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A MARTINGALE APPROACH TO SOME WIENER-HOPF PROBLEMS, II,

by

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To a large extent, this paper, II, may be read independently of Paper I.

Please see the introductory remarks to Paper I for a brief indication of the relationship between the two papers.

PART A. A NEW LOOK AT THE MARKOV-CHAIN CASE

1. Fluctuating clocks for Markov chains. Let E be a finite set, let X be an irreducible Markov chain on E with Q-matrix Q, and let m denote the unique invariant probability measure for X. For $x \in E$, P^x denotes the law of X when $X_0 = x$. Let v be a map $v:E\to\mathbb{R}\setminus\{0\}$, and put $E^+ \equiv v^{-1}(0,\infty)$, $E^{-1} \equiv v^{-1}(-\infty,0)$. We suppose that both E^+ and E^- are non-empty. For $t \geq 0$, define

$$(1.1)\quad \phi_t \equiv \int_0^t v(X_s)ds,\ \tau_t^+ \equiv \inf\{s:\phi_s > t\},\ \ \tau_t^- \equiv \inf\{s:-\phi_s > t\},$$

$$Y_t^+ \equiv X(\tau_t^+),\quad Y_t^- \equiv X(\tau_t^-).$$

It is elementary to prove that Y^+ and Y^- are Markov chains on E^+ and E^- respectively. We suppose that

$$(1.2)\quad Vm(E) > 0,$$

where Vm is the signed measure on E with $Vm(x) \equiv v(x)m(x)$. Then Y^+ has infinite lifetime, and Y^- has finite lifetime. Let G^+ be the $E^+ \times E^+$ matrix which is the Q-matrix of Y^+, and let G^- be the Q-matrix of Y^-. Let Π^+ and Π^- (respectively) be the $E^- \times E^+$ and $E^+ \times E^-$

matrices with entries:

(1.3) $\Pi^+(b,a) \equiv P^b[Y_0^+ = a]$, $\Pi^-(a,b) \equiv P^a[Y_0^- = b]$, for $a \in E^+$, $b \in E^-$.

Let V be the diagonal matrix $\mathrm{diag}(v(i))$, or, in other words, the operator of multiplication by v. The partitioning $E = E^+ \cup E^-$ of E induces the partitioning:

$$V^{-1}Q = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}$$

of the matrix $V^{-1}Q$. It is obvious from the probabilistic interpretation that

(1.4) $$G^+ = A + B\Pi^+, \quad \Pi^+ = \int_0^\infty e^{tD} C e^{tG^+} dt,$$

so that

(1.5) $$\Pi^+ G^+ = \int_0^\infty e^{tD} C e^{tG^+} G^+ dt = -C - D\Pi^+$$

(by integration by parts). On combining (1.4) and (1.5) with their 'minus' analogues, we obtain the 'Wiener-Hopf factorization' of $V^{-1}Q$:

(1.6) $$\begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} = \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix}.$$

It is easy to see that the matrix inverse occurring on the left-hand side of (1.6) exists because of assumption (1.2).

The existence of the factorization (1.6) was proved by a different (martingale) method in Barlow-Rogers-Williams [1]. It was also shown there that the factorization (1.6) is unique in the strong sense now to be explained.

Suppose that Π^+, Π^-, G^+ and G^- are any four matrices (on $E^- \times E^+$, $E^+ \times E^-$, $E^+ \times E^+$ and $E^- \times E^-$ respectively) such that every eigenvalue of G^+ has nonpositive real part, every eigenvalue of G^- has negative real part, and (1.6) holds. Then Π^+ and Π^- must be as at (1.3), and (hence) G^+ and G^- must be the Q-matrices of Y^+ and Y^- respectively.

So as not to look for difficulties, *we shall assume throughout the remainder of this paper that*

(1.7, *Assumption*) $V^{-1}Q$ *has* $(|E|)$ *distinct eigenvalues*. It is then easy to prove the uniqueness assertion for the factorization (1.6) stated above, for it hinges on the following lemma.

(1.8) LEMMA. *Let* $f = \binom{f^+}{f^-}$ *be a function on* E *(with restrictions* f^+ *and* f^- *to* E^+ *and* E^- *respectively.)*

(i) *If* $V^{-1}Qf = \alpha f$ *for some* α *with nonpositive real part, then*

$$f = \binom{I}{\Pi^+} f^+, \quad \text{and} \quad G^+f^+ = \alpha f^+.$$

(ii) *If* $V^-Qf = \beta f$ *for some* β *with strictly positive real part, then*

$$f = \binom{\Pi^-}{I} f^-, \quad \text{and} \quad G^-f^- = -\beta f^-.$$

It follows that $V^{-1}Q$ has exactly $|E^+|$ eigenvalues of non-positive real part, and $|E^-|$ eigenvalues of strictly positive real part.

2. *A spectral expansion for* Π^+.

Let

$$\beta_1, \beta_2, \ldots, \beta_k \qquad (k = |E^-|)$$

be the eigenvalues of $V^{-1}Q$ of strictly positive real part. From Lemma 1.8, we know that (with $\sigma(.)$ denoting spectrum):

$$(2.6)\qquad Z_t \equiv X(\rho_t), \quad \text{where} \quad \rho_t \equiv \inf\{u: \int_0^u |v(X_s)|ds > t\} .$$

We have shown that

(2.7) <u>for</u> $b \in E^-$, <u>the law</u> $\Pi^+(b,\cdot)$ <u>of</u> Y_0^+ <u>under</u> P^b <u>is a 'mixture' of Laplace transforms of entrance laws for</u> Z <u>from</u> E^- <u>to</u> E^+ <u>with Laplace-transform parameters</u> $\beta_1, \beta_2, \ldots, \beta_k$, <u>the eigenvalues of</u> $-G^-$.

It should be noted that the 'symmetric appearance' of (2.4) suggests that a dual form of the second equation at (1.4) must hold:

$$(2.8)\qquad \Pi^+ = \int_0^\infty e^{t(D+\Pi^+B)}Ce^{tA}dt ,$$

and that it is (2.8) which provides the motivation for Theorem 2.5. We shall now see that (2.8) is best proved by a time-reversal argument.

3. <u>Time-reversal</u>. Suppose now that X_0 is chosen according to the invariant measure m. Then the time-reversal X^* of X is a Markov chain with Q-matrix Q^* which is the adjoint of Q on $L^2(E,m)$:

$$(Q^*f)_x = \sum_{y\in E} q^*_{xy}f_y, \quad \text{where} \quad q^*_{xy} = m_y q_{yx} m_x^{-1} .$$

For an $E^- \times E^+$ matrix $H = (h_{ba})$, define H^* to be the $E^+ \times E^-$ matrix with $h^*_{ab} \equiv m_b h_{ba} m_a^{-1}$; and so on. By simple algebraic operations on (1.4),

$$(3.1)\quad \begin{pmatrix} I & (\Pi^+)^* \\ (\Pi^-)^* & I \end{pmatrix}^{-1} \begin{pmatrix} A^* & C^* \\ -B^* & -D^* \end{pmatrix} \begin{pmatrix} I & (\Pi^+)^* \\ (\Pi^-)^* & I \end{pmatrix} = \begin{pmatrix} (A+\Pi^-C)^* & 0 \\ 0 & -(D+\Pi^+B)^* \end{pmatrix} ,$$

(2.1) $\sigma(G^-) = \{-\beta_1, -\beta_2, \ldots, -\beta_k\}$.

Now it follows quickly from (1.6) that

(2.2) $D + \Pi^+B = (I - \Pi^+\Pi^-)G^-(I - \Pi^+\Pi^-)^{-1}$,

so that the matrix $D + \Pi^+B$ is similar to the matrix G^-. By standard matrix theory, there exist $E^- \times E^-$ matrices $J_1, J_2, \ldots, J_k$ such that (with I^- denoting the identity $E^- \times E^-$ matrix):

(2.3) $J_i^2 = J_i, \quad \sum_i J_i = I^-, \quad J_i(D + \Pi^+B) = -\beta_i J_i$.

From (1.4) and (1.5),

(2.4) $C + D\Pi^+ + \Pi^+A + \Pi^+B\Pi^+ = 0$,

Hence,

$$J_iC + J_i\Pi^+A - \beta_iJ_i\Pi^+ = 0$$

and so

$$J_i\Pi^+ = J_iC(\beta_i - A)^{-1} .$$

Since $\Sigma J_i = I^-$, we have proved the following theorem.

(2.5) THEOREM.

$$\Pi^+ = \sum_i J_iC(\beta_i - A)^{-1} .$$

Now, for $\lambda > 0$, the matrix $C(\lambda - A)^{-1}$ has an obvious significance as the Laplace transform of an entrance law from E^- to E^+ for a chain Z with Q-matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

We obtain such a chain Z via the classical time-substitution:

and this is the unique Wiener-Hopf factorization of $V^{-1}Q^*$. The probabilistic interpretation now implies that

$$(\Pi^+)^* = \int_0^\infty e^{tA^*} C^* e^{t(D+\Pi^+B)^*} dt,$$

and equation (2.8) follows on taking adjoints on $L^2(E,m)$.

4. <u>The 'symmetrizable' case.</u> The most interesting case is that in which

(4.1) $Q = Q^*$,

that is, in which $m_x q_{xy} = m_y q_{yx}$ and X is 'symmetrizable' (identical in law to its time-reversal X^*).

<u>Assume now that (4.1) holds.</u> Then, on comparing (1.4) with (3.1), we see that

(4.2) $(\Pi^+)^* = \Pi^-$, $(\Pi^-)^* = \Pi^+$, $G^+ = (A + \Pi^- C)^*$, $G^- = (D + \Pi^+ B)^*$.

Let us now follow up some of the 'algebra' of the situation.

The operator $V^{-1}Q$ is self-adjoint relative to the <u>indefinite</u> inner-product defined as follows:

$$\langle f,g \rangle_{Vm} \equiv \sum_{x \in E} f_x \bar{g}_x v_x m_x ,$$

$\bar{g}_x$ denoting the complex conjugate of g_x. If f^+ and g^+ are vectors on E^+, define

$$\langle f^+,g^+ \rangle_+ \equiv \langle f,g \rangle_{Vm}, \quad \text{where } f \equiv \binom{I}{\Pi}^+ f^+ , \quad g \equiv \binom{I}{\Pi}^+ g^+ .$$

Then

$$\langle f^+,g^+ \rangle_+ = \langle f^+, (I - \Pi^-\Pi^+)g^+ \rangle_{(Vm)^+} ,$$

the inner product on the right being the inner product in $L^2(E^+,(Vm)^+)$, $(Vm)^+$ being the restriction of Vm to E^+. From (4.2), the matrix $(I - \Pi^-\Pi^+)$ is self-adjoint on $L^2(E^+,(Vm)^+)$. Further, since $\Pi^-\Pi^+$ is substochastic and definitely not stochastic, it is easily shown that $(I - \Pi^-\Pi^+)$ is strictly positive-definite. (For example, one can express $(I - \Pi^-\Pi^+)^{-\frac{1}{2}}$ by binomial expansion.) In particular, if f is an eigenvector of $V^{-1}Q$ corresponding to an eigenvalue α of nonpositive real part, then

$$\langle f,f\rangle_{Vm} = \langle f^+,f^+\rangle_+ > 0$$

and we can use the old-familiar argument to show that α is real:

$$\alpha\langle f,f\rangle_{Vm} = \langle V^{-1}Qf,f\rangle_{Vm} = \langle f,V^{-1}Qf\rangle_{Vm}$$

$$= \langle f,\alpha f\rangle_{Vm} = \bar{\alpha}\langle f,f\rangle_{Vm},$$

so that $\alpha = \bar{\alpha}$.

Analogously, if f^- and g^- are vectors on E^-, define

$$\langle f^-,g^-\rangle_- \equiv \langle f,g\rangle_{Vm}, \quad \text{where } f \equiv \binom{\Pi^-}{I}f^-, \quad g \equiv \binom{\Pi^-}{I}g^-,$$

so that

$$\langle f^-,g^-\rangle_- = -\langle f^-,(I - \Pi^+\Pi^-)g^-\rangle_{(Vm)^-}$$

where $(Vm)^-$ is <u>minus</u> the restriction of Vm to E^- (so that $Vm = (Vm)^+ - (Vm)^-$). Then $\langle\cdot,\cdot\rangle_-$ is a negative-definite inner-product, and we can show as above that if β is an eigenvalue of $V^{-1}Q$ with strictly positive real part, then $-\beta$ is an eigenvalue of G^- and β is real.

Thus, <u>every</u> eigenvalue of $V^{-1}Q$ is real, and the usual undergraduate method shows that eigenvectors of $V^{-1}Q$ corresponding to different eigenvalues are orthogonal for the inner product $\langle\cdot,\cdot\rangle_{Vm}$. As a consequence

G^- has real eigenvalues and eigenvectors which are orthogonal relative to the inner-product $\langle\cdot,\cdot\rangle_-$. Hence,

(4.3) G^- <u>is self-adjoint relative to the inner-product</u> $\langle\cdot,\cdot\rangle_-$. Since we know from (4.2) that $G^- = (D + \Pi^+ B)^*$, the result (4.3) does in fact follow immediately from (2.2). However, the inner-product concepts prove to give a helpful way of thinking about things.

Take m-adjoints in (2.3):

(4.4) $(J_i^*)^2 = J_i$, $\quad \Sigma J_i^* = I^-$, $\quad G^- J_i^* = -\beta_i J_i^*$.

For $1 \le i \le k$, let $\Psi_i = (\Psi_i(b) : b \in E^-)$ be a real eigenvector of G^- corresponding to $-\beta_i$, and normalise the Ψ_i so that

$$\langle \Psi_i, \Psi_j \rangle_- = -\delta_{ij} .$$

Then, for any vector η on E^-, we have

$$J_i^* \eta = -\langle \eta, \Psi_i \rangle_- \Psi_i = \Psi_i \Psi_i^* (I - \Pi^+ \Pi^-) \eta,$$

where, for $b \in E^-$, $\Psi_i^*(b) \equiv \Psi_i(b) m(b)$. Hence,

(4.5) $$J_i = (I - \Pi^+ \Pi^-) \Psi_i \Psi_i^* .$$

5. <u>Recapitulation</u>. Let us collect together some of the facts which we have established for the case when X is symmetrizable.

Firstly, we know that the eigenvalues $-\beta_1, -\beta_2, \ldots, -\beta_k$ of G^- are real and negative; and from (4.4) we have the resolvent expansion:

(5.1) $$(\lambda - G^-)^{-1} = \sum_{i=1}^{k} \frac{J_i^*}{\lambda + \beta_i} \qquad (\lambda > 0).$$

Secondly, we have:

(5.2) $$\Pi^+ = \sum_{i=1}^{k} J_i C(\beta_i - A)^{-1}$$

(restating (2.5)); and, since this is now an expansion of Π^+ in terms of real matrices, the 'interpretation' (2.6) is a little more meaningful. Finally, we have (4.5).

6. <u>A special case</u>. We now make the further assumption that E^- contains a 'special' state labelled 0 such that jumps from E^- to E^+ can be made only from state 0. It follows from the symmetrizability assumption that jumps from E^+ to E^- can only be made <u>to</u> state 0.

We shall prove under this assumption that

(6.1) $$J_i(0,0) \geq 0 \qquad (\forall i),$$

so that, from (5.2), $\Pi^+(0,\cdot)$ is a convex combination of the Laplace transforms of the entrance law for Z from 0 into E^+ of parameters $\beta_1, \beta_2, \ldots, \beta_k$. Moreover, we have, from (5.1) and the fact that $J_i^*(0,0) = J_i(0,0)$,

(6.2) $$(\lambda - G^-)^{-1}(0,0) = \sum_i \frac{J_i(0,0)}{\lambda + \beta_i} .$$

<u>Note on the 'Brownian' case</u>. In the 'Brownian' case considered in Paper I, the entrance law of Z from 0 into $(0,\infty)$ is of course identical to the entrance law of a reflecting Brownian motion from 0 into $(0,\infty)$. The Laplace transform of this entrance law with parameter $\beta = \frac{1}{2}r^2$ $(r > 0)$ is well known to be the measure

$$2e^{-rx}dx$$

(in terms of the standard local time at 0). An 'explanation' of the fundamental formula:

$$(6.3) \qquad p_4(dx) = dx \int e^{-rx} J(dr)$$

from Paper I - with J a (nonnegative)measure - is therefore now before us; and the way in which the measure J in (6.3) features in the resolvent of Y^- in equation (21) of Paper I corresponds exactly to our equation (6.2) □

Proof of inequality (6.1). Because of (4.5), we need only show that

(6.4) $(f^- - \Pi^+\Pi^-f^-)(0)$ and $f^-(0)$ have the same sign whenever $f = \binom{f^+}{f^-} = \binom{\Pi^-}{I}f^-$ is an eigenvector of $V^{-1}Q$ corresponding to an eigenvalue $\beta > 0$. (For we can take $\beta = \beta_i$ and $f^- = \Psi_i$.) Now since $Qf = \beta Vf$,

$$\exp(-\beta\phi_t)f(X_t)$$

is a local martingale under every P^y measure. Let

$$\eta \equiv \inf\{t : X_t \in E^-\}.$$

Let $x \in E^+$. Then, under P^x,

$$\exp(-\beta\phi_{t\wedge\eta})f(X_{t\wedge\eta})$$

is a bounded local martingale, and hence a martingale. Hence, for $x \in E^+$,

$$(6.5) \qquad f(x) = (\Pi^-f^-)(x) = E^x[\exp(-\beta\phi_\eta)f(X_\eta)] = E^x[\exp(-\beta\phi_\eta)]f(0)$$

since we are assuming that X can enter E^- from E^+ only at the point 0. Obviously, we need only prove (6.4) under the assumption that $f^-(0) > 0$. But then, from (6.5),

$$(\Pi^-f^-)(x) \le f^-(0) \qquad (\forall x \in E^+),$$

and, since Π^+ is substochastic (in fact, stochastic),

$$(\Pi^+\Pi^- f^-)(0) \le f^-(0),$$

and the result (6.4) follows. □

7. Another expression for π^+. Now that the reader has seen that the chain case can indeed throw light on the diffusion case, perhaps he will tolerate one further small rephrasing of the analytic form of π in the general 'symmetrizable' case for chains. The reader will appreciate that what is merely a trivial rephrasing in the finite-dimensional case may correspond to something deeper in the infinite-dimensional context.

So, let X be symmetrizable. Every eigenvalue of $V^{-1}Q$ is real. If $k = |E^-|$, then we can find k linearly independent real eigenvectors f_i $(1 \le i \le k)$ of $V^{-1}Q$ with corresponding eigenvalues positive. Each f_i has the form

$$f_i = \begin{pmatrix} \pi^- \\ I \end{pmatrix} \Psi_i$$

where Ψ_i is the restriction of f_i to E^-. We can (and do) choose the f_i so that

(7.1) $$\langle f_i, f_j \rangle_{Vm} = \langle \Psi_i, \Psi_j \rangle_- = -\delta_{ij} .$$

Let $\xi_i = (I - \pi^+\pi^-)\Psi_i$. Then

(7.2) $$\langle \Psi_i, \xi_j \rangle_{(Vm)^-} = -\langle \Psi_i, \Psi_j \rangle_- = \delta_{ij} ,$$

so that (ξ_i) is a dual basis for (Ψ_i) relative to the classical inner product:

$$\langle \Psi, \xi \rangle_{(Vm)^-} = \sum_{b \in E^-} \Psi(b)\xi(b)\,|V(b)|\,m(b)$$

for real vectors on E^-.

Moreover,

$$J_i = \xi_i \Psi_i^*$$

and

$$\pi^+ = \sum_i J_i \pi^+ = \sum_i \xi_i (\pi^- \Psi_i)^*.$$

Thus,

$$(7.3) \qquad \pi^+(b,a) = \left(\sum_{i=1}^{k} \xi_i(b) f_i(a)\right) m(a), \qquad (b \in E^-,\ a \in E^+).$$

PART B. DETAILED CALCULATIONS FOR A DIFFUSION EXAMPLE.

8. Let B be a Brownian motion, and let

$$\phi_t \equiv \int_0^t I_{(0,\infty)}(B_s)ds - \int_0^t I_{[-1,0]}(B_s)ds.$$

For $t \geq 0$, let

$$\tau_t^+ \equiv \inf\{u : \phi_u > t\}, \qquad \tau_t^- \equiv \inf\{u : \phi_u < -t\},$$

$$Y_t^+ = B(\tau_t^+), \qquad Y_t^- \equiv B(\tau_t^-).$$

Here, we clearly have a 'symmetrizable' situation with $m(dx) = dx$ on $[-1,\infty)$,

and

$$V = \begin{cases} 1 & \text{on } (0,\infty), \\ -1 & \text{on } [-1,0]. \end{cases}$$

The Operator A (say) corresponding to $V^{-1}Q$ is the operator with

$$f = \begin{cases} \frac{1}{2}f'' & \text{on } (0,\infty) \\ -\frac{1}{2}f'' & \text{on } [-1,0] \end{cases}$$

where

$$\mathcal{D}(\mathcal{A}) = C^1[-1,\infty) \cap C^2(-1,0) \cap C^2(0,\infty)$$
$$\cap \{f : f'(-1) = f'(-1+) = 0\}.$$

We hope that the reader will allow a notational shift which proves to be convenient. We shift $[-1,0]$ to $[0,1]$. Thus, we shall write:

y for a typical point of $(0,\infty)$

x in $[0,1]$ for the point which is really the point $x-1$ in $[-1,0]$.

This will become clear in a moment. Actually, there is rather more than mere notational convenience involved here ... .

With this understanding, note that

$\mathcal{A}$ has the bounded eigenfunction g, where

$g_\theta(x) = \cosh\theta x$ on $[0,1]$ (really, $\cosh\theta(x+1)$ on $[-1,0]$),

$g_\theta(y) = \cosh\theta\cos\theta y + \sinh\theta\sin\theta y$ on $(0,\infty)$,

corresponding to the NEGATIVE eigenvalue $-\frac{1}{2}\theta^2$.

By analogy with (1.8.i), or directly from the martingale argument in Paper I, we know that if

(8.1) $$\pi(x,dy) = P^x[Y_0^+ \in dy],$$

then

(8.2) $$\int_y \pi(x,dy)(\cosh\theta\cos\theta y + \sinh\theta\sin\theta y) = \cosh\theta x.$$

Moreover, we know from Part 3 of Paper I that

(8.3) the measure $\pi(x,\cdot)$ on $(0,\infty)$ is uniquely determined by the fact that (8.2) holds for all $\theta > 0$.

We note that:

$\mathcal{A}$ has the bounded eigenfunction f_n, where

$$f_n(x) = \cos \alpha_n x / \cos \alpha_n \quad \text{on} \quad [0,1],$$

$$f_n(y) = e^{-\alpha_n y} \quad \text{on} \quad (o,\infty),$$

<u>corresponding to the POSITIVE eigenvalue</u> $\frac{1}{2}\alpha_n^2$, <u>where</u>

$$\alpha_n = (n+\tfrac{1}{2})\pi, \quad n = 0,1,2,\ldots$$

We note that

$$\langle f_m, f_n \rangle_{Vm} = -\delta_{mn} .$$

From (7.3), we expect that

(8.4) $$\pi(x,dy) = \pi(x,y)dy$$

where

(8.5) $$\pi(x,y) = \sum_{n\geq 0} H_n(x) e^{-\alpha_n y}$$

and the 'dual basis' H_n satisfies:

(8.6) $$\int_0^1 H_m(x) \cos \alpha_n x dx = \delta_{mn} \cos \alpha_n .$$

Moreover, since $\pi^+ = (\pi^-)^*$, we expect that

$$P^y[Y_0^- \in dx] = dx \pi(x,y),$$

and, by analogy with (1.8.ii), that

(8.7) $$\int_0^1 \pi(x,y) \cos \alpha_n x \, dx = e^{-\alpha_n y} \cos \alpha_n .$$

The problem is that equation (8.2) for π, equation (8.7) for π, and equations (8.6) for the H_m, are none of them of conventional form.

It was only after very considerable effort - and a remarkable piece of luck - that we discovered that for $0 < x < 1$,

$$(8.8) \qquad \pi(x,y) = \frac{\cosh \tfrac{1}{2}\pi y (\cos \tfrac{1}{2}\pi x \sinh \tfrac{1}{2}\pi y)^{\frac{1}{2}}}{2^{\frac{1}{2}}(\sinh^2 \tfrac{1}{2}\pi y + \cos^2 \tfrac{1}{2}\pi x)} .$$

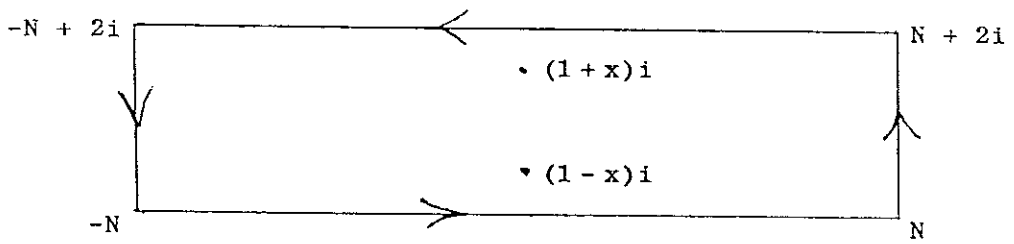
9. Evaluation of some integrals. Let us now verify that if π is defined as at (8.8), then

$$(9.1) \qquad \int (\cosh\theta \cos\theta y + \sinh\theta \sin\theta y)\pi(x,y)dy = \cosh\theta x, \quad \text{when} \quad 0 < x < 1, \; \theta > 0.$$

Consider the contour integral

$$\oint \frac{\cosh \tfrac{1}{2}\pi z (\sinh \tfrac{1}{2}\pi z)^{\frac{1}{2}}}{\sinh^2 \tfrac{1}{2}\pi z + \cos^2 \tfrac{1}{2}\pi x} e^{i\theta z} dz$$

around the contour:



There is no problem with the square root because

$\operatorname{Im} \sinh \tfrac{1}{2}\pi z \geq 0$ inside and on the contour. It is trivial that the contribution from the 'vertical' parts of the contour tends to 0 as $N \to \infty$, so that we can 'take $N = \infty$'.

The poles occur when

$$\sinh \tfrac{1}{2}\pi z = \pm i \cos \tfrac{1}{2}\pi x,$$

and so the only poles within the contour are at

$(1 + x)i$ and $(1 - x)i$.

The residues at these poles are found to be

$$\frac{e^{-\theta(1+x)}}{\pi(i\cos\frac{1}{2}\pi x)^{\frac{1}{2}}} \quad \text{and} \quad \frac{e^{-\theta(1-x)}}{\pi(i\cos\frac{1}{2}\pi x)^{\frac{1}{2}}} \quad \text{respectively.}$$

Because

$$\cosh\tfrac{1}{2}\pi(z+2i) = -\cosh\tfrac{1}{2}\pi z$$

$$\sinh\tfrac{1}{2}\pi(z+2i) = -\sinh\tfrac{1}{2}\pi z ,$$

the total contribution to $\oint$ from the horizontal parts of the contour is

$$(1+ie^{-2\theta})\int_{-\infty}^{\infty} \frac{\cosh\frac{1}{2}\pi z(\sinh\frac{1}{2}\pi z)^{\frac{1}{2}}}{\sinh^2\frac{1}{2}\pi z + \cos^2\frac{1}{2}\pi x}\, e^{i\theta z} dz.$$

It is now a straightforward exercise to deduce the desired result (9.1) from the Residue Theorem.

Because of (8.3), we have now determined the law of Y_0^+.

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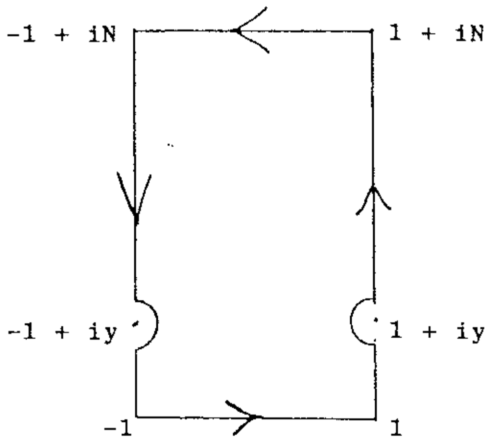
We can prove in a similar fashion that if π is again defined via (8.8), then (8.7) holds. This time, we evaluate

$$\oint \frac{(\cos\frac{1}{2}\pi z)^{\frac{1}{2}} e^{i\alpha_n z}}{\sinh^2\frac{1}{2}\pi y + \cos^2\frac{1}{2}\pi x}\, dz$$

around the contour



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Next, notice that if π is defined as at (8.8), then π has the form (8.5) as can be seen by binomial expansion; and then (8.6) follows from (8.7). It is interesting to note that

$$H_0(x) = (\cos \tfrac{1}{2}\pi x)^{\frac{1}{2}} .$$

10. A brief sketch of our route to (8.8). Some interesting complex analysis underlies this work, and it is very likely that it will be taken up by some of us in a further paper. In particular, the contours used in §9 relate to our problem in a fascinating way.

For now, we explain briefly how we arrived at the formula (8.8).

We began by solving for this example the problem considered in Paper I, namely, that of determining the law of Y^+. We assume now that the reader is familiar with the results of Paper I.

[At this point, DW apologizes - for the fault is his - for the fact that the notation in these two papers could have been better integrated. But there are not enough letters to go round, and one needs an enormous number of letters to describe certain Wiener-Hopf expansions which make essential companions to those mentioned in this paper.]

The bounded eigenfunction h (say) of A $(=\frac{1}{2}\mathrm{sgn}(x)D^2)$ corresponding to eigenvalue $-\frac{1}{2}\theta^2$ and normalized so that $h_\theta'(0) = 1$, is given on $(0,\infty)$ by

$$h_\theta(y) = \theta^{-1}\coth\theta\cos\theta y + \theta^{-1}\sin\theta y.$$

Hence, since p_1 and p_2 are obviously zero, we must find a measure J on $(0,\infty)$ such that

$$(10.1) \qquad \frac{\cosh\theta}{\theta\sinh\theta} = \frac{\displaystyle\int\frac{J(dr)}{r^2+\theta^2}}{\theta^2\displaystyle\int\frac{J(dr)}{r(r^2+\theta^2)}}$$

But one of the recurring themes of these two papers is that J must be supported by r-values such that $\frac{1}{2}r^2$ is a positive eigenvalue of A. Thus J must be supported by the set

$$\{\alpha_0,\alpha_1,\alpha_2,\ldots\}, \quad \text{where} \quad \alpha_n = (n+\tfrac{1}{2})\pi.$$

We shall write $\alpha_n H_n = J\{\alpha_n\}$. Then (10.1) takes the form:

$$\frac{\cosh\theta}{\sinh\theta} = \frac{\displaystyle\Sigma\frac{\alpha_n H_n}{\alpha_n^2+\theta^2}}{\displaystyle\theta\Sigma\frac{H_n}{\alpha_n^2+\theta^2}}.$$

Write $i\theta$ for θ and rearrange to get:

$$(10.2) \qquad \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \frac{\displaystyle\Sigma\frac{H_n}{\alpha_n+\theta}}{\displaystyle\Sigma\frac{H_n}{\alpha_n-\theta}}.$$

Note that the two sides have the same zeros in $\mathbb{C}$ and the same poles (with correct residues) in $\mathbb{C}$. It follows from Paper I that amongst non-negative sequences, the sequence H_n is unique modulo multiplication by a constant.

We discovered that

$$(10.3) \qquad H_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \qquad \text{(modulo a constant multiplier)}$$

by considering the Mittag-Leffler expansion of

$$\operatorname{cosec}(z-\tfrac{\pi}{4}) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{(k-\frac{1}{4})^2\pi^2}\right),$$

and then discovered that applied mathematicians had spotted (10.3). (But we have found the Mittag-Leffler technique useful in other probabilistic examples.) For (H_n) as at (10.3),

$$(10.4) \qquad \Sigma \frac{H_n}{\alpha_n + \theta} = \frac{\Gamma(\frac{1}{4}+\frac{\theta}{\pi})}{\Gamma(\frac{3}{4}+\frac{\theta}{\pi})} .$$

Since we now know the measure J, we can calculate the Lévy measure for Y^+. We find that

$$p_4(y,\infty) = \frac{\text{constant}}{\sqrt{(\sinh \frac{1}{2}\pi y)}} .$$

The calculation of $\pi(x,y)$ seems to be altogether more challenging.

Since we know the Lévy measure for Y^+, we can show by entirely standard arguments that

$$(10.5) \qquad P^0[Y_t^+ \in dy] = b_y(t)dy \qquad (t > 0,\ y > 0)$$

where, for $\gamma > 0$,

$$(10.6) \qquad \int_{(0,\infty)} e^{-\frac{1}{2}\gamma^2 t} b_y(t)\,dt = \frac{\sum_n \frac{2\alpha_n H_n}{\gamma^2 - \alpha_n^2}\left(e^{-\alpha_n y} - e^{-\gamma y}\right)}{\sum_m \frac{\gamma H_m}{\alpha_m + \gamma}}$$

From (10.6) and (10.2), it follows that if

$$(10.7) \qquad \gamma_k = (k + \tfrac{1}{2})\pi, \qquad k = 0,1,2,\ldots$$

$$(10.8) \qquad \int_{(0,\infty)} e^{-\frac{1}{2}\gamma_k^2 t} b_y(t)\,dt = \frac{1}{\gamma_k r_k} \sum_n \frac{2\alpha_n H_n}{\gamma_k^2 - \alpha_n^2} e^{-\alpha_n y},$$

where

$$r_k = \sum_n \frac{H_n}{\alpha_n + \gamma_k} = \frac{\Gamma(k+\frac{3}{4})}{\Gamma(k+\frac{5}{4})}.$$

But it is probabilistically obvious - compare the second equation at (1.4), and use the strong Markov theorem for rigour - that

$$(10.9) \qquad \pi(x,y) = \int_{(0,\infty)} a_x(t) b_y(t)\,dt \qquad (0 \le x < 1,\ y \ge 0)$$

where for $0 \le x < 1$,

$$(10.10) \qquad a_x(t)\,dt = P^x[\text{RBM first hits } 1 \text{ within } (t, t+dt)],$$

where RBM signifies a Brownian motion reflected at 0.

It is well known that, for $\gamma > 0$,

$$\int_{(0,\infty)} e^{-\frac{1}{2}\gamma^2 t} a_x(t)\,dt = \frac{\cosh \gamma x}{\cosh \gamma} = \sum_k \frac{(-1)^k \gamma_k \cos \gamma_k x}{\frac{1}{2}\gamma_k^2 + \frac{1}{2}\gamma^2}$$

$$= \int_{(0,\infty)} e^{-\frac{1}{2}\gamma^2 t} \Big(\sum_k R_k(x) e^{-\frac{1}{2}\gamma_k^2 t}\Big), \qquad R_k(x) = (-1)^k \gamma_k \cos \gamma_k x .$$

Hence

(10.11) $$a_x(t) = \sum_k R_k(x) e^{-\frac{1}{2}\gamma_k^2 t} .$$

On putting together (10.8), (10.9), and (10.11), we see that

(10.12) $$\pi(x,y) = \sum_n H_n(x) e^{-\alpha_n y}$$ (Cheers!)

where

(10.13) $$H_n(x) = 2\alpha_n H_n \sum_k \frac{(-1)^k \gamma_k \cos \gamma_k x}{\gamma_k r_k (\gamma_k^2 - \alpha_n^2)} .$$

But

$$\frac{\cos \alpha_n x}{\cos \alpha_n} = \sum_k \frac{(-1)^k 2\gamma_k \cos \gamma_k x}{\gamma_k^2 - \alpha_n^2} ,$$

so that

(10.14) $$H_n(x) = \alpha_n H_n \int_{-1}^{1} g(x-u) \frac{\cos \alpha_n u}{\cos \alpha_n}\, du ,$$

where

$$(10.15)\qquad g(x) = \sum_k \frac{1}{(k+\frac{1}{2})\pi} \frac{\Gamma(k+\frac{5}{4})}{\Gamma(k+\frac{3}{4})} \cos(k+\tfrac{1}{2})\pi x.$$

Do note the range of integration in (10.14).

As a consequence of a preposterous guess, we were able to sum this series and show that

$$(10.16)\qquad g(x) = \frac{\sqrt{\pi}}{8}\int_{(x,1)} (\sin\tfrac{1}{2}\pi u)^{-\frac{3}{2}}du \qquad (0 < x < 1).$$

Formula (8.8) now follows on putting together (10.12), (10.14), and (10.16).

Of course, you may well feel that there is just as much chance of guessing (8.8) as of guessing that the sum of (10.15) is given by (10.16); and you may well be right! But we have told it the way it happened. The fact that (10.15) and (10.16) do agree follows from our uniqueness theorems and the calculations in §9; but we intend to indicate a direct proof of this fact and related facts elsewhere.

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REFERENCES

1. M.T. BARLOW, L.C.G. ROGERS, and DAVID WILLIAMS, Wiener-Hopf factorization for matrices, Séminaire de Probabilités XIV, Springer Lecture Notes in Math. 784, 324-331, 1980.

2. H. DYM and H.P. McKEAN, Gaussian processes, function theory, and the inverse spectral problem, Academic Press, New York, 1976.

3. Priscilla GREENWOOD and Jim PITMAN, Fluctuating identities for Lévy processes and splitting at the maximum, Adv. Appl. Prob. 12, 893-902, 1980.

4. K. ITÔ and H.P. McKEAN, Diffusion processes and their sample paths, Springer, Berlin, 1965.

5. J.F.C. KINGMAN, Markov transition probabilities, II; completely monotone functions, Z. Wahrscheinlichkeitstheorie & verw. Geb. 6, 248-270, 1967.

6. L.C.G. ROGERS and DAVID WILLIAMS, Time-substitution based on fluctuating additive functionals (Wiener-Hopf factorization for infinitesimal generators), Séminaire de Probabilités XIV, Springer Lecture Notes in Math. 784, 332-342.

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