Volatility estimation with price quanta

L.C.G. Rogers*

University of Bath

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Abstract. Volatility estimators based on high, low, opening and closing prices have been developed, and perform well on simulated data, but on real data they frequently give lower values for volatility than the simple open-close estimator. This may be due to the fact that for real data, the maximum (or minimum) price is often at the beginning or end of the day. While this could not happen if the observed process was log Brownian, it *could* happen if the observed process were log Brownian, but observed only to the nearest penny. We develop the theory of such approximations to derive the corrected versions of the basic estimators.

KEY WORDS: Brownian motion, Rogers-Satchell estimator, Euler-Maclaurin expansion, Wiener-Hopf factorisation.

1 Introduction

However imperfect the log-Brownian model for the movement of a share price may be, this model is widely used in practice, and at least for long enough time scales

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the assumptions are not too bad (monthly returns look reasonably log-normal, but daily returns do not, for example). The crucial parameter of the model is the volatility σ, and various ways are proposed for estimating it. If one views the log-price process $X_t = \sigma B_t + ct$ at regular intervals of length δ, then the differences $X(n\delta) - X((n-1)\delta)$, $n = 1, ..., N$, should be independent Gaussians with common mean $c\delta$ and variance $\sigma^2\delta$. If one computes the sample variance to estimate $\sigma^2\delta$ (and thence σ^2), one comes up with an estimate of σ^2 which is *not* independent of δ, as recent empirical work of Joubert & Rogers [3] on tick data shows; typically, the estimate of σ^2 decreases with δ. This calls into question the usefulness of estimating from high-frequency data; it should be possible to exploit the wealth of information available in such data, but as yet it is not clear how to do this. Accordingly, it makes sense to make the best estimate one can from *daily* data, specifically the open and close prices. However, the high and low prices each day are also readily available in financial newspapers; and various estimators based on the open, close, high and low prices have been proposed; see, for example, Parkinson [5], Garman & Klass [1], Rogers & Satchell [6]. The present study arose from the empirical fact that the crude estimator of σ^2 based only on open and close prices is consistently higher than any of the estimators based on open, close, high and low; see Rogers, Satchell & Yoon [7]. One possible explanation for this may lie in the often-observed phenomenon that one of the end points of the interval [low, high] can be the open or close price, which would be impossible if we were truly observing Brownian motion with a drift. In this paper, we attempt to understand this apparent discrepancy by assuming that a log Brownian motion drives the share prices along, but we only observe the Brownian motion at times when it has moved a distance ε from where it was last observed[1]. More precisely, if the log share price is

$$X_t = \sigma B_t + ct$$

we define $\tau_0 = 0$ and

$$\tau_{n+1} \equiv \inf \{ t > \tau_n : |X_t - X(\tau_n)| > \varepsilon \} ,$$

[1]This is a natural assumption if we realise that a share price will only be quoted to an accuracy of a penny. Strictly, we should observe X only when $\exp(X)$ has changed by ε, but the linear approximation is acceptable in view of the relatively small percentage changes under discussion.

and

$$(1) \qquad X_t^{(\varepsilon)} \equiv \sum_{n \geq 0} X(\tau_n) I_{\{\tau_n \leq t < \tau_{n+1}\}}.$$

If $\bar{X}_t^{(\varepsilon)} \equiv \sup\{X_u^{(\varepsilon)} : 0 \leq u \leq t\}$, we shall copmute in Section 2 an expression for $E\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)})$ and examine its behaviour as $\varepsilon \downarrow 0$. The reason for being interested in this is found in the remarkable fact (see [6]) that

$$(2) \qquad E\bar{X}_t(\bar{X}_t - X_t) = \frac{\sigma^2 t}{2} \quad \forall\ t$$

for all drifts c! We find the small-ε asymptotics. In the case $c = 0$ (which is a comparatively innocent restriction in view of the typical values of σ and c), for fixed $t > 0$,

$$(3) \qquad E\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) \sim \frac{\sigma^2 t}{2} - \varepsilon\sigma\sqrt{\frac{2t}{\pi}} + \frac{5\varepsilon^2}{12} - \frac{\varepsilon^3}{\sqrt{72\pi\sigma^2 t}}.$$

In Section 3 we present and discuss the results of a simulation study which compares the quantum-adjusted Rogers-Satchell (RS) estimate with other adjustments of the estimator. The first thing one notices from the results is that (with a few exceptions when the mean number of price moves in a time interval is small) *all the adjustments of the RS estimator agree to two significant figures.* The second fact is that the sample standard deviation of the various adjusted RS estimators is similar for all, and is *of the order of 30-50 % of the sample standard deviation of the naive open-close estimator.* The adjusted RS estimated values are also *quite accurate*; of 60 estimates computed for shares that move on average at least 10 times per day, only 5 were more than 2 standard deviations away from the true value, and 22 were within 1 standard deviation. None of the results explain the empirical observations of Rogers, Satchell & Yoon [7]; but in the conclusions we discuss various possible explanations for this discrepancy.

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2 Asymptotic expansion of the mean value of the estimator

The original proof of (2) proceeded via the introduction of an independent exponential random variable T with mean λ^{-1}, and then invoking the Wiener-Hopf factorisation of the Lévy process X to compute $E\bar{X}_T(\bar{X}_T - X_T)$. While this is not the only approach to (2) (a direct calculation with Brownian transition densities is perfectly feasible), in the context studied here, the Wiener-Hopf approach pays considerable dividends. In particular, it is clear that the process $(X(\tau_n \wedge T))_{n\geq 0}$ is a random walk stopped at an independent geometric time, and therefore

(4) $\qquad \bar{X}_T^{(\varepsilon)}$ *and* $\bar{X}_T^{(\varepsilon)} - X_T^{(\varepsilon)}$ *are independent;*

(5) $\qquad -(\bar{X}_T^{(\varepsilon)} - X_T^{(\varepsilon)})$ *has the same law as* $\underline{X}_T^{(\varepsilon)} \equiv \inf\left\{X_u^{(\varepsilon)} : 0 \leq u \leq T\right\}$.

The proof of this may be found in Greenwood & Pitman [2] (see also Rogers & Williams [8], I.29). Thus we have

$$
\begin{aligned}
E\bar{X}_T^{(\varepsilon)}(\bar{X}_T^{(\varepsilon)} - X_T^{(\varepsilon)}) &= E\bar{X}_T^{(\varepsilon)} \quad E(\bar{X}_T^{(\varepsilon)} - X_T^{(\varepsilon)}) \\
&= E\bar{X}_T^{(\varepsilon)} \quad E(-\underline{X}_T^{(\varepsilon)}),
\end{aligned}
\tag{6}
$$

so our first goal is to find an expression for the mean of $\bar{X}_T^{(\varepsilon)}$ (and of $-\underline{X}_T^{(\varepsilon)}$, although this follows from the first, by changing the drift c to drift $-c$). But

$$
\begin{aligned}
E\bar{X}_T^{(\varepsilon)} &= \varepsilon \sum_{n\geq 1} P[\bar{X}_T^{(\varepsilon)} \geq n\varepsilon] \\
&= \varepsilon \sum_{n\geq 1} P(H_{n\varepsilon} \leq T),
\end{aligned}
\tag{7}
$$

where $H_a \equiv \inf\{t : X_t = a\}$. It is well known that for $X_t = \sigma B_t + ct$

$$
P(H_{n\varepsilon} \leq T) = E\exp(-\lambda H_{n\varepsilon}) = \exp\left[-n\varepsilon(\sqrt{c^2 + 2\lambda\sigma^2} - c)/\sigma^2\right],
\tag{8}
$$

and

$$
\begin{aligned}
P[H_{-n\varepsilon} \leq T] &= \exp\left[-n\varepsilon(\sqrt{c^2 + 2\lambda\sigma^2} + c)/\sigma^2\right] \\
&= e^{-2n\varepsilon c/\sigma^2} P[H_{n\varepsilon} \leq T].
\end{aligned}
\tag{9}
$$

Abbreviating $\gamma \equiv \sqrt{c^2 + 2\lambda\sigma^2}$, we have therefore

$$
\begin{aligned}
(10) \qquad E[\bar{X}_T^{(\varepsilon)}]E[-\underline{X}_T^{(\varepsilon)}] &= \varepsilon^2 \sum_{n\geq 1} e^{-n\varepsilon(\gamma-c)/\sigma^2} \sum_{m\geq 1} e^{-m\varepsilon(\gamma+c)/\sigma^2} \\
&= \varepsilon^2 \sum_{r=2}^{\infty} \left(\sum_{m=1}^{r-1} e^{-2m\varepsilon c/\sigma^2)} \right) e^{-r\varepsilon(\gamma-c)/\sigma^2} \\
&= \varepsilon^2 \sum_{r=1}^{\infty} \left\{ \frac{1-e^{-2r\varepsilon c/\sigma^2}}{1-e^{-2\varepsilon c/\sigma^2}} - 1 \right\} P[H_{r\varepsilon} \leq T],
\end{aligned}
$$

at least if $c \neq 0$. Now we can invert the Laplace transform by inspection of (6) and (10) to discover that (with $b \equiv 2c/\sigma^2$)

$$
(11) \qquad E\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) = \varepsilon^2 \sum_{r=1}^{\infty} \left\{ \frac{1-e^{-rb\varepsilon}}{1-e^{-b\varepsilon}} - 1 \right\} P[H_{r\varepsilon} \leq t].
$$

We introduce the notation for $x \in \mathbb{R}$, $t > 0$

$$
F(x,t) \equiv P[H_x \leq t] = \int_0^t \frac{|x|}{\sigma} \exp\left\{ -\frac{(x-cs)^2}{2\sigma^2 s} \right\} \frac{ds}{\sqrt{2\pi s^3}},
$$

which is the well-known Brownian first-passage distribution (see, for example Rogers & Williams [8], I.9). In what follows, we shall mostly be thinking of $t > 0$ as fixed, so we shall generally abbreviate $F(x,t)$ to $F(x)$. Clearly, $F(x)$ decreases from 1 to 0 as x increases from 0 to ∞. In terms of this notation, (11) becomes

$$
(12) \qquad E\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) = \frac{\varepsilon^2}{1-e^{-b\varepsilon}} \sum_{r\geq 1} (1-e^{-rb\varepsilon})F(r\varepsilon) - \varepsilon^2 \sum_{r\geq 1} F(r\varepsilon).
$$

The first term of the right-hand side is approximately

$$
\begin{aligned}
(13) \qquad & \frac{1}{b}\int_0^\infty (1-e^{-bx})F(x)dx \\
&= \frac{1}{b}\int_0^\infty (1-e^{-bx}) \left(\int_0^t \frac{x}{\sigma} \exp\left\{ -\frac{(x-cs)^2}{2\sigma^2 s} \right\} \frac{ds}{\sqrt{2\pi s^3}} \right) dx \\
&= \frac{1}{b}\int_0^t \frac{ds}{\sqrt{2\pi s^3}} \int_{-\infty}^\infty \frac{x}{\sigma} \exp\left(-\frac{(x-cs)^2}{2\sigma^2 s} \right) dx \\
&= \frac{\sigma^2 t}{2}
\end{aligned}
$$

which is in agreement with (2), since the second term on the right-hand side of (12) is (to first-order in ε)

$$
-\varepsilon \int_0^\infty F(x)dx.
$$

However, we can be much more precise than this; the sums appearing on the right-hand side of (12) are Riemann-sum approximations to certain Riemann integrals, and the closeness of approximation is the subject of the Euler-Maclaurin expansion (see, for example, Olver [4], Chapter 8). This says that for any integer $m \geq 1$, if $\varphi : [0, a] \to \mathbb{R}$ is C^{m+1}, then

$$\varphi(a) - \varphi(0) - \varepsilon \sum_{n=1}^{N} \varphi'(n\varepsilon) = \sum_{r=1}^{m} c_r \{\varphi^{(r)}(a) - \varphi^{(r)}(0)\} \varepsilon^r + \varepsilon^{m+1} R_{m+1} \tag{14}$$

where $\varepsilon \equiv a/N$ is tending to 0, and the remainder term R_{m+1} remains bounded. The coefficients c_r are determined by the relation

$$\begin{aligned} \sum_{r \geq 1} c_r \theta^r &= 1 - \theta - \frac{\theta}{e^\theta - 1} \\ &= 1 - \theta - \sum_{k \geq 0} B_k \frac{\theta^k}{k!} \end{aligned} \tag{15}$$

where $(B_k)_{k \geq 0}$ are the Bernoulli numbers, defined by the above relation (15). The first few non-zero Bernoulli numbers are

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad B_8 = -\tfrac{1}{30}, \quad B_{10} = \tfrac{5}{66}. \tag{16}$$

In fact, the present application requires the Euler-Maclaurin formula applied to the unbounded interval $[0, \infty)$, and for this we require some bounds on the behaviour at infinity of $F(x)$. The following result is much more than we need.

PROPOSITION 1 *The function F is C^∞ on $\mathbb{R}^+$, and for each $n \geq 0$*

$$\sup_{x \geq 0} |F^{(n)}(x)| \exp\{(x - ct)^2 / 3\sigma^2 t\} = K_n < \infty. \tag{17}$$

Proof We take first the region $0 \leq x \leq 2ct$ and write

$$\bar{F}(x) \equiv 1 - F(x) = \int_t^\infty \frac{x}{\sigma} \exp\left\{-\frac{(x - cs)^2}{2\sigma^2 s}\right\} \frac{ds}{\sqrt{2\pi s^3}}$$

as

$$\begin{aligned} e^{-cx/\sigma^2} \bar{F}(x) &= \int_t^\infty \frac{x}{\sigma} e^{-x^2/2\sigma^2 s} e^{-c^2 s/2\sigma^2} \frac{ds}{\sqrt{2\pi s^3}} \\ &= \sum_{n \geq 0} \int_t^\infty \frac{x}{\sigma} \left(\frac{-x^2}{2\sigma^2 s}\right)^n \frac{1}{n!} e^{-c^2 s/2\sigma^2} \frac{ds}{\sqrt{2\pi s^3}}, \end{aligned}$$

the power series being absolutely convergent because

$$\int_t^\infty \frac{x}{\sigma} \exp\left(\frac{x^2}{2\sigma^2 s}\right) e^{-c^2 s/2\sigma^2} \frac{ds}{\sqrt{2\pi s^2}} < \infty.$$

The analyticity of F in $[0, 2ct]$ follows.

For the region $x \geq 2ct$, we have

$$\begin{aligned} e^{-cx/\sigma^2} F(x) &= \int_0^t \frac{x}{\sigma} \exp\left[-\frac{x^2}{2\sigma^2 s} - \frac{c^2 s}{2\sigma^2}\right] \frac{ds}{\sqrt{2\pi s^3}} \\ &= \int_{x/t}^\infty \exp\left[-\frac{x}{2\sigma^2}(v + c^2/v)\right] \frac{\sqrt{x} dv}{\sqrt{2\pi\sigma^2 v}} . \end{aligned}$$

Hence

$$\begin{aligned} H(x) &\equiv x^{-1/2} e^{-cx/\sigma^2} F(x) \\ &= \int_{x/t}^\infty \exp\left[-\frac{x}{2\sigma^2}(v + c^2/v)\right] \frac{dv}{\sqrt{2\pi\sigma^2 v}} . \end{aligned}$$

The smoothness of F in $[2ct, \infty)$ is now evident, and the boundedness of $\exp\{(x - ct)^2/3\sigma^2 t\} F^{(n)}(x)$ is easy to deduce.

The Euler-Maclaurin expansion on the infinite interval reads

$$\varphi(\infty) - \varphi(0) - \varepsilon \sum_{n \geq 1} \varphi'(n\varepsilon) = \sum_{r \geq 1} c_r \varepsilon^r \{\varphi^{(r)}(\infty) - \varphi^{(r)}(0)\}. \tag{18}$$

If we now assume $c > 0$ and define functions φ_0 and φ_1 by $\varphi_0(0) = \varphi_1(0) = 0$, and

$$\varphi_0'(x) = b^{-1}(1 - e^{-bx}) F(x), \quad \varphi_1'(x) = F(x),$$

then using the Euler-Maclaurin expansion (18) and observing that $\varphi_0^{(n)}(\infty) = \varphi_1^{(n)}(\infty) = 0$ for all $n \geq 0$, we conclude from (12) that

$$\begin{aligned} E[\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)})] &= \frac{b\varepsilon}{1 - e^{-b\varepsilon}} \left\{ \frac{\sigma^2 t}{2} + \sum_{r \geq 1} c_r \varepsilon^r \varphi_0^{(r)}(0) \right\} \\ &\quad -\varepsilon \left\{ \int_0^\infty F(x) dx + \sum_{r \geq 1} c_r \varepsilon^r \varphi_1^{(r)}(0) \right\}. \end{aligned} \tag{19}$$

Taking the terms in the expansion up to order ε^4, and using the fact that

$$\int_0^\infty F(x) dx = \int_0^\infty P[\bar{X}_t > x] dx = E\bar{X}_t,$$

we obtain

$$
\begin{aligned}
(20) E\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) &= \frac{\sigma^2 t}{2} + \varepsilon\left[\frac{ct}{2} - E\bar{X}_t\right] + \frac{\varepsilon^2}{12}\left[\frac{2c^2 t}{\sigma^2} + 5\right] \\
& -\frac{\varepsilon^3}{12}\left[\frac{c}{\sigma^2} + \int_t^\infty e^{-c^2 s/2\sigma^2}\frac{ds}{\sigma\sqrt{2\pi s^3}}\right] - \frac{\varepsilon^4 c^2}{90\sigma^4}[2 + \frac{c^2 t}{\sigma^2}] \\
& +0(\varepsilon^5).
\end{aligned}
$$

We have in general that

$$
E\underline{X}_t^{(\varepsilon)}(\underline{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) = E\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}),
$$

so that the mean of the analogue of the Rogers-Satchell estimator $\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) + \underline{X}_t^{(\varepsilon)}(\underline{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)})$ is just twice the expression on the right-hand side of (20). A word of caution: do not use (20) for $c < 0$! The left-hand side of (20) should be (and is) the same for $-c$ as for $+c$, but the expression on the right-hand side is only correct for $c \geq 0$. The remedy is obvious; substitute $|c|$ throughout the right-hand side.

While no closed-form expression holds for $c \neq 0$, for the case $c = 0$ we can establish

$$
\begin{aligned}
(21) \qquad E\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) &= \frac{\sigma^2 t}{2} - \varepsilon\sigma\sqrt{\frac{2t}{\pi}} + \frac{5\varepsilon^2}{12} - \frac{\varepsilon^3}{6}(2\pi\sigma^2 t)^{-\frac{1}{2}} \\
& -\frac{\varepsilon^5}{360(2\pi\sigma^6 t^3)^{\frac{1}{2}}} + O(\varepsilon^6).
\end{aligned}
$$

Is it reasonable in practice to proceed on the assumption that $c = 0$? If not, we could always make some simple estimate of c and proceed using that, but the dependence of the integral terms on the right-hand side of (20) on c and σ complicates matters. Taking some typical values, $\sigma = 0.2$ and $c = 0.12$ would be reasonable for the annualised volatility and returns, and the unit of time $t = 1/250$ would correspond to a trading day. With these values, we consider the coefficients of the powers of ε on the right-hand side of (20). The term in ε has $ct/2 = 0.00024$, and $E\bar{X}_t = \sqrt{2t/\pi} = 0.05046$ when $c = 0$, so the contribution of $ct/2$ is unimportant. For the term in ε^2 we have $2c^2 t/\sigma^2 = 0.0029$, which is negligible compared with 5. The term in ε^3 has a contribution $c/\sigma^2 = 3$, to be compared with the integral, which is equal to $2/\sqrt{2\pi\sigma^2 t} = 63.08$ when computed with $c = 0$. It is clear then that for typical values of the parameters the assumption that $c = 0$ cannot do much damage, and will simplify the calculations.

What is the impact of this in practice? If we saw on each day $j (j = 1, ..., N)$ the high price S_j, low price I_j, opening price O_j and closing price C_j,[2] we could use these to form the Rogers-Satchell estimator of σ^2:

$$\hat{\sigma}^2_{\mathrm{RS}} = \frac{1}{Nt} \sum_{j=1}^{N} \{(S_j - O_j)(S_j - C_j) + (C_j - I_j)(O_j - I_j)\}, \tag{22}$$

using $t = 1/250$ since we are assuming 250 trading days per year. Knowing the size ε of the price-quantum, we would now modify this estimate to $\hat{\sigma}^2_\varepsilon$ which solves

$$\hat{\sigma}^2_\varepsilon = \hat{\sigma}^2_{\mathrm{RS}} + 2\varepsilon\hat{\sigma}_\varepsilon \sqrt{\frac{2}{\pi t}} - \frac{5\varepsilon^2}{6t} + \frac{\varepsilon^3}{3}(2\pi\hat{\sigma}^2_\varepsilon t^3)^{-\frac{1}{2}}. \tag{23}$$

This is an expansion in powers of $\varepsilon/\sqrt{t}$ (which should not be surprising!) Abbreviating $h \equiv \varepsilon/\sqrt{t}$, we have

$$\hat{\sigma}^2_\varepsilon = \hat{\sigma}^2_{\mathrm{RS}} + h\hat{\sigma}_\varepsilon \sqrt{\frac{8}{\pi}} - \frac{5h^2}{6} + \frac{h^3}{\hat{\sigma}_\varepsilon\sqrt{18\pi}}. \tag{24}$$

Observe that $\sqrt{8/\pi} = 1.596$ and $(18\pi)^{-1/2} = 0.1330$, so that $(\hat{\sigma}_\varepsilon\sqrt{18\pi})^{-1}$ will be of the order of 0.66. In order that this approximation should be good, we shall want h to be reasonably small. If we took a typical value for σ as being approximately 0.2, we would want h to be at most 0.1 to ensure that the order-h term was not too large compared with $\hat{\sigma}^2_{\mathrm{RS}}$, corresponding to ε being of the order of 10^{-3}. Such figures are reasonable; a \$ 10 share quoted to the nearest penny would give such an ε.

3 Numerical study

The effects of this correction and others were investigated numerically, as follows. Firstly, a value for σ is picked, and a unit of time τ fixed; we report in detail on the case where $\sigma = 0.2$ and $\tau = 1/250$, corresponding to a unit of time equal to one trading day. Next, a value of ε is chosen, in the light of two considerations. One is that ε should be a realistic proportion of a share to stand for a price quantum (and for this the range 10^{-2} to 10^{-4} looks appropriate); the other is that the expected

[2] We mean log prices, of course. The opening price now enters because we have up to now assumed that the process X begins at zero.

number of price movements in the day, $\tau\sigma^2/\varepsilon^2$, should also be reasonable. With $\tau = 1/250$, $\sigma = 0.2$, this ranges from 1.6 ($\varepsilon = 10^{-2}$) to 1600 ($\varepsilon = 10^{-4}$), which covers any imaginable real situation.

The distribution of $T \equiv \inf\{u : |B_u| = 1\}$ is not known in closed form, but expressions for it can be found; using one of these, the distribution function F of T was computed and stored numerically. The inverse distribution F^{-1} was also computed and stored numerically, and then a sequence $U_1, U_2, ...$ of pseudo-random $U(0,1)$ numbers was generated. From these, random variables $T_k = F^{-1}(U_k)$ were computed, and used as follows. The partial sums $S_n = \sum_{k=1}^{n}(\varepsilon/\sigma)^2 T_k$ are computed sequentially, and at the same time a random walk (ξ_n) is computed; the increments are independent of (S_n) and take values $\pm\varepsilon$ with equal probability. The running values of $\bar{\xi}_n = \sup_{k\leq n}\xi_k$ and $\underline{\xi}_n = \inf_{k\leq n}\xi_k$ are computed and held. The random walk is stopped once S_n exceeds the unit of time τ. At this point, the values of $(\xi, \bar{\xi}, \underline{\xi})$ are stored, and constitute the result of the first simulation. The whole procedure is repeated to produce a total of N simulations; let the outputs be denoted (X_j, H_j, L_j), $j = 1, ..., N$, with $H_j \geq X_j \geq L_j$.

For each period j, we compute estimates $v_{\alpha,j}$ of the variance σ^2, where the label α belongs to the index set $\Lambda = \{OC, RS, RS1, RS2, RSQ\}$. The five estimators are defined as follows.

(i) The open-close estimator, v_{OC}. This is the most primitive; we take

$$v_{OC,j} \equiv X_j^2/\tau.$$

If X_j were the value of σB at time τ, then we would know that $Ev_{OC} = \sigma^2$. Since it is formed from the embedded random walk, we know that in fact

$$Ev_{OC} < \sigma^2;$$

but we have no closed-form expression for the bias as a function of ε and τ (if we had, we would have a closed-form expression for the first-passage density of the BES (3) process). Accordingly, we let v_{OC} stand as a (biased) estimator of σ^2; as $\varepsilon \downarrow 0$, the bias decreases to zero.

(ii) The Rogers-Satchell estimator, v_{RS}. This time, we use the definition

$$v_{RS,j} \equiv \{H_j(H_j - X_j) + L_j(L_j - X_j)\}/\tau.$$

As before, as $\varepsilon \downarrow 0$ the bias decreases to zero.

(iii) The first adjusted Rogers-Satchell estimator, v_{RS1}. In [6], a simple correction to the Rogers-Satchell estimator was presented. The situation was that an underlying Brownian motion was viewed at time intervals of δ, and the sup and inf of the resulting embedded random walk were recorded – but this sup underestimates the true sup! To compensate for this, we find the positive root s of the quadratic

$$s^2 = v_{RS} + \frac{2(H-L)a_1\sqrt{\delta}}{\tau}\ s + \frac{2b_1\delta}{\tau}\ s^2$$

where $a_1 \equiv \sqrt{2\pi}\{\frac{1}{4} - (\sqrt{2}-1)/6\}$, $b_1 \equiv (4+3\pi)/48$, and then set

$$v_{RS1} = s^2.$$

Note that in [6], only the case where $\tau = 1$ was discussed, but the general story is easily deduced.

While this adjustment is inappropriate (the quantisation is happening here in the *spatial* variable, not time), we still compute it. What should be the value of δ? Surely we can do no other than choose the mean time between price moves, namely ε^2/σ^2; we simply take the *true* value for this. This seems the best we can do in the circumstances; in any application of the adjustment of Rogers & Satchell, we would need to rely on *some* additional information on the volume of trading, and what that information was would determine how we chose δ. Explicitly,

$$(v_{RS1})^{\frac{1}{2}} = \frac{a_1(H-L)\sqrt{\delta} + \{a_1^2(H-L)^2\delta + (H(H-X) + L(L-X))(\tau - 2b_1\delta)\}^{\frac{1}{2}}}{\tau - 2b_1\delta}.$$

(iv) The second adjusted Rogers-Satchell estimator, v_{RS2}. The adjustment of [6] was intended to compensate for the discrepancy Δ between the true sup of the Brownian motion and the sup of the embedded random walk. But in this case, since the discretisation is happening in space, the discrepancy Δ is simply *the sup* *of a Brownian motion conditioned to hit $-\varepsilon$ before it hits $+\varepsilon$.*[3] Routine calculations with the scale function give us

$$P[\Delta < x|\text{hit } -\varepsilon \text{ before } +\varepsilon] = \frac{2x}{\varepsilon + x}$$

so that

$$\begin{aligned} E[\Delta|\text{hit } -\varepsilon \text{ before } +\varepsilon] &= \varepsilon(2\log 2 - 1) \equiv \varepsilon a_2, \\ E[\Delta^2|\text{hit } -\varepsilon \text{ before } +\varepsilon] &= \varepsilon^2(3 - 4\log 2) \equiv \varepsilon^2 b_2. \end{aligned}$$

Since the sup of the Brownian motion would be $H^* \equiv H + \Delta$, and the inf would be $L^* \equiv L - \tilde{\Delta}$, where $\tilde{\Delta}$ is an independent copy of Δ, we get

$$\begin{aligned} H^*(H^* - X) + L^*(L^* - X) &= H(H - X) + L(L - X) + \Delta(2H - X) + \Delta^2 \\ &\quad -\tilde{\Delta}(2L - X) + \tilde{\Delta}^2 \end{aligned}$$

by taking expectations we get on the left-hand side $\sigma^2\tau$ and on the right-hand side

$$\tau E v_{RS} + 2E(H - L)E\Delta + 2E\Delta^2.$$

Accordingly, we define v_{RS2} by

$$\tau v_{RS2} = \tau v_{RS} + 2(H - L)\varepsilon a_2 + 2\varepsilon^2 b_2.$$

(v) The 'quantum' correction to the Rogers-Satchell estimate, v_{RSQ}. Recalling the relation (24), we find the unique positive s solving the (cubic)

$$s^2 = v_{RS} + hs\sqrt{\frac{8}{\pi}} - \frac{5h^2}{6} + \frac{h^3}{s\sqrt{18\pi}},$$

and then define

$$v_{RSQ} = s^2.$$

As before, $h \equiv \varepsilon/\sqrt{\tau}$.

We then compute $\bar{v}_\alpha \equiv N^{-1}\sum_{j=1}^N v_{\alpha,j}$ for each $\alpha \in \Lambda$, together with the associated sample standard deviation

$$SD_\alpha = \left\{ N^{-1}\sum_{j=1}^N v_{\alpha j}^2 - \bar{v}_\alpha^2 \right\}^{\frac{1}{2}}$$

[3] Strictly speaking, this would not be true if the sup were attained at the last move of the price, but we have no worries over ignoring this!

and the results obtained are shown in Table 1. Also in Table 1 we display the corresponding estimates for various aggregations of the simulation data, to investigate how the estimates react to changes in ε, or changes in τ. To form aggregates in ε, we pick some integer m bigger than 1, and only view the random walk when it moves across the grid $m \in \mathbb{Z}$ (the values H_j, L_j for this aggregated problem can be computed from the corresponding values for the original problem, but the simulation has to separately keep track of the X_j for each aggregation of interest). The time aggregation is done in the obvious way.

The results in Table 1 demonstrate a number of conclusions.

1. The standard deviations of adjusted RS estimators typically are about 30-50 % of the standard deviation of the simple estimator v_{OC};
2. All the adjusted RS estimators agree well, usually to at least two significant figures;
3. For high intensity conditions, all the high-low estimators are good (a 95 % confidence interval contained the true value in 55 cases out of 60 observations);
4. The estimators improve as ε decreases and τ increases;
5. Usually $v_{RS1} > v_{RS2}$;
6. With these simulations, we always find $v_{RSQ} > v_{RS1}$.

To explain points 5 and 6, we observe that to find v_{RS1}, we solve (for standard deviation s)

$$s^2 = v_{RS} + \frac{H-L}{\sqrt{\tau}} \,.\, 2a_1 \frac{\varepsilon}{\sqrt{\tau}} \,.\, \frac{s}{\sigma} + 2b_1 \,.\, \frac{\varepsilon^2}{\tau} \,.\, \frac{s^2}{\sigma^2} \tag{25}$$

to find v_{RS2} we solve

$$s^2 = v_{RS} + \frac{H-L}{\sqrt{\tau}} \,.\, 2a_2 \,.\, \frac{\varepsilon}{\sqrt{\tau}} + 2b_2 \,.\, \frac{\varepsilon^2}{\tau} \tag{26}$$

and to find v_{RSQ} we solve

$$s^2 = v_{RS} + s\sqrt{\frac{8}{\pi}} \,.\, \frac{\varepsilon}{\sqrt{\tau}} - \frac{5}{6}\frac{\varepsilon^2}{\tau} + \frac{1}{s} \,.\, \left(\frac{\varepsilon^2}{\tau}\right)^{\frac{3}{2}} \,.\, (18\pi)^{-\frac{1}{2}}. \tag{27}$$

We note that $E((H-L)/\sqrt{\tau}) = \sqrt{8/\pi}$, and $a_1 = 0.45361 > a_2 = 0.38629$. Also, $b_1 = 0.27968 > b_2 = 0.22741$. Thus if we replaced $\frac{s}{\sigma}$ throughout (25) by 1, (which should be approximately true), it is clear the estimator v_{RS1} would typically be bigger than v_{RS2}. Likewise, since $2a_1 = 0.90722 < 1$, we are not surprised to see (comparing terms up to first order in (25) and (27) that v_{RSQ} comes out typically higher than v_{RS1}. Also, since $(H-L)/\sqrt{\tau}$ in (25) gets replaced by its mean value in (27), we are not surprised to see the variance of v_{RSQ} slightly smaller. However, the low values of v_{OC}, even with small ε, are surprising, and demand explanation.

4 Conclusions

We have constructed an adjustment to the basic RS estimator introduced in [6] which performs well in simulation. Its behaviour is very similar to that of other adjustments of the RS estimator, as has been shown both theoretically and numerically. The simulation study demonstrates conclusively that we should always prefer to use one of these estimators rather than the crude open-close estimator, since the standard deviation of the open-close estimator is typically 2-3 times larger.

We have not found an explanation for the observation of Rogers, Satchell & Yoon [7], who found that when used on actual daily data, the open-close estimator gave values that were quite a lot higher than those from all estimators based on highs and lows as well. One possible explanation for this is as follows. Suppose that on different days the log share price is moving like a Brownian motion with a drift that may change from day to day (there may be justification for this, in that there tends to be some negative correlation between the price changes on successive days). Thus on day j, the log share price is $X_t = \sigma B_t + c_j t$, $(j-1)\delta < t \leq j\delta$, where the drift c_j is decided at the beginning of the day. Suppose for simplicity that the process c is stationary, with mean zero. Now if this were the case, then the mean of the open-close estimator would be $E[v_{OC}] = \sigma^2 + E[c_1^2]/\delta$, whereas the mean of the RS estimator would be σ^2, because of the remarkable fact that the RS estimator is an unbiased estimator of the variance for *all* values of the drift. If this were the explanation of the observations of Rogers, Satchell & Yoon, then we have an additional reason to use one of the adjusted RS estimators; because the variance that should be input to any calculation of option prices should be the value σ^2, not the higher value $E[v_{OC}]$. Set against this is the observation that any high-low estimator is making use of properties of the Brownian model much more heavily than the simple open-close estimator, which must be expected to be more robust to departures from the Brownian model.

Another possible interpretation of the failure to explain the observed data may be that the Brownian model of share prices is not a good model of reality, and a model with some sort of stochastic volatility may well do a better job; this is undoubtedly true, but it seems hard to think of a qualitative mechanism which would produce the observed tendency for high or low prices to occur at the beginning or end of a day.

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TABLE 1a. Estimates of $100\sigma^2$, with $\varepsilon = 4 \times 10^{-4}$

(Average 1000 price moves per day)

Days/block	v_{OC}	v_{RS}	v_{RS1}	v_{RS2}	v_Q
1	4.082	3.816	3.999	3.971	4.006
	(0.077)	(0.030)	(0.031)	(0.030)	(0.030)
5	3.941	4.050	4.135	4.120	4.138
	(0.153)	(0.067)	(0.068)	(0.067)	(0.067)
25	3.943	3.974	4.011	4.005	4.013
	(0.351)	(0.149)	(0.150)	(0.149)	(0.150)
60	3.179	4.285	4.309	4.304	4.311
	(0.391)	(0.243)	(0.244)	(0.243)	(0.244)
250	3.624	3.542	3.552	3.551	3.553
	(1.357)	(0.342)	(0.343)	(0.342)	(0.342)

TABLE 1b. Estimates of $100\sigma^2$, with $\varepsilon = 8 \times 10^{-4}$

(Average 250 price moves per day)

Days/block	v_{OC}	v_{RS}	v_{RS1}	v_{RS2}	v_Q
1	4.080	3.619	3.982	3.927	3.991
	(0.076)	(0.029)	(0.032)	(0.030)	(0.031)
5	3.933	3.958	4.126	4.097	4.131
	(0.153)	(0.066)	(0.069)	(0.067)	(0.068)
25	3.944	3.932	4.005	3.994	4.009
	(0.351)	(0.149)	(0.152)	(0.150)	(0.151)
60	3.169	4.252	4.301	4.291	4.304
	(0.389)	(0.242)	(0.244)	(0.242)	(0.243)
250	3.585	3.521	3.541	3.539	3.544
	(1.357)	(0.345)	(0.346)	(0.345)	(0.346)

TABLE 1c. Estimates of $100\sigma^2$, with $\varepsilon = 16 \times 10^{-4}$

(Average 62.5 price moves per day)

Days/block	v_{OC}	v_{RS}	v_{RS1}	v_{RS2}	v_Q
1	4.074	3.255	3.969	3.862	3.967
	(0.076)	(0.028)	(0.034)	(0.030)	(0.032)
5	3.944	3.777	4.110	4.055	4.118
	(0.154)	(0.065)	(0.070)	(0.067)	(0.068)
25	3.942	3.867	4.013	3.990	4.020
	(0.349)	(0.150)	(0.155)	(0.152)	(0.153)
60	3.189	4.224	4.323	4.304	4.328
	(0.387)	(0.240)	(0.245)	(0.241)	(0.243)
250	3.568	3.513	3.555	3.551	3.560
	(1.315)	(0.341)	(0.344)	(0.342)	(0.344)

TABLE 1d. Estimates of $100\sigma^2$, with $\varepsilon = 40 \times 10^{-4}$

(Average 10 price moves per day)

Days/block	v_{OC}	v_{RS}	v_{RS1}	v_{RS2}	v_Q
1	4.053	2.320	4.105	3.773	3.899
	(0.075)	(0.027)	(0.042)	(0.031)	(0.036)
5	3.936	3.316	4.139	3.997	4.129
	(0.154)	(0.063)	(0.076)	(0.066)	(0.070)
25	3.961	3.641	4.004	3.947	4.015
	(0.348)	(0.146)	(0.158)	(0.150)	(0.153)
60	3.183	4.084	4.329	4.282	4.341
	(0.385)	(0.236)	(0.248)	(0.240)	(0.243)
250	3.610	3.423	3.525	3.516	3.538
	(1.338)	(0.350)	(0.357)	(0.352)	(0.356)

TABLE 1e. Estimates of $100\sigma^2$, with $\varepsilon = 80 \times 10^{-4}$

(Average 2.5 price moves per day)

Days/block	v_{OC}	v_{RS}	v_{RS1}	v_{RS2}	v_Q
1	4.030	1.175	5.455	3.910	3.408
	(0.075)	(0.026)	(0.076)	(0.033)	(0.048)
5	3.914	2.628	4.276	3.954	4.098
	(0.154)	(0.059)	(0.087)	(0.066)	(0.076)
25	3.978	3.239	3.946	3.841	3.952
	(0.353)	(0.142)	(0.167)	(0.149)	(0.157)
60	3.191	3.865	4.355	4.259	4.369
	(0.393)	(0.224)	(0.248)	(0.231)	(0.238)
250	3.846	3.300	3.504	3.487	3.525
	(1.355)	(0.355)	(0.371)	(0.359)	(0.368)