The potential approach to the term structure of interest rates and foreign exchange rates

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Abstract

It is possible to specify a model for interest rates in various different ways, by giving the dynamics of the spot rate, or of the forward rates, for example. A less well developed approach is to specify the law of the state-price density process directly. In abstract, the state-price density process is a positive supermartingale, and the theory of Markov processes provides a rich framework for the generation of examples of such things. We will show how this can be done, and provide simple examples (some familiar, some new) where prices of derivatives can be computed very easily. One benefit of the potential approach is that it becomes very easy to model the yield curve in many countries at once, together with the exchange rates between them.

Running Title: Potential approach to interest and exchange rates.

Keywords: interest rates, foreign exchange, term structure, potential, Markov process, resolvent.

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*This work was supported in part by EPSRC grants GR/J82041 and GR/J97281. A first version of it was presented at the Isaac Newton Institute, Cambridge, in June 1995 at the end of the programme on Financial Mathematics. The author thanks participants in the meeting for their comments, and the Isaac Newton Institute for its hospitality. Thanks are also due to Phil Dybvig, Stewart Hodges, Kin Pang, Stefan Jaschke, Yong Yao, and other seminar participants at the University of Warwick (FORC), the University of Exeter, Birkbeck College, Humboldt Universität Berlin, the Bank of England, the University of Groningen and participants in the RISK Conference in Geneva, January 1996, and the Mathematical Finance Conference, Aarhus, June 1996. This article appears in *Mathematical Finance* **7**, 1997, 157–176.

1 Introduction

The arbitrage-pricing paradigm gives the time-t price of a contingent claim Y payable at time $T > t$ to be

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) Y | \mathcal{F}_t\right], \tag{1.1}$$

where $(r_t)_{t\geq 0}$ is the *spot rate of interest* process, and $\mathbb{E}(\cdot|\mathcal{F}_t) \equiv \mathbb{E}_t$ is the conditional expectation given the information $\mathcal{F}_t$ available at time t, taken with respect to a fixed risk-neutral measure. An important example is provided by the *zero-coupon bonds*, where $Y \equiv 1$:

$$P(t,T) \equiv \mathbb{E}_t \exp\left(-\int_t^T r_s ds\right) \tag{1.2}$$

which serves as a definition of the notation $P(t,T)$ for the time-t price of a bond which delivers 1 at time T.

Although it is unobservable in practice, the spot-rate process r is a helpful concept, and by directly modelling the process r, one can come up with expressions for the prices of zero-coupon bonds and other interest-rate derivatives; Vasicek [29], Cox, Ingersoll & Ross [7], Brennan & Schwartz [5], Schaefer & Schwartz [28], Richard [23], Black, Derman & Toy [4], Longstaff & Schwartz [21], Fong & Vasicek [13], Duffie & Kan [8], Beaglehole & Tenney [3], Hull & White [17] are just some of the many papers which study different models (often multi-factor) for the spot rate process, and explore the consequences of this model choice.

Another approach to interest-rate modelling began with the paper of Ho & Lee [16] and was thoroughly analysed in the continuous-time setting by Babbs [1] and Heath, Jarrow & Morton [15] (see also Dybvig [9] and Jamshidian [18] for the continuum limit of the discrete-time Ho & Lee model). The idea of this approach is to model the forward rate process (or, equivalently, to model the movements of the yield curve) directly. This is appealing because the yield curve is something more readily observable than the spot rate and some models which can be easily described in this framework cannot be easily described in terms of the spot rate. Nevertheless, $r_t = f_{tt}$, so the two specifications are descriptions of the same structure.

The approach we shall pursue here [1] is to model directly the state-price density process ζ_t; we imagine there is some reference probability $\tilde{\mathbb{P}}$ (equivalent to the risk-neutral probability $\mathbb{P}$) in terms of which

$$\zeta_t \equiv \exp\left(-\int_0^t r_s ds\right) \cdot \left.\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right|_{\mathcal{F}_t} \equiv \exp\left(-\int_0^t r_s ds\right) \cdot Z_t. \tag{1.3}$$

Using the state-price density ζ_t, a trivial reworking of (1.2) gives

$$P(t,T) = \tilde{\mathbb{E}}_t[\zeta_T]/\zeta_t. \tag{1.4}$$

The earliest published reference to this approach appears to be Constantinides [6], where it is used to generate a fairly general squared-Gaussian model; we discuss this as Example 2 in Section 3.

[1] We refer to it as the *potential approach* for reasons soon to be explained.

Assuming $r \geq 0$ (which we always shall, since we are concerned with *nominal* interest rates), the process ζ_t is a positive supermartingale. If additionally we assume the economically reasonable condition $\tilde{\mathbb{E}}\zeta_t \equiv P(0,t) \to 0$ as $t \to \infty$, then ζ is what is known as a *potential* (see, for example, Rogers & Williams [26, II.59]).

A positive supermartingale tending to 0 in expectation is called a potential because of the very close links with the Markov process concept of a potential. In Section 2 of this paper we discuss this further, and show that for any Markov process $(X_t)_{t\geq 0}$ with resolvent $(R_\lambda)_{\lambda>0}$, we can take $\alpha > 0$ and a positive function g on the statespace of X and make an interest-rate model by setting

$$\zeta_t = e^{-\alpha t} R_\alpha g(X_t), \quad r_t = g(X_t)/R_\alpha g(X_t). \tag{1.5}$$

In fact, all that is required to make an interest-rate model is to specify some function $h(t,x)$ such that $h(t, X_t)$ is a positive supermartingale, and this can be done in many ways; another generic example discussed in Section 2 is formed by taking

$$h(t,x) = \sum_{i=1}^{n} c_i e^{-\lambda_i t} \varphi_i(t,x), \tag{1.6}$$

where the φ_i are eigenfunctions of the generator,

$$G\varphi_i = \lambda_i \varphi_i$$

and the constants c_i are positive.

The scope for generating interest-rate models is immense; Section 3 gives many examples, and computes bond prices and T-forward measures as the basis for pricing of interest-rate derivatives.

The general technology of Markov processes works very well for time-homogeneous models, but there are natural extensions to time-inhomogeneous models which we also discuss in Section 4; in particular, we show how to make simple modifications of the modelling which allows any initial yield curve to be fitted. The scope for fitting the initial volatilities is a little more restricted, but we discuss what can be done.

Of course, (1.5) is not the only way in which one can represent a potential, and in a recent interesting paper, Flesaker & Hughston [12] have worked with another quite general representation, namely

$$\zeta_t = \tilde{\mathbb{E}}\left[A_\infty - A_t | \mathcal{F}_t\right], \tag{1.7}$$

where $A_t \equiv \int_0^t \eta_s \mu(ds)$ is an increasing adapted process. Flesaker & Hughston do not express their results in this way, working instead with a family $(M_{st})_{0\leq s\leq t}$ of positive $\tilde{\mathbb{P}}$-martingales in terms of which the bond prices are

$$P(s,t) = \frac{\int_t^\infty M_{su}\mu(du)}{\int_s^\infty M_{su}\mu(du)} \tag{1.8}$$

However, it is easy to see that this description is equivalent to (1.7) on taking $\eta_t \equiv M_{tt}$. They generate various examples, among which is the simple and tractable Rational Log Normal model, exemplified by taking $\zeta_t = a(t) + b(t)M_t$, where $M_t = \exp(\sigma W_t - \sigma^2 t/2)$, and a and b are positive decreasing functions.

One major advantage of the potential approach is the ease with which yield curves in many countries can be handled, along with the exchange rates between them. Indeed, a simple change-of-numeraire argument (see, for example, Saá-Requejo [27]) proves that (assuming complete markets) *the exchange rate between two countries must be the ratio of their state-price densities.*[2] This has the practical advantage that if one has adopted the potential approach to term-structure modelling, then once the term structure has been modelled in two countries, the exchange rate between them is determined; *no further Brownian motions are needed!* This is an important advantage. By modelling a single Markov process, and then defining a yield curve model in terms of a function of that Markov process, we are able to extend our model to incorporate additional countries, simply by taking a new function of the Markov process. Adding a new country does *not* mean adding a new source of randomness, thus keeping the working dimension under control as we bring in more and more countries; also, the calculations of prices for the original countries are clearly not affected by the new countries added! Alternatively, if it is felt that there really *is* some source of noise specific to each new country added, then this could easily be modelled as a Markov process independent of the base Markov process. This way, the base Markov process supports all the *global* randomness, while the independent Markov processes explain the *country-specific* randomness.

We illustrate this in Section 5, where among the examples we give is one in which the spot rates in each country are squared-Gaussian processes, and the log exchange rates between countries are Brownian motions.

2 The Markovian context

Throughout this section we shall take some Markov process $(X_t)_{t\geq 0}$ with state space $\mathcal{X}$ and resolvent[3] $(R_\lambda)_{\lambda>0}$. We represent the positive supermartingale ζ as $\zeta_t = h(t, X_t)$ in various different ways.

Generic Approach I. Fixing some function $g : \mathcal{X} \to [0, \infty)$ and $\alpha > 0$, we can make a positive supermartingale by setting

$$\zeta_t = e^{-\alpha t} R_\alpha g(X_t)/R_\alpha g(X_0). \tag{2.1}$$

To see that this *is* a supermartingale, we just need to consider the martingale

$$\begin{aligned} M_t &\equiv \tilde{\mathbb{E}}\left[\int_0^\infty e^{-\alpha s} g(X_s)ds | \mathcal{F}_t\right] \\ &= \int_0^t e^{-\alpha s} g(X_s)ds + e^{-\alpha t} R_\alpha g(X_t), \end{aligned} \tag{2.2}$$

[2]The simple proof of this is given in the appendix to this paper.

[3]The resolvent $(R_\lambda)_{\lambda>0}$ is defined by $(R_\lambda f)(x) = \mathbb{E}^x[\int_0^\infty e^{-\lambda t} f(X_t)dt]$ for any $x \in \mathcal{X}, \lambda > 0$, bounded measurable $f : \mathcal{X} \to \mathcal{X}$. This is a common object in the study of Markov processes and you will find a full account in Chapter III of Rogers & Williams [26], among many other places.

using the Markov property of X. The increasing process $A_t \equiv \int_0^t e^{-\alpha s} g(X_s)ds$ compensates the supermartingale $R_\alpha g(X_0)\zeta_t$; from (2.2),

$$R_\alpha g(X_0)\zeta_t = M_t - A_t,$$

exhibiting ζ as a supermartingale. The positivity of ζ follows from the positivity of the operator R_α and the assumed non-negativity of g.

Note from the definition (1.3) of ζ that also

$$d\zeta_t = \zeta_t(-r_t dt + Z_t^{-1} dZ_t), \tag{2.3}$$

so that *if we are told ζ, we can work out what r is.* We can therefore apply the recipe (2.3) to recover the multiplicative decomposition of ζ;

$$\begin{aligned} \zeta_{t-}^{-1} d\zeta_t &= \zeta_{t-}^{-1}(dM_t - e^{-\alpha t} g(X_t)dt) \\ &= \zeta_{t-}^{-1} dM_t - \frac{g(X_t)}{R_\alpha g(X_t)} dt \end{aligned}$$

so if we abbreviate $r_t \equiv g(X_t)/R_\alpha g(X_t)$, we see that

$$Z_t \equiv \zeta_t \exp\left(\int_0^t r_s ds\right) \quad \textit{is a local martingale.}$$

Thus *if we were using ζ as the state-price density relative to the probability space of the Markov process X, the spot rate process is*

$$r_t = \frac{g(X_t)}{R_\alpha g(X_t)} \tag{2.4}$$

as claimed in the Introduction at (1.5).

In applications, it may not be very easy to specify the resolvent of a given Markov process in a usable closed form (for example, even the resolvent of an Ornstein-Uhlenbeck process in one dimension has only an integral expression). However, the heuristic[4] interpretation of the resolvent as

$$R_\lambda = (\lambda - G)^{-1}, \tag{2.5}$$

where G is the generator of the Markov process, gives us another avenue to approach the problem; we firstly pick some positive function $f : \mathcal{X} \to (0, \infty)$ and then define g via[5]

$$g = (\alpha - G)f. \tag{2.6}$$

Thus we have $R_\alpha g = f$, and, *provided g is everywhere non-negative* we have the situation described at (2.4);

$$r_t = \frac{g(X_t)}{R_\alpha g(X_t)} = \frac{(\alpha - G)f(X_t)}{f(X_t)}. \tag{2.7}$$

[4]Technical problems arise in the interpretation (2.5) because G is not defined on *all* functions. While these points are important at a theoretical level, when it comes to applications, they really don't matter, as we shall see.

[5]If G were the generator of a diffusion, a second-order differential operator, then the right-hand side of (2.6) is undefined if f is not twice differentiable. But when modelling, it is unlikely that we would want to work with f which were not smooth.

The demand that g should be non-negative arises because we are modelling nominal interest rates, not real ones. The framework we are dealing with here is certainly broad enough to handle real rates of interest too. Indeed, we may simply take some other positive function $\tilde{g}$ and model the 'real' state-price density as $\tilde{\zeta}_t = e^{-\alpha t} R_\alpha \tilde{g}(X_t)/R_\alpha \tilde{g}(X_0)$. Everything lives on the same Markov process!

Generic Approach II. As another approach to the construction of examples, note the following: if $\varphi_i (i = 1, ..., n)$ are eigenfunctions of G,

$$G\varphi_i = \lambda_i \varphi_i$$

and $g \equiv \sum_{i=1}^n c_i \varphi_i$ is non-negative, then if $\alpha > \max \lambda_i$

$$f \equiv R_\alpha g = \sum_{i=1}^n c_i (\alpha - \lambda_i)^{-1} \varphi_i$$

provides us with an example of potential type, for which bond prices can simply be written down. Indeed,

$$P(0,T) = \frac{e^{-\alpha T} \sum_{i=1}^n \tilde{c}_i e^{\lambda_i T} \varphi_i(X_0)}{\sum_{i=1}^n \tilde{c}_i \varphi_i(X_0)},$$

where $\tilde{c}_i \equiv c_i/(\alpha - \lambda_i)$. While it may appear that this approach is much more tractable than Generic Approach I, in that we only need to find the eigenfunctions but do not need to compute the transition density, this is illusory; if we wanted to find the price of a caplet, even though we have a simple expression for the bond price, we will *still* need to know the law of X_t given starting value X_0 in order to compute the price. Nevertheless, Generic Approach II is better suited to time-inhomogeneous extensions, as we shall see.

Forward measures. When it comes to valuing contingent claims, we are faced with computing expressions such as $\tilde{\mathbb{E}}[\zeta_t Y]$, where Y is $\mathcal{F}_t$-measurable. A useful concept for these calculations is the *T-forward measure*, $\mathbb{P}_T^*$, defined by

$$\mathbb{E}_T^*(Y) \equiv \tilde{\mathbb{E}}(\zeta_T Y)/\tilde{\mathbb{E}}(\zeta_T) \tag{2.8}$$

for bounded $\mathcal{F}_T$–measurable Y; see Jamshidian [19] and Geman [14] for early uses of this concept. If we can find a reasonably simple description of the T-forward measure, the pricing of various path-dependent contingent claims often becomes easier. When we work in this Markovian framework, the process X in the T-forward measure is a (time-inhomogeneous) Markov process which we can characterise reasonably simply using the martingale-problem formulation as follows.

To begin with, we note that since ζ_t is a function of t and X_t

$$\begin{aligned} \left.\frac{d\mathbb{P}_T^*}{d\tilde{\mathbb{P}}}\right|_{\mathcal{F}_t} &= \tilde{\mathbb{E}}(\zeta_T|\mathcal{F}_t)/\tilde{\mathbb{E}}(\zeta_T|\mathcal{F}_0) \\ &= \varphi(t, X_t), \end{aligned} \tag{2.9}$$

say. Now if G is the generator of X, and $\psi(t,x)$ is any (reasonably nice) function, then

$$\psi(t, X_t) - \int_0^t \left\{ \frac{\partial \psi}{\partial t} + G\psi \right\} (s, X_s) ds \quad \textit{is a } \tilde{\mathbb{P}}\textit{−martingale.} \tag{2.10}$$

Taking $\psi = \varphi$, we learn that[6]

$$\frac{\partial \varphi}{\partial t} + G\varphi = 0. \tag{2.11}$$

For a fairly general test function ψ,

$$\psi(X_t) - \int_0^t \theta_s ds \textit{ is a } \mathbb{P}_T^* - \textit{martingale} \tag{2.12}$$

iff

$$Y_t \equiv \varphi(t, X_t) \left\{ \psi(X_t) - \int_0^t \theta_s ds \right\} \textit{ is a } \tilde{\mathbb{P}} - \textit{martingale.}$$

Now we apply Itô's formula to Y and obtain[7]

$$dY_t \doteq \left(\frac{\partial}{\partial t} + G \right) (\varphi\psi)(t, X_t)dt - \theta_t \varphi(t, X_t)dt$$

from which we see that if

$$\begin{aligned} \theta_t &\equiv \varphi^{-1} \left[G(\varphi\psi) + \psi \frac{\partial \varphi}{\partial t} \right] (t, X_t) \\ &= \varphi^{-1}[G(\varphi\psi) - \psi G\varphi](t, X_t) \end{aligned}$$

(using (2.11)) then (2.12) holds. Thus the generator G_T^* of X under the T-forward measure $\mathbb{P}_T^*$ is given by

$$G_T^*\psi(t, x) = \left[\frac{G(\varphi\psi) - \psi G\varphi}{\varphi} \right] (t, x) \tag{2.13}$$

for suitable test functions $\psi : \mathcal{X} \to \mathbb{R}$.

We can similarly compute the law of the process X under the risk-neutral measure $\mathbb{P}$ rather than the reference measure $\tilde{\mathbb{P}}$; the generator becomes G^*,

$$G^*\psi = \frac{1}{h}\{G(h\psi) - \psi Gh\},$$

where $Z_t = h(t, X_t)$.

3 Some explicit examples

We present here eight examples which illustrate some of the points raised in a more abstract setting earlier. The first five examples take as the underlying Markov process the Gaussian diffusion in $\mathbb{R}^d$ given by

$$dX_t = dW_t - BX_t \, dt, \tag{3.1}$$

where B is a general $d \times d$ matrix . The distribution of X_T is $N(e^{-TB}X_0, \ V_T)$, where we define

$$V_T \equiv \int_0^T e^{-sB}(e^{-sB})^T ds. \tag{3.2}$$

[6] A continuous finite-variation martingale is constant; see, for example, Rogers & Williams [25, IV.30.4].

[7] The symbol "$\doteq$" signifies that the two sides differ by a $\tilde{\mathbb{P}}$-local martingale.

The generator is

$$Gf(x) \equiv \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}\,(x) - \sum_{i=1}^{d}(Bx)_i \frac{\partial f}{\partial x_i}\,(x). \tag{3.3}$$

Example 1 (exponential-linear example). Take $f(x) \equiv R_\alpha g(x) = \exp(a \cdot x)$ as the function in Generic Approach I, where a is some fixed non-zero element of $\mathbb{R}^d$. Then

$$g(x) = (\alpha - G)f(x) = e^{a \cdot x}(\alpha - \frac{1}{2}|a|^2 + a \cdot Bx)$$

giving an interest rate process

$$r_t = \alpha - \frac{1}{2}|a|^2 + a \cdot BX_t$$

which is a multifactor Gaussian model. Such models are widely used because of their tractability, but we reject them because they allow nominal interest rates to go negative – and this drawback is practical as well as theoretical, as Rogers [24] explains. In any case, we can in the potential description have simultaneously the non-negativity of spot rates *and* tractability if we simply modify our choice of f, as the next examples show.

Example 2 (exponential-quadratic example). Here we use Generic Approach I with function

$$f(x) = \exp\left(\frac{1}{2}(x-c)^T Q(x-c)\right) \tag{3.4}$$

where $c \in \mathbb{R}^d$ and Q is $d \times d$ positive-definite symmetric. Now we obtain

$$\begin{aligned} g(x) &\equiv (\alpha - G)f(x) \\ &= f(x)\left[\frac{1}{2}(x - S^{-1}v)^T S(x - S^{-1}v) + \alpha - \frac{1}{2}trQ - \frac{1}{2}|Qc|^2 - \frac{1}{2}v^T S^{-1}v\right] \end{aligned} \tag{3.5}$$

where

$$S \equiv B^T Q + QB - Q^2, \quad v \equiv (B^T - Q)Qc. \tag{3.6}$$

Now the choice of α, Q, c is at our disposal, and we shall take Q small enough to ensure that S is positive-definite, and take

$$\alpha = \frac{1}{2}tr\ Q + \frac{1}{2}|Qc|^2 + \frac{1}{2}v^T S^{-1} v \tag{3.7}$$

which reduces (3.5) to

$$g(x) = f(x)\ .\ \frac{1}{2}(x - S^{-1}v)^T S(x - S^{-1}v). \tag{3.8}$$

It follows that the spot rate process (r_t) is simply

$$r_t = \frac{1}{2}(X_t - S^{-1}v)^T S(X_t - S^{-1}v)$$

which is a squared-Gaussian process; such processes are known to provide tractable examples (Cox, Ingersoll & Ross [7], El Karoui, Myneni & Viswanathan [10], Beaglehole & Tenney [3],

...). Using the potential description, we can easily compute bond prices, and the T-forward measure. Indeed,

$$
\begin{aligned}
P(0,T) &= \tilde{\mathbb{E}}[e^{-\alpha T}R_\alpha g(X_T)|\mathcal{F}_0]/R_\alpha g(X_0) \\
&= e^{-\alpha T}\tilde{\mathbb{E}}\left[\exp\frac{1}{2}(X_T-c)^TQ(X_T-c)|\mathcal{F}_0\right]/R_\alpha g(X_0), \\
&= e^{-\alpha T}\det(I-QV_T)^{-\frac{1}{2}}\exp\left(\frac{1}{2}\mu_T^T(I-QV_T)^{-1}Q\mu_T-\frac{1}{2}\mu_0^TQ\mu_0\right) \qquad (3.9)
\end{aligned}
$$

where $\mu_T \equiv e^{-TB}X_0 - c$.

As at (2.9), we can now extract the T-forward measure; a few calculations give

$$
\begin{aligned}
\phi(t,X_t) = \frac{\det(I-QV_\tau)^{-\frac{1}{2}}}{\det(I-QV_T)^{\frac{1}{2}}}\exp\Big[&\frac{1}{2}\xi_t^T(I-QV_\tau)^{-1}Q\xi_t-\frac{1}{2}(X_t-c)^TQ(X_t-c) \\
&-\frac{1}{2}\mu_T^T(I-QV_T)^{-1}Q\mu_T+\frac{1}{2}(X_0-c)^TQ(X_0-c)\Big]
\end{aligned}
$$

where $\tau \equiv T-t$, $\xi_t \equiv e^{-\tau B}X_t - c$. From here, we may identify the T-forward measure; the generator, G_T^*, of X under $\mathbb{P}_T^*$ is given by

$$G_T^*f = Gf + (\nabla f).(\nabla \log\phi) \qquad (3.10)$$

where explicitly

$$\nabla\log\phi(t,x) = e^{-\tau B^T}(I-QV_\tau)^{-1}Q(e^{-\tau B}x-c) - Q(x-c). \qquad (3.11)$$

Example 3 (quadratic example). The first two examples were intended to show how familiar classes of models appear and can be handled within the potential specification. Here is an example of novel type, which can again be handled very simply in this framework.

We take

$$f(x) = \gamma + \frac{1}{2}(x-c)^TQ(x-c), \qquad (3.12)$$

so that

$$g(x) = \alpha\gamma - \frac{1}{2}\mathrm{tr}\,Q + \frac{1}{2}\alpha c^TQc + \frac{1}{2}(x-v)^TS(x-v) - \frac{1}{2}v^TSv \qquad (3.13)$$

where

$$S = \alpha Q + B^TQ + QB \qquad (3.14)$$

and

$$v = S^{-1}(\alpha Qc + B^TQc) \qquad (3.15)$$

If we choose γ so that

$$\gamma = \frac{\mathrm{tr}\,Q + v^TSv}{2\alpha} - \frac{1}{2}c^TQc, \qquad (3.16)$$

then the spot rate process is given by

$$r(X_t) = \frac{g(X_t)}{f(X_t)} = \frac{(X_t-v)^TS(X_t-v)}{2\gamma + (X_t-c)^TQ(X_t-c)} \quad , \qquad (3.17)$$

the zero-coupon bond prices are given by

$$P(0,t) = \frac{\exp\{-\alpha\, t\}}{f(x_0)}(\gamma + \frac{1}{2}(\mathrm{tr}(Q\,V_t) + \mu_t^T Q \mu_t)) \tag{3.18}$$

(with μ_t as in Example 2) and the state price density is given by

$$\zeta_{0,t} = \exp\{-\alpha\, t\}\frac{\gamma + \frac{1}{2}(X_t - c)^T Q (X_t - c)}{\gamma + \frac{1}{2}(X_0 - c)^T Q (X_0 - c)}. \tag{3.19}$$

The T-forward measure can be computed just as easily; we conclude that

$$G_T^* f = Gf + \frac{\mu_\tau^T Q e^{-\tau B}\nabla f}{\gamma + \frac{1}{2}\mu_\tau \cdot Q \mu_\tau + \frac{1}{2}\mathrm{tr}(Q\,V_\tau)} \tag{3.20}$$

where $\tau = T - t$.

Example 4 (cosinh example). For this example, we shall assume that $B = \beta I$, and take

$$f(x) = \cosh \gamma \cdot (x + c) \tag{3.21}$$

for some fixed $c, \gamma \in \mathbb{R}^d$. Evidently we lose no generality in taking $d = 1$, which we now do for notational simplicity. In this case,

$$g(x) = (\alpha - G) f(x) = \alpha \cosh \gamma(x + c) - \frac{1}{2}\gamma^2 \cosh \gamma(x + c) + \beta\gamma x \sinh \gamma(x + c)$$

which will be non-negative if and only if α is large enough. Especially interesting is the case where $\inf g(x) = 0$, for then the spot rate process is

$$r_t = \beta\gamma X_t \tanh \gamma(X_t + c) + k. \tag{3.22}$$

In the special case $c = 0$, we have $k = 0$ and

$$r_t = \beta\gamma X_t \tanh \gamma X_t,$$

which looks like a Gaussian model for X large, squared-Gaussian for X small. We can easily compute bond prices:

$$P(0,t) = \frac{\cosh \gamma(X_0 e^{-\beta t} + c)}{\cosh \gamma(X_0 + c)} \exp\left[-\alpha t + \frac{\gamma^2}{2}\,\frac{1 - e^{-2\beta t}}{2\beta}\right]. \tag{3.23}$$

In the T-forward measure, the generator of X becomes G_T^*, where

$$G_T^* f = Gf + \gamma e^{-\beta\tau} \tanh \gamma(x e^{-\beta\tau} + c)\ \ f'(x), \tag{3.24}$$

where $\tau \equiv T - t$ as before.

Example 5 (harmonic oscillator example). We use the diffusion (3.1), specialised to one dimension for simplicity, with generator $G = \frac{1}{2}D^2 - \beta x D$. Defining functions $\phi_k(\cdot)$ via

$$\sum_{n\geq 0} \theta^k \phi_k(x) \equiv \exp\left[\beta(x + \theta)^2\right] \equiv F(x, \theta),$$

we see that $\phi_k(x)\exp(-\beta x^2)$ is a polynomial of degree k, and

$$\begin{aligned}\sum_{n\geq 0}\theta^k G\phi_k(x) &= \beta\left[2\beta\theta(x+\theta)+1\right]F(x,\theta)\\ &= \beta\left\{\theta\frac{\partial F}{\partial\theta}(x,\theta)+F(x,\theta)\right\}.\end{aligned}$$

From this,

$$G\phi_k = \beta(k+1)\phi_k,$$

as is well known to any physicist. As before, we can make spot-rate processes by taking $g(x) = \sum_{i=1}^n c_i\phi_i(x)$; the special case $Q = 1$, $c = 0$ and $\gamma = 1/(2\alpha)$ which we studied in Example 3 is of this form.

For the next two examples, we shall take the underlying Markov process to be the d-dimensional diffusion

$$dX_t^i = 2\sqrt{x_t^i}\,dW_t^i + \left(a_i + \sum_j b_{ij}x_t^j\right)dt, \quad (i = 1, ..., d) \tag{3.25}$$

where $a_i > 0$ for all i, $b_{ij} \geq 0$ for $i \neq j$, and $b_{ii} \leq 0$ for all i. Here, the W^i are independent standard Brownian motions. We could have inserted variance parameters in front of the dW^i terms, but since we work with functions of x, this is unnecessary; the normalisation we have chosen is taken because of the connection with squared Bessel processes (see Revuz & Yor [22, Chapter XI]).

Example 6. We take some $\gamma \in \mathbb{R}^d$ fixed and consider

$$f(x) = e^{\gamma^T x}, \tag{3.26}$$

which gives us

$$g(x) \equiv (\alpha - G)f(x) = e^{\gamma^T x}\left[\alpha - 2\sum_{i=1}^d \gamma_i^2 x_i - \gamma^T(a + Bx)\right].$$

For non-negativity, we require

$$(B^T\gamma)_i + 2\gamma_i^2 \leq 0 \tag{3.27}$$

for all i.

In cases where (3.27) holds, and the inequality is strict for some i, we have an interesting example where the spot rate process is an affine function of the diffusion x, and, for $\alpha = \gamma^T a$, it is even linear in x. Such examples are discussed by Duffie & Kan [8]; they recall the well-known fact that the expectations we have to compute to find bond prices,

$$f(x_0)\,.\,P(0,t) = \tilde{\mathbb{E}}_0[e^{-\alpha t}R_\alpha g(x_t)] = \tilde{\mathbb{E}}_0[e^{-\alpha t}\exp(\gamma^T x_t)],$$

can seldom be written down in closed form, although by solving a matrix Ricatti equation the bond prices can be rapidly and accurately evaluated numerically.

Example 7 (multi-type branching diffusion example). We take the same underlying Markov process as in the previous example but now choose

$$f(x) = \gamma^T x, \tag{3.28}$$

where $\gamma \in \mathbb{R}^d, \gamma_i > 0$ for all i. This time,

$$g(x) = (\alpha - G)f(x) = -\gamma^T a + x^T(\alpha - B^T)\gamma$$

so we shall also require of γ that

$$(\alpha - B^T)\gamma \geq 0, \quad -\gamma^T a \geq 0 \tag{3.29}$$

for non-negativity of the spot rate. The spot rate process is a rational function,

$$r(x) = \frac{-\gamma^T a + x^T(\alpha - B^T)\gamma}{\gamma^T x}$$

which can take arbitrarily large values if $-\gamma^T a > 0$, and can take arbitrarily small values if $\gamma^T(\alpha - B)_i = 0$ for some i. The case where $\gamma^T a = 0$ results in a bounded function r.

This time, the bond prices can be computed more easily; as we readily see,

$$\tilde{\mathbb{E}}[x_t|\mathcal{F}_0] = -B^{-1}a + e^{tB}(x_0 + B^{-1}a),$$

which can be easily computed. Hence the bond prices are given by

$$P(0,t) = \frac{e^{-\alpha t}\gamma^T\{-B^{-1}a + e^{tB}(x_0 + B^{-1}a)\}}{\gamma^T x_0} .$$

The T-forward measure can again be computed; the generator gains the additional first-order term

$$\sum_i \frac{4(\gamma^T e^{(T-t)B})_i x_i}{\gamma^T(-B^{-1}a + e^{(T-t)B}(x + B^{-1}a))} D_i .$$

Example 8 (Rational Log-normal Model). For our final example, we consider an example which appears in [12]. We take the underlying process to be Brownian motion, and note that for any θ, $\phi(x) \equiv \exp(\theta x)$ is an eigenfunction of the generator $\frac{1}{2}D^2$, with eigenvalue $\frac{1}{2}\theta^2$. Thus we may take

$$g(x) = \sum_{i=1}^n c_i \exp(\theta_i x)$$

provided this is everywhere positive, and we find that for $\alpha > \max(\frac{1}{2}\theta_i^2)$

$$f(x) = R_\alpha g(x) = \sum_{i=1}^n c_i \left(\alpha - \frac{1}{2}\theta_i^2\right)^{-1} \exp(\theta_i x) \equiv \sum_{i=1}^n \tilde{c}_i e^{\theta_i x}.$$

In this example, the bond prices are simply

$$P(0,t) = \frac{e^{-\alpha t}\sum_{i=1}^n \tilde{c}_i e^{\theta_i x_0 + \theta_i^2 t/2}}{\sum_{i=1}^n \tilde{c}_i e^{\theta_i x_0}},$$

and the T-forward measure can be described also by saying that the generator gains the additional first-order term

$$\frac{\sum_{i=1}^n \tilde{c}_i\theta_i \exp[\theta_i x + \frac{1}{2}\theta_i^2(T-t)]}{\sum_{i=1}^n \tilde{c}_i \exp[\theta_i x + \frac{1}{2}\theta_i^2(T-t)]} D.$$

4 Time-inhomogeneous models

This section is quite short, and only begins to discuss the possibilites. We recall the Generic Approach II, where we represent the state-price density as

$$\zeta_t = \sum_{i=1}^{n} c_i M_t^i, \tag{4.1}$$

where the martingales $M_t^i \equiv e^{-\lambda_i t}\varphi(X_t)$ are defined in terms of the eigenfunctions φ_i of G, $G\varphi_i = \lambda_i \varphi_i$. For a time-homogeneous variant of this, we could make the coefficients c_i in (4.1) to depend deterministically on time. This would allow us to fit *any* initial yield curve. This is trivial, but more interesting is to decide whether we can fit a given initial covariance structure of yields of different maturities.

Let us suppose that the underlying Markov process X is a diffusion in some Euclidean space,

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt, \tag{4.2}$$

where W is a d-dimensional Brownian motion, and write $Y(t,T) \equiv -(T-t)^{-1}\log P(t,T)$ for the yield at time t on the bond maturing at later time T. Since the bond can be expressed as

$$P(t,T) = \frac{\sum_{i=1}^{n} c_i(T)M_t^i}{\sum_{i=1}^{n} c_i(t)M_t^i}, \tag{4.3}$$

we have that

$$dY(t,T)\Big|_{t=0} = T^{-1}\sum_{i=1}^{n}\Big(c_i(0) - \frac{c_i(T)}{P(0,T)}\Big)\, dM_0^i \; + \text{finite-variation terms}, \tag{4.4}$$

after some calculations. If we abbreviate $v_i(T) \equiv T^{-1}(c_i(0) - c_i(T)/P(0,T))$, we see that the covariation of yields of maturities T and T' is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{n} v_i(T)v_j(T')\nabla\varphi_i(X_0)\cdot(\sigma\sigma^T)_{ij}\nabla\varphi_j(X_0). \tag{4.5}$$

One consequence of this is that if we select some maturities $T_1, T_2, \ldots, T_m$ and consider the covariation of the yields of those m maturities, *this can have rank at most d.* A consequence of this is that we cannot expect to fit the initial covariations of a yield curve; this is a feature of *any* model with just a few driving Brownian motions. Do we conclude from this that we should be working with a model with many Brownian motions, so as to fit the covariance structure of yields? No; the covariance structure is really only a meaningful concept in the context of diffusion-based models of the term-structure, and we may use *any* Markov process as the base process. The recent work of Babbs & Webber [2] for example, exhibits a convincing model of term structure which steps away from the well-ploughed field of diffusions, and it may well turn out that many of the problems associated with fitting interest-rate models are arising because of the rigidities imposed by the assumption of continuous paths.

5 Foreign exchange rates: an example

In some sense, we have already said all that needs to be said about foreign exchange rates; the exchange rate is given as the ratio of state-price densities, and Section 3 described general methods for creating examples. Here, we will concentrate on the example studied (as Example 2) in the previous section:

$$dX_t = dW_t - BX_t dt, \tag{5.1}$$

taking

$$f_i(x) = \exp\left[\frac{1}{2}(x-c_i)^T Q_i (x-c_i)\right]. \tag{5.2}$$

As before, we take

$$\alpha_i = \frac{1}{2} trQ_i + \frac{1}{2}|Q_i c_i|^2 + \frac{1}{2} v_i S_i^{-1} v_i, \tag{5.3}$$

where $S_i \equiv B^T Q_i + Q_i B - Q_i^2$, $v_i \equiv (B^T - Q_i)Q_i c_i$, and we obtain

$$\begin{aligned} g_i(x) &\equiv (\alpha_i - G) f_i(x) \\ &= f_i(x)\ .\ \frac{1}{2}(x-a_i)^T S_i (x-a_i) \end{aligned} \tag{5.4}$$

where $a_i = S_i^{-1} v_i$. This gives us a spot rate for country i which is again a squared-Gaussian process.

To illustrate the scope of this class, we shall look in more detail at two cases: (a) $Q_i = Q$ for all i; (b) $c_i = 0$ for all i.

Case (a): $Q_i = Q$ for all i. By applying a rotation to X, we can (and shall) suppose that Q is diagonal. The state-price density in country i is

$$\zeta_t^i = \exp\left[-\alpha_i t + \frac{1}{2}(X_t - c_i)^T Q (X_t - c_i) - \frac{1}{2}(X_0 - c_i)^T Q (X_0 - c_i)\right], \tag{5.5}$$

so the fact that we have picked the same Q for all i gives

$$\frac{Y_t^{ij}}{Y_0^{ij}} = \frac{\zeta_t^j}{\zeta_t^i} = \exp[(\alpha_i - \alpha_j)t + (c_i - c_j)^T Q (X_t - X_0)], \tag{5.6}$$

which is a particularly simple log-Brownian motion. Thus we have the appealing structure that *exchange rates between countries are log-Brownian, and all domestic spot-rates are squared-Gaussian*; in addition, underlying everything is a Gaussian diffusion, so this class is likely to be about as tractable as we might wish.

To illustrate this, suppose an agent in country i wishes to buy a European call option with strike k and maturity T on a unit of currency j. He will value it at

$$\tilde{\mathbb{E}}[\zeta_T^i (Y_T^{ij} - k)^+] = \tilde{\mathbb{E}}[(Y_0^{ij} \zeta_T^j - k\zeta_T^i)^+].$$

While this might seem to require integration over $\mathbb{R}^d$, we can reduce the problem by defining a new probability measure $\tilde{\mathbb{P}}_T$ by[8]

$$\tilde{\mathbb{E}}_T Y \equiv \tilde{\mathbb{E}}\left[Y \exp\left(\frac{1}{2} X_T^T Q X_T\right)\right] / \tilde{\mathbb{E}} \exp\left(\frac{1}{2} X_T^T Q X_T\right). \tag{5.7}$$

[8]This is in fact the T-forward measure for a country in which $c = 0$.

As we saw, under $\tilde{\mathbb{P}}$, $X_T \sim N(\mu, V)$, where $\mu_T \equiv e^{-TB}X_0$ and $V_T \equiv \int_0^T e^{-sB}(e^{-sB})^T ds$, so under $\tilde{\mathbb{P}}_T$,

$$X \sim N(\tilde{\mu}_T, \tilde{V}_T), \quad \tilde{\mu}_T \equiv (I - V_T Q)^{-1}\mu, \quad \tilde{V}_T \equiv (V_T^{-1} - Q)^{-1}.$$

Thus the value of the option to the agent can be expressed as

$$\tilde{\mathbb{E}} \exp\left(\frac{1}{2}X_T^T Q X_T\right) \; . \; \tilde{\mathbb{E}}_T\left[\{Y_0^{ij}\exp(\rho_j - c_j^T Q X_T) - k\exp(\rho_i - c_i^T Q X_T)\}^+\right]$$

where $\rho_j = -\alpha_j T + \frac{1}{2}c_j^T Q c_j - \frac{1}{2}(X_0 - c_j)^T Q (X_0 - c_j)$. This is now a lot easier to deal with, as it is the expectation of a function of a *pair* of correlated Gaussian variables.

For this example, we computed at (3.9) the bond prices, so the yield curve is

$$\begin{aligned} y_t^i &\equiv -\frac{1}{t}\log P(0,t) \\ &= \alpha_i + \frac{1}{2t}\log\det(I - QV_t)\frac{1}{2t}\left[(x - c_i)^T Q (x - c_i) - \xi_t^T (I - QV_t)^{-1} Q \xi_t\right], \end{aligned} \tag{5.8}$$

where $\xi_t \equiv e^{-tB}x - c_i$, and we write x in place of X_0. We can compare two yield curves; after some simplification, we find that

$$y_t^i - y_t^j = \alpha_i - \alpha_j + \frac{1}{2t}(c_i - c_j)Q(I - V_t Q)^{-1}\{V_t Q(2x - c_i - c_j) - 2(I - e^{-tB})x\}. \tag{5.9}$$

Differentiation of (5.8) with respect to x gives us the term structure of volatility; if $\mathrm{vol}_i(t,x)$ is the volatility of the maturity–t yield when $X_0 = x$, we obtain

$$\mathrm{vol}_i(t,x) = \frac{1}{t}\|Q(x - c_i) - (e^{-tB})^T (I - QV_t)^{-1} Q (e^{-tB}x - c_i)\|. \tag{5.10}$$

Not much more can be said at this level of generality, and it is necessary to investigate concrete examples numerically.

Case (b): $c_i = 0$ **for all** i. This simplification centres all the quadratic forms which appear, and we find that the yield curve is

$$y_t^i = \alpha_i + \frac{1}{2t}\log\det(I - Q_i V_t) + \frac{1}{2t}[x^T Q_i x - (e^{-tB}x)^T (I - Q_i V_t)^{-1} Q_i \, e^{-tB}x],$$

and the term structure of volatility is

$$\mathrm{vol}_i(t,x) = \frac{1}{t}\|Q_i x - (e^{-tB})^T (I - Q_i V_t)^{-1} Q_i \, e^{-tB}x\|.$$

Once again, not much more can be said at a theoretical level, and numerical investigation is the route to take.

6 Appendix: exchange rates in terms of state-price densities

What follows is essentially a change of numéraire argument, similar to others in the literature (for example, El Karoui, Geman & Rochet [11]).

If $(S_t)_{t\geq 0}$ is the price process of some traded asset which pays no dividends, then $(\zeta_t S_t)_{t\geq 0}$ is a $\tilde{\mathbb{P}}$-local martingale; in the simplest situation of a complete market ζ is (up to multiples) the only process for which ζS is a $\tilde{\mathbb{P}}$-local martingale for every traded asset S. If we put ourselves in this situation then, but imagine that there is more than one country, the assets of one country being freely exchangeable for those of another at the prevailing exchange rate, then if $S^i_t(S^j_t)$ is the price process of a traded asset in country i (country j) measured in that country's currency, and at time t, 1 unit of currency j can be exchanged for $Y^{ij}(t)$ units of currency i, then

$$Y^{ij}_t S^j_t \quad \textit{is a traded asset in country } i,$$

being the country-i value of the asset whose price in country j is S^j. Thus if ζ^i is the state-price density for country-i assets, then

$$\zeta^i_t Y^{ij}_t S^j_t \quad \textit{is a } \tilde{\mathbb{P}}\textit{-local martingale} \tag{6.1}$$

but on the other hand we also know that

$$\zeta^j_t S^j_t \quad \textit{is a } \tilde{\mathbb{P}}\textit{-local martingale} \tag{6.2}$$

and so we conclude that

$$Y^{ij}_t = Y^{ij}_0 \zeta^j(t)/\zeta^i(t), \tag{6.3}$$

at least in the complete situation. This is a simple and memorable result; *at time t a unit of currency j can be exchanged for $Y^{ij}_0\zeta^j(t)/\zeta^i(t)$ units of currency i.*

This is the situation which prevails if the market is complete. If the market is incomplete, (6.3) may still hold (and for modelling purposes one probably would assume it held in the absence of strong reasons to the contrary), but other possibilities may arise; for example, we could have

$$Y^{ij}_t = N_t \zeta^j_t / \zeta^i_t$$

where N is a bounded positive martingale strongly orthogonal to all the local martingales of the form $\zeta^j_t S^j_t$.

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