategy super replicates the payoff of the digital.

If $\zeta_2 > \zeta_1$ then the relevant inequality is

$$\begin{aligned} 1_{\{\bar{M}_2 \geq y\}} \leq & \frac{(M_2 - \zeta_2)^+}{y - \zeta_2} + \frac{(M_1 - \zeta_1)^+}{y - \zeta_1} - \frac{(M_1 - \zeta_2)^+}{y - \zeta_2} \\ & + 1_{\{\bar{M}_1 \geq y\}} \frac{y - M_1}{y - \zeta_1} + 1_{\{\bar{M}_1 \geq y, M_1 \geq \zeta_2\}} \frac{M_1 - M_2}{y - \zeta_2} + 1_{\{\bar{M}_1 < y, \bar{M}_2 \geq y\}} \frac{y - M_2}{y - \zeta_2} . \end{aligned} \tag{16}$$

We interpret this inequality as the following strategy, we initially buy $1/(y - \zeta_2)$ maturity 2 calls with strike ζ_2, sell $1/(y - \zeta_2)$ maturity 1 calls with strike ζ_2, and buy $1/(y - \zeta_1)$ maturity 1 calls with strike ζ_1. Then if the underlying first reaches y before time 1 then sell forward $1/(y - \zeta_1)$ units of the underlying to time 1. If at time 1 the underlying has already reached the level y and the current price is greater or equal to ζ_2 then sell forward $1/(y - \zeta_2)$ units of the underlying to time 2. Finally if the underlying first reaches the level y after time 1, sell forward $1/(y - \zeta_2)$ units of the asset to time 2, when the underlying first reaches y.

Clearly from the above inequality this is a super-replicating strategy. The cost of this strategy, since selling forward is a costless transaction, is $c_2(\zeta_2)/(y - \zeta_2) + 1_{\{\zeta_2 > \zeta_1\}}[c_1(\zeta_2)/(y - \zeta_2) + c_1(\zeta_1)/(y - \zeta_1)]$. If we minimise over all ζ_1, ζ_2, then from the proof of Theorem 3.3 we know there exist $\xi_1(y), \xi_2(y)$ such that the minimum is attained. Without further assumptions there can be no cheaper super replicating strategy since the martingale constructed in Theorem 3.3 achieves this cost.

A super-replicating strategy for a lookback option can be constructed using the hedging strategy for digitals as its building blocks. The required payoff is $\bar{M}_2$, so we buy, for every $y > 0$, (dy) units of the hedging strategy for a digital with payoff $1_{\{\bar{M}_2 \geq y\}}$ and hold M_0 in cash. The hedging strategy is clear and must be the cheapest possible since the martingale constructed in Theorem 3.3 has fair price of the lookback option equal to the cost of this strategy.

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