Brownian motion in a wedge with variable skew reflection: II

by

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1. Introduction. Let \( D = \{ z = re^{i\theta} \in \mathbb{C} : \ r > 0, \ 0 < \theta < \xi \} \) be a wedge in the complex plane, and suppose given on \( \partial D \) a vector field pointing into \( D \), which will specify the direction of reflection of Brownian motion in \( D \) when it hits the boundary. There are several natural questions which one can ask about this process, in particular;

(1.i) Can one construct the process, at least until it first reaches 0?

(1.ii) Does the process approach 0?

(1.iii) If so, will the process reach 0 in finite time?

(1.iv) If so, can the process be extended beyond its first hit on 0 so as to have continuous paths, and to spend no time in 0?

(1.v) If so, is such an extension unique?

In the case where the directions of reflection are constant on each side of the wedge, these questions were completely answered in [VW] (see also [R1] for an excursion-theoretic derivation of the results.) The case of variable direction of reflection is considerably more difficult, and has been begun in [R2], to which this paper is a sequel.

In order to state the main results of this paper, we review the notation and results of the earlier paper [R2]. The construction (1.i) is sufficiently straightforward as to need no comment here. For convenience, we may assume that the directions of reflection on \( \partial D \) are normal outside a neighbourhood of 0, since the answers to (1.ii-v) are determined by what happens in a neighbourhood of 0. The first step in [R2] is to transform \( D \) by the map \( z \mapsto -1/z^{\pi/\xi} \), which takes \( D \) to \( H \), mapping 0 to \(+\infty\), and preserving the directions of reflection on the boundary; thus (1.ii) is equivalent to the question, 'Is the skew-reflecting Brownian motion in \( H \) transient?' (Aside: the 'skew-reflecting Brownian mapping theorem', to which we have just appealed, is an 'obvious fact', but, thanks to Chris Burdzy's insistence, I have set down a proper statement and proof of the result in the appendix!)
The reflection vector field on $\partial H$ is specified by a function $\theta : \mathbb{R} \to (-\pi/2, \pi/2)$, the reflection at $x \in \mathbb{R}$ being in a direction making angle $\theta(x)$ in a clockwise sense with the inward-pointing normal. In [R2], the function $\theta$ was taken to be $C^1$ with bounded derivatives, satisfying $|\tan \theta(x)| \leq A(1 + |x|)$ for some $A$. These conditions are unnecessarily restrictive - local Hölder $\alpha$ for some $\alpha \in (0, 1]$ would suffice, and even weaker conditions work in some sense, as we shall see.

Define now the analytic function $\psi : H \to \mathbb{C}$ by

$$
\psi(z) = \exp \left[ \int_{-\infty}^{\infty} \frac{\theta(x)dx}{\pi} \left\{ \frac{1}{x - z} - \frac{x}{1 + x^2} \right\} \right].
$$

We shall insist that $\psi$ can be extended continuously to $\overline{H}$ (a local Hölder condition on $\theta$ will ensure this). The essential property of $\psi$ is that its argument at $x \in \mathbb{R}$ is $\theta(x)$. To see why this is relevant, suppose that $h : \overline{H} \to \mathbb{R}$ is $C^2$. Then a simple Itô-formula calculation (carried out in [R2]) shows that, if $Z$ is the skew reflecting Brownian motion, then $h(Z_t)$ is a local martingale if and only if

$$
(3.1) \quad \Delta h = 0 \quad \text{in} \quad H
$$

$$
(3.ii) \quad \tan \theta(x) \frac{\partial h}{\partial x}(x) + \frac{\partial h}{\partial y}(x) = 0 \quad \forall x \in \mathbb{R}
$$

If now we take $g$ to be the conjugate function to $h$, so that $f \equiv g + ih$ is analytic in $H$, the boundary condition (3.ii) can be restated as

$$
(4) \quad \text{Re}(f'(x)/\psi(x)) = 0 \quad \text{for all} \quad x \in \mathbb{R}.
$$

The key to the study of this problem appears to be the construction of suitable analytic $f$ satisfying the boundary condition (4).

To state the main results of this paper, we need some simple facts about analytic functions $\phi : H \to H$. This class of functions is called the class of Pick functions by Donoghue [Do], from whose book we quote the following facts.

Every Pick function $\phi$ has the representation

$$
(5) \quad \phi(z) = \int_{-\infty}^{\infty} \frac{\mu(dx)}{\pi} \left[ \frac{1}{x - z} - \frac{x}{1 + x^2} \right] + c_1 z + c_2,
$$

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where $c_1, c_2$ are real constants, $c_1 \geq 0$, and $\mu$ is a measure satisfying the integrability condition
\[
\int \frac{\mu(dx)}{1 + x^2} < \infty.
\]
If $\text{Im} \phi$ can be extended continuously to $\overline{H}$, then we may take $\mu(dx) = \text{Im} \phi(x) dx$. The constant $c_1$ is identified as
\[
(6) \quad c_1 = \lim_{b \to \infty} \frac{\phi(i b)}{i b}.
\]
Notice that $i \psi$ is a Pick function, so has a representation (5).

Define
\[
\phi_+(z) \equiv \exp\left[\int_0^{\infty} \frac{\theta(x) + \pi/2}{\pi} dx \left\{ \frac{1}{x - z} - \frac{x}{1 + x^2} \right\} \right],
\]
\[
\phi_-(z) \equiv i \psi(z) / \phi_+(z).
\]
Here then are the main results of this paper.

**THEOREM 1.** If
\[
(7.i) \quad \int_1^{\infty} \text{Im} \phi_+(x) dx < \infty,
\]
then the skew-reflecting Brownian motion is transient, and may escape to $\infty$ without ever hitting $(-\infty, 0)$. If the integral $\int_1^{\infty} \text{Im} \phi_+ dx$ is divergent, then the skew-reflecting Brownian motion is certain eventually to hit $(-\infty, 0)$. The corresponding test for escape to $\infty$ down $(-\infty, 0)$ without ever hitting $\mathbb{R}^+$ is
\[
(7.ii) \quad -\int_{-\infty}^{-1} \text{Im} (\phi_-(x)/x) dx < \infty.
\]
If either of (7.i) or (7.ii) obtains, then the original skew-reflecting Brownian motion in the wedge $D$ reaches 0 in finite time almost surely.

The proof of this result is in §2. Having decided when a one-sided escape is possible, and what happens in that case, we turn our attention to the remaining cases.

**THEOREM 2.** Assume that neither of the conditions (7.i), (7.ii) holds. If $c_1 > 0$, then the skew-reflecting Brownian motion in $H$ is recurrent. If $c_1 = 0$, then the skew-reflecting Brownian motion in $H$ is recurrent if and only if
\[
(8) \quad \int_{-\infty}^{\infty} \frac{\text{Im} \psi(x)}{1 + |x|} dx = +\infty.
\]
This result is proved in §3. The constant $c_1$ is the constant appearing in the Pick function representation (5) of $i\psi$.

Theorem 2 improves on Theorem 2 in [R2], where the function $\Psi(z) \equiv \int_0^z \psi(\omega) d\omega$ was used to map $H$ to the domain $\Psi(H)$; the process was transient if $\Psi(H)$ was contained in a wedge of angle $< \pi$, and recurrent if $\Psi(H)$ contains a non-vertical half-plane. Evidently, there is a gap between these conditions, and Burdzy and Marshall have refined the complex mapping techniques of [R2] to obtain a result equivalent to ours. In fact, the skew-reflecting Brownian motion in $H$ can with advantage be defined via a time change of the vertically reflecting Brownian motion in $\Psi(H)$, mapped back under $\Psi^{-1}$. As Burdzy remarks, this allows us to define the skew-reflecting Brownian motion in cases where the direction of reflection is much less regular; an account of this work (with D. Marshall) is in preparation.

Let us make a few remarks on the methodology used here to prove Theorem 2. We shall be making use of the same classical techniques of complex analysis which Dynkin [D] used (see also Malyutov [M]), with the aim of constructing a Green's function for the process. Let $G(\cdot, \cdot)$ be the Green's function for the process (assumed for the moment to be transient). Now it is clear that we may decompose

\begin{equation}
G(z, z_0) = G^0(z, z_0) + h(z, z_0),
\end{equation}

where $G^0$ is the Green's function for Brownian motion in $H$ killed on first hitting $R$,

\begin{equation}
G^0(z, z_0) = \frac{1}{\pi} \log \left| \frac{z - z_0}{z - \overline{z}_0} \right| = \text{Im} \frac{i}{\pi} \{\log(z - \overline{z}_0) - \log(z - z_0)\},
\end{equation}

and where formally

\begin{equation}
h(z, z_0) = E^z \left[ \int_0^\tau \delta_{z_0}(Z_t) dt \right],
\end{equation}

where $\tau \equiv \inf\{u : Z_u \in R\}$. It is easy to see that $h(\cdot, z_0)$ is harmonic, and non-negative. If we fix $z_0$ for the time being, and drop it from the notation, we may take the conjugate function $g$ to $h$ to form $f = g + i h$, and then

\begin{equation}
G(z) = \text{Im} \left( \frac{i}{\pi} \{\log(z - \overline{z}_0) - \log(z - z_0)\} + f(z) \right)
\end{equation}
Since $G$ must satisfy the boundary condition (3.ii), we have from (4) that

$$
\text{Im} \frac{1}{\psi(x)} \left\{ \frac{i}{\pi} \left( \frac{1}{x - z_0} - \frac{1}{x - z_0} \right) + f'(x) \right\}
$$

(12)

$$
= \text{Im} \frac{1}{\psi(x)} \left\{ \frac{2y_0}{\pi|x - z_0|^2} + f'(x) \right\} = 0 \quad \forall x \in \mathbb{R},
$$

where $z_0 \equiv x_0 + iy_0 \in \mathbb{H}$.

While it is not clear that a Green's function with sufficient regularity properties to justify the above steps should exist, we shall instead use the condition (12) to build a candidate Green's function, and use the properties of the function constructed to decide recurrence or transience.

It is slightly disappointing that the clean complex mapping techniques of [R2] must give way to the analysis of various integral expressions, but if you want an integral test for transience, you must expect to deal with integrals ....! Nonetheless, those complex mapping techniques serve us well in the proof of Theorem 1.

In the final section of this paper, we turn our attention to the question (1.iii) – will the process in the wedge reach 0 in finite time? We are unable to give a complete answer to this question, but the following result shows that the question is an interesting one.

**THEOREM 3.** If the angle $\xi$ of the wedge is greater than $\pi/2$ and if the skew-reflecting Brownian motion $Z$ can approach 0, then $Z$ will reach 0 in finite time.

If the angle $\xi$ of the wedge is less than $\pi/2$, then it is possible for $Z$ to approach 0 but never reach 0 in finite time.

We then proceed to analyse the Green's function in more detail, to obtain a criterion for the mean passage time to 0 to be finite. Recall that the wedge $D$ has been mapped to $\mathbb{H}$ by the function $z \mapsto -1/z^{\pi/\xi}$, and that the Green's function defined by (9) is defined for skew-reflecting Brownian motion $\zeta$ in $\mathbb{H}$, which is the time-change of the image of skew-reflecting Brownian motion $Z$ in $D$. Thus to go back from skew-reflecting Brownian motion $\zeta$ in $\mathbb{H}$ to $D$, we must firstly change time by the additive functional

$$
\int_0^t (\xi/\pi)^2 |\zeta_s|^{2-2\xi/\pi} ds,
$$

and then map $\zeta \mapsto (-1/\zeta)^{\xi/\pi}$. Thus $Z$ will reach 0 in finite time if and only if

$$
\int_0^\infty |\zeta_s|^{2-2\xi/\pi} I_{\{|\zeta_s| \geq 1\}} ds < \infty.
$$

The final result gives a necessary and sufficient condition for the mean of this random variable to be finite.
THEOREM 4. Assume that neither of (7.i), (7.ii) holds, and that $Z$ approaches 0.
For $\xi < \pi/2$, the mean time spent by $Z$ in $\{z : |z| \leq 1\}$ before reaching 0 is finite if
and only if

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \frac{\text{Im}(i\psi(x))\text{Im}(-1/i\psi(v))}{1 + |v|^{1+2\xi}/\pi} \text{tan}^{-1}(x) - \text{tan}^{-1}(v) < \infty.$$ 

2. Escape down a side. Let us define the Pick functions

$$\phi_+(z) \equiv \exp \left[ \int_0^z \frac{\theta(x) + \pi/2}{\pi} dx \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) \right],$$

$$\phi_-(z) \equiv \exp \left[ \int_z^{\infty} \frac{\theta(x) + \pi/2}{\pi} dx \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) \right],$$

so that $\phi_+\phi_- = i\psi$, and $\phi_+$ (respectively, $\phi_-$) is real positive on $\mathbb{R}^-$ (respectively $\mathbb{R}^+$).

We emulate the complex mapping methods of [R2] by defining the analytic functions

$$\Phi_+(z) \equiv \int_i^z \phi_+(\omega)d\omega,$$

$$\Phi_-(z) \equiv \int_i^z (-\phi_-(\omega)/\omega)d\omega.$$ 

These are 1-1 on $H$ (see Lemma 1 of [R2]). The analysis of the two is very similar, so we shall concentrate here on $\Phi_-$, which is slightly more involved. The effect of choosing the integrand $(-\phi_-(\omega)/\omega)$ for $\Phi_-$ is to consider the original skew-reflection problem but replacing $\theta$ in $\mathbb{R}^+$ by the constant value $\pi/2$. The effect of this is that $\Phi_-$ sends $\mathbb{R}^+$ to a horizontal half line; indeed, $\phi_-(\omega)/\omega$ is integrable near zero, and for $t > 0$, $\partial\Phi_-/\partial t = -\phi_-(t)/t$ is real negative. Thus the region $\Phi_-(H)$ looks like:

![Diagram](image)

(cf. [R2], where the pictures are like these, but rotated through $-\pi/2$. The negative half line has mapped to the curved part of the boundary, and the directions of reflection have become horizontal in the new system of coordinates.)
Since the argument of $\Phi_-$ on $\mathbb{R}^-$ agrees with the argument of $i\psi$, it follows that $\text{Im}\Phi_-$ satisfies (3) except for the boundary condition on $\mathbb{R}^+$. If $Z$ denotes the skew-reflecting Brownian motion in $H$, and if $\tau \equiv \inf\{u : Z_u \in \mathbb{R}^+\}$, then certainly the process $\text{Im}\Phi_-(Z_{t \wedge \tau})$ is a continuous local martingale, and is bounded above, so is almost surely convergent. If $\text{Im}\Phi_-$ were bounded below, then the local martingale is bounded, so is closed on the right by its terminal value $\text{Im}\Phi_-(Z_\tau)$. This implies that $P(\tau < \infty) < 1$. On the other hand, if $\text{Im}\Phi_-$ were not bounded below, then the continuous local martingale $\text{Im}\Phi_-(Z_{t \wedge \tau})$ converges, to some value at most $\alpha \equiv \sup \text{Im}\Phi_-(x)$. If it converged to some value strictly less than $\alpha$, then, since $Z$ keeps on hitting $\mathbb{R}$, we would conclude that $Z$ ultimately kept hitting $\mathbb{R}$ in some small neighbourhood, which is impossible. To summarize, then,

$$P(\tau < \infty) = 1 \text{ if and only if } \text{Im}\Phi_- \text{ is unbounded below.}$$

But in view of the explicit form of $\Phi_-$, this can be restated as

$$P(\tau < \infty) < 1 \text{ if and only if } \int_{-\infty}^{-1} \text{Im}\phi_-(x) \frac{dx}{x} < \infty. \quad (14)$$

An analogous reasoning for $\mathbb{R}^+$ establishes that

$$P(Z_t \notin (-\infty,0) \text{ for all } t) > 0 \text{ if and only if } -\int_{1}^{\infty} \text{Im}\phi_+(x) dx < \infty. \quad (15)$$

In either of the situations (14), (15), then, it is possible for the skew-reflecting Brownian motion to make a one-sided escape to infinity. Any such escape must be very rapid indeed, for consider the process $\log Z_t$, which is a skew-reflecting Brownian motion in the strip $\{z : 0 \leq \text{Im}(z) \leq \pi\}$, run with clock $\int_{0}^{t} |Z_s|^{-2} ds$. Now if $Z$ makes a one-sided escape to infinity (along $\mathbb{R}^+$, say) it has to be that $\text{Im}\log Z_t$ converges to zero (because $\text{Im}\log Z_t$ is the time change of a reflecting Brownian motion in $[0,1]$) and so the clock must be almost surely convergent:

$$\int_{0}^{\infty} |Z_s|^{-2} ds < \infty \text{ a.s.} \quad (16)$$

However, the map $z \mapsto (1/z)^{\eta}$, $\eta \equiv \xi/\pi$ takes $H$ back to skew-reflecting Brownian motion in $D$ with the clock $\int_{0}^{t} |Z_s|^{-2-2\eta} ds$. Because of (16) and the transience of $Z$, this ensures that $\int_{0}^{\infty} |Z_s|^{-2-2\eta} ds < \infty$ a.s., and so skew-reflecting Brownian motion in $D$ reaches $0$ in finite time whenever it reaches $0$ down one side only.
3. Proof of Theorem 2. Let us fix \( z_0 \equiv x_0 + iy_0 \in \mathbb{H} \). We shall construct some analytic \( f : \mathbb{H} \rightarrow \mathbb{C} \) which is \( C^1 \) on \( \mathbb{H} \), and satisfying

\[
\text{Im} \left[ \frac{f'(x)}{i\psi(x)} \right] = \text{Im} \left[ \frac{-1}{i\tilde{\psi}(x)} \right] \frac{2y_0}{x|y_0|} \quad \text{for} \quad x \in \mathbb{R},
\]

which is simply the condition (11) replaced. If we now define \( \tilde{\psi}(z) \equiv \psi(y_0 z + x_0) \), \( \tilde{f}(z) \equiv \frac{\pi}{2} f(y_0 z + x_0) \), then (17) becomes

\[
\text{Im} \left[ \frac{\tilde{f}'(x)}{i\tilde{\psi}(x)} \right] = \text{Im} \left[ \frac{-1}{i\tilde{\psi}(x)} \right] \frac{1}{1 + x^2} \quad \text{for} \quad x \in \mathbb{R},
\]

effectively simplifying (17) to the case \( z_0 = i \). Notice that \(-1/i\tilde{\psi}\) is a Pick function so for some real \( c'_1 \geq 0, c'_2 \), we may express

\[
\frac{-1}{i\tilde{\psi}(z)} = \int \frac{\gamma(x)dx}{\pi} \left\{ \frac{1}{x - z} - \frac{x}{1 + x^2} \right\} + c'_1 z + c'_2,
\]

where \( \gamma(x) \equiv \text{Im}(-1/i\tilde{\psi}(x)) \). The first step is to form the analytic extension into \( \mathbb{H} \) of the right-hand side of (18). Since \(-1/i\tilde{\psi}\) is a Pick function, it follows that \( \gamma(x)/(1 + x^2) \) is integrable, so the analytic extension may be given as

\[
\rho(z) \equiv \int \frac{\gamma(x)dx}{\pi(1 + x^2)(x - z)}.
\]

Now observe that

\[
(1 + z^2)\rho(z) = \int \frac{\gamma(x)dx}{\pi} \left( \frac{1}{x - z} - \frac{x}{1 + x^2} \right) - z \int \frac{\gamma(x)dx}{\pi(1 + x^2)}
\]

\[
= \left\{ -\frac{1}{i\tilde{\psi}(z)} - c'_1 z - c'_2 \right\} - z \text{Im} \left( -\frac{1}{i\tilde{\psi}(i)} - c'_1 \right),
\]

in view of the representation (19);

\[
= -\frac{1}{i\tilde{\psi}(z)} - c'_2 - z \text{Im} \left( -\frac{1}{i\tilde{\psi}(i)} \right).
\]

If we now write \(-1/i\tilde{\psi}(i) \equiv \alpha + i\beta\), then \( c'_2 = \alpha \) (from (19)), and we have

\[
(1 + z^2)\rho(z) = -\frac{1}{i\tilde{\psi}(z)} - \alpha - z\beta
\]

\[
= \frac{1}{i\tilde{\psi}(i)} - \frac{1}{i\tilde{\psi}(z)} - \beta(z - i).
\]

In view of (18), this leads us to define \( \tilde{f} \) to within an additive constant by

\[
\tilde{f}'(z) = i\tilde{\psi}(z)\rho(z)
\]

\[
= \left( \frac{i\tilde{\psi}(z)}{i\tilde{\psi}(i)} - 1 \right)/(1 + z^2) - \frac{\beta i\tilde{\psi}(z)}{z + i}.
\]
To understand better what \( \tilde{f} \) must be, we integrate the two terms of \( \tilde{f}' \) separately.

\[
\int^z \frac{d\omega}{1 + \omega^2} \left( \frac{i\tilde{\psi}(\omega)}{i\tilde{\psi}(i)} - 1 \right) = \frac{1}{i\tilde{\psi}(i)} \int^z \frac{d\omega}{1 + \omega^2} \left\{ \int \frac{\text{Im}\tilde{\psi}(x)}{\pi} dx \left( \frac{1}{x - \omega} - \frac{1}{x - i} \right) + c_1(\omega - i) \right\} \\
= \frac{1}{i\tilde{\psi}(i)} \left[ \int \frac{\text{Im}\tilde{\psi}(x) dx}{\pi(1 + x^2)} \int^z d\omega \left( \frac{1}{\omega + i} - \frac{1}{\omega - x} \right) + c_1 \log(z + i) \right] \\
= \frac{1}{i\tilde{\psi}(i)} \left[ \int \frac{\text{Im}\tilde{\psi}(x) dx}{\pi(1 + x^2)} \{ \log(z + i) - \log(z - x) \} + c_1 \log(z + i) \right],
\]

(21)

and integrating the second term yields

\[
- \beta \int^z \frac{d\omega}{\omega + i} \left\{ \int \frac{\text{Im}\tilde{\psi}(x)}{\pi} dx \left( \frac{1}{x - \omega} - \frac{x}{1 + x^2} \right) + c_1 \omega + c_2 \right\} \\
= - \beta \int \frac{\text{Im}\tilde{\psi}(x) dx}{\pi(1 + x^2)} \{ (x - i)(\log(z + i) - \log(z - x)) - x \log(z + i) \}
\]

(22)

\[ - \beta c_1 z - \beta(c_2 - ic_1) \log(z + i). \]

Using the fact \(-1/i\tilde{\psi}(i) = \alpha + i\beta\), adding (21) and (22) gives us that to within an additive constant

\[
\tilde{f}(z) = \int \frac{\text{Im}\tilde{\psi}(x)}{\pi(1 + x^2)} dx \{ (\alpha + \beta x) \log(z - x) - \alpha \log(z + i) \} \\
- c_1 \alpha \log(z + i) - c_1 z - \beta c_1 z - \beta c_2 \log(z + i).
\]

This simplifies when we notice that

\[
\int \frac{\text{Im}\tilde{\psi}(x) dx}{\pi(1 + x^2)} = \text{Im}(i\tilde{\psi}(i)) - c_1,
\]

removing the terms in \( \log(z + i) \) and leaving

\[
\tilde{f}(z) = \int \frac{\text{Im}\tilde{\psi}(x)}{\pi(1 + x^2)}(\alpha + \beta x) \log(z - x) dx - \beta c_1 z.
\]

(23)

Since it is only the imaginary part which will interest us, we can write \( z = \alpha + ib \) and deduce that

\[
\text{Im}(\tilde{f}(z) - \tilde{f}(i)) = \int \frac{\text{Im}\tilde{\psi}(x)}{\pi(1 + x^2)}(\alpha + \beta x) \left\{ \tan^{-1} \left( \frac{x - a}{b} \right) - \tan^{-1}(x) \right\} dx - \beta c_1(b - 1).
\]

(24)

Theorem 2 will follow from an analysis of the properties of \( \text{Im}\tilde{f} \) as given by (24). Recall the hypothesis of that theorem, that there is no one-sided escape to infinity. In particular, this means that \( \arg(Z_t) \) oscillates back and forth between 0 and \( \pi \).
The argument which gave (12) shows that if we add the function $\log |z+i| - \log |z-i|$ to $\text{Im} \tilde{f}$, then we obtain a function harmonic in $H \setminus \{i\}$, continuous in $\overline{H}$, and satisfying the boundary condition (3.ii) on $R$.

Case 1: $c_1 = 0$, \[ \int \text{Im} \psi(x)(1 + |x|)^{-1} dx < \infty. \]

In this case, it is evident from (24) that $\text{Im} \tilde{f}$ is bounded, and so the function $h^*(z) \equiv \text{Im} \tilde{f}(z) + \frac{1}{2} \log |z+i| - \frac{1}{2} \log |z-i|$ is harmonic in $H \setminus \{i\}$, satisfies the boundary condition (3.ii) on $R$ and is bounded below, hence $h^*(Z_t)$ is a local martingale bounded below, hence $h^*(Z_t)$ is convergent almost surely, forcing $|Z_t| \to \infty$ a.s..

Case 2: $c_1 > 0$ or \[ \int \text{Im} \psi(x)(1 + |x|)^{-1} dx = +\infty. \]

Define the function \[ K(x) \equiv \int_0^x \frac{t \text{Im} \tilde{\psi}(t)}{\pi(1 + t^2)} dt. \]

Clearly, $K$ is non-negative, increasing in $(0, \infty)$, and decreasing in $(-\infty, 0)$. Moreover it is easy to see that $K(x)/|x| \to 0$ as $|x| \to \infty$.

Now consider the coefficient of $\beta$ in the right-hand side of (24):

\[
\int \frac{x \text{Im} \tilde{\psi}(x)}{\pi(1 + x^2)} dx \left\{ \tan^{-1} \left( \frac{x-a}{b} \right) - \tan^{-1}(x) \right\}
= - \int K(x) \left\{ \frac{b}{b^2 + (x-a)^2} - \frac{1}{1 + x^2} \right\} dx
\leq \int K(x) \frac{dx}{1 + x^2},
\]

which is finite, because
\[
\int \frac{K(x)}{x^2} dx = \int \frac{\text{Im} \tilde{\psi}(t)}{\pi(1 + t^2)} dt
\]
is finite, $i \hat{\psi}$ being a Pick function. Thus

\[
\text{Im}(\tilde{f}(z) - \tilde{f}(i)) = \beta \int K(x) \left\{ \frac{1}{1 + x^2} - \frac{b}{b^2 + (x-a)^2} \right\} dx
- \beta c_1 (b-1) + o(1)
\]
is a harmonic function in $H$, bounded above. Moreover, it is easy to see that $\text{Im}(\tilde{f}(ib)) \to -\infty$ as $b \uparrow \infty$, either because $c_1 > 0$, or else because $c_1$ is zero and the integral $\int \text{Im} \psi(x)(1 + |x|)^{-1} dx$ is divergent. This last implies that $K$ is unbounded, and monotonicity of $K$ in each half of $R$ gives the desired conclusion.
We now consider as before the function \( h^*(z) = \text{Im}\tilde{f}(z) + \frac{1}{2} \log |z + i| - \frac{1}{2} \log |z - i| \). As before, \( h^*(Z_t) \) is a continuous local martingale, and so a.s. either converges or oscillates between \(+\infty\) and \(-\infty\). But away from some neighbourhood of \( i \), \( h^* \) is bounded above, and tends to \(-\infty\) along the ray \( i\mathbb{R}^+ \). If the process \( Z \) were transient, since the argument of \( Z \) oscillates back and forth between 0 and \( \pi \), it must be that \( Z \) keeps crossing \( i\mathbb{R}^+ \) at an arbitrarily large distance from 0. Thus \( h^*(Z_t) \) cannot be bounded below. Hence \( h^*(Z_t) \) cannot be bounded above, which can only be happening if \( Z \) keeps entering a neighbourhood of \( i \), implying that \( Z \) is recurrent, a contradiction.

4. Reaching the vertex of the wedge in finite time. The first objective of this section is to prove Theorem 3. Let \( Z \) be skew-reflecting Brownian motion in the wedge \( D \), and let \( g(z) = z^{\pi/\xi} \) be the obvious analytic map taking \( D \) to \( \mathbb{H} \), fixing 0. By the skew-reflecting Brownian mapping theorem, \( g(Z) \) is a time-change of skew-reflecting Brownian motion in \( \mathbb{H} \). We could therefore alternatively start with a skew-reflecting Brownian motion \( \zeta \) in \( \mathbb{H} \), time-change it by the additive functional \( A_t \equiv \int_0^t (\xi/\pi)^2 |\zeta_s|^2 (\xi/\pi)^{-2} ds \), and map back to \( D \) by \( g^{-1} \) to obtain a skew-reflecting Brownian motion in \( D \). The skew-reflecting Brownian motion in \( D \) will reach 0 in finite time if and only if

\[
\int_0^\tau |\zeta_s|^{2(\xi/\pi)^{-2}} ds < \infty
\]

where \( \tau = \inf\{t : \zeta_t = 0\} \). But if \( \xi \in (\pi/2, \pi] \), then \( \beta \equiv 2(\xi/\pi) - 2 \in (-1, 0] \), and so

\[
\int_0^\tau |\zeta_s|^\beta ds \leq \int_0^\tau (\text{Im} \, \zeta_s)^\beta ds < \infty,
\]

since \( \text{Im} \, \zeta_s \) is just a reflecting Brownian motion on \( \mathbb{R}^+ \) and the additive functional \( \int_0^t (\text{Im} \, \zeta_s)^\beta ds = \int_0^\infty L(t, x)x^\beta dx \) is evidently finite-valued (where \( \{L(t, x) : t \geq 0, x \geq 0\} \) is the jointly continuous local time of \( \text{Im} \, \zeta \)). For \( \xi > \pi, \beta = 2(\xi/\pi) - 2 > 0 \), to the finiteness of \( A_t \) is trivial. All that remains is to notice that \( \tau \) must be finite a.s. in the case where 0 can be approached - because if \( \zeta \) were to approach zero but never reach it, then \( \text{Im} \, \zeta_t \) would tend to 0 as \( t \to \infty \), which is not a property of reflecting Brownian motion on \( \mathbb{R}^+ \)!

To complete the proof of Theorem 3, we must build an example of a wedge \( D \) with opening \( \zeta < \pi/2 \) where \( Z \) will approach but never reach 0. The example is best
understood by firstly transforming $D$ by log, taking $D$ to $S \equiv \{(x+iy): 0 \leq y \leq \xi\}$, and then specifying the directions of reflection on $\partial S$.

The strip $S$ is going to be split into an infinite sequence of ‘boxes’ by the reals $\beta_0 = 0 > \alpha_0 > \beta_1 > \alpha_1 > \cdots$; in the boxes $\{(x+iy): \alpha_n < x \leq \beta_n, \ 0 \leq y \leq \xi\}$ the reflection at the upper and lower edges will be normal, and in the boxes $\{(x+iy): \beta_n < x \leq \alpha_{n-1}, 0 \leq y \leq \xi\}$, the reflection at the upper and lower edges will be degenerate, pushing to the left. (It may seem improper to allow a tangential reflection. An example could doubtless be constructed where the reflection was extremely and increasingly close to tangential between $\beta_n$ and $\alpha_{n-1}$, and also varying smoothly, but this is a pointless embellishment; the real point is that the qualitative behaviour described is possible.) We refer to boxes of the first kind as ‘pools’, and boxes of the second kind as ‘valves’. We envisage that the lengths of the boxes are going to infinity, but that $\beta_n - \alpha_n \gg \alpha_{n-1} - \beta_n$; a pool is much longer than the valve immediately to the right. The name ‘valve’ is very appropriate, because if one considers Brownian motion in a valve, as soon as it touches one of the horizontal sides it is immediately swept to the left-hand end. Thus it is very easy for Brownian motion to pass through a valve from right to left, but almost impossible for it to pass through from left to right. The purpose of ‘pools’ is to provide a place for Brownian motion to swim around and spend time, so that the total time for the skew-reflecting Brownian motion in $D$ to reach 0 will accumulate to $+\infty$. This suggests that one makes the length $\beta_n - \alpha_n$ of the pool very large; but not so large that the probability that the process escapes through the valve to the right before it escapes through the valve to the left becomes too big. It is not clear that these conflicting aims can be achieved simultaneously; but, if $\xi < \pi/2$, we shall see that they can.

We begin with the analysis of valves. We shall consider a valve of length $l \equiv \alpha_{n-1} - \beta_n$, and estimate the rate of excursions from the left-hand end of the valve which reach the right-hand end. For simplicity, let us suppose that the left-hand end of the box is on
the imaginary axis. It is easy to see that for \( 0 \leq y \leq \xi, \quad x > 0 \) small

\[ P^{x+iy} \) (reach right-hand end of valve before any other edge) \]

\[ \leq P^{x+i\xi/2} \) (reach right-hand end of valve before any other edge) \]

and it is not too hard to persuade oneself that this last quantity varies like const. \( e^{-\pi l/\xi} \) \( \sinh(\pi x/\xi) \), by considering the harmonic function \( h(x, y) = \sinh(\pi x/\xi) \sin(\pi y/\xi) \) which vanishes on three sides of the valve. An application of the optional sampling theorem gives

\[ E^{x+i\xi/2}(\sin(\pi Y_{\tau}/\xi); \quad X_{\tau} = 1) = \sinh(\pi x/\xi)/\sinh(\pi l/\xi) \]

where \( \tau \) is the exit time from the valve. This argument is not conclusive, but a firm proof can be given proceeding via another route. One firstly transforms \( \{x + iy : x \geq 0, \quad 0 \leq y \leq \xi\} \) to \( H \) and then observes what has happened to the edge \( \text{Re} \) \( z = l \). This has been mapped to a somewhat complicated curve in \( H \), but this curve is contained between two concentric circular arcs with centre on \( R \). The probability of leaving the valve firstly through the right-hand side is at most the probability that Brownian motion in \( H \) hits the outer of the two circular arcs before it hits \( R \), and this is a straightforward matter to compute. (The Brownian motion starts at a point exterior to both circular arcs). We now deduce an upper bound for the measure of excursions starting at the left-hand end of the valve and reaching the right end; we divide by \( 2x \) (the measure of Brownian excursions from \( 0 \in R \) which get above \( x \)) and let \( x \downarrow 0 \). The conclusion is that the measure of excursions from the left end of the box which get through to the right end is at most

\[ 25\pi \frac{e^{-\pi l/\xi}}{2\xi} \]

The other part of the analysis of valves is concerned with showing that if one starts at the right end of the valve, the measure of excursions which get through to the left end is bounded below. For simplicity, suppose the right end is on the imaginary axis. The ingredients of the argument are similar; the worst possible case is when one starts on the line \( y = \xi/2 \), and now the probability of leaving the valve anywhere but the right end is at least the probability of leaving the semi-infinite strip \( \{x + iy : x \geq 0, \quad 0 \leq y \leq \xi\} \) anywhere but the right end. This is easier to deal with than the other case, and leads
to a lower bound for the measure of excursions from the right end of the valve which get through to the left end; it must be at least \( \frac{\pi}{2\xi} \).

Thus if we define

\[
P_n \equiv \sup_{0 \leq y \leq \xi} P^{\beta_n + iy}(X \text{ reaches } \beta_{n-1} \text{ before } \beta_{n+1}),
\]

\[
P'_n \equiv \sup_{0 \leq y \leq \xi} P^{\alpha_n + iy}(X \text{ reaches } \beta_n \text{ before } \beta_{n+1}),
\]

\[s_{n-1} \equiv \beta_{n-1} - \alpha_{n-1}, \quad l \equiv \alpha_{n-1} - \beta_n\]

(where \( Z_t \equiv X_t + iY_t \) is the skew-reflecting Brownian motion in \( S \)), we can estimate

\[
(26) \quad P'_n \leq \frac{1}{2s_n} \left( \frac{1}{2s_n} + \frac{\pi}{2\xi} \right)^{-1} = \xi(\xi + \pi s_n)^{-1}
\]

because the rate of excursions of \( X \) from \( x = \alpha_n \) which get to \( x = \beta_n \) before returning is \( 1/(2s_n) \), and the rate of excursions of \( X \) from \( x = \alpha_n \) which escape through to \( x = \beta_{n+1} \) before returning to \( x = \alpha_n \) is, as we have just seen, at least \( \pi/2\xi \).

By considering the situation when \( X \) starts at \( \beta_n \), and decomposing at the times of reaching the ends of boxes, one estimates similarly

\[
(27) \quad P_n \leq \frac{2\pi \cdot 2^{5/6} e^{-\pi l/\xi} (p'_n - (1 - p'_n)p_n) + \frac{1}{2s_n} p'_n p_n}{2\pi \cdot 2^{5/6} e^{-\pi l/\xi} + \frac{1}{2s_n}}.
\]

Rearrangement, and use of the estimate (26), lead finally to the estimate

\[
(28) \quad P_n \leq \left( 1 + \frac{\xi + \pi s_{n-1}}{\xi + \pi s_n} \cdot \frac{e^{\pi l/\xi}}{25} \right)^{-1}
\]

We shall aim to choose the \( \alpha_n, \beta_n \) in such a way that always \( P_n \leq (2 + \delta)^{-1} < \frac{1}{2} \), because then if one looks at the sequence of values which \( Z \) visits, this sequence goes almost surely to \(-\infty\), by comparison with simple random walk. The estimate (28) ensures that this will happen whenever

\[
(29) \quad \frac{\xi + \pi s_{n-1}}{\xi + \pi s_n} \cdot \frac{e^{\pi(\alpha_{n-1} - \beta_n)/\xi}}{25} \geq 1 + \delta \quad \text{for all } n.
\]

We turn now to the analysis of pools, which is considerably simpler. The pools are the places where the process accumulates 'time' but 'time' measured by the additive functional \( \int_0^t \exp(2X_s)ds \), which is the clock used to transform the time-scale into the time-scale for Brownian motion in \( D \). If we take the \( n^{th} \) pool, abbreviating \( \alpha_n, \beta_n \) to \( \alpha, \beta \), we shall obtain

\[
E^{\beta + iy} \left[ \exp\left( -\frac{1}{2} \int_0^{t_n} e^{2X_s} ds \right) \right]
\]

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where \( \tau_\alpha \equiv \inf \{ t : X_t = \alpha \} \). This is a simple problem in one-dimensional diffusion theory; it is not hard to prove that

\[
E^{\beta + i \gamma} \left[ \exp(-\frac{1}{2} \int_0^{\tau_\alpha} e^{2X_s} ds) \right]
\]

\[
= \frac{I_0'(e^\beta)K_0(e^\beta) - K_0'(e^\beta)I_0(e^\beta)}{I_0'(e^\alpha)K_0(e^\alpha) - K_0'(e^\alpha)I_0(e^\alpha)}.
\]

(30)

Using the asymptotics \( I_0'(x) \sim x/2, K_0(x) \sim -\log x, K_0'(x) \sim -1/x \) for small \( x \), the numerator of (30) is, for \( \beta \ll 0 \), asymptotically like \( e^{-\beta} \), and the denominator is asymptotically like \( e^{-\beta} - \frac{1}{2} \alpha e^\beta \). Thus as \( \beta \to \infty \) we have that

\[
E^{\beta + i \gamma} \left[ \exp(-\frac{1}{2} \int_0^{\tau_\alpha} e^{2X_s} ds) \right]
\]

\[
\sim (1 + (-\alpha/2)e^{2\beta})^{-1}.
\]

(31)

Now the total 'time' taken to reach 0, \( \int_0^\infty \exp(2X_s)ds \), is at least the sum of the contributions between first hitting \( \beta_n \) and first hitting \( \alpha_n \). Thus

\[
E \exp(-\frac{1}{2} \int_0^\infty e^{2X_s} ds) \leq \prod_{n=0}^\infty E^{\beta_n + i\xi/2} \left[ \exp(-\frac{1}{2} \int_0^{\tau(\alpha_n)} e^{2X_s} ds) \right]
\]

= 0

if

\[
\sum |\alpha_n|e^{2\beta_n} = +\infty,
\]

(32)

from the asymptotics (31).

Thus we have to choose \( \alpha_n, \beta_n \) in such a way that (29) and (32) are satisfied. We start with \( \beta_0 = 0, \alpha_0 = -1 \), and will always have \( \alpha_n, \beta_n \) related by \( \alpha_n = -e^{-2\beta_n} \). This will guarantee (32), but can we also have (29)? If we have determined \( \alpha_{n-1}, \beta_{n-1}, \ldots, \alpha_0, \beta_0 \), we determine \( \beta_n \) so as to satisfy (29); we want

\[
\frac{e^{\pi(\alpha_{n-1}-\beta_n)/\xi}}{\xi + \pi(\beta_n - \alpha_n)} \geq \frac{25(1 + \delta)}{\xi + \pi(\beta_{n-1} - \alpha_{n-1})},
\]

but, since \( \alpha_n = -e^{-2\beta_n} \), this is the same as

\[
\frac{e^{\pi(\alpha_{n-1}-\beta_n)/\xi}}{\xi + \pi(\beta_n + e^{-2\beta_n})} \geq \frac{25(1 + \delta)}{\xi + \pi(\beta_{n-1} - \alpha_{n-1})}
\]

but
and since $\pi/\xi > 2$ by assumption, we can always achieve this inequality by letting $\beta_n$ go far enough to the left.

The proof of Theorem 3 is complete.

The last item to deal with now is the proof of Theorem 4. For this, we return to the analysis of §3, and study the Green’s function more closely, assuming that one-sided escape to $\infty$ is impossible (i.e., that neither of (7.i), (7.ii)) holds) and that the process is transient, that is

$$c_1 = 0, \quad \int \frac{\text{Im} \psi(x)}{1 + |x|} \, dx < \infty.$$  \hfill (33)

Now consider the expression (24) for $\text{Im} \tilde{f}$, which simplifies here to

$$\text{Im} \tilde{f}(z) = \int \frac{\text{Im} i\tilde{\psi}(x)}{\pi(1 + x^2)}(\alpha + \beta x) \tan^{-1}\left(\frac{x-a}{b}\right) \, dx + \text{const.},$$  \hfill (34)

in view of (33). If we consider this function for $a \in \mathbb{R}$, the limit of $\text{Im}(\tilde{f}(a) - \tilde{f}(-a))$ as $a \to \infty$ is

$$-\pi \int \frac{\text{Im} i\tilde{\psi}(x)}{\pi(1 + x^2)}(\alpha + \beta x) \, dx = -\pi \{ \alpha \text{ Im} i\tilde{\psi}(i) + \beta \int \frac{\text{Im} i\tilde{\psi}(x)x \, dx}{\pi(1 + x^2)} \}$$

$$= -\pi \{ \alpha \text{ Im} i\tilde{\psi}(i) + \beta \text{ Re}(i\tilde{\psi}(i) - c_2) \},$$  \hfill (35)

where we refer to the Pick function representation (5) of $i\tilde{\psi}$;

$$i\tilde{\psi}(z) = \int \frac{\text{Im} i\tilde{\psi}(x) \, dx}{\pi(x - z)} + c_2.$$  

Now $\alpha + i\beta = -1/i\tilde{\psi}(i)$ by definition, so (35) simplifies to $\beta \pi c_2$, and $\beta > 0$. But recall that one-sided escape to $\infty$ is ruled out, so that the process $Z$ must keep on visiting $(0, \infty)$ and $(-\infty, 0)$ arbitrarily far from 0. Moreover, $h^*(z) = \text{Im} \tilde{f}(z) + \frac{i}{2} \log |z + i| - \frac{1}{2} \log |z - i|$ is harmonic in $H \backslash \{i\}$, satisfies the boundary condition on $\mathbb{R}$ and is bounded below, so $h^*(Z_t)$ converges almost surely. But this cannot happen if

$$\lim_{a \to \infty} \text{Im}(\tilde{f}(a) - \tilde{f}(-a)) \neq 0;$$  

so the conclusion is that $c_2$ must be zero, and

$$i\tilde{\psi}(z) = \int \frac{\text{Im} i\tilde{\psi}(x)}{\pi(x - z)} \, dx.$$  \hfill (36)

We are now going to return to (17) and derive another expression for the difference $h(z, z_0) = G(z, z_0) - G^0(z, z_0)$ of the Green’s function and the Green’s function for the
killed process. Holding \(z_0 = x_0 + iy_0\) fixed for now, we shall build \(f\) to satisfy (17) by defining

\[
f'(z) \equiv i\psi(z) \int \frac{\text{Im}(-1/i\psi(x))}{\pi^3 |x-x_0|^2} \frac{2y_0}{\pi |x-z_0|^2} \, dx
\]

\[
= \int \frac{\text{Im} i\psi(v)dv}{\pi(v-z)} \int \frac{\text{Im}(-1/i\psi(x))}{\pi(x-z)} \frac{2y_0}{\pi |x-z_0|^2} \, dx
\]

so that

\[
f(z) - f(i) = 2y_0 \int \int \frac{\text{Im} i\psi(v)\text{Im}(-1/i\psi(x))}{\pi^3 |x-x_0|^2(v-x)} \left[ \log(w-v) - \log(w-x) \right]^2 \, dv \, dx
\]

and hence

\[
\text{Im} (f(z) - f(i)) = \int \int \frac{\text{Im} i\psi(v)\text{Im}(-1/i\psi(x))2y_0}{\pi^3 |x-x_0|^2(v-x)} \, dx \, dv \{ \tan^{-1}(\frac{v-a}{b}) - \tan^{-1}(\frac{x-a}{b}) \\
- \tan^{-1}(v) + \tan^{-1}(x) \}.
\]

The arbitrary constant in the definition of \(f\) is determined in such a way that

\[
\text{Im} f(z) = \int \int \frac{\text{Im} i\psi(v)\text{Im}(-1/i\psi(x))2y_0}{\pi^3 |x-x_0|^2} \tan^{-1} \left( \frac{v-a}{b} \right) - \tan^{-1} \left( \frac{x-a}{b} \right)
\]

the integral being absolutely convergent in view of the integrability condition (33), the integrability of \((1 + x^2)^{-1}\text{Im}(-1/i\psi(x))\), and the elementary estimate;

\[
0 \leq \frac{\tan^{-1}(v) - \tan^{-1}(x)}{v-x} \leq c.(1 + |x| + |v|)^{-1}.
\]

Now, comparing the definitions (20) and (37) of \(\tilde{f}\) and \(f\), we see that \(\tilde{f} = \frac{\pi}{2}y_0 f'(y_0 z + x_0)\), implying that \(\tilde{f}(z) - \frac{\pi}{2} f(y_0 z + x_0)\) is constant. It follows from (34), (38) that

\[
\int \frac{\text{Im} i\tilde{\psi}(x)}{\pi(1 + x^2)}(\alpha + \beta x)^2 \tan^{-1} \left( \frac{x-a}{b} \right) \, dx
\]

\[
= \int \int \frac{\text{Im} i\psi(v)\text{Im}(-1/i\psi(x))}{\pi^3 |x-x_0|^2} \frac{2y_0}{\pi |x-z_0|^2} \tan^{-1} \left( \frac{v-a}{b} \right) - \tan^{-1} \left( \frac{x-a}{b} \right) \, dx \, dv \frac{1}{v-x}.
\]

the two sides differing by a constant, which is seen to be 0 if we let \(b \to \infty\), while \(a = 0\). The right-hand side is evidently non-negative, the left-hand side is evidently a bounded harmonic function, Now, as we have seen, \(\tilde{f}(z) + \frac{\pi}{2} G^\theta(z, z_0)\) is harmonic, satisfies the boundary condition, and is bounded below; from (40) therefore

\[
G(z, z_0) - G^\theta(z, z_0) = \int \int \frac{\text{Im} i\psi(v)\text{Im}(-1/i\psi(x))}{\pi^3 |x-x_0|^2} \frac{2y_0}{\pi |x-z_0|^2} \tan^{-1} \left( \frac{v-a}{b} \right) - \tan^{-1} \left( \frac{x-a}{b} \right) \, dx \, dv \frac{1}{v-x}.
\]
(41)

If we specialize to $z = i$, we have that

$$G(i, z_0) - G^0(i, z_0) = \int \int 2y_0 \frac{\text{Im} \psi(v) \text{Im} \left( -\frac{1}{i\psi(x)} \right)}{\pi^3 |x - z_0|^2} \, dv \, dx \frac{\tan^{-1}(v) - \tan^{-1}(x)}{v - x},$$

and hence $E^i [\int_0^\infty \phi(z_s) \, ds] < \infty$ if and only if

$$\int \int \text{Im} \psi(v) \text{Im} \left( -\frac{1}{i\psi(x)} \right) dv \, dx \frac{\tan^{-1}(v) - \tan^{-1}(x)}{v - x} \int \int \frac{y_0}{(x - z_0)^2} \phi(z_0) \, dx \, dy < \infty.$$

(42)

The case of major interest to us is the case where $\phi(z) = |z|^{-2-2\xi/\pi} \Pi_{|z| \geq 1}$, as was explained in the Introduction; the finiteness of the integral (42) in this case will guarantee that the process in the wedge will reach the vertex in finite time. It is straightforward to calculate for $v > 0, \eta > 0$, that

$$\int_{|z| \geq 1} |z|^{-2-\eta} \text{Im} \left( \frac{1}{v - z} \right) \, dx \, dy$$

$$= 2v^{-1-\eta} \int_{1/v}^\infty (\log |1 + y| - \log |1 - y|) y^{-1-\eta} \, dy,$$

and as $v \to \infty$ this behaves like $(1 + v^{1+\eta})^{-1}$ for $\eta < 1$, and like $(1 + v^2)^{-1}$ for $\eta > 1$. The case we are particularly concerned with corresponds to $\eta = 2\xi/\pi$ ($<1$ by assumption), and so the finiteness condition (42) becomes the condition stated in Theorem 4, concluding the proof.

Appendix: A skew-reflecting Brownian mapping theorem

Let $D_1, D_2$ be two open simply-connected domains in $C$, with closures $\overline{D}_1, \overline{D}_2$, and suppose that $f : \overline{D}_1 \to \overline{D}_2$ is a $C^2$ diffeomorphism, whose restriction to $D_1$ is analytic.

Let $v$ be a vector field defined on $\partial D_1$, everywhere inward-pointing and non-vanishing, and let $\tilde{v}$ be the corresponding vector field on $\partial D_2$, defined by $\tilde{v}(f(z)) = f'(z)v(z), z \in \partial D_1$. Suppose that $Z$ is a skew-reflecting Brownian motion in $\overline{D}_1$ with the vector field $v$ specifying the directions of reflection;

$$Z_t = Z_0 + W_t + \int_0^t v(Z_s) \, dL_s$$

where $W$ is a standard Brownian motion in $C$, $L$ is a continuous increasing process satisfying $\int_0^t I_{D_1}(Z_s) \, dL_s = 0$, and $Z$ spends no time on $\partial D_1(\int_0^t I_{\partial D_1}(Z_s) \, ds = 0)$. 

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Then $f(Z_t) \equiv \tilde{Z}_t$ is a skew-reflecting Brownian motion in $\overline{D}_2$, run with the clock $\int_0^t |f'(Z_s)|^2 ds$.

**Proof.** Applying Itô’s formula to $f(Z_t)$, we obtain

$$d\tilde{Z}_t \equiv df(Z_t) = f'(Z_t) dW_t + \tilde{v}(\tilde{Z}_t) dL_t.$$ 

Thus if $A_t \equiv \int_0^t |f'(Z_s)|^2 ds$, $(\tau_t)$ is the inverse to $A$, we deduce that

$$\zeta_t \equiv \tilde{Z}_{\tau_t} = \tilde{Z}_0 + \beta_t + \int_0^t \tilde{v}(\zeta_s) d\lambda_s,$$

where $\lambda_t \equiv L_{\tau_t}$, and $\beta$ is a complex Brownian motion. Moreover, $\int_0^t I_{D_1}(\zeta_s) d\lambda_s = \int_0^t I_{D_2}(Z_s) dL_s = 0$, and $\int_0^t I_{\partial D_2}(\zeta_s) ds = \int_0^\tau I_{\partial D_1}(Z_u)|f'(Z_u)|^2 du = 0$, which establishes that $\zeta$ is skew-reflecting Brownian motion in $\overline{D}_2$. 

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