The relaxed investor and parameter uncertainty

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Running title: The relaxed investor

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Published in *Finance and Stochastics*. The original publication is available on LINK at http://lin.springer.de.

[1]Supported partly by EPSRC grants GR/J97281 and GR/L10000.

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Summary. We firstly consider an investor faced with the classical Merton problem of optimal investment in a log-Brownian asset and a fixed-interest bond, but constrained only to change portfolio (and, if relevant, consumption) choices at times which are a multiple of h. We show that the cost of this constraint can be well described by a power series expansion in h, the first few terms of which we determine explicitly. Typically, this cost is not too large. We then compare this with the cost of parameter uncertainty, as modelled by supposing that the rate of return on the share has a prior Gaussian distribution. We find that the effect of parameter uncertainty is typically bigger than the effects of infrequent policy review.

JEL Classification: C61,D81,D90,G11

AMS (1991) Subject Classification: 90A09, 90A10, 90A43

Keywords: Merton consumption problem, Merton investment problem, time lag, parameter uncertainty.

1 Introduction.

In the classical Merton investment/consumption model (Merton (1969)), an agent who may invest in a log-Brownian share[3] and a constant interest rate money market account seeks to maximise the expected integrated utility of consumption, subject to the constraint that wealth remains non-negative at all times. For the case where the utility is constant relative risk aversion (that is, $U(c) = c^{1-R}/(1-R)$ for some $R > 0$), Merton finds that the optimal behaviour of the agent is always to invest a fixed proportion of wealth in the risky asset, and always to consume at a rate proportional to wealth. The constants of proportionality have explicit expressions in terms of the parameters of the problem[4].

One feature of this solution is that the agent trying to implement it will be continuously adjusting the share holding and the consumption rate, which is clearly impractical. A more realistic scenario would be that the agent chooses to review

[2] Supported partly by EPSRC grants GR/J97281 and GR/L10000.

[3] We consider only a single share throughout this paper; multiple shares can be handled similarly, with slightly more tiresome notation.

[4] Merton solves the problem by solving the associated HJB equation. This technique is restricted in its scope, but the problem can be solved in much greater generality; see, for example, Karatzas (1989)

his investment and consumption decisions on a regular basis, at intervals of some fixed positive h; think of the agent as lazy (or relaxed!) if you wish, or perhaps as having other things to do with his life. Alternatively, the agent may be reluctant to rebalance often due to the presence of transactions costs, and we can consider this study as an attempt to decouple the two effects which act in a transactions-costs problem; the losses due to imperfect portfolio balance induced by infrequent portfolio revision, and the losses due to transactions costs. We shall have more to say on this in the conclusions section.

For brevity, we call an agent constrained in this way an h-investor. The class of investment/consumption policies available to the h-investor is a subset of those available to the Merton investor, so his objective will be smaller than that of the Merton investor - but by how much? It appears to be impossible to answer this question in closed form, but by deriving a series expansion in the (small) parameter h, we shall find remarkably good approximations which allow us to conclude that the cost of a relaxed approach is surprisingly small, at least if the Merton proportion is in $[0,1]$. At first sight, we might expect that the Merton solution should be the limit as $h \downarrow 0$ of the solution for the h-investor, but this turns out to be true only if the Merton proportion is in $[0,1]$; the h-investor will never borrow to buy the share, nor sell the share short, because in a time interval of length h the share price could move disastrously, and the investor would be wiped out. The Merton investor on the other hand would simply adjust his holding of the share as the price moved and would keep his wealth non-negative. Thus if the Merton proportion is not in [0,1], there is a difference between the Merton investor and the 0+-investor, and this is the main effect. It is worth remarking that situations where the Merton proportion is not in [0,1] are not common in practice.

The h-investor's problem reduces to a problem in discrete time, the form of whose solution is known from the work of Hakansson (1970); nevertheless, this solution is less explicit than the solution of Merton, in that it involves the solution of a one-step optimisation problem which cannot be found in closed form. In this paper, we take the view that the parameter h is small, and we find a series expansion for the optimal solution. It turns out that the first three terms of this expansion are already enough for the selection of situations we took for numerical examples.

The paper is organised as follows. In Section 2, we briefly review the two classical Merton problems, the wealth problem (where the agent's objective is to maximise the expected utility of wealth at some fixed terminal time), and the consumption problem (where the objective is to maximise the expected integrated utility of consumption over all future time). Then in Section 3 we investigate the impact of the lag on the wealth problem, and find a power-series expansion in powers of h for the loss of efficiency, as well as the optimal investment policy. [5] In Section 4, we perform the corresponding analysis for the Merton consumption problem. In both cases, we check our results against exact numerically computed answers, and find

[5] There is a growing literature on approximate hedging under finite rebalancing restrictions; see, for example, the preprint of Martini & Patry (2000) and the references therein.

that the power series is very accurate, and also that the magnitude of the effect is small. Section 5 discusses why this should be a small effect; the main reason is that the payoff of a fixed-proportion rule is quite insensitive to the chosen proportion in a neighbourhood of the Merton proportion. However, the neighbourhood where the payoff is affected little is actually quite *small* compared to the confidence interval for the rate-of-return parameter which one would get from a typical length of data; so this leads us in Sections 6 and 7 to investigate the effect of uncertainty in the rate-of-return parameter. We assume that this parameter has a Gaussian prior distribution, and then use techniques of filtering theory to derive the optimal rules. The use of such techniques in financial economics is far from new; Bawa, Brown & Klein (1979), Klein & Bawa (1976), Klein & Bawa (1977), Dothan & Feldman (1986), Feldman (1989), Gennotte (1986), Lakner (1995), Browne & Whitt (1996), Lakner (1998), Karatzas & Zhao (1998), Brennan (1998) is a selection of references dealing with classical economic optimisation problems complicated by a Bayesian updating of unknown parameter distributions. Browne & Whitt (1996) solve the Merton wealth problem in the case of logarithmic utility, and Lakner (1995) deals with general utility.

Finally, in Section 8 we conclude, and discuss the relationship between our results and earlier results on transactions costs.

2 The classical Merton wealth and consumption problems.

An investor may invest in two assets, a money market account with constant interest rate r, and a share with price process $(S_t)_{t\geq 0}$ satisfying

$$dS_t = S_t(\sigma dW_t + \alpha dt) \tag{2.1}$$

for constants σ and α, where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

In the *Merton wealth problem*, the investor chooses to hold θ_t in the share at time t, so that his wealth evolves as

$$dw_t = rw_t dt + \theta_t(\sigma dW_t + (\alpha - r)dt), \tag{2.2}$$

and he aims to maximise his objective

$$E\, U(w_T), \tag{2.3}$$

where $T > 0$ is some fixed time horizon, and the utility U is constant relative risk aversion (CRRA):

$$U(c) = \frac{c^{1-R}}{1-R} \tag{2.4}$$

for some positive R, $(R \neq 1)$.[6] The optimal behaviour is to invest a fixed multiple of wealth in the risky asset at all times: $\theta_t = \pi w_t$ for some constant π. A few lines of calculations shows that if the agent does indeed follow the rule $\theta_t = \pi w_t$, then the value of his objective is

$$U(w_0) \exp\Big[(1-R)T\{\pi(\alpha - r) + r - \frac{1}{2} R\pi^2\sigma^2\} \Big], \tag{2.5}$$

which is optimised by taking $\pi = \pi_*$, where

$$\pi_* = \frac{\alpha - r}{\sigma^2 R} \tag{2.6}$$

is the so-called *Merton proportion*, though we note that in general it does not need to lie in $[0,1]$. The optimised value is

$$U(w_0) \exp((1-R)T(r + \frac{1}{2}\sigma^2 R \pi_*^2)). \tag{2.7}$$

In the *Merton consumption problem*, the investor chooses to hold θ_t in the share at time t and to consume at rate c_t at time t, so that his wealth at time t, w_t, evolves according to the wealth equation

$$dw_t = (rw_t - c_t)dt + \theta_t(\sigma dW_t + (\alpha - r)dt). \tag{2.8}$$

Subject to the constraint that $w_t \geq 0$ for all t, his objective is to maximise

$$E \int_0^\infty e^{-\rho t} U(c_t) dt \tag{2.9}$$

where $\rho > 0$ is constant, and the utility U is as at (2.4) Denoting the value function for this problem by V_M, Merton [5] finds the following explicit form for the solution:

$$V_M(w) = \gamma_*^{-R} U(w), \tag{2.10}$$
$$\theta_t = \pi_* w_t, \tag{2.11}$$
$$c_t = \gamma_* w_t, \tag{2.12}$$

where π_* is the Merton proportion (2.6), and

$$\gamma_* = \frac{\rho + (R-1)(r + (\alpha - r)^2/2R\sigma^2)}{R} \tag{2.13}$$
$$= \frac{\rho + (R-1)(r + \frac{1}{2}\sigma^2 R \pi_*^2)}{R}. \tag{2.14}$$

There is no need to repeat Merton's proof here. In view of the scale-invariant nature of the problem, the form of the solution is not perhaps surprising; under the

[6] The case of $R = 1$ corresponds to logarithmic utility and needs to be treated separately, but the results remain very similar; the details of this case are left to the interested reader.

assumption that the investor is going to choose to hold $\theta_t = \pi w_t$ and consume at rate γw_t for *some* constants π and γ, the wealth of the investor is easily seen to be

$$w_t = w_0 \exp(\pi\sigma W_t + (r - \gamma + \pi(\alpha - r) - \pi^2\sigma^2/2)t)$$

and his payoff is easily computed to be

$$E \int_0^\infty e^{-\rho t} U(\gamma w_t) dt = \frac{(\gamma w_0)^{1-R}}{1-R} \left[\rho + (R-1)(r - \gamma + \pi(\alpha - r) - R\pi^2\sigma^2/2) \right]^{-1}. \quad (2.15)$$

Once again, a little calculus shows that the optimal choices of π and γ are as stated.

3 The h-investor's wealth problem.

Suppose we take the Merton wealth problem for the h-investor with a fixed time horizon $T = Nh$; the investor is only going to change his portfolio at times which are multiples of h and aims to maximise the objective (2.3). His problem is a discrete-time dynamic programming problem, whose value function $V_n(w) \equiv \sup E[U(w_T)|w_{nh} = w]$ has the form

$$V_n(w) = A^{N-n} U(w), \quad (3.1)$$

where

$$A = (1-R) \sup_{p \in [0,1]} E \, \frac{(pZ + (1-p)e^{rh})^{1-R}}{1-R} \equiv (1-R)\kappa. \quad (3.2)$$

Here, $Z \equiv \exp(\sigma W_h + (\alpha - \sigma^2/2)h)$ is the random return on the risky asset in a time interval of length h. Notice that A is positive whatever the value of R.

PROOF OF (3.1). The Bellman equation links V_n and V_{n+1} by

$$V_n(w) = \sup_{p \in [0,1]} E \, V_{n+1}(pwZ + (1-p)we^{rh}); \quad (3.3)$$

the agent chooses the proportion p of his wealth to invest in the risky asset (and this proportion must be in $[0,1]$ as discussed in the introduction), and aims to maximise the expected derived utility of wealth at the next time point. We know that $V_N = U$, and so by induction if (3.1) holds for all $n > m$, we have from (3.3)

$$\begin{aligned} V_m(w) &= \sup_{p \in [0,1]} E \, A^{N-m-1} U(pwZ + (1-p)we^{rh}) \\ &= A^{N-m-1} w^{1-R} \sup_{p \in [0,1]} E \, \frac{(pZ + (1-p)e^{rh})^{1-R}}{1-R} \\ &= U(w) A^{N-m} \end{aligned}$$

as required. ∎

We therefore have an expression $A^N U(w_0)$ for the optimal value of the objective of the h-investor which we can compare with the value (2.6) of the Merton investor. But how should we compare the two? The natural measure is the *efficiency*.

Definition 1 *The* efficiency $\Theta(h)$ *of the h-investor relative to the Merton investor is the amount of wealth at time 0 which the Merton investor would need to obtain the same maximised expected utility at time h as the h-investor who started at time 0 with unit wealth:*

$$\Theta(h) = A^{1/(1-R)} \exp(-h(r + \frac{1}{2}\sigma^2 R \pi_*^2)). \tag{3.4}$$

It appears that there is no simple closed-form expression for the efficiency for this problem, but as we shall see it is perfectly possible to derive a good expansion for Θ in powers of h. The only part of the expression (3.4) for the efficiency which is mysterious is

$$\kappa \equiv \sup_{p \in [0,1]} E \; \frac{(pZ + (1-p)e^{rh})^{1-R}}{1-R},$$

which we aim to explain. Thinking of h as a small parameter, the two terms Z and e^{rh} in the expression for κ are both close to 1, so we make a binomial expansion of $(pZ + (1-p)e^{rh})^{1-R}$. Defining

$$\begin{aligned} X &\equiv p(Z-1) + (1-p)(e^{rh} - 1) \\ &= p(\exp(\sigma W_h + (\alpha - \sigma^2/2)h) - 1) + (1-p)(e^{rh} - 1), \end{aligned}$$

we have for any $N \geq 0$

$$\begin{aligned} E \; \frac{(1+X)^{1-R}}{1-R} &= E \Big[\frac{1}{1-R} \sum_{j=0}^{2N} \frac{\Gamma(2-R)}{\Gamma(2-R-j)} \; \frac{X^j}{j!} \\ &\quad + \frac{1}{1-R} \; \frac{\Gamma(2-R)}{\Gamma(2-R-2N-1)} \; (1+\theta X)^{-R-2N} \; \frac{X^{2N+1}}{(2N+1)!} \Big], \end{aligned} \tag{3.5}$$

where the random variable θ takes values in $[0,1]$. In order to control the remainder term, we need the estimation provided by the following result.

Proposition 1 *For each $N \in \mathbb{Z}^+$, there exists $C_N = C_N(R, \sigma, \alpha, r)$ such that for every $0 \leq h \leq 1$, for any random variable θ with values in $[0,1]$, and for any $p \in [0,1]$ we have*

$$E \; |(1+\theta X)^{-R-2N} \, X^{2N+1}| \leq C_N h^{(2N+1)/2}.$$

PROOF. See Appendix. ∎

Hence the Taylor expansion (3.5) reduces to the statement

$$E\ \frac{(1+X)^{1-R}}{1-R} = \frac{1}{1-R}\sum_{j=0}^{2N}\frac{\Gamma(2-R)}{\Gamma(2-R-j)}\ \frac{EX^j}{j!} + O(h^{(2N+1)/2}), \tag{3.6}$$

and all of the expectations appearing in this expression can be evaluated in closed form, since X is raised to an integer power.

When $\pi_* \notin (0,1)$, the optimal choice for p to pick the point of $[0,1]$ nearest to π_*, and this is covered in Theorems 1 and 3 below. The interesting case is when $\pi_* \in (0,1)$, Theorem 2, when we express the optimal p as an analytic function[7] of h, $p(h) = \sum_{n\geq 0} p_n\, h^n$, and to use (3.6) to obtain as many terms in the expansion as are required. The calculations were done using Maple, and we present below the expansion of p up to terms in h of order 2, as well as the corresponding expansion of the optimal consumption rate, and the efficiency. There are three cases to be considered, according as the Merton proportion π_* is less than 0, in [0,1], or greater than 1, and we present the three cases separately.

Theorem 1 *In the case $\pi_* \leq 0$, the optimal choice of p is 0. The efficiency is*

$$\Theta(h) = \exp(-\frac{1}{2}Rh\sigma^2\pi_*^2). \tag{3.7}$$

Theorem 2 *In the case $0 < \pi_* < 1$, the expansion $\Theta(h) = \sum_{n\geq 0}\Theta_n h^n$, of the efficiency is given up to order 3 by*

$$\Theta_0 = 1 \tag{3.8}$$
$$\Theta_1 = 0 \tag{3.9}$$
$$\Theta_2 = -\frac{1}{4}R\pi_*^2\sigma^4(1-\pi_*)^2 \tag{3.10}$$
$$\Theta_3 = -R\pi_*^2\sigma^6(1-\pi_*)^2(4R\pi_*^2 - 4R\pi_* - 1)/24 \tag{3.11}$$

The optimal proportion has a power series expansion the first three terms of which are

$$p_0 = \pi_* \tag{3.12}$$
$$p_1 = \sigma^2\pi_*(\pi_* - \frac{1}{2})(1-\pi_*) \tag{3.13}$$
$$p_2 = -\sigma^4\pi_*(2\pi_* - 1)(\pi_* - 1)(6R\pi_*^2 - 6R\pi_* - 1)/12 \tag{3.14}$$

[7] The analyticity of the optimal p as a function of h follows from the analyticity of κ in a neighbourhhod of $p = \pi_*$, $h = 0$, by the Implicit Function Theorem.

Theorem 3 *In the case $\pi_* \geq 1$, the optimal choice of p is 1. The efficiency is*

$$\Theta(h) = \exp(-\frac{1}{2}Rh\sigma^2(1-\pi_*)^2). \tag{3.15}$$

In the situation of Theorem 1, the optimal choice of p is 0, and so from (3.2) we find that $A = e^{rh}$, and the stated expression (3.7) for the efficiency follows from the definition (3.4).

Similarly, in the situation of Theorem 1, the optimal choice of p is 1, and so from (3.2) we find that $A = \exp(\alpha h - \sigma^2 Rh/2)$; substituting into the definition (3.4) of the efficiency yields (3.15), using the definition (2.6) of π_*.

The expansion reported in Theorem 2 was obtained using Maple. As a check, we computed the exact values of κ for the lags $h = 2^{-i/2}$, $i = -4, \ldots, 6$ and compared them with the asymptotics of Theorem 2. The parameter values used were $\sigma = 0.35$, $r = 0.1$, $\alpha = 0.18$ and $R = 4$, resulting in the value 0.1632653 for π_*, and Figure 1 displays a graph of the numerically-computed values (marked as dots) together with the asymptotic values, computed from the power series expansion up to terms in h^5.

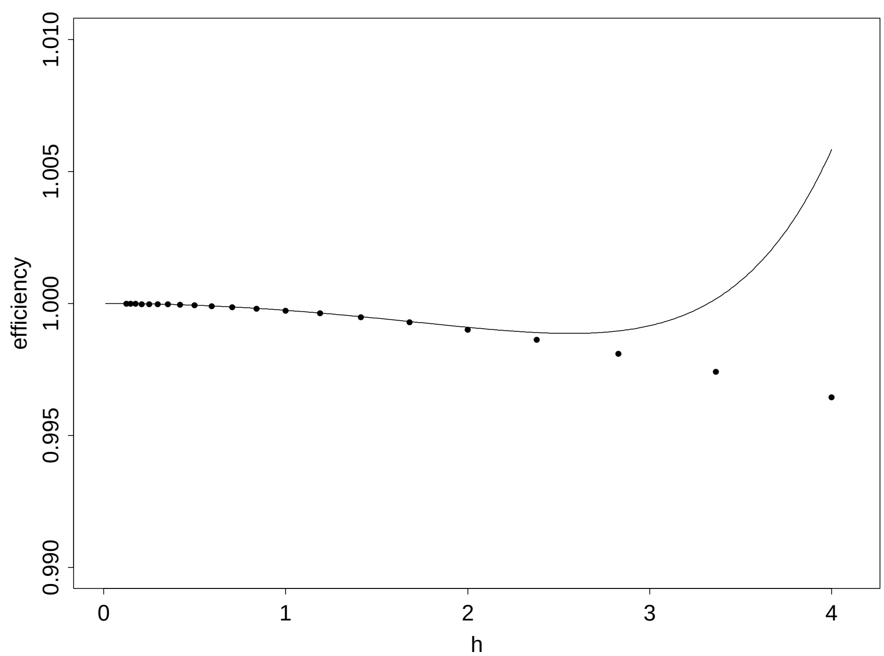

Figure 1: Efficiency as a function of lag h for the wealth problem. Parameters for Case B.

We see that the asymptotic is very accurate out to lags of 2 years, which would be a long time for most investors to leave their portfolio unchanged. We also see that the numerical values of the efficiency are *very* close to unity; even for a 4-year lag, the loss of efficiency is only 0.36 %.

In Theorem 2 we find that the loss of efficiency is $O(h^2)$. What we typically have in mind is that the investor has a time horizon $T > 0$ and will review the portfolio N times before the time horizon, at intervals of length $h = T/N$. Thus if we compare the performance over $[0, T]$ of the Merton investor and the h-investor, we shall find that the efficiency of the h-investor is $\Theta(h)^N$, and therefore to leading order the loss of efficiency is $\frac{1}{4}R\pi_*^2\sigma^4(1-\pi_*)^2Th$. This $O(h)$ loss of efficiency is seen also in the effect on the Merton consumption problem.

4 The h-investor's consumption problem.

In this Section, we turn to the consumption problem of Merton.The h-investor makes a sequence of decisions at the times nh, $n \in \mathbb{Z}$. Specifically, at time nh he sets aside an amount ch of wealth to be consumed at rate c during the interval $[nh, nh+h)$, and then chooses a proportion $p \equiv 1-q$ of his remaining wealth $w-ch$ to be invested in the risky asset until the next decision time $nh+h$. Of course, all of the decisions have to be non-anticipating. We call such an investment/consumption strategy an *h-discrete policy*, and define the value function as

$$V_h(w) \equiv \sup E[\int_0^\infty e^{-\rho t} U(c_t) dt | w_0 = w],$$

the supremum being taken over h-discrete policies. We may now write down the Bellman equation for the h-investor:

$$V_h(w) = \sup_{c \geq 0, p \in [0,1]} U(c) \, \frac{1-e^{-\rho h}}{\rho} + e^{-\rho h} E[V_h((w-ch)(pZ + (1-p)e^{rh}))], \tag{4.1}$$

where $Z \equiv \exp(\sigma W_h + (\alpha - \sigma^2/2)h)$ is the random return on the risky asset in a time interval of length h. As in the original Merton problem, scale-invariance implies that

$$V_h(w) = a(h) U(w)$$

for some constant $a(h) > 0$, and our goal is to understand the dependence of $a(h)$ on h.

As before, we shall describe the effect of the constraint in terms of the *efficiency* of the h-investor, defined as

$$\Theta(h) \equiv (a(h)\gamma_*^R)^{1/(1-R)}. \tag{4.2}$$

The interpretation of the efficiency is that if the Merton investor were given $\Theta(h)$ at time 0, and the h-investor were given 1 at time 0, then both would achieve the same maximised payoff.

Using the scaling of V_h, the Bellman equation for the problem becomes

$$\frac{a(h)}{1-R} = \sup_{\gamma \geq 0, p \in [0,1]} \tilde{h}\frac{\gamma^{1-R}}{1-R} + a(h)e^{-\rho h}(1-\gamma h)^{1-R} E \; \frac{(pZ + (1-p)e^{rh})^{1-R}}{1-R}, \quad (4.3)$$

where $\tilde{h} = (1 - e^{-\rho h})/\rho$. Inspection of (4.3) reveals a decoupling of the maximisation problem; indeed, if we take as before

$$\kappa \equiv \sup_{p \in [0,1]} E \; \frac{(pZ + qe^{rh})^{1-R}}{1-R}, \quad (4.4)$$

then the Bellman equation (4.3) can be expressed more simply as

$$\frac{a(h)}{1-R} = \sup_{\gamma \geq 0} \left[\tilde{h}\frac{\gamma^{1-R}}{1-R} + a(h)e^{-\rho h}(1-\gamma h)^{1-R}\kappa \right],$$

and this can be maximised explicitly over γ. We find that

$$a(h)^{1/R} = \frac{h(\tilde{h}/h)^{1/R}}{1 - (\kappa(1-R)e^{-\rho h})^{1/R}}, \quad (4.5)$$

$$\gamma = h^{-1}\left[1 - (\kappa e^{-\rho h}(1-R))^{1/R}\right] \quad (4.6)$$

$$\equiv \gamma(h). \quad (4.7)$$

Now we saw in the previous section how to deal with κ, by supposing that the optimal proportion has an analytic expansion in powers of h, and computing the leading terms in that expansion. Once again, there are three cases to be considered, according as the Merton proportion π_* is less than 0, in [0,1], or greater than 1, and we present them separately.

Theorem 4 *In the case $\pi_* \leq 0$, the optimal choice of p is 0, and the optimal κ is* $\exp(r(1-R)h)/(1-R)$. *Using (4.2), (4.4), (4.5), and (4.6), the efficiency $\Theta(h)$ and consumption rate $\gamma(h)$ have explicit expressions, with expansions $\Theta(h) = \sum_{n\geq 0}\Theta_n h^n$ and $\gamma(h) = \sum_{n \geq 0}\gamma_n h^n$ given up to order 2 by*

$$\Theta_0 = \left[1 + \frac{(\sigma\pi_*)^2 R(R-1)}{2(\rho + (R-1)r)}\right]^{R/(1-R)} \quad (4.8)$$

$$\Theta_1 = -r\Theta_0/2 \quad (4.9)$$

$$\Theta_2 = \frac{4r^2R - (r-\rho)^2}{24R}\Theta_0 \quad (4.10)$$

and

$$\gamma_0 = \frac{\rho + (R-1)r}{R} \quad (4.11)$$

$$\gamma_1 = -\frac{\gamma_0^2}{2} \quad (4.12)$$

$$\gamma_2 = \frac{\gamma_0^3}{6} \quad (4.13)$$

Theorem 5 *In the case $0 < \pi_* < 1$, the expansions $\Theta(h) = \sum_{n\geq 0}\Theta_n h^n$, and $\gamma(h) = \sum_{n\geq 0}\gamma_n h^n$ of the efficiency up to order 2, and the consumption rate up to order 1 are given by*

$$\Theta_0 = 1 \tag{4.14}$$

$$\Theta_1 = -\frac{1}{2}r - \frac{1}{4}\pi_*^2\sigma^2 R - \frac{\pi_*^2\sigma^4 R(1-\pi_*)^2}{4\gamma_*} \tag{4.15}$$

$$\begin{aligned}\Theta_2 = {} & \frac{4r^2 - R(r-\gamma_*)^2}{24} \\ & + \frac{R\sigma^2\left(-R\gamma_* r + \sigma^4 + 3\,r\sigma^2 + R{\gamma_*}^2 + 3\,\gamma_*\sigma^2 + 4\,\gamma_* r\right){\pi_*}^2}{24\gamma_*} \\ & - \frac{R\sigma^4\left(-2\,R\sigma^2 + 3\,r + 3\,\gamma_* + \sigma^2\right){\pi_*}^3}{12\gamma_*} \\ & - \frac{R\sigma^4\left(42\,\gamma_* R\sigma^2 - 12\,{\gamma_*}^2 + R^2{\gamma_*}^2 - 3\,\sigma^4 - 4\,\gamma_*\sigma^2 - 12\,\gamma_* r - 4\,R{\gamma_*}^2\right){\pi_*}^4}{96{\gamma_*}^2} \\ & + \frac{R\sigma^6\left(3\,R\gamma_* - \sigma^2\right){\pi_*}^5}{8{\gamma_*}^2} - \frac{R\sigma^6\left(5\,R\gamma_* - 9\,\sigma^2\right){\pi_*}^6}{48{\gamma_*}^2} - \frac{R{\pi_*}^7\sigma^8}{8{\gamma_*}^2} + \frac{R{\pi_*}^8\sigma^8}{32{\gamma_*}^2}\end{aligned} \tag{4.16}$$

and

$$\gamma_0 = \gamma_* \tag{4.17}$$

$$\gamma_1 = -\frac{1}{2}\gamma_*^2 - \frac{1}{4}\sigma^4\pi_*^2(1-\pi_*)^2(R-1) \tag{4.18}$$

The expansion of the optimal proportion to invest in the share is as given in Theorem 2.

Theorem 6 *In the case $\pi_* \geq 1$, the optimal choice of p is 1, and the resulting expression for κ is $\exp(h(1-R)(\alpha - R\sigma^2/2)/(1-R)$. Using (4.2), (4.4), (4.5) and (4.6), the efficiency $\Theta(h)$ and consumption rate $\gamma(h)$ have explicit expressions with expansions $\Theta(h) = \sum_{n\geq 0}\Theta_n h^n$ and $\gamma(h) = \sum_{n\geq 0}\gamma_n h^n$ given up to order 2 by*

$$\Theta_0 = \left[1 - \frac{\frac{1}{2}(R-1)\sigma^2(\pi_*-1)^2}{\gamma_*}\right]^{R/(R-1)} \tag{4.19}$$

$$\Theta_1 = -\frac{1}{4}\Theta_0(\sigma^2 R(2\pi_* - 1) + 2r) \tag{4.20}$$

$$\begin{aligned}\Theta_2 = \Theta_0\Bigg[& \frac{R(4R-1)\sigma^4 - 4R\sigma^2(\gamma_* + 3r) + 16r^2 - 4R(\gamma_* - r)^2}{96} \\ & - \frac{\pi_* R\sigma^2(4R\sigma^2 - 6r - \sigma^2 - 2\gamma_*)}{24} \\ & + \frac{\pi_*^2 R\sigma^2(3\sigma^2(3R-1) + 2(R-1)(\gamma_* - r))}{48} \\ & - \frac{\pi_*^3 R\sigma^4(R-1)}{24} - \frac{\pi_*^4 R\sigma^4(R-1)^2}{96}\Bigg]\end{aligned} \tag{4.21}$$

and

$$\gamma_0 = \gamma_* - \frac{1}{2}\sigma^2(\pi_* - 1)^2(R-1) \tag{4.22}$$

$$\gamma_1 = -\frac{1}{8}(\sigma^2(\pi_* - 1)^2(R-1) - 2\gamma_*)^2 \tag{4.23}$$

$$\gamma_2 = -\frac{(\sigma^2(\pi_* - 1)^2(R-1) - 2\gamma_*)^3}{48} \tag{4.24}$$

Remarks. In Theorem 4, the main effect on efficiency is in the limit as $h \downarrow 0$, when we find the efficiency Θ_0 which is no greater than 1, and will be equal to 1 in the limiting case $\pi_* = 0$. Notice that in this case we also have $\gamma_0 = \gamma_*$. Apart from that effect on efficiency, the linear term is given by $\Theta_0(1 - \frac{1}{2}rh)$, which will not be very far from Θ_0 since $\frac{1}{2}r$ is usually in the range 0–10 %.

The most interesting case is Theorem 5, where we see the effect on efficiency dominated by the linear term Θ_1. Notice that Θ_1/Θ_0 is actually a C^1 function of π_*, for $\pi_* \in \mathbb{R}$. The effect on the optimal proportion invested in the share is also noteworthy: if π_* exceeds $\frac{1}{2}$, then the h-investor invests *more* in the risky asset, else he invests less. And irrespective of the value of π_*, the h-investor will consume more slowly than the Merton investor. Both p_1 and γ_1 depend continuously on π_*, but the dependence is not C^1.

Theorem 6 once again reveals a fixed cost for the h-investor, since Θ_0 is again no more than 1. The linear term in the cost, $\Theta_1 h/\Theta_0$, is greater than the corresponding cost in Theorem 1. Generally, the higher-order terms in the expansion do not admit such clear interpretations, and are reported more for completeness. Numerical results make it clear that we may indeed need more than just the linear term in the expansion.

As stated previously, Theorems 4, 5 and 6 were obtained using Maple. What should constitute a proof of these statements? A conventional proof would be far too laborious to check; alternatively, the sequence of Maple commands used could be regarded as proof (and these are available to any interested reader on application to the author). What we shall do is present the results of an independent verification of the solution. This involved a separate (Fortran) computation of κ as defined at (3.4) for a range of parameter values. In all cases, we took $\sigma = 0.35$, $r = 0.1$, and for α and R we took the three pairs

Case	α	R	π_*
A	0.08	4.0	-0.0408163
B	0.18	4.0	0.1632653
C	0.18	0.5	1.3061224

We display in Figure 2 the surface of the efficiency as a function of (h, μ) (where $\mu \equiv 1/\rho$ - the 'mean impatience' - is a more meaningful parameter than ρ).

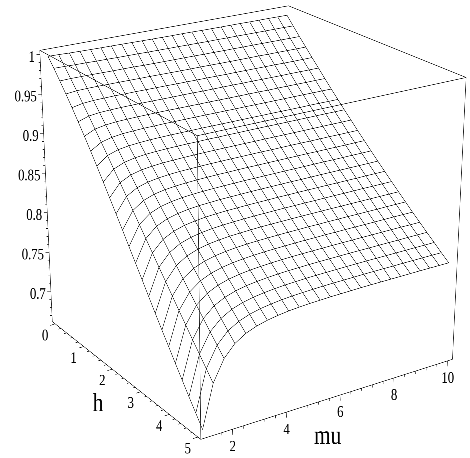

Figure 2: Plot of efficiency against $\mu = 1/\rho$ and h for the Merton consumption problem in case A

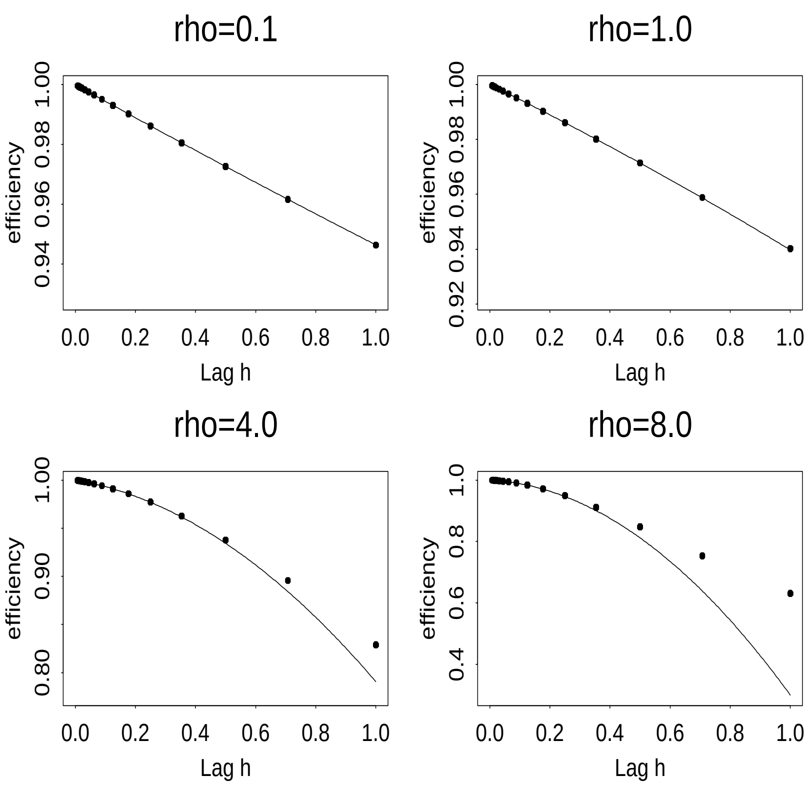

Figure 3: Plot of efficiency against lag, showing true values (marked as large dots) and the values obtained from the asymptotic formula using terms up to and including order h^2 (plotted as a continuous curve), for the Merton consumption problem in case B, with four different values of ρ

In this case, Theorem 4 applies, and the efficiency is given in closed form. Notice how for $h < 1$ we are getting of the order of 95 % efficiency, and even for h up to 2 the efficiency is of the order of 90 % .

The next pictures Figure 3 show the efficiency in case B for four values of ρ: 0.1, 1, 4 and 8. The numerically computed values are given by the dots, and the asymptotic is shown by the continuous line. For the first two cases, the asymptotic is extremely good. For the third and fourth, the asymptotic is good certainly out to $2/\rho$ years, which is a long time in terms of the impatience parameter ρ. Notice how the numerically computed values exhibit pronounced curvature, in contrast to the first two cases. This shows that there is sometimes need of higher-order terms in the expansion than just the first. It is of interest to compare the size of the loss of efficiency for this case with the previous case of the wealth problem. There we lost 0.36% for lag of 4, here we lose *ten* times as much, 3.6%, for lag 0.6 when $\rho = 0.1$ or 1, for lag 0.33 when $\rho = 4$, and for lag 0.20 when $\rho = 8$. Thus the h-effect on the consumption investor is much greater than on the wealth investor, though not very great even so.

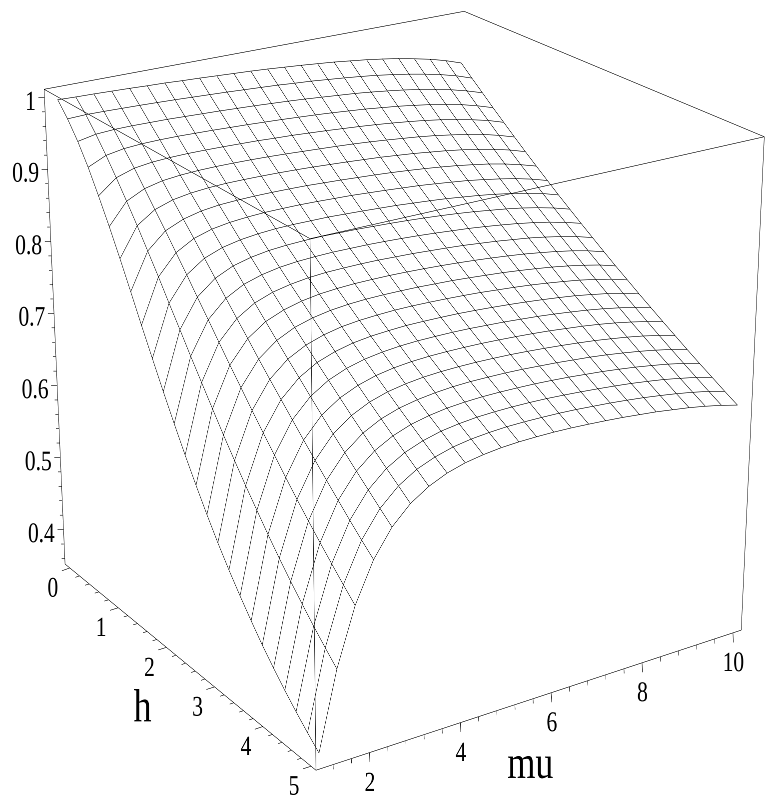

Figure 4: Plot of efficiency against $\mu = 1/\rho$ and h for the Merton consumption problem in case C

Figure 4 displays the efficiency as a function of (h, μ) for case C. The effect of h up to a year is no worse than a 10 % loss of efficiency. An interesting feature is that the efficiency is not monotone in ρ.

5 Why are the effects so small?

In both the Merton consumption problem and especially the Merton wealth problem, we find that the effect of the h-delay is numerically very small; the relaxed investor is not losing very much. Why should this be?

To understand this better, we display in Figure 5 the efficiency of a Merton consumption investor *who uses the wrong value for α in the Merton policy*. Specifically, we suppose the parameters are as for Case B with $\rho = 0.1$, and that the investor puts a constant proportion $(a-r)/\sigma^2 R$ of wealth into the share, and consumes at rate $(\rho + (R-1)(r + (a-r)^2/2R\sigma^2)/R$ times wealth. If $a = 0.18$, the true value of α, then the investor is following the Merton policy, otherwise he is behaving sub-optimally. The expression for the efficiency is easily derived from the expression (2.7) for the maximum payoff. Figure 5 shows also the levels 0.9 and 0.95. We see that the efficiency drops off quite slowly from the maximum, being over 0.95 in the interval [.10819,.23000] and over 0.9 in the interval [.07904,.24761].

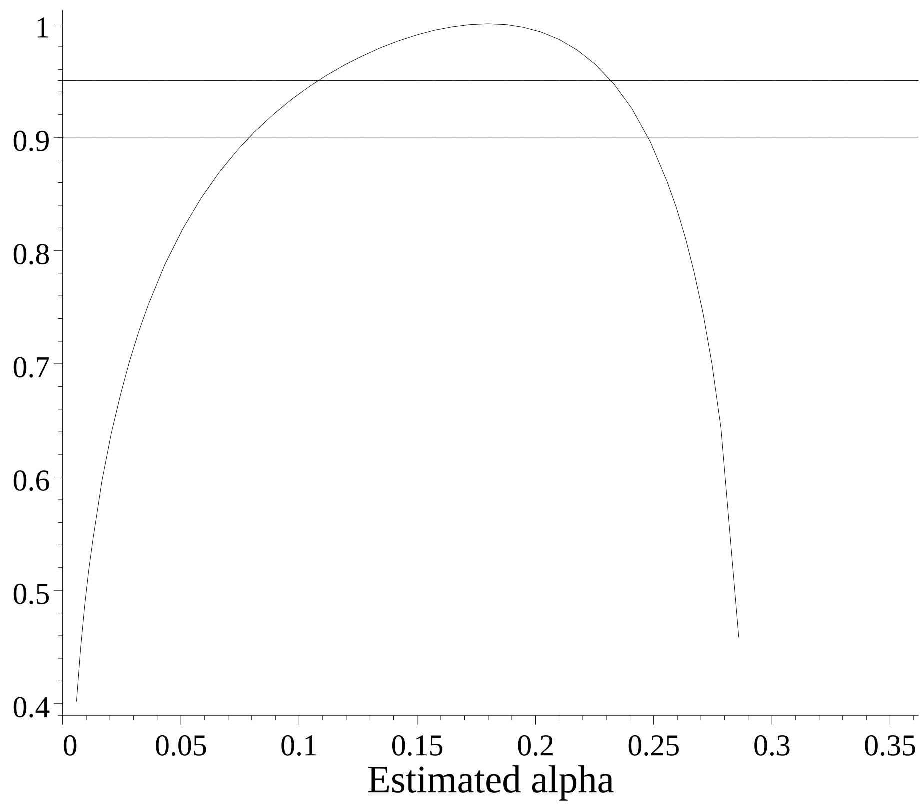

Figure 5: Plot of efficiency against estimated α for the Merton consumption problem in case B, with $\rho = 0.1$

This then is a qualitative explanation of the small magnitude of the loss of efficiency for the h-investor; even for moderately large values of h, it is unlikely that the portfolio balance will be so far away from the Merton proportion to cost too much. It is obvious once one sees it, but the efficiency as displayed in Figure 5 is a smooth function of a, and so near the maximum looks like a quadratic - a departure from the Merton proportion of δ only incurs a loss of order δ^2. This is relevant also to the issue of transactions costs, as we shall briefly discuss in Section 8.

One might from Figure 5 conclude that it should not matter too much if the Merton consumption investor has poor knowledge of the rate of return α on the share. However, typically the rate of return is known with very little precision, and the range [-0.00206, 0.36206] for a which was used in Figure 5 represents a *90 % confidence interval for α based on ten years' observations of the share price process!!*

The magnitude of the error in our estimate of α is huge; it is a point made before, but well worth reiterating. Suppose we have daily observations on the price of an asset which has volatility $\sigma = 20\%$ and rate of return $\alpha = 20\%$. As we observe more and more days, our estimates $\hat{\sigma}$ and $\hat{\alpha}$ improve, but how long would we need to observe in order to be 95% certain that we knew the values to within 5% (that is, $|\hat{\sigma} - \sigma| \leq 0.01$, $|\hat{\alpha} - \alpha| \leq 0.01$)? For our estimate of σ, the answer is about 3 years; but after 3 years we only have 95% confidence that α *is positive!* To obtain the desired precision in our estimate of α, we would need to go on observing the share *for another 1530 years!!* And this level of uncertainty is intrinsic to the problem; we could sharpen up our estimates of σ by observing more frequently than daily, but this would do nothing to improve the estimate of α!

Having seen that effects of parameter uncertainty are likely to be much greater than the effects of delay, we now turn to a study of the effects of uncertainty in α.

6 The impact of uncertainty on the wealth problem.

In this section, we consider a Merton investor who may invest in the money-market account with constant interest rate r, and the share (2.1), but with this difference: *the return parameter α is a random variable.* We shall suppose that distribution of $\lambda \equiv \alpha/\sigma$ is $N(\lambda_0, v_0)$, and in order that the problem be well posed we have to assume that $R > 1$. The case of the Merton wealth problem is treated in a paper by Lakner (1995); we shall briefly summarize the relevant results below. Much of the essential of the problem is covered also by Browne & Whitt (1996) in the case of logarithmic utility. Brennan (1998) derives the dynamics in the observation filtration, and finds an expression for the optimal investment rule in terms of a solution of a HJB equation. We shall make the optimal rule explicit in what follows.

We shall investigate the cost of uncertainty, by comparing the maximised expected utility obtained by the investor who does not know the value of α (and so has to filter it from observations of the share) with the maximised expected utility obtained by an investor who is told at time 0 what the (random) value of α is, and who then follows the Merton optimal rule for that case.

The dynamics of the share price can be expressed as

$$\begin{aligned} dS_t &= \sigma S_t(dW_t + \lambda dt) \\ &= \sigma S_t dX_t, \end{aligned} \tag{6.1}$$

so the uninformed investor attempts to filter the value of λ from observations of $X_t \equiv W_t + \lambda t$. We assume that λ and W are independent. This is a standard Kalman filtering problem, whose solution is given by

$$dX_t = d\hat{W}_t + \frac{v_0 X_t + \lambda_0}{1 + v_0 t}\, dt \tag{6.2}$$

where $\hat{W}$ is a Brownian motion in the observation filtration $(\mathcal{X}_t)$. The distribution of λ conditional on $(\mathcal{X}_t)$ is Gaussian with mean

$$\hat{\lambda}_t \equiv \frac{v_0 X_t + \lambda_0}{1 + v_0 t}$$

and variance

$$v_t \equiv \frac{v_0}{1 + v_0 t}.$$

For more detail on the derivation of these results, see for example Bawa, Brown & Klein (1979). The effect of this is that the investor is now investing in a risky asset whose dynamics are given by (6.1) and (6.2). The change-of-measure martingale which converts X in $(\mathcal{X}_t)$ into a Brownian motion with drift r/σ is

$$Z_t \equiv (1+v_0 t)^{1/2} \exp\left[-\frac{1}{2}\, \frac{v_0}{1 + v_0 t}\, X_t^2 + \left(\frac{r}{\sigma} - \frac{\lambda_0}{1 + v_0 t} \right) X_t + \frac{t}{2}\left(\frac{\lambda_0^2}{1 + v_0 t} - \frac{r^2}{\sigma^2} \right) \right]. \tag{6.3}$$

As is well known (see, for example, Karatzas (1989)), the optimal solution to the Merton wealth problem with fixed horizon T is found by taking the marginal utility of terminal wealth w_T^* to be a multiple of the state-price density process $\zeta_t \equiv e^{-rt} Z_t$:

$$U'(w_T^*) = \gamma \zeta_T$$

for some γ chosen to match the initial wealth condition. In the case of CRRA utility $U(c) = c^{1-R}/(1-R)$, we can compute everything explicitly [8], to obtain

$$\sup E[U(w_T) | w_0 = w] = U(w) \left(E\zeta_T^{(R-1)/R)} \right)^R. \tag{6.4}$$

[8]The value function at an intermediate time will be a function of w_t and $\hat{\lambda}_t$. Since $\hat{\lambda}_0 = \lambda_0$ is deterministic, there is no need to make it explicit in the notation

Now a few calculations with the explicit expression (6.3) lead to

$$E\zeta_T^{1-(1/R)} = e^{-r(R-1)T/R}\frac{(1+v_0T)^{(R-1)/2R}}{(1+cv_0T)^{1/2}}\exp\Big[-\frac{(\lambda_0-(r/\sigma))^2cT}{2R(1+cv_0T)}\Big], \tag{6.5}$$

where $c \equiv 1-(1/R)$, so that

$$\begin{aligned}\sup E[U(w_T)|w_0=w] &= U(w)V(T,\lambda_0,v_0)\\ &\equiv U(w)\frac{(1+v_0T)^{(R-1)/2}}{(1+cv_0T)^{R/2}}\cdot\\ &\qquad \exp\Big[-\frac{(\lambda_0-(r/\sigma))^2cT}{2(1+cv_0T)}-r(R-1)T\Big].\end{aligned} \tag{6.6}$$

The optimal wealth process w_t will satisfy

$$w_t = \frac{\gamma^{-1/R}}{\zeta_t}\,E_t\,\zeta_T^{1-(1/R)},$$

with optimal proportion π_t to be invested in the risky asset being given as the coefficient of $d\hat{W}$ in the expansion of dw/w. Here, E_t denotes conditional expectation given $(\mathcal{X}_t)$. From this we find that

$$\pi_t = \frac{(\sigma\hat{\lambda}_t-r)(1+v_0t)}{\sigma^2R(1+v_0T-(T-t)v_0/R)}. \tag{6.7}$$

As remarked by Brennan (1998), when $R=1$ we find the standard Merton rule, confirming the results of Feldman (1992). We also find that if the Merton proportion is positive, then the derivative of π_t with respect to R at $R=1$ is negative, exactly as one would expect, and in agreement with Brennan; the more risk averse the agent, the less he will invest in the risky asset under uncertainty. Notice also that as $t\uparrow T$, the investor is following more and more closely the Merton rule, and as $T\to\infty$, the proportion of wealth in the risky asset goes to zero (unless $R=1$); both of these conclusions seem intuitively reasonable.

Let us now compare with the investor who is told the value of λ at time 0, and who therefore follows the Merton rule. By time T he has made maximised expected utility (see (2.7))

$$U(w)\exp\Big[-T(R-1)(r+\frac{(\lambda\sigma-r)^2}{2\sigma^2R})\Big] \tag{6.8}$$

which now has to be averaged over the random parameter λ which has a $N(\lambda_0,v_0)$ distribution. After integrating, we find that the informed investor has maximised expected utility

$$U(w)\left(\frac{R}{R+(R-1)Tv_0}\right)^{1/2}\exp\Big[-\frac{T(R-1)((\lambda_0\sigma-r)^2+2r\sigma^2\{R+T(R-1)v_0\}}{2\sigma^2(R+T(R-1)v_0)}\Big].$$

The efficiency $\Theta_I(T)$ of the uninformed investor relative to the informed one is seen to be

$$\Theta_I(T) = \left(\frac{v_0T(R-1)+R}{R(1+v_0T)}\right)^{1/2} = \left(1-\frac{T}{R(\tau+T)}\right)^{1/2} \tag{6.9}$$

after some calculations. Here, we write $\tau \equiv 1/v_0$. In words, the agent who is informed at time 0 of the true value of α can do as well with wealth $\Theta_I(T)$ as the investor who starts with wealth 1, but is given no information other than what he can deduce from watching the price process.

Notice that the efficiency depends only on the time horizon T and the variance v_0 of the initial estimate of λ_0. What is the order of magnitude of the efficiency? Imagine the situation where an investor observes the price of the share for a time period of length τ before investing, and then uses the historical data to estimate λ (we make the situation easier by supposing that σ is known exactly). Then it is not hard to show that the variance of the estimator of λ_0 is τ^{-1}. So a typical value for v_0 would be somewhere in the range 0.5 to 5; observing for longer would not necessarily help, since the mean return would not necessarily be constant over long periods of time.

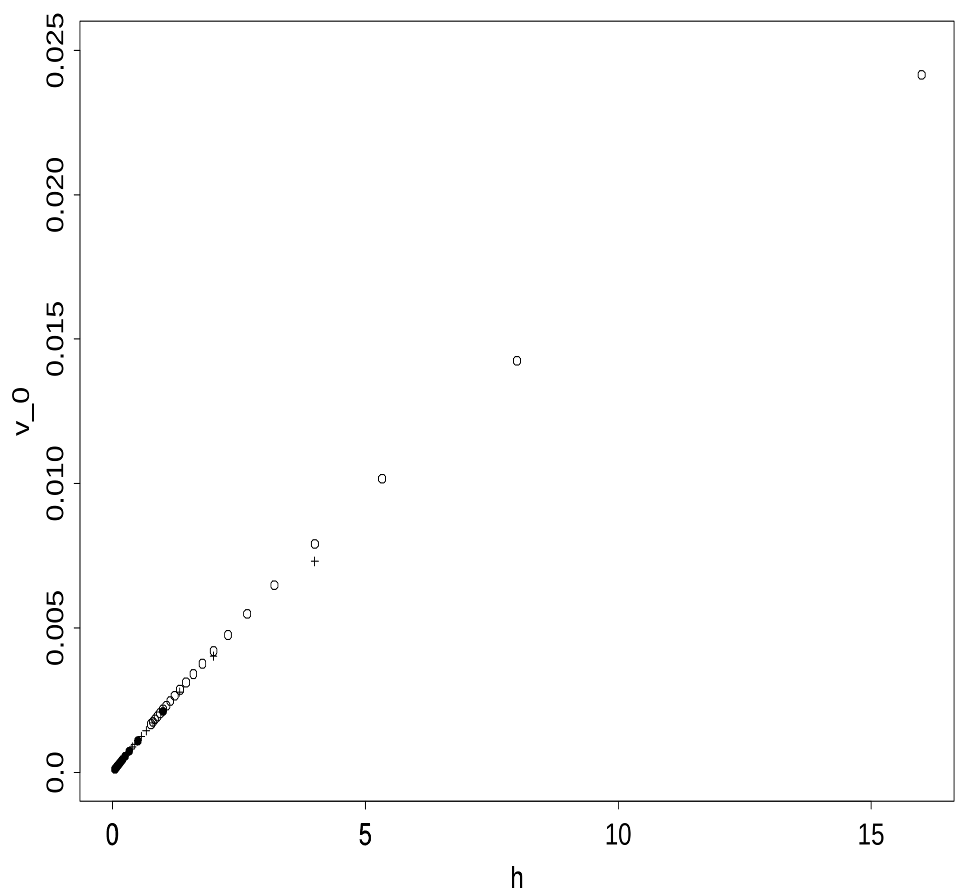

Figure 6: Plot of the values of v_0 required to give the same efficiency as the h-investor solving the wealth problem, with horizons $T = 16$ years (open circle symbol), $T = 4$ years (+) and $T = 1$ year (•). Parameters as in Case B.

Figure 6 shows for a range of values of h the prior variance v_0 that would be required to give the same efficiency as the h-investor for the Merton wealth problem. The plot contains results for time horizons $T = 1, 4, 16$ years. As expected, the prior variance at which the efficiencies match increases with h, but the main thing to notice is the extremely small size of the values of v_0 - even the highest point on the plot would require a prior variance of about 0.025, corresponding to 40 years' worth of estimation data. For a 1-year horizon, with $h = 1$, we would need a prior variance of about 0.002, corresponding to about 500 years' worth of estimation data. This shows that the effect of parameter uncertainty on the Merton wealth problem is far more significant than the h-effect.

7 The impact of uncertainty on the consumption problem.

The dynamics in the observation filtration of the asset price is once again given by (6.1)–(6.2), with the state-price density once again given by the change-of-measure martingale (6.3) multiplied by the discount factor e^{-rt}. As is well known (see, for example, Karatzas (1989)), the optimal solution to the problem is described in terms of the state-price density process ζ as

$$e^{-\rho t} U'(c_t^*) = \gamma \zeta_t \tag{7.1}$$

for some constant γ chosen to match the budget constraint:

$$E \int_0^\infty \zeta_t c_t dt = w_0. \tag{7.2}$$

The budget constraint reworks to give

$$\begin{aligned} w_0 &= \gamma^{-1/R} E \int_0^\infty e^{-\rho t/R} \zeta_t^{(R-1)/R} dt \\ &\equiv \gamma^{-1/R} \varphi(\lambda_0, v_0) \end{aligned}$$

say, where

$$\varphi(\lambda_0, v_0) = \int_0^\infty \frac{(1+v_0 t)^{(R-1)/2R}}{(1+cv_0 t)^{1/2}} \exp\left[- \frac{(\lambda_0 - (r/\sigma))^2 ct}{2R(1+cv_0 t)} - \frac{\rho + r(R-1)}{R} t \right] dt.$$

There appears to be no simpler expression for φ. If we next consider the situation of the investor who is told at time 0 the true value $\alpha = \sigma\lambda$ of the rate of return of the share, then the maximised objective of this agent will be obtained by firstly conditioning on the value of α and then averaging over that value. What results is

$$U(w_0) \int_{-\infty}^\infty \frac{\exp(-(x-\sigma\lambda_0)^2/2\sigma^2 v_0)}{\sqrt{2\pi\sigma^2 v_0}} \left[\frac{R}{\rho + (R-1)(r + (x-r)^2/2R\sigma^2)} \right]^R dx,$$

which once again seems to be impossible to express in any simpler form.

In Figure 7, we plot the surface of the efficiency of the uninformed consumption investor relative to the informed consumption investor, as a function of $\mu = 1/\rho$ and the prior variance v_0. The efficiency is seen to be falling with increasing v_0, but (perhaps a little more surprisingly) falling with increasing μ also.

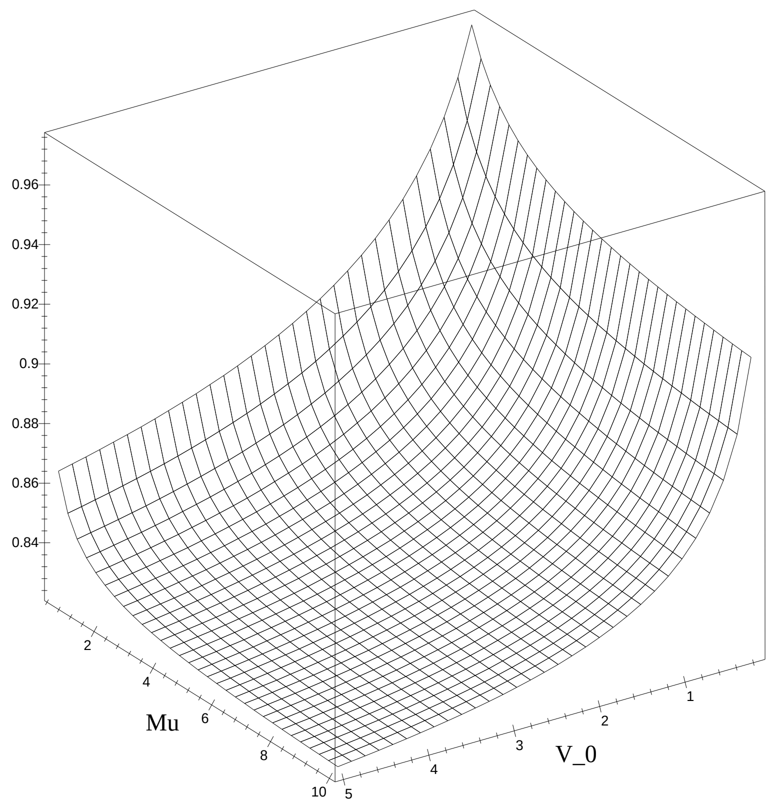

Figure 7: Plot of the efficiency of the uninformed consumption investor against the informed consumption investor, as a function of the prior variance v_0 and the mean impatience $\mu = 1/\rho$. Parameters as in Case B.

One way to understand this is that for μ very short, the investor is only concerned with what happens in the very near future, so if the investor makes poor decisions he does not have to live with them for long. On the other hand, for large μ, early mistakes may be impossible to rectify.

Figure 8 presents for a range of values of lag parameter h the prior variance for which the loss of efficiency of the uninformed consumption investor is exactly the same as the loss of efficiency of the h-investor. The four panels show four different values of ρ. In contrast to the similar picture Figure 6 for the wealth problem, we find that the values of v_0 are not unimaginably small: taking $v_0 = 0.2$ (corresponding to an estimate based on 5 years' data) we find that the loss of efficiency is the same as for $h = 1.6$ for $\rho = 0.1$, $h = 0.76$ for $\rho = 1$, $h = 0.2$ for $\rho = 4$, and $h = 0.09$ for $\rho = 8$.

As we see, these values are not unrealistic; $h = 0.09$ would correspond to an investor who would review his portolio about once a month. So the effect of uncertainty and the h-effect are of similar order here, in contrast to the situation of the previous section, where we found that the effect of uncertainty was far larger. Even so, we find that the effect of uncertainty is somewhat larger than the h-effect.

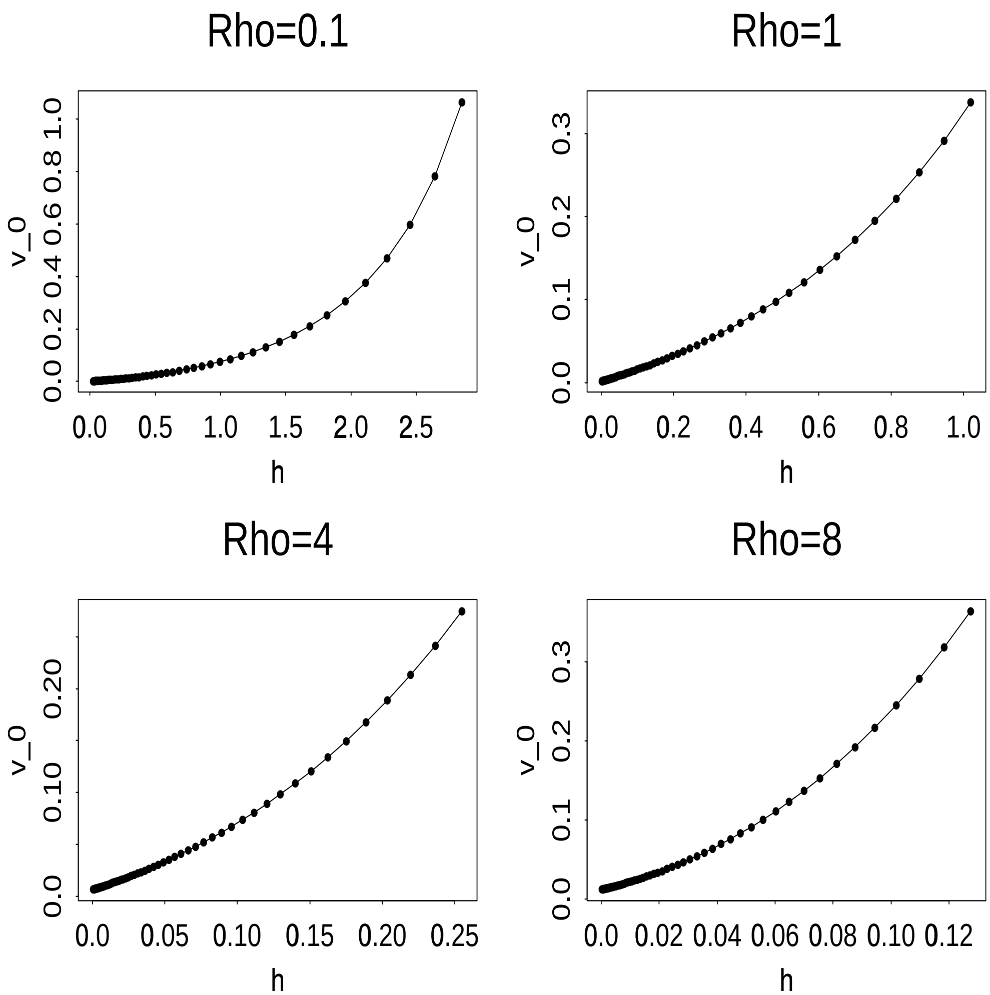

Figure 8: Plots of the values of v_0 required to give the same efficiency as the h-investor solving the consumption problem for four different values of ρ. Parameters as in Case B.

8 Conclusions and discussion.

In this paper, we have modelled the effect of infrequent policy review, by allowing the agent to change his portfolio and consumption only at times which are a multiple of some positive h. [9] Taking as the natural yardstick the efficiency of the h-investor (that is, the quantity of money at time 0 which the ideal Merton investor would require to gain the same payoff), we have shown that:

(8.1) the effect of infrequent policy review can be well approximated by a power series expansion in h, for both the wealth problem and the consumption problem;

(8.2) the magnitude of the effect is quite small in the consumption problem, and very small for the wealth problem;

The small size of the effect is explained to some extent by the relative insensitivity of the payoff in the standard Merton problems to the policy used, as we have seen. This is related to results on transactions costs (see for example Constantinides (1986) Davis & Norman (1990), Morton & Pliska (1995), Dumas & Luciano (1991)) where one finds that introducing a small transaction cost means that the optimal policy for the Merton consumption investor involves very infrequent portfolio rebalancing. As is explained in Rogers (1999), the loss in the classical problem of Davis & Norman (1990) is made up of two components, the loss due to the transactions costs, and the loss due to being imperfectly invested, which means that the value of the portfolio is not growing as fast as it would in the ideal Merton situation. It turns out that these two losses are of comparable size; our analysis here shows that the loss due to imperfect portfolio balance is typically *small*, and so therefore will be the loss due to transactions costs. This explains qualitatively the fact that optimal rebalancing in the transactions costs situation is infrequent.[10]

The small size of the effect of imperfect portfolio balance led us to compare with the magnitude of the effect of parameter uncertainty, especially in the rate of return. Here we were able to come up with the explicit form of the optimal portfolio choice (apparently for the first time) in the case where there is a prior Gaussian distribution over the parameter.

By comparing the losses of efficiency due to the lag effect and due to parameter uncertainty, we showed that:

(8.3) losses due to parameter uncertainty were far higher than the losses due to the lag effect for the wealth problem, requiring prior estimates based on many decades of data to match the lag effect for lags of several years;

(8.4) losses due to parameter uncertainty for the consumption problem were larger than (but comparable with) the lag effect losses.

[9] It might be argued that we should allow the agent to change consumption between reviews of the portfolio - the times when reviews take place are the only times the agent has access to the market, but he can observe in the meantime what is going on. A problem of this kind is treated by Rogers & Zane (1998).

[10] If there is a proportional transaction cost δ, then the loss due to transactions costs in the problem considered by Davis & Norman (1990) is $O(\delta^{2/3})$ - see Shreve (1995) and Rogers (1999).

It appears then that the Merton wealth investor can be very relaxed; his losses due to infrequent portfolio review are far smaller than likely losses due to parameter uncertainty. On the other hand, the Merton consumption investor can be fairly relaxed about reviewing his portfolio, but he cannot neglect this effect in comparison with parameter uncertainty.

Appendix: proof of Proposition 1. Through the proof, $C_N = C_N(R,\sigma,\alpha,r)$ denotes some constant whose value changes from line to line. By the Cauchy-Schwartz inequality, we have

$$E\,|(1+\theta X)^{-R-2N}\,X^{2N+1}| \le \left(E\,|1+\theta X|^{-2R-4N}\right)^{1/2}\,\left(E\,|X|^{4N+2}\right)^{1/2}.$$

This gives us two terms to estimate. For the first,

$$\begin{aligned} E\,|1+\theta X|^{-2R-4N} &= E\left[|1+\theta X|^{-2R-4N} : X \ge 0\right] + E\left[|1+\theta X|^{-2R-4N} : X < 0\right] \\ &\le 1 + E\left[(p\exp(\sigma W_h + (\alpha-\sigma^2/2)h) + qe^{rh})^{-2R-4N}\right] \\ &\le 1 + pE\,\exp(-(2R+4N)(\sigma W_h + (\alpha-\sigma^2/2)h)) + qe^{-(2R+4N)rh} \\ &\le C_N(R,\sigma,\alpha,r). \end{aligned}$$

For the second, we write $\varphi(h) \equiv p(e^{\alpha h}-1)+q(e^{rh}-1)$. Notice that $\sup_{0\le h\le 1} h^{-1}|\varphi(h)| < \infty$. Thus

$$\begin{aligned} E\,|X|^{4N+2} &= |pe^{\alpha h}(e^{\sigma W_h - \sigma^2 h/2}-1) + \varphi(h)|^{4N+2} \\ &\le C_N(R,\sigma,\alpha,r)h^{2N+1}, \end{aligned}$$

for all $0 \le h \le 1$, since by the Burkholder-Davis-Gundy inequalities (see, for example, Rogers & Williams (1987), IV.42) $E|e^{\sigma W_h - \sigma^2 h/2} - 1|^j \le c_j h^{j/2}$ for all $h \in [0,1]$.
□

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