(30)

Multiple points of Markov processes in a complete metric space

by

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1. Introduction.

Let (S,d) be a complete metric space with Borel σ-field $\mathcal{S}$, and let $(X_t)_{t\geq 0}$ be an S-valued strong Markov process whose paths are right continuous with left limits. We ask

(Q) Is $P(X_{t_1} = \cdots = X_{t_k}$ for some $0 < t_1 < \cdots < t_k) > 0$?

This is equivalent to the question

(Q′) Is $P(X(I_1) \cap \cdots \cap X(I_k) \neq \emptyset) > 0$ for some disjoint compact intervals $I_1, ..., I_k$?

We shall find conditions sufficient to ensure that X has k-multiple points with positive probability, and we will apply this to Lévy processes, providing another proof of a result of LeGall, Rosen and Shieh [6], and its improvement due to Evans [3]. However, it is advantageous to begin with the easier question

($\bar{\text{Q}}$) Is $P(\bar{X}(I_1) \cap \cdots \cap \bar{X}(I_k) \neq \emptyset) > 0$ for some disjoint compact intervals $I_1, ..., I_k$?

Here, $\bar{X}(I_j) \equiv$ closure $(\{X_s : s \in I_j\})$, a compact subset of S. In recent years, much effort has been devoted to a study of (Q), usually in the form of constructing some non-trivial random measure on the set $\{(t_1, ..., t_k) : X_{t_1} = \cdots = X_{t_k}\}$ from which the existence of common points in the ranges $X(I_j)$ follows immediately. We mention only the work of Dynkin [1] and Evans [2] on symmetric Markov processes, of Rosen [8], [9], Geman, Horowitz and Rosen [4], LeGall, Rosen and Shieh [6] and Evans [3] on more concrete Markov processes in $\mathbb{R}^n$, as a sample of recent activity. Typically, one studies the random variables

(1) $$Z_\varepsilon \equiv \int_C I_U(X_{t_1}) F_\varepsilon(X_t)\, dt ,$$

where $C = I_1 \times \cdots \times I_k$, with the I_j disjoint compact intervals in $\mathbb{R}^+$, $U \in \mathcal{S}$, and

(2) $$F_\varepsilon(x_1, ..., x_k) \equiv \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1}) ,$$

(A) $\mu(B_{2\varepsilon}(x)) \le K\,\mu(B_\varepsilon(x)) \qquad \forall\ \varepsilon \in (0,\eta],\ \forall\ x \in V$;

(B) $\int_{V\times V} g_{0,T}(x,y)^k\,\mu(dx)\,\mu(dy) < \infty$;

(C) for each $\delta \in (0,2T)$,

$$\sup_{x,y\in V} g_{\delta,2T}(x,y) < \infty;$$

(D) for each $0 < a < b < \infty$, $g_{a,b}(\bullet,\bullet)$ is lower semicontinuous on $V \times V$;

(E) for some $\xi \in U$ and $\tau \in (0,T)$,

$$g_{0,\tau}(\xi,\xi) > 0.$$

Remarks on conditions (A)-(E). Condition (A) seems fairly mild; it is trivially satisfied for Lebesgue measure on Euclidean space. The purpose of (A) is to let us take

$$(6) \qquad f_\varepsilon(x,y) \equiv \mu(B_\varepsilon(x))^{-1}\, \mathrm{I}_{\{d(x,y)\le\varepsilon\}}$$

and estimate

$$(7) \qquad \begin{aligned} f_\varepsilon(x,y) &\le K\,\mu(B_{2\varepsilon}(x))^{-1}\, \mathrm{I}_{\{d(x,y)\le\varepsilon\}} \\ &\le K\,\mu(B_\varepsilon(y))^{-1}\, \mathrm{I}_{\{d(x,y)\le\varepsilon\}} \\ &= K\,f_\varepsilon(y,x). \end{aligned}$$

Condition (B) is the 'folklore' condition for k-multiple points. Condition (C) may appear severe, but is frequently satisfied. Conditions (A)-(C) will give us (3.i), and conditions (D) and (E) will give us (3.ii). We may (and shall) suppose that the τ appearing in (E) is a point of increase of $g_{0,\bullet}(\xi,\xi)$.

THEOREM 1. *Assuming conditions* (A), (B), *and* (C), *the family* $\{Z_\varepsilon : 0<\varepsilon<\eta/k\}$ *is bounded in* L^2. *Assuming also conditions* (D) *and* (E), *there exist initial distributions such that for some disjoint compact intervals* $\mathrm{I}_1, \ldots, \mathrm{I}_k$

$$P(\overline{X}(\mathrm{I}_1) \cap \cdots \cap \overline{X}(\mathrm{I}_k) \ne \varnothing) > 0.$$

Proof. (i) Let m be the law of X_0. For ease of exposition, we shall suppose that X has a transition density $p_t(\bullet,\bullet)$ with respect to μ; the result remains true without this assumption though.

exploiting (6), integrating out $x_1, y_1, \ldots, x_{j-1}, y_{j-1}$ to leave as an upper bound

$$K^{2j-2} \int \mathrm{I}_V(x_j)\, \mathrm{I}_V(y_j)\, g(x_j, y_j)^k \, \mu(dx_j)\, \mu(dy_j)$$

which is finite, by assumption (B). Hence for $0 < \varepsilon < \eta/k$, $E(Z_\varepsilon^2)$ is bounded above by a finite constant independent of ε, which proves the first statement.

(ii) We next exploit (D) and (E) to give us (3.ii). By the choice of the set C, we have that for some small enough $\theta > 0$,

$$C \supseteq C_0 = \{(t_1, \ldots, t_k) : |t_i - t_{i-1} - \tau| < \theta \quad \text{for} \quad i = 1, \ldots, k\},$$

where $t_0 = 0$. Hence

$$\begin{aligned} EZ_\varepsilon &\geq E\left[\int_{C_0} dt\, \mathrm{I}_U(X_{t_1})\, F_\varepsilon(X_t)\right] \\ &= \int m(dx_0)\, \mathrm{I}_U(x_1) \prod_{i=1}^{k} g(x_{i-1}, x_i) \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1})\, \mu(dx), \end{aligned}$$

where we write g as an abbreviation for $g_{\tau-\theta,\, \tau+\theta}$. Since τ is a point of increase of $g_{0,\cdot}(\xi, \xi)$, we know that $g(\xi, \xi) > 0$. Thus

(8) $$EZ_\varepsilon \geq \int m(dx_0)\, \mathrm{I}_U(x_1)\, g(x_0, x_1)\, \underline{g}_\varepsilon(x_1)^{k-1} \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1})\, \mu(dx),$$

where

$$\underline{g}_\varepsilon(x_1) \equiv \inf\{g(x, y) : d(x, x_1) \leq k\varepsilon,\, d(y, x_1) \leq k\varepsilon\},$$

which, in view of (D), increases as $\varepsilon \downarrow 0$ to $g(x_1, x_1)$. By integrating out the variables $x_k, x_{k-1}, \ldots, x_2$ in (8), we obtain the lower bound

$$EZ_\varepsilon \geq \int m(dx_0)\, \mathrm{I}_U(x_1)\, g(x_0, x_1)\, \underline{g}_\varepsilon(x_1)^{k-1}\, \mu(dx_1),$$

and hence the estimate

$$\liminf_{\varepsilon \downarrow 0} EZ_\varepsilon \geq \int m(dx_0)\, \mathrm{I}_U(x_1)\, g(x_0, x_1)\, g(x_1, x_1)^{k-1}\, \mu(dx_1).$$

By lower semi-continuity and the fact that $g(\xi, \xi) > 0$, we know that $g(x, y)$ is positive in a neighbourhood of (ξ, ξ) and so taking $m = \delta_\xi$, for example, yields

$$\liminf_{\varepsilon \downarrow 0} EZ_\varepsilon > 0.$$ ◊

since $\overline{R}_K \backslash R_K \subset \bigcup_{j=1}^{K} (\overline{X}(I_j) \backslash X(I_j))$, and $\overline{X}(I_j) \backslash X(I_j)$ is contained in the (countable) set of left endpoints of jumps of X during time interval I_j, it follows from (F) that the set $\overline{R}_K \backslash R_K$ is *polar*, contradicting (10). ◊

3. Multiple points of Lévy processes. Let X be a Lévy process in $\mathbb{R}^n$, with resolvent $(U_\lambda)_{\lambda > 0}$. We shall assume that the resolvent is strong Feller (equivalently, that each $U_\lambda(x,.)$ has a density with respect to Lebesgue measure - see Hawkes [5]), in which case there is for each $\lambda > 0$ a λ-excessive lower semi-continuous function u_λ such that

$$U_\lambda f(x) = \int u_\lambda(y) f(y+x)\, dy .$$

To establish sufficient conditions for k-multiple points, we shall need three lemmas on Lévy processes of interest in their own right.

LEMMA 1. *The resolvent* $(U_\lambda)_{\lambda > 0}$ *is strong Feller if and only if for every* $0 \le a < b < \infty$ *the kernel* $G_{a,b}$ *has a density* $g_{a,b}$.

If this happens, the densities $g_{a,b}(.)$ *may be chosen so that*

(i) $g_{a,b}(.)$ *is lower semicontinuous for each* $0 \le a < b < \infty$;

(ii) $(a,b) \to g_{a,b}(x)$ *is left-continuous increasing in* b *and right-continuous decreasing in* a *for each* x;

(iii) *for all* $0 \le a < b < \infty$ *and all* $x \in \mathbb{R}^n$

$$g_{a,b}(x) = \lim_{\delta \downarrow 0} \delta^{-1} \int g_{0,\delta}(y)\, g_{a,b-\delta}(x-y)\, dy .$$

LEMMA 2. *For a Lévy process with a strong Feller resolvent, the following are equivalent:*

(i) *for some* $\varepsilon, T > 0$,

$$\int_{\{|x| \le \varepsilon\}} g_{0,T}(x)^k\, dx < \infty ;$$

whence $g_{\delta,T}(.)$ is bounded globally (exploiting lower semi-continuity).

This completes the proof that (11.i-ii) implies that X has k-multiple points with positive probability, and hence, by Borel-Cantelli, there are almost surely k-multiple points.

Proof of Lemma 1. The arguments used are similar to those of Hawkes [5], so we will just give an outline. The first statement of the lemma is immediate. To get good versions of the densities $g_{a,b}$, firstly take any densities $g'_{p,q}(.)$ for $G_{p,q}$, $0 \le p < q < \infty$ rational, then define

$$g''_{a,b}(x) \equiv \sup\{g'_{p,q}(x) : a < p < q < b\},$$

which have property (ii) (which remains preserved under the subsequent modifications). Next, for $n > (b-a)^{-1}$ define

$$\tilde{g}^n_{a,b}(x) = n \int g_{0,\delta}(y)\, g_{a,b-\delta}(x-y)\, dy, \qquad (\delta \equiv n^{-1})$$

which is lower semicontinuous in x (it is the increasing limit as $M \uparrow \infty$ of

$$n \int g_{0,\delta}(y)\, (M \wedge g_{a,b-\delta}(x-y))\, dy,$$

which are continuous by the strong Feller property of $G_{0,\delta}$). Finally, we take

$$g_{a,b}(.) \equiv \sup\{\tilde{g}^n_{a,b}(.) : n > (b-a)^{-1}\}.$$

Since, for fixed $a < b$, $\tilde{g}^n_{a,b}$ is increasing almost everywhere to a version of the density of $G_{a,b}$, this provides a version with the desirable properties (i) - (iii). ◊

Proof of Lemma 2. The implications (iii) => (iv) => (i) are trivial. The implication (ii) => (iii) follows easily from the estimate

$$\begin{aligned} \int g_{a,a+T}(x)^k\, dx &= \int \left(\int P_a(dy)\, g_{0,T}(x-y)\right)^k dx \\ &\le \int dx \int P_a(dy)\, g_{0,T}(x-y)^k \\ &= \int g_{0,T}(z)^k\, dz\,. \end{aligned}$$

So, finally, we assume (i) and prove (ii). Specifically, let K denote the cube

$$K \equiv \{x \in \mathbb{R}^n : |x_i| \le \tfrac{1}{2} \quad \text{for} \quad i = 1, \ldots, n\},$$

and assume without loss of generality that

Remarks. (i) It is evident that (11.ii) is equivalent to the condition

(9.ii) for some $\lambda > 0$, $u_\lambda(0) > 0$.

Hence, in view of Lemma 2, the conditions (11) are equivalent to those imposed by Evans [3].

(ii) Similar techniques can be used to study the problem of the existence of common points in the ranges of k independent Markov processes, a technically easier problem.

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