Multiple points of Markov processes in a complete metric space

by
L.C.G. Rogers

1. Introduction.

Let \((S,d)\) be a complete metric space with Borel \(\sigma\)-field \(S\), and let \((X_t)_{t \geq 0}\) be an \(S\)-valued strong Markov process whose paths are right continuous with left limits. We ask

\((Q)\) Is \(P(X_{t_1} = \cdots = X_{t_k} \text{ for some } 0 < t_1 < \cdots < t_k) > 0\) ?

This is equivalent to the question

\((Q')\) Is \(P(\bar{X}(I_1) \cap \cdots \cap \bar{X}(I_k) \neq \emptyset) > 0\) for some disjoint compact intervals \(I_1, \ldots, I_k\) ?

We shall find conditions sufficient to ensure that \(X\) has \(k\)-multiple points with positive probability, and we will apply this to Lévy processes, providing another proof of a result of LeGall, Rosen and Shieh [6], and its improvement due to Evans [3]. However, it is advantageous to begin with the easier question

\((\bar{Q})\) Is \(P(\bar{X}(I_1) \cap \cdots \cap \bar{X}(I_k) \neq \emptyset) > 0\) for some disjoint compact intervals \(I_1, \ldots, I_k\) ?

Here, \(\bar{X}(I_j) \equiv \text{closure} \{(X_s : s \in I_j)\}\), a compact subset of \(S\). In recent years, much effort has been devoted to a study of \((Q)\), usually in the form of constructing some non-trivial random measure on the set \(\{(t_1, \ldots, t_k) : X_{t_1} = \cdots = X_{t_k}\}\) from which the existence of common points in the ranges \(X(I_j)\) follows immediately. We mention only the work of Dynkin [1] and Evans [2] on symmetric Markov processes, of Rosen [8], [9], Geman, Horowitz and Rosen [4], LeGall, Rosen and Shieh [6] and Evans [3] on more concrete Markov processes in \(\mathbb{R}^n\), as a sample of recent activity. Typically, one studies the random variables

\[(1)\quad Z_\varepsilon = \int_C I_U(X_{t_1}) F_\varphi(X_t) \, dt ,\]

where \(C = I_1 \times \cdots \times I_k\), with the \(I_j\) disjoint compact intervals in \(\mathbb{R}^+\), \(U \in \mathcal{S}\), and

\[(2)\quad F_\varphi(x_1, \ldots, x_k) = \prod_{i=1}^{k-1} f_\varphi(x_i, x_{i+1}) ,\]
(A) \[ \mu(B_{2\varepsilon}(x)) \leq K \mu(B_{\varepsilon}(x)) \quad \forall \varepsilon \in (0,\eta], \forall x \in V; \]

(B) \[ \int_{V} \frac{g_{0,T}(x,y)^k}{\mu(dx) \mu(dy)} < \infty; \]

(C) for each \( \delta \in (0,2T), \)

\[ \sup_{x,y \in V} g_{\delta,2T}(x,y) < \infty; \]

(D) for each \( 0 < a < b < \infty, g_{a,b}(\cdot,\cdot) \) is lower semicontinuous on \( V \times V; \)

(E) for some \( \xi \in U \) and \( \tau \in (0,T), \)

\[ g_{0,\tau}(\xi,\xi) > 0. \]

Remarks on conditions (A)-(E). Condition (A) seems fairly mild; it is trivially satisfied for Lebesgue measure on Euclidean space. The purpose of (A) is to let us take

\[ f_{\varepsilon}(x,y) \equiv \mu(B_{\varepsilon}(x))^{-1} I_{d(x,y) \leq \varepsilon} \]

and estimate

\[ f_{\varepsilon}(x,y) \leq K \mu(B_{2\varepsilon}(x))^{-1} I_{d(x,y) \leq \varepsilon} \]

\[ \leq K \mu(B_{\varepsilon}(y))^{-1} I_{d(x,y) \leq \varepsilon} \]

\[ = K f_{\varepsilon}(y,x). \]

Condition (B) is the ‘folklore’ condition for \( k \)-multiple points. Condition (C) may appear severe, but is frequently satisfied. Conditions (A)-(C) will give us (3.i), and conditions (D) and (E) will give us (3.ii). We may (and shall) suppose that the \( \tau \) appearing in (E) is a point of increase of \( g_{0,\tau}(\xi,\xi) \).

THEOREM 1. Assuming conditions (A), (B), and (C), the family \( \{Z_\varepsilon : 0 < \varepsilon < \eta/k\} \) is bounded in \( L^2 \). Assuming also conditions (D) and (E), there exist initial distributions such that for some disjoint compact intervals \( I_1, \ldots, I_k \)

\[ P(\overline{X}(I_1) \cap \cdots \cap \overline{X}(I_k) \neq \emptyset) > 0. \]

Proof. (i) Let \( m \) be the law of \( X_0 \). For ease of exposition, we shall suppose that \( X \) has a transition density \( p_t(\cdot,\cdot) \) with respect to \( \mu \); the result remains true without this assumption though.
exploiting (6), integrating out $x_1, y_1, \ldots, x_{j-1}, y_{j-1}$ to leave as an upper bound

$$K^{2j-2} \int I_V(x_j) I_V(y_j) g(x_j, y_j)^k \mu(dx_j) \mu(dy_j)$$

which is finite, by assumption (B). Hence for $0 < \varepsilon < \eta/k$, $E(Z_2^2)$ is bounded above by a finite constant independent of $\varepsilon$, which proves the first statement.

(ii) We next exploit (D) and (E) to give us (3.ii). By the choice of the set $C$, we have that for some small enough $\theta > 0$,

$$C \supseteq C_0 = \{(t_1, \ldots, t_k) : |t_i - t_{i-1} - \tau_i| < \theta \text{ for } i = 1, \ldots, k\},$$

where $t_0 = 0$. Hence

$$E Z_\varepsilon \geq E \left[ \int_{C_0} dt I_U(X_{t_i}) F_\varepsilon(X_t) \right]$$

$$= \int m(dx_0) I_U(x_1) \prod_{i=1}^k g(x_{i-1}, x_i) \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1}) \mu(dx),$$

where we write $g$ as an abbreviation for $g_{\tau, \eta, \varepsilon, \xi, \xi}$, since $\tau$ is a point of increase of $g_0, (\xi, \xi)$, we know that $g(\xi, \xi) > 0$. Thus

$$E Z_\varepsilon \geq \int m(dx_0) I_U(x_1) g((x_0, x_1)) \bar{g}_\varepsilon(x_1)^{k-1} \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1}) \mu(dx),$$

where

$$\bar{g}_\varepsilon(x_1) \equiv \inf \{g(x, y) : d(x, x_1) \leq k\varepsilon, d(y, x_1) \leq k\varepsilon\},$$

which, in view of (D), increases as $\varepsilon \downarrow 0$ to $g(x_1, x_1)$. By integrating out the variables $x_k, x_{k-1}, \ldots, x_2$ in (8), we obtain the lower bound

$$E Z_\varepsilon \geq \int m(dx_0) I_U(x_1) g((x_0, x_1)) \bar{g}_\varepsilon(x_1)^{k-1} \mu(dx_1),$$

and hence the estimate

$$\liminf_{\varepsilon \downarrow 0} E Z_\varepsilon \geq \int m(dx_0) I_U(x_1) g((x_0, x_1)) g(x_1, x_1)^k \mu(dx_1).$$

By lower semi-continuity and the fact that $g(\xi, \xi) > 0$, we know that $g(x, y)$ is positive in a neighbourhood of $(\xi, \xi)$ and so taking $m = \delta_\xi$, for example, yields

$$\liminf_{\varepsilon \downarrow 0} E Z_\varepsilon > 0.$$
since $\overline{R}_K \setminus R_K \subseteq \bigcup_{j=1}^{K} (\overline{X}(I_j) \setminus X(I_j))$, and $\overline{X}(I_j) \setminus X(I_j)$ is contained in the (countable) set of left endpoints of jumps of $X$ during time interval $I_j$, it follows from (F) that the set $\overline{R}_K \setminus R_K$ is polar, contradicting (10).

3. Multiple points of Lévy processes. Let $X$ be a Lévy process in $\mathbb{R}^n$, with resolvent $(U_\lambda)_\lambda > 0$. We shall assume that the resolvent is strong Feller (equivalently, that each $U_\lambda(x,.)$ has a density with respect to Lebesgue measure - see Hawkes [5]), in which case there is for each $\lambda > 0$ a $\lambda$-excessive lower semi-continuous function $u_\lambda$ such that

$$U_\lambda f(x) = \int u_\lambda(y) f(y + x) \, dy.$$

To establish sufficient conditions for $k$-multiple points, we shall need three lemmas on Lévy processes of interest in their own right.

**Lemma 1.** The resolvent $(U_\lambda)_\lambda > 0$ is strong Feller if and only if for every $0 \leq a < b < \infty$ the kernel $G_{a,b}$ has a density $g_{a,b}$.

If this happens, the densities $g_{a,b}(.)$ may be chosen so that

(i) $g_{a,b}(.)$ is lower semicontinuous for each $0 \leq a < b < \infty$;
(ii) $(a,b) \to g_{a,b}(x)$ is left-continuous increasing in $b$ and right-continuous decreasing in $a$ for each $x$;
(iii) for all $0 \leq a < b < \infty$ and all $x \in \mathbb{R}^n$

$$g_{a,b}(x) = \lim_{\delta \downarrow 0} \delta^{-1} \int g_{0,\delta}(y) g_{a,b-\delta}(x-y) \, dy.$$

**Lemma 2.** For a Lévy process with a strong Feller resolvent, the following are equivalent:

(i) for some $\epsilon$, $T > 0$,

$$\int_{|x| \leq \epsilon} g_{0,T}(x) \, dx < \infty;$$

7
whence \( g_{a,T}(.) \) is bounded globally (exploiting lower semi-continuity).

This completes the proof that (11.i-ii) implies that \( X \) has \( k \)-multiple points with positive probability, and hence, by Borel-Cantelli, there are almost surely \( k \)-multiple points.

**Proof of Lemma 1.** The arguments used are similar to those of Hawkes [5], so we will just give an outline. The first statement of the lemma is immediate. To get good versions of the densities \( g_{a,b} \), firstly take any densities \( g'_{p,q}(.) \) for \( G_{p,q} \), \( 0 \leq p < q < \infty \) rational, then define

\[
\tilde{g}'_{a,b}(x) = \sup \{ g'_{p,q}(x) : a < p < q < b \},
\]

which have property (ii) (which remains preserved under the subsequent modifications). Next, for \( n > (b-a)^{-1} \) define

\[
\tilde{g}_{a,b}^n(x) = n \int g_{0,\delta}(y) g_{a,b-\delta}(x-y) \, dy, \quad (\delta \equiv n^{-1})
\]

which is lower semicontinuous in \( x \) (it is the increasing limit as \( M \uparrow \infty \) of

\[
n \int g_{0,\delta}(y) (M \wedge g_{a,b-\delta}(x-y)) \, dy,
\]

which are continuous by the strong Feller property of \( G_{0,\delta} \). Finally, we take

\[
g_{a,b}(.) = \sup \{ \tilde{g}_{a,b}^n(.) : n > (b-a)^{-1} \}.
\]

Since, for fixed \( a < b \), \( \tilde{g}_{a,b}^n \) is increasing almost everywhere to a version of the density of \( G_{a,b} \), this provides a version with the desirable properties (i) - (iii). \( \diamond \)

**Proof of Lemma 2.** The implications (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i) are trivial. The implication (ii) \( \Rightarrow \) (iii) follows easily from the estimate

\[
\int g_{a,a+r}(x)^k \, dx = \int (\int P_a(dy) g_{0,r}(x-y))^k \, dx
\]

\[
\leq \int dx \int P_a(dy) g_{0,r}(x-y)^k
\]

\[
= \int g_{0,r}(z)^k \, dz.
\]

So, finally, we assume (i) and prove (ii). Specifically, let \( K \) denote the cube

\[
K = \{ x \in \mathbb{R}^n : |x_i| \leq \frac{1}{2} \quad \text{for} \quad i = 1, ..., n \},
\]

and assume without loss of generality that
Remarks. (i) It is evident that (11.ii) is equivalent to the condition

(9.ii) \( \lambda > 0, \quad u_{\lambda}(0) > 0. \)

Hence, in view of Lemma 2, the conditions (11) are equivalent to those imposed by Evans [3].

(ii) Similar techniques can be used to study the problem of the existence of common points in the ranges of \( k \) independent Markov processes, a technically easier problem.

Acknowledgements. It is a pleasure to thank my hosts at the Laboratoire de Probabilités, especially Marc Yor, for numerous stimulating discussions on these and other subjects during my visit to Paris in October 1987; and a referee for helpful criticisms on the first draft of this paper.