A GUIDED TOUR THROUGH EXCURSIONS

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1. *Introduction*

Imagine you play a game with a friend in which you toss a fair coin repeatedly, winning 1 each time the coin falls heads, losing 1 each time it falls tails. Thus if X_n denotes your winnings on the nth toss of the coin, the X_n are independent random variables with common distribution

$$P(X_n = 1) = \tfrac{1}{2} = P(X_n = -1),$$

and your net winnings after n tosses of the coin are

$$S_n \equiv X_1 + \ldots + X_n \ (n \geqslant 1), \quad S_0 \equiv 0.$$

The process $(S_n)_{n \geqslant 0}$ is the classical *symmetric simple random walk*, so dear to probabilists; a good introduction to its properties is to be found in Feller **[14]**, also dear to probabilists! Let us look at some properties of (S_n) which are almost too obvious for Feller to dwell on. Let $T_0 \equiv 0$, and let T_n be the nth time that the random walk returns to zero; formally,

$$T_{n+1} \equiv \inf\{k > T_n : S_k = 0\}, \ n \geqslant 0.$$

Figure 1 shows a typical outcome of the game in the form of the graph of S_n against n, linearly interpolated.

When the coin has been tossed T_n times, your net winnings are zero. It is therefore obvious that:

(1.i) *what happens after T_n is independent of what happened before;*
(1.ii) *the evolution of the game from T_n on, $\{S_{T_n+k} : k \geqslant 0\}$, is just like the evolution of the original game $\{S_k : k \geqslant 0\}$.*

The whole game starts afresh at time T_n! Thus if we define the nth excursion ξ_n by

$$\xi_n \equiv \{S_k : T_{n-1} \leqslant k \leqslant T_n\}, \ n \in N,$$

(1.i) can be re-expressed as saying $\xi_1, \ldots, \xi_n$ *are independent of* $\xi_{n+1}, \xi_{n+2}, \ldots$; because of (1.ii), ξ_{n+1} has *the same distribution as* ξ_1. Hence

(2) *the excursions $\xi_1, \xi_2, \ldots$ are independent identically distributed.*

Now that you understand this, you understand the essential of excursion theory.

To show how excursion theory (= (2)!) can be used to do calculations, let us find the distribution of the number of returns to 0 before the time $\tau \equiv \inf\{k : S_k = -2\}$. Trivially,

$$\begin{aligned} P(\xi_1 \text{ visits } -2) &\equiv P(S_k = -2 \text{ for some } k \text{ with } 0 < k < T_1). \\ &= P(S_1 = -1, S_2 = -2) \\ &= \tfrac{1}{4}, \end{aligned}$$

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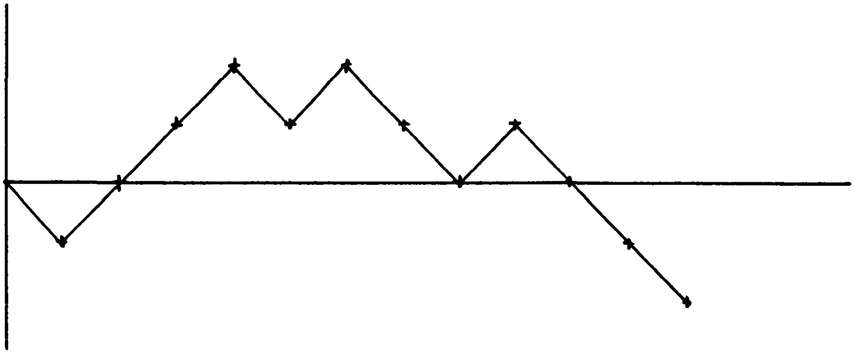

FIG. 1

so that

$$
\begin{aligned}
&P(\text{no. of returns to 0 before } \tau \text{ is at least } k)\\
&= P(\text{excursions } \xi_1, \ldots, \xi_k \text{ do not visit } -2)\\
&= P(\xi_1 \text{ does not visit } -2)^k, \quad \text{by (2);}\\
&= (\tfrac{3}{4})^k.
\end{aligned}
$$

(*Puzzle:* how would this calculation be different if the coin were not fair?)

The ideas of excursion theory can be applied to any continuous-time Markov process which has some recurrent state, and the theory can be developed in great generality. Such a development is not the purpose of this article. (See Chapter VI of Rogers and Williams [**62**] for more details and references.) We intend here to concentrate on what excursion theory has to say for Brownian motion, thereby avoiding technicalities through the concreteness of the examples. We choose to study Brownian motion because it illustrates perfectly the kind of technical problems which have to be overcome in general (Brownian motion spends (Lebesgue) almost no time in zero, yet returns to zero immediately), and because the applications of excursion theory, such as the Ray–Knight theorem, and the arc-sine law, are both simple to prove and powerful. From time to time we shall look at the general setting when general ideas arise, but we shall hurriedly move on with a few references whenever the technicalities threaten to slow our progress.

Here is a brief outline of the contents of the rest of the paper. Section 2 introduces Brownian motion, assuming no more knowledge of probability than is contained in most first courses; even so, quite surprising results fall out easily. The first deep result on Brownian motion is Trotter's theorem, which forms the starting point of the discussion in Section 3 of Brownian local time and excursions. The decomposition of the Brownian motion into a Poisson point process of excursions is the key result of this section, and together with some minimal information on the excursion measure, it allows us to prove the Ray–Knight theorem on local time. A few definitions and basic ideas of Markov process theory occupy Section 4 before the analysis of the excursion measure can be carried further in Section 5. The picture of the typical excursion—evolving like the Markov process, but 'kicked in' by the σ-finite entrance law—emerges here, and the ideas are used to prove the arc-sine law for Brownian motion. Another striking application of excursion methods, David Williams' path decompositions, is the subject of Section 6. In Section 7, we draw back a while from applications to consider what extensions of the killed process are possible, and then proceed to characterise the extensions in some interesting concrete examples, such as Feller Brownian motions, and Brownian motion in a wedge with skew reflection along the boundaries. Finally, in Section 8 we review the theory of excursions from a set. Though obviously much more involved, there are few features which will surprise anyone who has digested the earlier parts of the paper!

The emphasis throughout is on the parts of the subject which appear to me to be interesting and useful. I have tried to include in the references a wide variety of papers in the area which reflect the developments and diversity of excursion theory, but am well aware that any such list is sure to overlook important contributions. Likewise, many of the main ideas have been discovered and rediscovered so often that it is dangerous to attempt to single out the 'first' appearance of the idea. The following remarks on the history of the subject must suffice. In the early days of the developments of Markov processes, the emphasis was on the analytic techniques of generators and semigroups. This was not because those working on Markov processes did not know about the sample-path picture behind the analysis (the fact that they negotiated the analysis at all proves this), but rather because the links between analysis and sample paths had not been sufficiently strongly made to be trustworthy. Latterly, sample path techniques have risen to prominence because these links *have* been firmly established, and also because it is now realised that very many of the probabilistically interesting results can be proved *entirely* by sample-path techniques ('all you need is Itô's formula!'). This trend is seen also in the development of excursion theory, which first appeared in analytic guise as 'last-exit decompositions'. See Rogers and Williams [**62**, VI.42], for more on the history and background.

It is hard to know where this all began, but the rise to prominence of sample path methods can be attributed to Itô [**25**], who first introduced the Poisson point process of excursions. Since then, effort has been devoted to analysis of excursions from sets (see, for example, Dynkin [**12**, **13**], Jacobs [**27**], Kaspi [**30**], Maisonneuve [**37**], Motoo [**49**]), resulting in quite complete theoretical representations, but disappointingly few spectacular applications. The other main direction of development has been into applications, and the richness and variety of the use of excursion ideas in papers such as Barlow [**1**], Bass and Griffin [**3**], Pitman and Yor [**54**], Greenwood and Pitman [**22**], Greenwood and Perkins [**20**], Millar [**46**], Rogers [**59**, **60**, **61**] confirms the importance of excursion ideas as a probabilistic device.

To those whose work on excursions receives little or no mention in this paper, I offer my apologies; the treatment is eclectic, not exhaustive. And to Professors Chris Lance and Peter Vámos I give my thanks; their enthusiasm and encouragement caused this paper to be written.

2. *Brownian motion*

Brownian motion (on the real line) is a real-valued stochastic process $(B_t)_{t\geq 0}$ with the following defining properties:

(3.i) *the paths* $t \to B_t(\omega)$ *are continuous;*

(3.ii) *for* $0 \leq s \leq t$, $B_t - B_s$ *has a zero-mean normal (or Gaussian) distribution with variance* $t-s$:

$$P(B_t - B_s \in dx) \equiv \gamma_{t-s}(x)\,dx = (2\pi(t-s))^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2(t-s)}\right) dx;$$

(3.iii) *for each* t, $\{B_{t+u} - B_t : u \geq 0\}$ *is independent of* $\{B_u : 0 \leq u \leq t\}$.

REMARKS. Virtually every class of real-valued process studied by probabilists contains Brownian motion, explaining in some way its importance. Here are a few examples.

(a) Brownian motion is a continuous *Gaussian process* (that is, for any $t_1, \ldots, t_n$, the random variables $B_{t_1}, \ldots, B_{t_n}$ have a jointly Gaussian distribution). The law of the Gaussian process B is then specified by its moments of first two orders. These are given by

$$EB_t = 0, \quad EB_tB_s = s \wedge t, \quad s, t \geq 0, \tag{4}$$

as is easily checked.

(b) For $t_0 = 0 < t_1 < \ldots < t_n$, the increments $B(t_k) - B(t_{k-1}), k = 1, \ldots, n$ of B are independent, in view of (3.iii). Moreover, the distribution of the increment $B_t - B_s$ depends only on the length $t-s$ of the interval $(s, t]$, and not on its starting point s. Thus B is a process with *stationary independent increments*, commonly called a *Lévy process*. We shall return briefly to Lévy processes later. There are many examples of such processes, but, remarkably, if the Lévy process has *continuous* paths it must be of the form $\sigma B_t + ct$; see, for example, Theorem I.9 of Williams [75].

Every Lévy process is a strong Markov process; in this context, this means that

(5) $\{B_{T+t} - B_T : t \geq 0\}$ *is a Brownian motion independent of* $\{B_u : u \leq T\}$

whenever T is a *stopping time*.

A stopping time is a random variable with values in $[0, \infty]$ such that the event $\{T < t\}$ is measurable with respect to $\mathscr{F}_t^\circ \equiv \sigma(\{B_s : s \leq t\})$. The first time that B enters a set is a stopping time, as is the first time after a stopping time that B enters a set.

(c) It is easy to see that for $s \leq t$

$$E(B_t \mid \mathscr{F}_s^\circ) = B_s$$

since the increment $B_t - B_s$ is independent of $\mathscr{F}_s^\circ$ and has zero mean. Thus B is a *continuous martingale*.

Similarly,

$$E[(B_t - B_s)^2 \mid \mathscr{F}_s^\circ] = t - s$$

from which we conclude that

(6) $B_t, B_t^2 - t$ *are continuous martingales*, $B_0 = 0$.

It is a remarkable result of Lévy that any process B with properties (6) *is* Brownian motion! Doob [9] gives a classical proof; for a proof via stochastic calculus, see Kunita and Watanabe [33]. It is even true that any continuous martingale is a time-change of Brownian motion (Dubins and Schwarz [10])! See Rogers and Williams [62, Chapter IV, Part 6], for these and many other fascinating properties of Brownian motion.

One of the very few deep results about Brownian motion to which we appeal is its existence; after that, most of the properties we need follow quite easily.

PROPOSITION 1. (i) *For any* $c \neq 0$, *the process* $\{cB_{t/c^2} : t \geq 0\}$ *is a Brownian motion. This is called the* scaling property *of Brownian motion.*

(ii) *The process* $(\tilde{B}_t)_{t \geq 0}$ *defined by*

$$\begin{aligned} \tilde{B}_t &= tB_{1/t} \quad (t > 0) \\ &= 0 \qquad (t = 0) \end{aligned}$$

is Brownian motion.

(iii) $P(\sup_t B_t = +\infty, \inf_t B_t = -\infty) = 1.$

(iv) *Almost surely, for every* a, $\{t: B_t = a\}$ *is unbounded.*

Proof. (i) It is immediate that (cB_{t/c^2}) is a continuous Gaussian process, and the check of (4) is a triviality.

(ii) It is immediate that $(\tilde{B}_t)_{t>0}$ is a zero-mean Gaussian process with continuous sample paths and the same covariance structure as $(B_t)_{t>0}$. But we know that

$$P(B_t \longrightarrow 0 \text{ as } t \downarrow 0) = 1$$

since B has continuous paths and therefore

$$P(\tilde{B}_t \longrightarrow 0 \text{ as } t \downarrow 0) = 1,$$

and $\tilde{B}$ is continuous at $t = 0$.

(iii) Let $Z \equiv \sup_t B_t$. By scaling, for any $c > 0$ the law of cZ is the same as the law of Z, whence Z must be either 0 or $+\infty$. However, if Z is zero, it must be that $B_1 \leqslant 0$, and that the supremum of the Brownian motion $\{B_{t+1} - B_1 : t \geqslant 0)$ must be zero (it cannot be $+\infty$!). Thus

$$\begin{aligned} p &\equiv P(Z = 0) \\ &\leqslant P(B_1 \leqslant 0, \sup_t (B_{t+1} - B_1) = 0) \\ &= P(B_1 \leqslant 0) \cdot p, \quad \text{using (3.iii)}; \\ &= \tfrac{1}{2}p \end{aligned}$$

since $B_1 \sim N(0, 1)$. This implies that $p = 0$. By the scaling property with $c = -1$, we have that $-B$ is Brownian motion, so its supremum is almost surely ∞.

(iv) Since, with probability 1, B is not bounded above or below, it must be that $B_t = a$ for some t. If there were a last time, T, at which B visited a, then after time T, would always remain above (say) a, and the infimum of B could not be $-\infty$.

REMARKS. Because of (iv), B_t must change sign infinitely often in any interval (n, ∞). Thus, by (ii), B must change sign infinitely often in any interval $(0, 1/n)$! So B is oscillating back and forth for small times; the exact oscillation is given by the celebrated law of the iterated logarithm:

$$\limsup_{t \downarrow 0} \frac{B_t}{(2t \log\log 1/t)^{\frac{1}{2}}} = +1 \text{ a.s.}$$

3. *Brownian local time and excursions*

Because the path of Brownian motion is continuous, the set $\{t: B_t \neq 0\}$ is open, and can be expressed as a disjoint countable union of maximal open intervals $\bigcup_j (a_j, b_j)$, during each of which B makes an *excursion* away from zero. Just as with symmetric simple random walk, the idea is to break the path of B into its excursions, but there are technical problems. We have seen that B oscillates wildly back and forth for small t, so, in particular, *there is no first excursion* from zero; before any $t > 0$, the Brownian motion has made infinitely many excursions away from zero. Moreover,

$$\text{Leb}(\{t: B_t = 0\}) = 0 \text{ a.s.},$$

as a trivial use of Fubini's theorem shows;

$$E[\mathrm{Leb}(\{t: B_t = 0\})] = E\left[\int_0^\infty I_{\{0\}}(B_t)\,dt\right]$$
$$= \int_0^\infty P(B_t = 0)\,dt$$
$$= 0.$$

So the zero set $\mathscr{Z} \equiv \{t: B_t = 0\}$ *has zero Lebesgue measure* almost surely! The first of these problems is the more fundamental, but the key to both is an understanding of *Brownian local time.*

The central result on Brownian local time has a deceptively simple statement.

THEOREM (Trotter). *There exists a jointly continuous process $\{L(t,x): t \geqslant 0, x \in \mathbb{R}\}$ such that for all bounded measurable f, and all $t \geqslant 0$*

$$\int_0^t f(B_s)\,ds = \int_{-\infty}^\infty f(x)\,L(t,x)\,dx. \tag{7}$$

In particular, for any Borel set A

$$\int_0^t I_A(B_s)\,ds = \int_A L(t,x)\,dx,$$

so L is an occupation density.

REMARKS. (i) For each x, the map $t \mapsto L(t,x)$ is continuous and increasing, and the growth set of $L(\cdot,x)$ is a.s. the Lebesgue-null set $\{t: B_t = x\}$, explaining why it is called *local* time at x. The local time $L(\cdot,x)$ is a random Cantor function.

(ii) The existence of a jointly continuous occupation density is in some sense concerned only with the 'real analysis' of the continuous function $t \mapsto B_t$. Such functions do not feature largely in first-year analysis courses, though; the following trivial result explains why.

COROLLARY 1. *Almost surely, there do not exist $0 \leqslant s < t$ and $C < \infty$ such that*

$$|B_{s+h} - B_s| \leqslant Ch \text{ for all } h \text{ such that } 0 \leqslant h \leqslant t-s.$$

Proof. Suppose that such s, t and C existed, and $h \leqslant t-s$. Then

$$h = \int_{-\infty}^\infty \{L(s+h,x) - L(s,x)\}\,dx, \quad \text{from (7);}$$
$$= \int_{B_s-y}^{B_s+y} \{L(s+h,x) - L(s,x)\}\,dx,$$

where $y = \sup\{|B_u - B_s| : s \leqslant u \leqslant s+h\}$, since for x outside $[B_s - y, B_s + y]$ the local time $L(\cdot,x)$ cannot increase during $[s, s+h]$ because B does not visit x during $[s, s+h]$. By hypothesis, $y \leqslant Ch$, so we have

$$h \leqslant 2Ch \sup\{L(s+h,x) - L(s,x): x \in \mathbb{R}\} = o(h),$$

a contradiction.

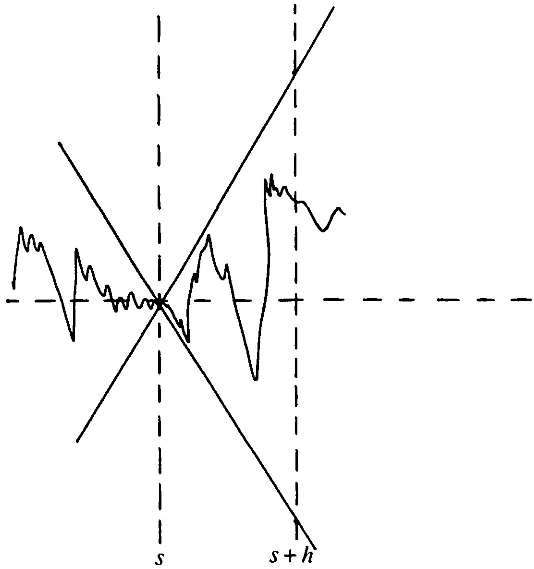

FIG. 2

Thus any continuous function which has a jointly continuous local time is *nowhere differentiable*, and does not even have finite one-sided upper and lower derivatives! Bare-handed construction of such a function is not a simple matter. The 'real analysis' of local times is the point of view from which the excellent survey article of Geman and Horowitz ('Occupation densities', *Ann. Probab.* 8 (1980) 1–67) proceeds. You will find many interesting and useful results there (including Corollary 1!).

(iii) The original proof of Trotter's theorem does not greatly appeal to contemporary palates, which prefer the proof via stochastic integrals and Tanaka's formula—see, for example, McKean [**36**], Meyer [**43**], Rogers and Williams [**62**].

To see how we obtain the excursion decomposition of the Brownian path, a pretty result of Lévy's is very helpful. Let us abbreviate $L(t,0)$ to L_t.

THEOREM 2 (Lévy). *Let* $S_t \equiv \sup\{B_s : s \leqslant t\}$. *Then*

$$(S_t, S_t - B_t)_{t \geqslant 0} \overset{\mathscr{D}}{=} (L_t, |B_t|)_{t \geqslant 0}$$

(*where* $\overset{\mathscr{D}}{=}$ *signifies that the laws of the two bivariate processes coincide*).

A simple proof of this result is available if one uses the Tanaka-formula definition of L, but more important for now is the picture, Figure 3, which shows a Brownian path with $S.$ drawn in dashed, *or* the path of $L. - |B.|$ with $L.$ drawn in dashed.

As we have said, the representation $\{t : B_t \neq 0\} = \bigcup_j (a_j, b_j)$ of $\mathscr{Z}^c$ as a disjoint union of open intervals allows us to tear the path of B apart into its *excursions*

$$\{B_{(t+a_j)\wedge b_j} : t \geqslant 0\},$$

each of which is an element of the *excursion space*

$$U \equiv \{\text{continuous } f\colon \mathbb{R}^+ \longrightarrow \mathbb{R} \text{ such that } f^{-1}(\mathbb{R}\backslash\{0\}) = (0,\zeta) \text{ for some } \zeta > 0\}.$$

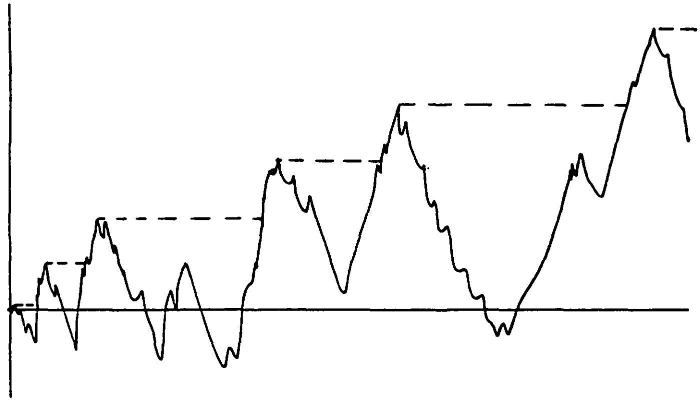

FIG. 3

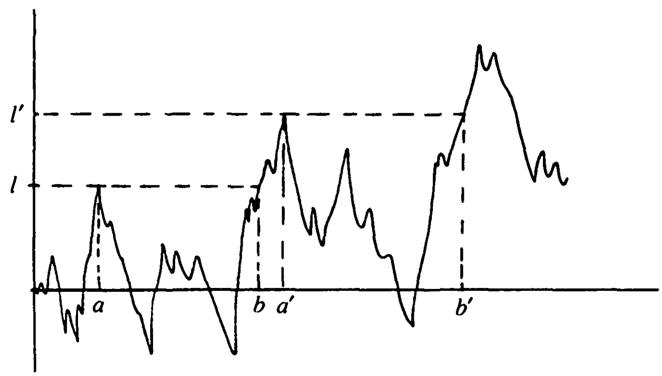

FIG. 4

Now although we may not speak of the first, second, third,... excursions, there is a *complete ordering* of the excursions; the excursion in the interval (a_j, b_j) comes before the excursion in (a_k, b_k) if $b_j < a_k$. And although this ordering cannot be captured by $\mathbb{N}$, it can be captured by $\mathbb{R}^+$, *using the local time* L. Thus in Figure 4 we can talk of the excursion straddling the interval (a, b) as the excursion at local time l, which is before the excursion straddling (a', b'), an excursion at local time $l' > l$. This allows us to tear apart the Brownian sample path into its excursions, and represent the path as a point process Ξ in $\mathbb{R}^+ \times U$, there being a point (t, ξ) in the point process Ξ if and only if Brownian motion makes an excursion ξ at local time t. Figure 5 illustrates this decomposition (representing U as a half-line is not entirely satisfactory, but nothing better springs to mind!).

There are only countably many points in this point process, but there are infinitely many in $(a, b) \times U$ for $0 \leqslant a < b$. This decomposition of the Brownian path into excursions can easily be reversed, because if we were given the excursion point process, we would know exactly what excursions to stick together in what order so as to recover the Brownian path.

So why has it helped us to represent the (reasonably comprehensible) continuous Brownian path by a point process in a somewhat complicated space? The explanation is the following.

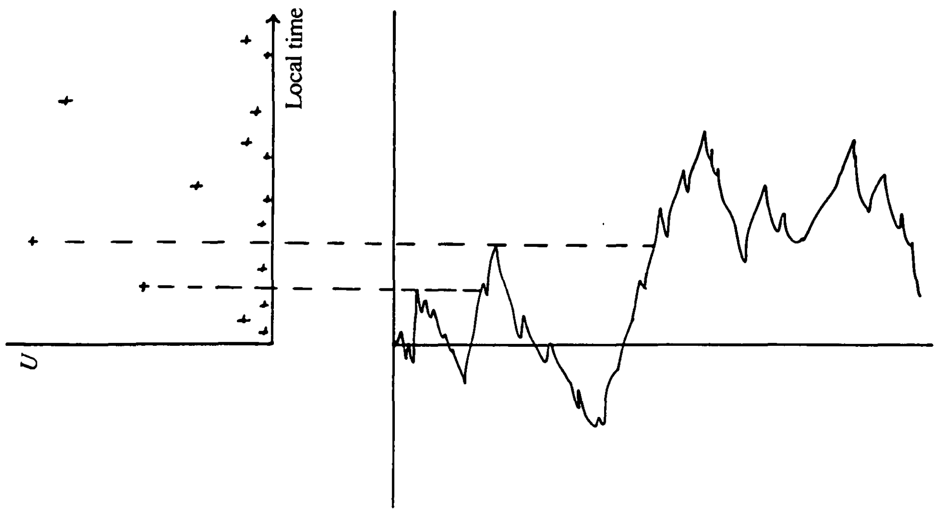

FIG. 5

THEOREM (Itô). *The excursion point process is a POISSON point process with excursion measure* Lebesgue $\times\, n$, *where the σ-finite measure n on U is called the* excursion *measure, or* characteristic *measure.*

REMARKS. We can regard the excursion point process Ξ in two different ways. Firstly, it can be viewed as a process $(\Xi_t)_{t\geqslant 0}$ with values in $U \cup \{\delta\}$ such that $\Xi_t = \delta$ for all but countably many t (δ is a 'graveyard' state). Secondly, it can be thought of as a random measure,

$$\Xi(A) = \text{no. of points of } \Xi \text{ in } A$$

for any $A \subset \mathbb{R}^+ \times U$. We use whichever description is most convenient at the time.

Proof. By slight abuse of notation, let $\theta_t \Xi$ be the point process $(\theta_t \Xi)_s \equiv \Xi_{t+s}$. If now

$$\gamma_t \equiv \inf\{u : L_u > t\}$$

is the right-continuous inverse to L, then each γ_t is a stopping time, and, since γ_t is a point of right increase of L, $B(\gamma_t) = 0$. Thus by the strong Markov property,

$$\{B(\gamma_t + u) : u \geqslant 0\} \textit{ is a Brownian motion independent of } \{B_u : u \leqslant \gamma_t\}.$$

But the restriction of Ξ to $(0, t) \times U$ is determined by $\{B_u : u \leqslant \gamma_t\}$—it is just the point process of excursions before local time t—and $\theta_t \Xi$ is determined by $\{B(\gamma_t + u) : u \geqslant 0\}$. Hence $\theta_t \Xi$ *is independent of* $\Xi|_{(0,t)\times U}$, *and has the same law as* Ξ. So in particular, for any $A \subseteq U$,

$$N_t(A) \equiv \Xi((0, t) \times A)$$

is a simple Poisson process, because it has stationary independent increments, and is increasing by unit jumps. A slight extension of this argument yields the result (see Itô [25] for more details).

The computational impact of this theorem follows from two elementary facts about Poisson point processes:

(8.i) if $A \subset U$, $n(A) < \infty$ then

$$P(\Xi \text{ puts no point in } (0, t) \times A) = \exp(-tn(A));$$

(8.ii) if $A_1, \ldots, A_k \subset U$ are disjoint, $n(A_i) < \infty$ for all i, and A is the union of the A_i, then

$$P(\Xi_\tau \in A_i) = n(A_i)/n(A),$$

where $\tau \equiv \inf\{u : \Xi_u \in A\}$.

The statement (8.i) is an immediate consequence of the fact that the number of points of Ξ in $(0, t) \times A$ is a Poisson random variable with parameter

$$(\mathrm{Leb} \times n)((0, t) \times A) = tn(A).$$

Thus *the (local) time τ at which there is first an excursion in A is exponentially distributed with parameter $n(A)$.* As for the second statement, the random variables $\tau_i \equiv \inf\{u : \Xi_u \in A_i\}$ are independent exponentials with parameters $n(A_i)$, and (8.ii) is an elementary consequence of this.

So to do calculations, we apply (8.i–ii) together with (a little) knowledge of n. The following information about n is enough to take us quite a long way.

PROPOSITION 2. *$n(\{f \in U : \sup_t f(t) > a\}) = (2a)^{-1}$ for each $a > 0$.*

Proof. We write $A = \{f \in U : \sup_t |f(t)| > a\}$ as the disjoint union $A = A_+ \cup A_-$, with $A_\pm = \{f \in U : \sup_t \pm f(t) > a\}$. By symmetry, $n(A_+) = n(A_-)$, so we just need to calculate $n(A)$. Now if $\tau = \inf\{t : \Xi_t \in A\}$, we know that τ has an $\exp(n(A))$ distribution, so to find $n(A)$ it would be enough to calculate $E\tau$. But τ is the local time at zero when $|B|$ first reaches a. By Lévy's theorem, $|B| - L$ is a martingale (in fact, a Brownian motion) so by the optional sampling theorem (see, for example, Williams [75, §II.53]),

$$0 = E(|B_0| - L_0) = E(|B_H| - L_H) = a - EL_H,$$

where $H \equiv \inf\{t : |B_t| = a\}$. Hence $EL_H = E\tau = a$, and the result follows.

EXAMPLE. As a first use of excursion theory, let us solve a problem (whose solution is easy by any other method!) exploiting the Poisson point process of excursions. For $y \in \mathbb{R}$, let

$$H_y \equiv \inf\{t > 0 : B_t = y\}.$$

We calculate for $a, b > 0$ that $P(H_a < H_{-b}) = b/(a+b)$. How do we see this? Let us define

$$A \equiv \{f \in U : \sup_t f(t) > a\},$$

$$B \equiv \{f \in U : \inf_t f(t) < -b\},$$

two disjoint sets. Now we can translate the event $\{H_a < H_{-b}\}$ into a statement about Ξ; it is, quite simply, the event that Ξ has a point in A before a point in B. From (8.ii) then,

$$\begin{aligned} P(H_a < H_{-b}) &= P(\Xi \text{ has a point in } A \text{ before a point in } B) \\ &= n(A)/(n(A) + n(B)) \\ &= b/(a+b), \end{aligned}$$

by Proposition 2.

Now we have a result whose proof by excursion theory is only a little less trivial, even though the original proofs were quite complicated—the celebrated Ray–Knight theorem on Brownian local time. We only state one part of the result; see, for example, Rogers and Williams [**62**, p. 428] for the full statement. The method of proof is very similar, and we are only interested in conveying the excursion ideas, not on chronicling their ultimate development.

THEOREM 4 (Ray, Knight). *Let* $\tau \equiv \inf\{t : B_t = -1\}$. *Then the process* $\{L(\tau, x): x \geqslant -1\}$ *is a time-inhomogeneous diffusion, behaving like a* BESQ(2) *process for* $-1 \leqslant x \leqslant 0$, *and like a* BESQ(0) *process for* $x > 0$.

REMARKS. It is not important here just what a BESQ(n) process may be. We shall actually obtain the transition mechanism of these (strong Markov) processes, so the exact terminology used by probabilists to label them is therefore irrelevant for present purposes.

Proof. We give the argument only for $(-1, 0)$, the argument for $(0, \infty)$ being similar. Fix some $a \in (-1, 0)$ and consider the excursions of B away from a. Of course, there is the first 'excursion' $\{B_t : 0 \leqslant t \leqslant H_a\}$, but by the strong Markov property this is independent of $\{B_t : H_a \leqslant t\}$, so we may lay it to one side for now. Now the Poisson point process of excursions into (a, ∞) is independent of the Poisson point process of excursions into $(-\infty, a)$; so given the value l of $L(\tau, a)$, the restrictions of Ξ to $(0, l) \times U_{a+}$ and $(0, l) \times U_{a-}$ are (conditionally) independent. (Here U_{a+} is the set of excursions into (a, ∞).)

Moreover, $\{L(\tau, x) - L(H_a, x) : x \geqslant a\}$ is determined by the restriction of Ξ to $(0, l) \times U_{a+}$, and $\{L(\tau, x): -1 \leqslant x \leqslant a\}$ is determined by the restriction of Ξ to $(0, l) \times U_{a-}$. Hence we see that $\{L(\tau, x) - L(H_a, x) : x \geqslant a\}$ *and* $\{L(\tau, x): -1 \leqslant x \leqslant a\}$ *are conditionally independent given* $L(\tau, a) = l$. Since the first excursion is independent of everything, we have that $\{L(\tau, x) : x \geqslant a\}$ is conditionally independent of

$$\{L(\tau, x): -1 \leqslant x \leqslant a\}$$

given $L(\tau, a)$. But this amounts to saying that $\{L(\tau, x) : x \geqslant -1\}$ is a Markov process.

The strong Markov property will follow by general results once we have identified the transition mechanism. But this is quite easy to do. Taking some $b = a + \delta \in (a, 0)$, the number of excursions from a which get up to b before local time l elapses at a is a Poisson variable with mean

$$l \times n(\{f : \sup_t f(t) > b\}) = l/2\delta,$$

by Proposition 1. Each time an excursion from a gets up as far as b, it will contribute to the local time at b before returning to a. How much will it contribute? By the strong Markov property, everything starts afresh when the excursion gets to b, so the contribution has the same distribution as the local time at zero when the Brownian motion first reaches $-\delta$. But this we saw was $\exp((2\delta)^{-1})$. Thus the local time $L(\tau, b)$ at b is the sum of a $\mathscr{P}(l/2\delta)$ number of independent $\exp(2\delta)$ random variables—plus one more contribution from the 'first excursion'. Hence the Laplace transform of $L(\tau, b)$ is

$$\sum_{k \geqslant 0} \left(\frac{l}{2\delta}\right)^k \frac{e^{-l/2\delta}}{k!} \left(\frac{1}{1+2\delta\lambda}\right)^{k+1} = (1 + 2\delta\lambda)^{-1} \exp\{-l\lambda/(1+2\delta\lambda)\},$$

which specifies the transition mechanism (of the BESQ(2) diffusion).

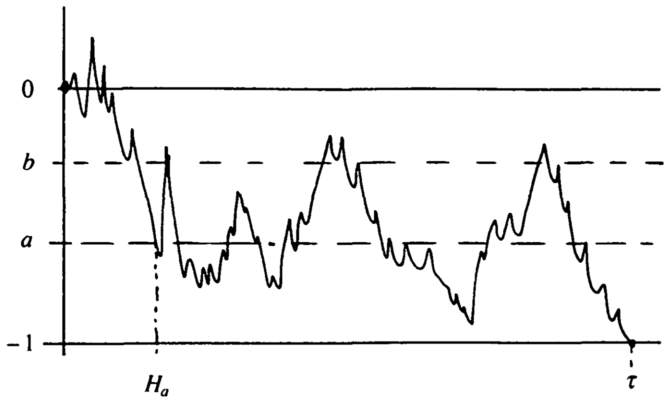

FIG. 6

REMARKS. So now you understand *why* $\{L(\tau, x): -1 \leqslant x \leqslant 0\}$ is a Markov process; and even if you have not committed the form of its transition mechanism to memory, you do not *need* to because you can reproduce the derivation in a few lines of trivial calculation! Such transparency and simplicity are the hallmarks of excursion theory, and should be the aim of all mathematicians; *not* because we strive for elegance, but rather because if something is simple, we can *use* it easily and powerfully, both to calculate details and, more importantly, to understand structure without the need to calculate details!

Illustrative aside. Not only is every continuous martingale essentially Brownian motion, but every (regular) one-dimensional diffusion is also essentially Brownian motion.

Suppose that we have a regular diffusion in natural scale on $[0, \infty)$ with speed measure m; this means that we can represent the diffusion as

$$X_t \equiv B(\sigma_t),$$

where σ is the right-continuous inverse to the additive functional

$$A_t \equiv \int_{[0,\infty)} L(t, x)\, m(dx),$$

and m is the so-called *speed measure* of the diffusion, a measure with the property that $0 < m((a, b)) < \infty$ for all $0 < a < b < \infty$. (We suppose here that B is a Brownian motion started at 1, say, with local time process L.) *Can the diffusion X reach zero in finite time*? This will happen if and only if

$$\lim_{t \uparrow H_0} A_t < \infty,$$

which happens if and only if

$$\int_{[0,1]} L(H_0, x)\, m(dx) < \infty, \tag{9}$$

since $L(H_0, x) = 0$ for all large enough x, and $m([1, x)) < \infty$ for all $x > 1$. Now a sufficient condition for (9) is

$$E \int_{[0,1]} L(H_0, x)\, m(dx) < \infty,$$

which, by the Ray–Knight theorem, can be re-expressed as

$$\int_{[0,1]} EL(H_0, x)\, m(dx) = \int_{[0,1]} 2xm(dx) < \infty.$$

This explains the condition

$$\int_0^1 xm(dx) < \infty$$

as the condition for the boundary point 0 to be accessible. It is also a necessary condition, as you will see from any account of one-dimensional diffusion theory; Rogers and Williams [**62**] contains quite an up-to-date one.

4. *Markov processes*

As yet, we know almost nothing about the Brownian excursion measure n, beyond what was shown in Proposition 1. Before we get any more results on Brownian motion using excursion theory, we have to get a clearer picture of the measure n; but this picture is clearest in a more general context, so we now take a little time to discuss (in a very superficial fashion) the theory of Markov processes. The reader will find many thorough treatments of the theory; Dellacherie and Meyer [**8**] or Chapter III of Williams [**75**] would be suitable places to start looking, and *General theory of Markov processes* by M. J. Sharpe (Academic Press, 1988) is a definitive account. Our aim is to give enough of the gist of things for the non-expert to follow through the subsequent discussion at an intuitive level; those who prefer may simply skim this section for notation and thereafter specialise every Markov process to Brownian motion!

A (continuous-time) Markov process is a stochastic process $(X_t)_{t\geqslant 0}$ which moves round its state space E in a random 'memoryless' way; more precisely, given that at time t it is at $x \in E$, its behaviour after time t is independent of its history up to time t. The dynamics of the process are determined by its *transition function* $P_t(x, A)$, interpreted as the probability that, if the process is now at $x \in E$, it will be in A at time t later. Thus repeated application of the 'memorylessness' property yields

$$\text{(10)}\quad \mathbb{P}^x(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \int_{A_1} P_{s_1}(x, dy_1) \int_{A_2} P_{s_2}(y_1, dy_2) \ldots \int_{A_n} P_{s_n}(y_{n-1}, dy_n),$$

where $s_j \equiv t_j - t_{j-1} \geqslant 0$, and $A_i \subseteq E$. (We use $\mathbb{P}^x$ to denote the distribution of the process under the condition $X_0 = x$, and $\mathbb{E}^x$ to denote expectation with respect to $\mathbb{P}^x$.)

EXAMPLE. For Brownian motion, the transition function has a density with respect to Lebesgue measure for each $t > 0$:

$$P_t(x, A) = \int_A (2\pi t)^{-\frac{1}{2}} \exp\{-(x-y)^2/2t\}\, dy.$$

For concreteness, we shall always suppose that the state space E is a complete separable metric space with its Borel sets, that the process X is defined on the *canonical sample space*

$$\Omega = \{\text{right-continuous } f\colon \mathbb{R}^+ \longrightarrow E \text{ with left limits}\}$$

of right-continuous left-limits paths, and that X is the *canonical process*;

$$X_t(\omega) = \omega(t), \quad t \geqslant 0, \ \omega \in \Omega.$$

The canonical sample space has the *canonical filtration*

$$\mathscr{F}_t^\circ \equiv \sigma(\{X_s : s \leqslant t\}), \quad t \geqslant 0,$$

an increasing family of σ-fields, and the *shift maps* θ_t defined by $(\theta_t \omega)(s) \equiv \omega(t+s)$. Informally, $\mathscr{F}_t^\circ$ contains all of the information about the process up to time t. We can re-express the Markov property (3) more compactly as

$$\mathbb{E}^x(YZ \circ \theta_T) = \mathbb{E}_x(Y \mathbb{E}^{X(T)}(Z)) \tag{11}$$

where Y and Z are bounded, measurable with respect to $\mathscr{F}_T^\circ$ and $\mathscr{F}_\infty^\circ$ respectively. Here, T is any fixed positive real. The random variable $Z \circ \theta_T$ is determined by what the process does from time T onwards, and Y is determined by what it does up to time T.

We also require (11) to hold for (a.s. finite) *stopping times* T, the *strong Markov property*. As before, T is a stopping time if $\{T < t\} \in \mathscr{F}_t^\circ$ for all t. The first time a process enters a given set is a stopping time, as is the first time after a stopping time when a process enters a given set. The last time a process is in a given set before time 1 is in general not a stopping time though.

Thus the strong Markov property says that the behaviour of the process after the stopping time T is independent of what led up to the stopping time T. The times $T_0, T_1, \ldots$ defined in Section 1 were stopping times (in a discrete setting).

If the Markov process being considered is a *Ray process* (and every interesting Markov process is!) then all of the above properties hold; *X takes values in a complete separable metric space, has the strong Markov property, and has right continuous paths with left limits.*

This is now ample background on general Markov processes for all that follows.

Before moving on to more concrete things, we record some remarks.

(a) The transition function may be sub-Markovian, that is $P_t(x, E) < 1$ for some x, t. This allows the process to die out. This is no great restriction, because by adding a 'coffin' state we can recover $P_t(x, E) = 1$ for all x, t; see p. 107 of Williams [75].

(b) Though the transition function is fundamental, it is often easier to work with the *resolvent*

$$R_\lambda f(x) \equiv \mathbb{E}^x\left[\int_0^\infty e^{-\lambda t} f(X_t)\, dt\right] = \int_0^\infty e^{-\lambda t} P_t f(x)\, dt, \tag{12}$$

$\lambda > 0$, f bounded measurable, $x \in E$. Sometimes the limit as $\lambda \downarrow 0$ exists and is finite for suitable f; in this case, R_0 is usually called the *Green kernel*.

Two probabilistic interpretations of the resolvent are possible and useful. If ζ is an $\exp(\lambda)$ random variable (that is, $P(\zeta > t) = e^{-\lambda t}$) independent of X, then we have

$$\lambda R_\lambda f(x) = \mathbb{E}^x[f(X_\zeta)] \tag{13}$$

as one interpretation, and

$$R_\lambda f(x) = \mathbb{E}^x\left[\int_0^\zeta f(X_s)\, ds\right] \tag{14}$$

as the other. The second of these says that if we *kill* the process at rate λ (that is, despatch it to a 'graveyard' at the time ζ) then $R_\lambda I_A(x)$ is *the expected amount of time spent in A before death*, starting at x.

EXAMPLE. For Brownian motion,

$$R_\lambda f(x) = \int_{-\infty}^{\infty} \theta^{-1} e^{-\theta|y|} f(x+y)\, dy,$$

where $\theta \equiv \sqrt{2\lambda}$.

When discussing excursion theory for a Markov process, we shall pick some distinguished state $a \in E$, and consider excursions from a. We shall always assume that a is *regular*, that is

$$\mathbb{P}^a(H = 0) = 1,$$

where $H \equiv \inf\{t > 0 : X_t = a\}$. Under this assumption, there is a local time L at a, a continuous additive functional whose growth set is the closure of the set $\{t : X_t = a\}$, and the decomposition of the path of X into its point process of excursions can be accomplished just as for Brownian motion. See Rogers and Williams **[62]** for more details.

We shall use the notation $({}_aP_t)_{t \geqslant 0}$ for the transition semigroup of X killed when it first reaches a:

$${}_aP_t f(x) \equiv \mathbb{E}^x[f(X_t) : t < H],$$

and ${}_a\mathbb{P}^x$ for the law of the process $(X(t \wedge H))_{t \geqslant 0}$ started at x. We set

$$U \equiv \{\text{right continuous } f : \mathbb{R}^+ \longrightarrow E \text{ with left limits such that for some } \zeta > 0, f^{-1}(E\backslash\{a\}) \cap (0, \infty) = (0, \zeta)\}$$

for the space of excursions, and refer to the $\zeta = \zeta(f)$ appearing in the definition of U as the *lifetime* of the excursion f; for $t > 0$,

$$f(t) = a \text{ if and only if } t \geqslant \zeta.$$

5. *The Markovian nature of excursions, marked excursions, and last exits*

Let us now define for $t > 0$ the measure n_t on $E\backslash\{a\}$ by

$$n_t(A) = n(\{f \in U : f(t) \in A, t < \zeta\}).$$

What we mean by saying that excursions are Markovian is explained by the following result.

THEOREM 5. *For* $0 < t_1 < \ldots < t_k, A_1, \ldots, A_k \subset E\backslash\{a\}$,

$$n(\{f : f(t_j) \in A_j, j = 1, \ldots, k, t_k < \zeta\}) = \int_{A_1} n_{t_1}(dx_1) \int_{A_2} {}_aP_{s_2}(x_1, dx_2) \ldots \int_{A_k} {}_aP_{s_k}(x_{k-1}, dx_k).$$

(Here, $s_j \equiv t_j - t_{j-1}$.*)*

Thus the excursion 'begins like n_t, and then evolves like the Markov process killed on hitting a'. It is an immediate consequence that for $t, s > 0$,

$$\int n_t(dx)\, {}_aP_s(x, A) = n_{t+s}(A);$$

$(n_t)_{t>0}$ is an *entrance law* for the subMarkovian transition semigroup $({}_aP_t)_{t\geq 0}$. See Rogers and Williams [**62**, Theorem VI.48.1] for a proof of Theorem 5. It should not surprise you that the essential idea is an application of the strong Markov property of X. Indeed, the same argument carries over to show that the excursions are *strongly* Markovian; for example, if $T(f) \equiv \inf\{t: f(t) \notin G\}$, G some fixed open neighbourhood of a, then ${}_a\mathbb{P}^{f(T)}$ is a regular conditional distribution of $n \circ \theta_T^{-1}$ on $\{f: T(f) < \infty\}$.

Thus the key to understanding the excursion measure n is to *find the entrance law* $(n_t)_{t>0}$ (the semigroup $({}_aP_t)_{t\geq 0}$ being supposed known). To understand this, and to see the links between the Poisson process (sample-path) expression of excursion theory and the last-exit decomposition (analytic) expression, we have to look at *marked excursions*.

The process X and the Poisson point process Ξ of excursions run on different time scales, the first in real time, the second in local time. The real time 1 is not readily identifiable on the local time scale, a fact which can cause problems. The way round this is to use *random* (*exponential*) *times* to link the real and local time scales. Here is how it works. Take a Poisson counting process $(N_t)_{t\geq 0}$ of rate $\lambda > 0$ on the real time axis, independent of X. Thus $N_0 = 0$, the paths of N are increasing $\mathbb{Z}^+$-valued, and increase by jumps of size 1 at the event times $0 < T_1 < T_2 < \ldots$. If the intervals $I_1, \ldots, I_k$ are disjoint, then the numbers $N(I_1), \ldots, N(I_k)$ of points (event times) in the respective intervals are independent Poisson random variables with means $\lambda|I_1|, \ldots, \lambda|I_k|$ respectively. An alternative way of specifying the distribution of N is to say that $(T_j - T_{j-1})_{j\geq 1}$ are independent $\exp(\lambda)$ random variables. We now think of the real time axis as having a *mark* at each of the times $T_1, T_2, \ldots$, so, when we break the path up into its excursions, *some of them will contain a mark*. But there is another way to carry out this marking procedure, which works because the number of points of N in disjoint intervals are independent Poisson variables; we break the (unmarked) path into its excursions, and then *mark each excursion with an independent Poisson process of rate* λ. Thus we end up decomposing the path into a Poisson process of marked excursions (which must take values in a space of marked excursions, etc.; we refer the reader to Rogers and Williams [**62**, §VI.49] for the formal details of the set-up, and proceed here in a more intuitive fashion). We can then say, for example, that the excursion measure of marked excursions is

$$\int_U n(df)(1 - e^{-\lambda\zeta(f)})$$

(because an interval of length ζ contains at least one mark with probability $1 - e^{-\lambda\zeta}$). Likewise, we can define the Laplace transform of the entrance law $(n_t)_{t>0}$ by

$$n_\lambda(dx) \equiv \int_0^\infty e^{-\lambda t}\, dt\, n_t(dx)$$

(no confusion should arise between the Laplace transform $(n_\lambda)_{\lambda>0}$ and the entrance law $(n_t)_{t>0}$ because we shall always index the first with a Greek letter, the second with a Roman letter); and then the excursion measure of those marked excursions whose first mark occurs when the process is in A is simply $\lambda n_\lambda(A)$. Thus the excursion measure of marked excursions can be expressed as

$$\lambda n_\lambda 1 = \int_E \lambda n_\lambda(dx) \equiv \int_U n(df)(1 - e^{-\lambda\zeta(f)}),$$

where 1 denotes the constant function equal to 1 everywhere.

Let us look at the example of Brownian motion to see how we would calculate any of this. Firstly, we shall obtain $\lambda n_\lambda 1$, the rate of marked excursions. By property (8.i), the local time when the first marked excursion appears is an exponential random variable with parameter $\lambda n_\lambda 1$. *But this is also the local time when the first point of the Poisson process N of marks on the real time axis appears*! So all we have to do is to find the distribution of $L(T_1)$. This follows easily from Theorem 2, Lévy's characterisation of reflecting Brownian motion, because

$$\begin{aligned} P(L(T_1) > a) &= P(S(T_1) > a) \\ &= P(H_a < T_1) \\ &= E(e^{-\lambda H_a}) \\ &= e^{-\theta a} \end{aligned}$$

where $\theta = \surd 2\lambda$. This last equality is a well-known result; see, for example, §II.58 of Williams [75]. Hence

$$\lambda n_\lambda 1 = \surd(2\lambda) \tag{15}$$

and when we invert the Laplace transform we discover that

$$n_t(U) = n(\{f\colon \zeta(f) > t\}) = 2(2\pi t)^{-\frac{1}{2}}.$$

We still want to know the entrance law $(n_t)_{t>0}$ (or, equivalently, its Laplace transform $(n_\lambda)_{\lambda>0}$). But the excursion measure of those marked excursions whose first mark occurs when the process is in A, is, as we have seen, $\lambda n_\lambda(A)$; thus by (8.ii),

$$P(\text{first mark occurs when the process is in } A) = \frac{\lambda n_\lambda(A)}{\lambda n_\lambda 1}.$$

But from the definition of the resolvent,

$$\begin{aligned} &P(\text{first mark occurs when the process is in } A) \\ &= \int_0^\infty \lambda\, e^{-\lambda t} P_t(0, A)\, dt \\ &= \lambda R_\lambda I_A(0) \\ &= \int_A \tfrac{1}{2}\theta\, e^{-\theta|y|}\, dy \end{aligned}$$

using the explicit form of the Brownian resolvent. Assembling all this gives

$$n_\lambda(A) = \int_A e^{-\theta|y|}\, dy,$$

from which

$$n_t(dx) = |x|\, (2\pi t^3)^{-\frac{1}{2}} e^{-x^2/2t}\, dx,$$

the entrance law for Brownian motion.

An exactly analogous argument in the general setting gives that for any bounded measurable g

$$\lambda R_\lambda g(a) = \frac{\lambda n_\lambda g}{\lambda n_\lambda 1},$$

at least in the case where the process spends no time in a (as happens with Brownian motion). If the process spends a positive amount of time in a, then the first mark could occur at an instant when the process was at a, and the formula must be modified to

$$\lambda R_\lambda g(a) = \frac{\lambda\gamma g(a) + \lambda n_\lambda g}{\lambda\gamma + \lambda n_\lambda 1} \tag{16}$$

where γ is some positive constant. It is usual to take $\gamma = 1$, because it is most natural to take as the local time at a the process

$$L_t \equiv \int_0^t I_{\{a\}}(X_s)\,ds.$$

Any positive multiple of a local time performs just as well as a local time, changing n by a constant factor. This one degree of freedom bedevils the theory of Brownian local time, since a commonly-used definition of Brownian local time is half that which we have taken. So beware factors of 2 when moving from one account to another!

Let us now reinterpret what we have just derived using the sample-path approach (Poisson point process of excursions, independent marking process) in the analytic language of last exits. The excursion measure of marked excursions is $\lambda\gamma + \lambda n_\lambda 1$, so $L(T_1)$ has an exponential distribution with parameter $\lambda\gamma + \lambda n_\lambda 1$ (assuming $X_0 = a$, of course). Thus

$$\begin{aligned} EL(T_1) &\equiv E\int_0^\infty \lambda e^{-\lambda t} L_t\,dt \equiv E\int_0^\infty e^{-\lambda t}\,dL_t \\ &= (\lambda\gamma + \lambda n_\lambda 1)^{-1}, \end{aligned}$$

so we can interpret (16) as the *last-exit decomposition*

$$\mathbb{P}^x(X_t \in A) = \mathbb{E}^x\left[\int_0^t n_{t-s}(A)\,dL_s\right], \quad A \subseteq E\backslash\{a\}, \tag{17}$$

after inverting the Laplace transform. Taking an analytic approach to Markov processes, the formula (16) for the resolvent has to be the expression of the last-exit decomposition, because the local time L has no place in the analytic setting. But (17) is much closer to being interpretable, and, of course, the sample-path picture of marked excursions is the clearest of all!

We close this section with a simple proof of the celebrated arc-sine law for Brownian motion.

THEOREM 6. *Let* $A_t \equiv \int_0^t I_{\{B_s > 0\}}\,ds$. *Then*

$$P(A_t \leqslant s) = (2/\pi)\arcsin\left[(s/t)^{\frac{1}{2}}\right] \quad (s \leqslant t).$$

Proof. Let T be an exponential random variable with parameter $\lambda > 0$. For $\alpha > 0$, we shall compute $E\exp(-\alpha A_T)$; for the theorem to be correct, this should be $Ee^{-\alpha A_T} = \gamma/\beta$, where $\gamma \equiv \sqrt{(2\lambda)}$, $\beta \equiv \sqrt{(2\lambda + 2\alpha)}$, as a few lines of calculus will confirm. This is what we prove.

Imagine that we insert red marks into the path of B at rate λ (that is, we take a Poisson process of rate λ and mark B at each jump time of the Poisson process). Imagine also that we mark the path of B independently with blue marks at rate $\alpha I_{[0,\infty)}(B_t)$. This just means that we have an independent blue Poisson process of rate

α and we mark B with a blue mark at every event time of the Poisson process *which occurs when B is in* $\mathbb{R}^+$. Any event time of the blue Poisson process which occurs when B is in $(-\infty, 0)$ is ignored. Then

$$E\exp(-\alpha A_T) = P(\text{the first red mark appears before the first blue mark}).$$

But we can construct this in another way. We can mark B with orange marks at rate $\lambda I_{(-\infty,0)}(B_t)$ and with green marks at rate $(\alpha+\lambda) I_{[0,\infty)}(B_t)$, and subsequently recolour the green marks, blue with probability $\alpha/(\alpha+\lambda)$, red with probability $\lambda/(\alpha+\lambda)$, and recolour all orange marks red. Then

$$\begin{aligned} &P(\text{first red mark appears before first blue mark}) \\ &\quad = P(\text{first orange mark appears before first green mark}) \\ &\qquad + P(\text{first green mark appears before first orange mark}) \frac{\lambda}{\lambda+\alpha}. \end{aligned}$$

Now the excursion measure of green-marked excursions is $\frac{1}{2}\beta$, and the excursion measure of orange-marked excursions is $\frac{1}{2}\gamma$—this comes from (15). Hence by (8.ii)

$$\begin{aligned} E\exp(-\alpha A_T) &= \frac{\gamma}{\gamma+\beta} + \frac{\beta}{\gamma+\beta} \cdot \frac{\lambda}{\lambda+\alpha} \\ &= \gamma/\beta, \end{aligned}$$

as required.

6. *David Williams' path-decomposition result*

One of the most celebrated and appealing results on Brownian motion and diffusions is David Williams' path decomposition [73]. There are many equivalent statements of it, but we give here one which is perfectly suited to an excursion interpretation and proof. If $X_t \equiv B_t + ct$, where $c > 0$, then the upward-drifting Brownian motion X tends to $+\infty$ as $t \to \infty$, in marked contrast to B. Since the path of X is continuous, it is bounded below and attains its minimum (uniquely, in fact). Williams' result decomposes X at its minimum.

THEOREM 7 (Williams). *Set up on some probability space three independent random elements:*

(i) $\{V_t : t \geqslant 0\}$, *a Brownian motion with drift* $-c$*;*

(ii) γ, *an exponential random variable with parameter* $2c$*;*

(iii) $\{Z_t : t \geqslant 0\}$, *a diffusion in* $\mathbb{R}^+$ *with generator*

$$\frac{1}{2}\frac{d^2}{dx^2} + c\coth cx \frac{d}{dx}, \tag{18}$$

started at zero.

Now let $\tau \equiv \inf\{t : V_t = -\gamma\}$ *and define*

$$\tilde{X}_t \equiv \begin{cases} V_t & (0 \leqslant t \leqslant \tau), \\ \gamma + Z_{t-\tau} & (\tau \leqslant t). \end{cases}$$

Then $(\tilde{X}_t)$ *has the same law as* $(X_t) = (B_t + ct)$.

REMARKS. The situation is illustrated in Figure 7. We run a downward drifting Brownian motion until it hits γ, and then run an independent diffusion with generator (18) from there. The significance of (18) is that it is the generator of *X conditioned not to hit zero*; see Williams [73] for more on this idea, which is valid for any one-dimensional diffusion. We shall now sketch an excursion proof of Williams' path decomposition, referring the reader to §VI.55 of Rogers and Williams [62] for a complete account.

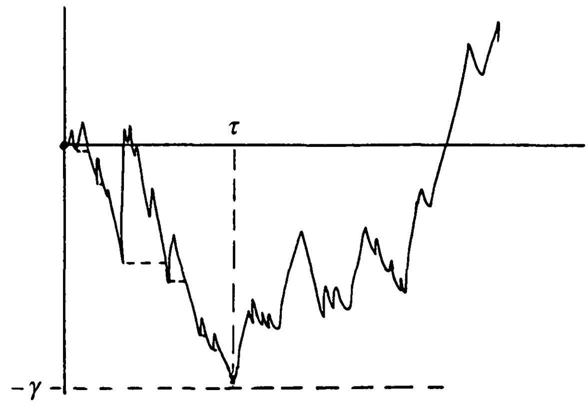

FIG. 7

Proof. The idea is to consider the strong Markov process $Y_t \equiv X_t + L_t$, where $L_t \equiv \sup\{-X_s : s \leqslant t\}$. To begin with, we shall not assume that $c > 0$; it could be that c is negative, or even zero. The process L is the local time at zero of Y, and the process Y killed on hitting zero has the transition density

$$_0p_t^c(x, y) = (2\pi t)^{-\frac{1}{2}} e^{c(y-x)-\frac{1}{2}c^2 t}[e^{-(x-y)^2/2t} - e^{-(x+y)^2/2t}]$$

for $x, y, t > 0$. (These facts are well known for $c = 0$, and the general case follows by the Cameron–Martin change of measure for $c \neq 0$; see, for example, §§IV.38–39 of Rogers and Williams [62].)

Now in the case $c > 0$, a situation arises which we have not considered before, namely, the possibility that there may be an excursion with infinite lifetime. We let U_∞ denote the collection of excursions with infinite lifetime, U_0 the collection of excursions with finite lifetime. In view of the Poisson point process description of the Brownian excursion process, it will surprise no one to learn that if we decompose the path of Y into its excursions (assuming still that $c > 0$) then we see *a Poisson point process of excursions in U_0 which stops at the instant when the first excursion in U_∞ appears.* This first infinite excursion appears after an exponential amount γ of local time has elapsed. Thus, given γ, the pieces $\{Y_t : 0 \leqslant t \leqslant \tau\}$ and $\{Y_t : t \geqslant \tau\}$ are independent, because the first is determined by the excursion process in U_0, which is independent of the second (which actually *is* the excursion process in U_∞!). Here, of course, $\tau = \sup\{u : L_u < \gamma\}$.

So excursion theory has already given us a lot of Williams' result; we know already that $\sup\{-X_s : s \geqslant 0\} \equiv \gamma$ *is exponentially distributed*, and, *conditional on* γ, $\{X_t : 0 \leqslant t \leqslant \tau\}$ *and* $\{X_{t+\tau} + \gamma : t \geqslant 0\}$ *are independent.*

All that remains is to identify the laws of the two path fragments, and for this we have to find the entrance laws of the excursion measure. Calculations using the reflection principle and Cameron–Martin, which you will find in Rogers and Williams [**62**, Lemma VI.55.1], yield the entrance law

(19) $$n_t^c(dx) = 2x(2\pi t^3)^{-\frac{1}{2}}\exp[-(x-ct)^2/2t]\,dx, \quad x, t > 0.$$

In the case where the drift c is positive, the probability that the process Y, started at $x > 0$, will ever reach 0 is $\exp(-2cx)$. Thus

$$\begin{aligned} &n^c(\{f: f(t_i) \in dx_i, i = 1, \dots, n, \zeta < \infty\}) \\ &= n_{t_1}^c(dx_1) \cdot \prod_{i=2}^n {}_0p_{s_i}^c(x_{i-1}, x_i)\,dx_i \cdot e^{-2cx_n} \end{aligned}$$

where $s_i \equiv t_i - t_{i-1} > 0$;

$$\begin{aligned} &= n_{t_1}^c(dx_1)\,e^{-2cx_1} \cdot \prod_{i=2}^n {}_0p_{s_i}^c(x_{i-1}, x_i)\,e^{-2c(x_i - x_{i-1})}\,dx_i \\ &= n_{t_1}^{-c}(dx_1) \prod_{i=2}^n {}_0p_{s_i}^{-c}(x_{i-1}, x_i)\,dx_i. \end{aligned}$$

Thus the restriction of n^c to the space U_0 of excursions which return to 0 is n^{-c}. Turning now to excursions which escape forever,

$$\begin{aligned} n^c(\{f: \zeta = \infty\}) &= \int n_t^c(dx)(1 - e^{-2cx}) \\ &= 2c, \end{aligned}$$

so $n^c(U_\infty) = 2c$; *the excursion measure of infinite excursions is* $2c$, explaining why γ should be exponential with parameter $2c$. Finally, for the distribution of the infinite excursion we have

$$\begin{aligned} &n^c(U_\infty)^{-1} n^c(\{f: f(t_i) \in dx_i, i = 1, \dots, n, \zeta = \infty\}) \\ &= (2c)^{-1} n_{t_1}^c(dx_1) \cdot \prod_{i=2}^n {}_0p_{s_i}^c(x_{i-1}, x_i)\,dx_i \cdot (1 - e^{-2cx_n}) \\ &= (2c)^{-1} n_{t_1}^c(dx_1)(1 - e^{-2cx_1}) \prod_{i=2}^n {}_0p_{s_i}^c(x_{i-1}, x_i)(1 - e^{-2cx_i})(1 - e^{-2cx_{i-1}})^{-1}\,dx_i. \end{aligned}$$

This can be identified as the distribution of the diffusion with generator (18). All the pieces of the proof can now be assembled.

REMARKS. No proof of Williams' path decomposition escapes entirely from the need to calculate, but this excursion proof delivers *immediately* the conditional independence of the pre-τ and post-τ pieces of the path.

Aside. The Williams' path decomposition is a decomposition of a particular Lévy process at its minimum. A similar decomposition at the minimum of a general Lévy process is possible, and yields the celebrated *Wiener–Hopf factorisation*; you, will find the whole story in Greenwood and Pitman [**22**], but we sketch the line of the argument here.

Let $(X_t)_{t \geqslant 0}$ be a (real-valued) Lévy process, with characteristic exponent ψ. This means that for $t \geqslant 0$, $\theta \in \mathbb{R}$,

$$E\,e^{i\theta X_t} = \exp(t\psi(\theta)).$$

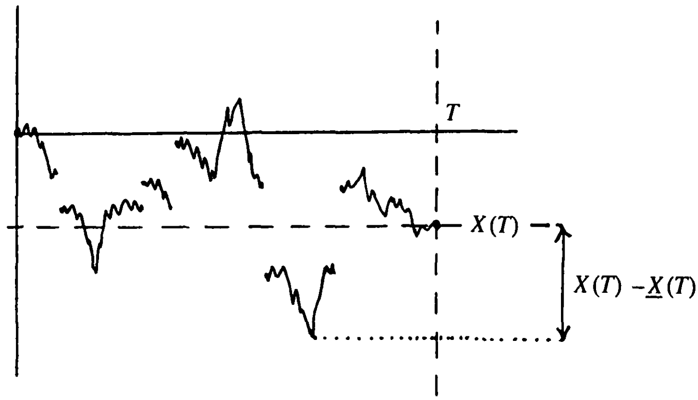

FIG. 8

The Lévy–Khinchin representation of ψ is so well known that we will not even bother to quote it—it is not relevant to the present discussion in any case. Let T be an exponential random variable of parameter λ, independent of X, and let

$$\bar{X}_t \equiv \sup\{X_s : s \leqslant t\},\ \underline{X}_t \equiv \inf\{X_s : s \leqslant t\}.$$

The Wiener–Hopf factorisation says here that

$$E e^{i\theta X(T)} = \lambda(\lambda - \psi(\theta))^{-1} = E e^{i\theta \bar{X}(T)} \cdot E e^{i\theta \underline{X}(T)}. \tag{20}$$

The idea is to consider the excursion process of $Y \equiv X - \underline{X}$ away from zero. Since an excursion of Y could end abruptly by the process X jumping to a level lower than any visited so far, the excursion space for Y needs slight modification to keep track of the final jump of the excursion, but this is a minor point. The main point is that $X_T - \underline{X}_T$ *is a functional of the first marked excursion* and so is independent of the point process of unmarked excursions. It is also independent of the local time at which the first marked excursion appears. However, $\underline{X}_T$ is a functional of the unmarked excursion process and of the local time at which the first marked excursion appears; *it is therefore independent of* $X_T - \underline{X}_T$, proving

$$E e^{i\theta X(T)} = E e^{i\theta \underline{X}(T)} \cdot E e^{i\theta(X(T) - \underline{X}(T))}.$$

To complete the proof of (20), we look at a picture of the path of $\{X_t : 0 \leqslant t \leqslant T\}$; Figure 8 shows a Brownian motion to which has been added a compound Poisson process (anything more complicated is impossible to draw!).

Now turn the picture upside-down and see what happens to the random variable $X(T) - \underline{X}(T)$ (go on!). By turning the picture upside-down, you are observing (relative to the dashed axes) $\tilde{X}_t \equiv X(T) - X(T-t)$, $0 \leqslant t \leqslant T$, and $X(T) - \underline{X}(T)$ *is* $\sup\{\tilde{X}_t : t \leqslant T\}$. But a moment's thought will show that $\{\tilde{X}_t : 0 \leqslant t \leqslant T\}$ has the same distribution as $\{X_t : 0 \leqslant t \leqslant T\}$ (convince yourself of this firstly if X was a discrete-time random walk). Therefore $\sup_{t \leqslant T} \tilde{X}_t$ has the same distribution as $\bar{X}_t$, and the proof of (20) is complete!

Of course, a bit more care is needed in a number of places, as you will see from Greenwood and Pitman [**22**], but this is the essential content of the result.

7. *Extending the process killed at a*

When we decomposed the paths of the (right-continuous, strong Markov) process X into the excursions from a, we found that the excursion measure n was determined by the transition function killed at a, $({}_aP_t)_{t\geqslant 0}$, and an entrance law $(n_t)_{t>0}$ for $({}_aP_t)$; the n-distribution of the path was specified by the Markovian-type measure of cylinder sets (see Theorem 5).

It is a natural and very interesting question to ask whether, *given some entrance law* $(n'_t)_{t>0}$ *for* $({}_aP_t)_{t\geqslant 0}$, *there exists some Markov process* X' *which is governed by* (n'_t) *and* $({}_aP_t)$ *in the same way as* X *is governed by* (n_t) *and* $({}_aP_t)$. Such a process X' would behave like X until it reached a, and then it would 'leave a' in some different way.

A moment's thought suggests an approach; we use the given entrance law to build a new excursion measure n' on U, we realise a Poisson point process on $(0,\infty)\times U$ using Lebesgue $\times\, n'$ as the characteristic measure, and then we 'stick the excursions together' to make the process. A few more moments' thought reveals at least one big problem: *how can we verify that the process so constructed is Markovian*? This is a very messy problem, because the constant times (in terms of which the Markov property is phrased) are *bad* times for the excursion point process; the problem is very similar to the problem of verifying the Markov property for the 'jump-hold' construction of a continuous-time Markov chain from a discrete-time chain and a sequence of independent exponential random variables. Freedman [**15**] attacks the latter problem, Salisbury [**63**] the former, but in each case the reader is likely to end up exhausted and sceptical!

While the sample-path approach is in general preferable for its greater clarity, there are cases where another approach is better, and this seems to be one. We already know that exponential times are 'good' times for the process and for the Poisson process of excursions, and that the Markov process at exponential times is captured in the *resolvent*. Thus it is preferable to construct the resolvent of some new process, using the given entrance law (n'_t). How do we know that what we construct is a resolvent, and, more importantly, that it corresponds to some right-continuous strong Markov process with left limits? The verification of the resolvent property reduces to some quite elementary algebraic manipulations, but to get the process itself, we need some general result which says that if a resolvent satisfies conditions such-and-such, then there exists a strong Markov process with right-continuous left-limits paths with this resolvent. Without going into technicalities, one shows that the resolvent constructed is a *Ray* resolvent, and then a deep theorem applies to tell us of the existence of a right-continuous left-limits process with the strong Markov property. See Rogers [**60**] for this, and a string of other (mostly earlier) papers (Chung [**5**, **6**], Dynkin [**11**, **12**], Lamb [**34**], Neveu [**50**, **51**], Pittenger [**55**, **56**], Reuter [**57**, **58**], Blumenthal [**4**], Salisbury [**63**, **64**], Williams [**72**], ...) for the same collection of ideas in different packaging. This resolvent approach seems to be clearer and quicker than the sample-path approach. Moreover, in the Ray setting, one sees the 'counter-examples' of Salisbury [**63**] for what they are, namely the by-products of choosing an inappropriate compactification, or topology, instead of the Ray–Knight compactification with the Ray–Knight topology.

We shall now briefly describe the form which the results take, and then go on to discuss some completely concrete examples. Recalling the notation

$$H \equiv \inf\{t>0 : X_t = a\},$$

we define the resolvent $({}_aR_\lambda)_{\lambda>0}$ of $({}_aP_t)_{t\geqslant 0}$ by

$$\begin{aligned}{}_aR_\lambda f(x) &\equiv \mathbb{E}^x\left[\int_0^H e^{-\lambda t} f(X_t)\,dt\right]\\ &= \int_0^\infty e^{-\lambda t}\,{}_aP_t f(x)\,dt,\end{aligned}$$

and we define for $\lambda > 0$

$$\psi_\lambda(x) \equiv \mathbb{E}^x[e^{-\lambda H}].$$

Then by applying the expectation operator to both sides of the obvious formula

$$\int_0^\infty e^{-\lambda t} f(X_t)\,dt = \int_0^H e^{-\lambda t} f(X_t)\,dt + e^{-\lambda H}\int_0^\infty e^{-\lambda s} f(X_{H+s})\,ds$$

and using the strong Markov property at H, we deduce the identity

(21) $$R_\lambda f(x) = {}_aR_\lambda f(x) + \psi_\lambda(x)\,R_\lambda f(a),$$

where, according to (16), we have the expression

(22) $$R_\lambda f(a) = \frac{\gamma f(a) + n_\lambda f}{\lambda\gamma + \lambda n_\lambda 1}$$

for $R_\lambda f(a)$ in terms of the Laplace transform of the entrance law. If now E is a compact metric space (the Ray–Knight compactification of the original space, for example) and $({}_aR_\lambda)_{\lambda>0}$ is a Ray resolvent on $C(E)$, *then* $(R'_\lambda)_{\lambda>0}$ *is again a Ray resolvent*, where we have the analogous identities

$$R'_\lambda f(x) = {}_aR_\lambda f(x) + \psi_\lambda(x)\,R'_\lambda f(a),$$

$$R'_\lambda f(a) = \frac{\gamma' f(a) + n'_\lambda f}{\lambda\gamma' + \lambda n'_\lambda 1}.$$

(See Rogers [60] for more details of this; see Getoor [17] or Williams [75] for a discussion of Ray processes.) This then is the main result which allows us to construct a new Ray process from an entrance law (n'_t) for the killed semigroup $({}_aP_t)$. Let us now look at some examples.

EXAMPLE: *skew Brownian motion.* If U is the space of Brownian excursions from 0, $U^+ = \{f\in U: f(t) \geqslant 0 \text{ for all } t\}$, $U^- = U\backslash U^+$, and $n^\pm$ denotes the restriction of n to $U^\pm$, then we can consider the extension of Brownian motion killed at 0 obtained when we 'make positive and negative excursions in an asymmetric manner'. In more detail, we deduce from the entrance law for $|B|$ that n^+ and n^- are given by the entrance laws

$$n_t^+(dx) = I_{(0,\infty)}(x)\,\tfrac{1}{2}x(2\pi t^3)^{-\frac{1}{2}}e^{-x^2/2t}\,dx,$$

and n_t^- defined analogously. As is well known, the resolvent of Brownian motion killed at zero is given by (writing θ for $\sqrt{(2\lambda)}$)

$${}_0R_\lambda(x,dy)/dy = \theta^{-1}\{e^{-\theta|x-y|} - e^{-\theta(x+y)}\}\,I_{(y>0)}$$

for $x > 0$, with a symmetric definition for $x < 0$; this follows from the reflection principle. We can now make a new resolvent (R'_λ) by the recipe

$$R'_\lambda f(x) = {}_0R_\lambda f(x) + e^{-\theta|x|}R'_\lambda f(0)$$

where

$$R'_\lambda f(0) = \frac{a n_\lambda^+ f + b n_\lambda^- f}{a\lambda n_\lambda^+ 1 + b\lambda n_\lambda^- 1}$$
$$= 2(a n_\lambda^+ f + b n_\lambda^- f)/(a+b)\,\theta,$$

and a, b are non-negative constants, $a+b=1$. The case $a=b$ is just the usual Brownian motion, the case $b=0$ gives the resolvent of reflecting Brownian motion, and for $0<a<1$ we get *skew Brownian motion*. Since the resolvent of skew Brownian motion is Ray, the process is strongly Markov, and continuous, since the excursion law is concentrated on the space of excursions which start from zero. It is therefore a one-dimensional diffusion on $\mathbb{R}$, which is easily seen to be regular. We can compute the scale function trivially because for $\xi > 0 > \eta$

$$\mathbb{P}^0(\text{hit } \xi \text{ before } \eta) = \frac{n(\text{excursions reaching } \xi)}{n(\text{excursions reaching } \xi \text{ or } \eta)}$$
$$= \frac{a/2\xi}{(a/2\xi) + (b/2|\eta|)}$$
$$= \frac{-a\eta}{b\xi - a\eta}$$

so we have the piecewise linear scale function

$$s(x) = bx/a \quad (x \geqslant 0)$$
$$= x \quad (x \leqslant 0).$$

EXAMPLE: *Feller Brownian motions.* A Feller Brownian motion is a strong Markov process in $[0,\infty)$ which behaves like Brownian motion when away from 0. The problem is to characterise all such processes, and, as we know, this is equivalent to the problem of *characterising all possible entrance laws for* $({}_0P_t)_{t\geqslant 0}$, the killed Brownian transition semigroup. This question can equally well be posed in a general setting, but it is helpful to see what happens in this specific context first. We know that the (Laplace transform of) an entrance law satisfies

$$(\mu - \lambda)\, n_\lambda\, {}_0R_\mu = n_\lambda - n_\mu, \tag{23}$$

and easy manipulations yield

$$\lambda n_\lambda\left((1-\psi_1)\frac{{}_0R_\mu f}{1-\psi_1}\right) = \frac{\lambda}{\lambda-\mu}(n_\mu - n_\lambda) f. \tag{24}$$

The point of this is that

$$\lambda n_\lambda(1-\psi_1) = \frac{\lambda}{\lambda-1}(n_1 1 - n_\lambda 1)$$

which remains bounded as $\lambda \to \infty$, so the measures

$$f \longmapsto \lambda n_\lambda((1-\psi_1) f) \equiv m_\lambda f$$

have bounded norm. Moreover, looking at the right-hand side of (23) we see that

$$n_\mu f = \lim_{\lambda\to\infty} m_\lambda\left(\frac{{}_0R_\mu f}{1-\psi_1}\right).$$

Thus *if we compactify* $(0,\infty)$ *in such a way that each of the functions* $(1-\psi_1)^{-1}{}_0R_\mu f$ *extends to a bounded continuous function*, the measures $(m_\lambda)_{\lambda>0}$ *have a weak limit as a measure on the compactification*, $\lambda\uparrow\infty$; and then the entrance law n_μ can be represented by an integral over this compactification. This idea is the fundamental idea used in the construction of the Martin boundary, and has been used countless times (Dynkin [**13**], Neveu [**50**], Pittenger [**55**], ...; see Rogers [**60**] for an account close in style to the present exposition (!)).

What then *is* the compactification in this case? Not too surprisingly, it is simply $[0,\infty]$, and for $f\in C([0,\infty])$,

$$\lim_{x\downarrow 0}\frac{{}_0R_\mu f(x)}{1-\psi_1(x)}=\sqrt{2}\int_0^\infty e^{-y\sqrt{(2\mu)}}f(y)\,dy,$$

$$\lim_{x\uparrow\infty}\frac{{}_0R_\mu f(x)}{1-\psi_1(x)}=\mu^{-1}f(\infty).$$

Thus the most general entrance law has the representation

$$n_\mu f=m(0)\sqrt{2}\int_0^\infty e^{-y\sqrt{(2\mu)}}f(y)\,dy+\int_{(0,\infty]}m(dy)(1-\psi_1(y))^{-1}{}_0R_\mu f(y), \tag{25}$$

where m is a finite measure. The mass $m(\{\infty\})$ corresponds to an excursion which starts at $+\infty$ and stays there, so, in effect, an excursion to the graveyard. The first term in (25) corresponds to the *continuous exits* of zero, the second corresponds to *jumping in to* $(0,\infty)$. Notice that $n(\{f:f(0)\in dy\})=m(dy)(1-\psi_1(y))^{-1}$ may be a σ-finite but *unbounded* measure; in terms of the sample paths, *the process started at* 0 *may make infinitely many* (*very small*) *jumps into* $(0,\infty)$ *in any time interval* $(0,\delta)$, very much as ordinary Brownian motion makes infinitely many excursions from 0 before any $\delta>0$. To summarise then, the most general extension of Brownian motion in $[0,\infty)$ killed at 0 has resolvent satisfying for $f\in C_K([0,\infty))$

$$R_\lambda f(0)=\frac{2p_2\int_0^\infty e^{-x\sqrt{(2\lambda)}}f(x)\,dx+p_3f(0)+\int_{(0,\infty)}p_4(dx)\,{}_0R_\lambda f(x)}{p_1+p_2\sqrt{(2\lambda)}+\lambda p_3+\int_{(0,\infty)}p_4(dx)(1-e^{-x\sqrt{(2\lambda)}})}$$

where $p_1,p_2,p_3\geqslant 0$, $\int_{(0,\infty)}p_4(dx)(1-e^{-x})<\infty$. The constant p_1 is the rate of *killing*, p_3 is the *stickiness of* 0, p_2 is the rate of *continuous exits*, and $p_4(\cdot)$ is the *jump-out measure*. The switch in notation is to conform with the statement of the result (proved via differential equations methods) in Itô–McKean [**26**, p. 186].

EXAMPLE: *exit non-entrance boundary for a diffusion.* We consider here a diffusion in natural scale on $[0,\infty)$ with speed measure m, which starts at some point in $(0,\infty)$. As we recalled in §3, this diffusion can be realised as a time-change of Brownian motion; the purpose of looking at this example is to show that *it may be impossible to make an extension* (of the process killed at zero) *which has continuous sample paths*. Why is this? Well, any such process would be a diffusion, and as such could be realised as a time-change of Brownian motion;

$$X_t=B(\sigma_t),$$

where $\sigma_t\equiv\inf\{u:A_u>t\}$, $A_u\equiv\int_{[0,\infty)}L(u,x)\,m(dx)$. Now suppose that

$$\int_{(0,1]}xm(dx)<\infty=\int_{(0,1]}m(dx),$$

and consider what happens to the additive functional A_t as $t \uparrow\uparrow H_0$. We saw that $\lim_{t\uparrow\uparrow H_0} A_t < \infty$, but *we also have now that*

$$A_t = +\infty \text{ for } t > H_0.$$

This is because for $t > H_0$, $L(t,0) > 0$, and, by continuity of L, $L(t,x) > 0$ for some x in a neighbourhood of 0. Hence for some $\varepsilon > 0$, $A_t \geqslant \varepsilon \int_{(0,\varepsilon)} m(dx) = +\infty$.

In this example, then, it is impossible for an extension to get out of 0 continuously; intuitively, it takes the process too long to get to any $\varepsilon > 0$ for such an extension to be possible.

EXAMPLE: *Brownian motion in a wedge with skew reflection.* Our final example is a very attractive one which has received the attention of a number of mathematicians, most recently Varadhan and Williams [**69**], Williams [**76**], ... We shall follow Varadhan and Williams very closely in some places, and in others our arguments will differ significantly from theirs.

The account given here is necessarily abbreviated in some respects, but none of the points skipped over is difficult; the interested reader should be able to supply the details to his or her own satisfaction in the places where more seems to be needed.

The problem concerns planar Brownian motion in the wedge

$$D = \{re^{i\theta} : 0 \leqslant \theta \leqslant \alpha, r \geqslant 0\}$$

of angle $\alpha \in (0, 2\pi)$ with constant directions of reflection on each of the boundaries as indicated:

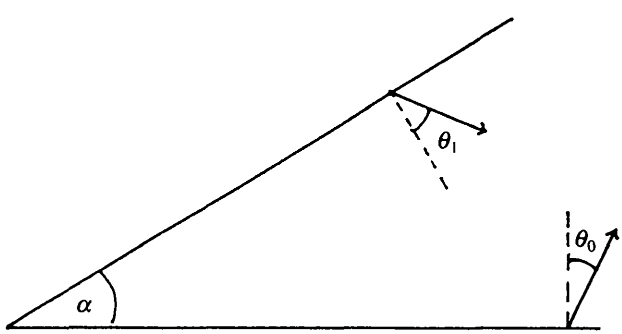

FIG. 9

The angle of reflection θ_0 on $\mathbb{R}^+$ is measured from the inward-pointing normal, the positive sense corresponding to a direction pointing *away* from 0, as in Figure 9. The angle of reflection θ_1 is similarly defined. Evidently, $\theta_0, \theta_1 \in (-\pi/2, \pi/2)$. [A planar Brownian motion is a process $Z_t \equiv X_t + iY_t$, where X and Y are independent (real-valued) Brownian motions. To understand the notion of a direction of reflection, consider first the case of a half-plane ($\alpha = \pi$) with a constant direction of reflection ($\theta_0 = -\theta_1$). Then if B is a Brownian motion, $L_t \equiv -\inf\{B_s : s \leqslant t\}$, and B' is a Brownian motion independent of B, we define the Brownian motion with skew reflection at angle θ_0 as

$$Z_t = B'_t + (\tan \theta_0) L_t + i(B_t + L_t);$$

the horizontal component gets a push whenever the vertical component is zero.]

We shall make frequent use of the fact that if f is analytic, and Z is planar Brownian motion, then $f(Z)$ *is a time-change of a planar Brownian motion*. In more detail, if

$$A_t \equiv \int_0^t |f'(Z_s)|^2 \, ds, \quad \tau_t \equiv \inf\{u : A_u > t\},$$

then $Z'_t \equiv f(Z_{\tau_t})$, $0 \leqslant t < A_\infty$, *is a Brownian motion* (killed at the stopping time A_∞ if this should be finite). This can also be expressed as

$$f(Z_t) = Z'(A_t) \equiv Z'\left(\int_0^t |f'(Z_s)|^2 \, ds\right). \tag{26}$$

A consequence of this is that if W is Brownian motion in some domain Ω with skew reflection on the boundary, then $f(W)$ is a time-change of Brownian motion in $f(\Omega)$ with the corresponding skew reflection on the boundary, since an analytic map is conformal.

A number of natural questions arise about the skew-reflecting Brownian motion in D.

(i) *Does it approach* 0? (It does if and only if $\theta_0+\theta_1 < 0$.)

(ii) *If it does, does it reach* 0 *in finite time*? (Yes.)

(iii) *If so, is it possible to extend the process in a non-trivial fashion beyond the first hit on* 0 *so that the paths are continuous*? (Yes, if and only if $2\alpha > -\theta_0-\theta_1 > 0$.)

(iv) *If so, what continuous extensions are possible*? (Modulo killing at 0, and stickiness of 0, just one.)

We shall prove firstly that the process approaches 0 if and only if $\theta_0+\theta_1 < 0$. Suppose that $\theta_0+\theta_1 < 0$. We are going to consider some function $z \mapsto z^\gamma$ (where $\gamma > 0$ will be chosen presently), which is analytic in $\mathbb{C}\backslash[0, \infty)$ and (we shall assume) fixes $\mathbb{R}^+ + i0$. The effect of this function is to open up the wedge D to a wedge of a different angle. By choosing $\gamma = -(\theta_0+\theta_1)/\alpha$, we ensure that the angle of the transformed wedge is $\alpha\gamma \in (0, \pi)$, and that *the two directions of reflection are facing each other*. By rotating the transformed wedge, we obtain a domain D' as in Figure 10, where the two directions of reflection are vertical.

By the invariance of exit distributions of Brownian motion under analytic transformations, the original process reaches 0 if and only if Brownian motion in the

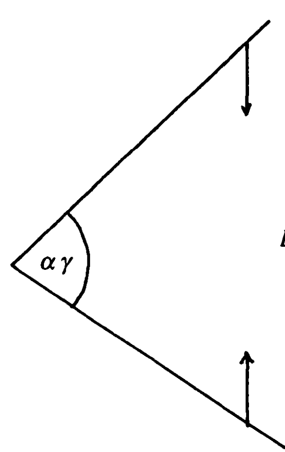

FIG. 10

transformed wedge reaches zero. But the x-component of this Brownian motion is (a time change of) a (one-dimensional) Brownian motion, bounded below by zero since $\gamma\alpha < \pi$, since the directions of push at the reflecting boundaries are vertical. Thus the x-component will approach zero, and thus the original process will approach zero; either it will get there in finite time, or it will tend to zero as t tends to infinity. We shall decide shortly which of these possibilities occurs.

For the converse, we firstly dispose easily of the case $\theta_0+\theta_1 > 0$. By applying the analytic function $z \mapsto 1/z$, we change θ_0 to $-\theta_1$, θ_1 to $-\theta_0$ while keeping the angle α of the wedge unchanged. This reduces the problem to the case already considered where the process reaches 0, and so in the case $\theta_0+\theta_1 > 0$, the process tends to infinity. The critical case $\theta_0+\theta_1$ also succumbs quickly to an analytic function, this time $z \mapsto \log z$, which converts the wedge D into the strip $D_0' \equiv \{z: 0 \leqslant \operatorname{Im}(z) \leqslant \alpha\}$ with opposed reflection on the boundaries:

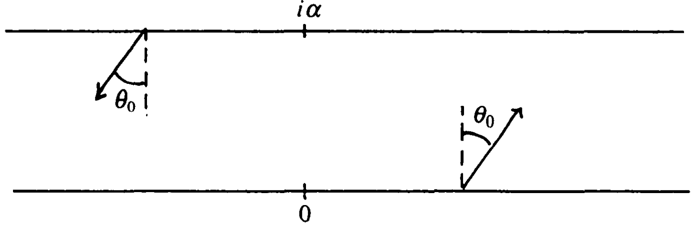

FIG. 11

But if $Z_t = X_t + iY_t$ is Brownian motion in this strip D_0 with reflection as shown, the process $X_t - \tan\theta_0\, Y_t$ is a constant multiple of a Brownian motion, implying that X is not bounded above or below, and the original process in D cannot reach 0, or tend to 0.

Let us now suppose that we are in the interesting case $\theta_0+\theta_1 < 0$, and attack the questions (ii) and (iii). Let Z denote the skew-reflecting Brownian motion in D, and let f denote the analytic function taking D to D'; $f(z) = e^{-i\theta_0}z^{\gamma}$. From (26) we know that the process $\zeta \equiv f(Z)$ is a time-change of Brownian motion;

$$\zeta_t = Z'\left(\int_0^t |f'(Z_s)|^2\, ds\right)$$

$$= Z'\left(\int_0^t \gamma^2|\zeta_s|^{2\beta}\, ds\right),$$

where $\beta \equiv (\gamma-1)/\gamma$, and Z' is a Brownian motion in D' with vertical reflection off the boundaries. Another way to express this is that

$$\zeta_t = Z'(\tau_t)$$

where $\tau_t \equiv \inf\{u: A_u > t\}$, $A_u \equiv \int_0^u \gamma^{-2}|Z_s'|^{-2\beta}\, ds$. So ζ is a time-change of a Brownian motion Z' in D'. Now because the angle of D' is less than π, and D' is contained in the right half-plane, there exists $\varepsilon > 0$ such that

$$\varepsilon \leqslant \operatorname{Re}(z)/|z| \leqslant \varepsilon^{-1}$$

for all non-zero $z \in D'$. So the additive functional

$$\tilde{A}_t \equiv \int_0^t \gamma^{-2} \operatorname{Re}(Z_s')^{-2\beta}\, ds$$

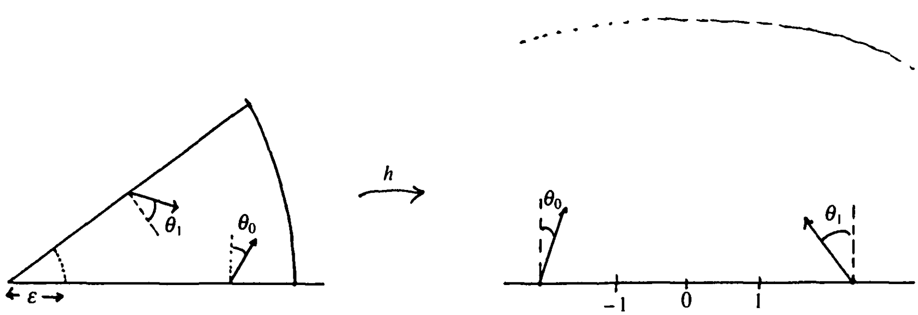

FIG. 12

is very similar to A_t; the ratio of their gradients is bounded away from 0 and ∞, so A tends to infinity if and only if $\tilde{A}$ tends to infinity, A reaches infinity in finite time if and only if $\tilde{A}$ reaches infinity in finite time.

So *to decide whether* ζ *reaches* 0 *in finite time, we could equally well try to decide whether* $\tilde{\zeta}_t \equiv Z'(\tilde{\tau}_t)(\tilde{\tau}_t \equiv \inf\{u: \tilde{A}_u > t\})$ *reaches* 0 *in finite time.* The merit in considering $\tilde{\zeta}$ instead of ζ is that $\mathrm{Re}(\tilde{\zeta})$ *is an autonomous diffusion*; it is the diffusion in natural scale with speed measure $m(dx) \equiv \gamma^{-2}x^{-2\beta}\,dx$. This trivialises the problem of whether we hit zero in finite time; we have already seen the condition for this as $\int_{0+} xm(dx) < \infty$, which translates into the condition $\beta < 1$, which is *always satisfied.* Thus an infinitely protracted approach to 0 is not possible.

Similarly, if 0 is an exit non-entrance boundary for $\mathrm{Re}(\tilde{\zeta})$, then no extension beyond the hitting time of 0 is possible; this translates into the condition $\beta \geqslant \frac{1}{2}$, equivalently $\gamma \geqslant 2$.

All that remains is to decide in the case $\gamma < 2$ whether an extension *is* possible, and, if so, *what* extensions are possible. To understand that, we investigate the distribution of Z_T, where $T \equiv \inf\{u: |Z_u| = 0 \text{ or } 1\}$, conditional on $|Z_T| = 1$, with initial position Z_0 distant ε from 0. Here, Z is Brownian motion in D with the specified skew reflection at the edges. The analytic function log takes $\Omega = D \cap \{|z| \leqslant 1\}$ into the half-strip $\{z: 0 \leqslant \mathrm{Im}\, z \leqslant \alpha, \mathrm{Re}\, z \leqslant 0\}$, which is mapped in turn to the upper half-plane $\mathbb{H}$ by the analytic function $z \mapsto \cosh(\pi(i\alpha - z)/\alpha)$. The composition h carries the curved part of the boundary of Ω into $[-1, 1]$, takes the segment of $\partial\Omega$ lying along $\mathbb{R}$ into $(-\infty, -1]$, and the remaining part of $\partial\Omega$ into $[1, \infty)$. See Figure 12. The advantage of making these transformations is that we can investigate the exit distribution from $\mathbb{H}$ of Brownian motion with the boundary reflections shown. The Brownian motion can only leave at some point of $[-1, 1]$. It does not take long to construct the function (now with the convention that z^α is cut along $(-\infty, 0]$)

$$\psi(z) = \frac{(1+z)^{\alpha_0}(1-z)^{\alpha_1}}{\xi - z}\frac{1}{\pi}(1+\xi)^{-\alpha_0}(1-\xi)^{-\alpha_1},$$

where $z \in \mathbb{H}$, $\xi \in (-1, 1)$ is fixed, and $\alpha_i = (\pi + 2\theta_i)/2\pi$, $i = 0, 1$. This is analytic in $\mathbb{H}$, and its imaginary part v has a pole at ξ of unit strength, and satisfies the boundary conditions

$$\begin{cases} \dfrac{\partial v}{\partial y}\cos\theta_0 + \dfrac{\partial v}{\partial x}\sin\theta_0 = 0 & \text{on } (-\infty, -1) \\[2ex] \dfrac{\partial v}{\partial y}\cos\theta_1 - \dfrac{\partial v}{\partial x}\sin\theta_1 = 0 & \text{on } (1, \infty). \end{cases}$$

Hence

$$\mathbb{P}^{h^{-1}(z)}[h(Z_T)\in d\xi]/d\xi = \operatorname{Im}\left\{\frac{(1+z)^{\alpha_0}(1-z)^{\alpha_1}}{\pi(\xi-z)}\right\}(1+\xi)^{-\alpha_0}(1-\xi)^{-\alpha_1}.$$

If we take the starting point $h^{-1}(z) = \varepsilon e^{i\phi}$, then

$$z = h(\varepsilon e^{i\phi}) = \cosh\left(\frac{\pi}{\alpha}\log\frac{1}{\varepsilon}\right)\cos\frac{\pi(\alpha-\phi)}{\alpha} + i\sinh\left(\frac{\pi}{\alpha}\log\frac{1}{\varepsilon}\right)\sin\frac{\pi(\alpha-\phi)}{\alpha},$$

which varies as $\frac{1}{2}\varepsilon^{-\pi/\alpha}\exp(i\pi(\alpha-\phi)/\alpha)$ as $\varepsilon\downarrow 0$. Thus

(27) $\mathbb{P}^{\varepsilon e^{i\phi}}[h(Z_T)\in d\xi]/d\xi$

$$\sim C\cdot\varepsilon^{-(\theta_0+\theta_1)/\alpha}\cdot\sin\left(\frac{\pi}{2}+\theta_0\frac{\alpha-\phi}{\alpha}-\theta_1\frac{\phi}{\alpha}\right)\cdot(1+\xi)^{-\alpha_0}(1-\xi)^{-\alpha_1},$$

as $\varepsilon\downarrow 0$. The most important conclusion to be drawn from (27) is that as $\varepsilon\downarrow 0$, conditional on $|Z_T| = 1$, *the distribution of Z_T started from $\varepsilon e^{i\phi}$ is the same whatever ϕ*. Thus if there *does* exist an excursion law for a continuous extension, then on any excursion from 0 which gets out as far as $\{|z| = 1\}$, the distribution of the position at the time of first hitting $\{|z| = 1\}$ is *determined*:

$$\frac{n(\{f: T(f) < \infty, h(f_T)\in d\xi\})}{n(\{f: T(f) < \infty\})} = C\cdot(1+\xi)^{-\alpha_0}(1-\xi)^{-\alpha_1}\,d\xi, \tag{28}$$

where $T(f) \equiv \inf\{t: |f_t| = 1\}$. The scaling properties of the process imply that the same law holds for the argument of the excursion f when it first crosses $\{|z| = r\}$ for any $r > 0$, and so, if there *is* an excursion law giving rise to a continuous extension, *there can only be one*. It is now fairly clear how we are going to go about constructing the excursion law. We shall insist that

$$n(\{f: \sup|f(t)| > r\}) = r^{(\theta_0+\theta_1)/\alpha}$$

and that, if $T_r(f) \equiv \inf\{t: |f(t)| > r\}$, then the distribution of the argument of $f(T_r)$, when $T_r < \infty$, is given by (28), and after T_r the process evolves like Brownian motion in D with skew reflection. The only thing which could go wrong is that as we add up the times taken to get from radius 2^{-n-1} to radius 2^{-n}, we might get an infinite sum; *the process could be moving too slowly near* 0. This does not happen, though, because the timescale of the process we are attempting to construct is rigidly linked to the timescale of the $\tilde{\zeta}$ process, whose real part is an autonomous diffusion in natural scale with speed measure $\gamma^{-2}x^{-2(\gamma-1)/\gamma}\,dx$; for this latter, the times taken to get from 2^{-n-1} to 2^{-n} do *not* accumulate to give an infinite sum, because a continuous extension of that one-dimensional diffusion is possible!

This rapid sketch oversimplifies the work required to provide a proper proof of any of these results. It does serve, however, to illustrate the use of excursion ideas (in conjunction with other techniques) to tackle a completely concrete problem, and to provide clear insight into the workings of that problem. The powerful applicability of excursion theory to such concrete problems is, to my mind, one of its most attractive features. Beautiful and powerful applications of the final extension of excursion theory which we shall consider are sadly rare; nonetheless, the ideas arising in the study of excursions from a *set* have so much in common with excursion theory from a singleton that the theory at least looks superficially similar. We turn now to that topic.

8. *Excursions from a general set*

Let E be the state space of our (strong Markov, right-continuous-with-left-limits) process; if the process is assumed Ray, we can even think of E as compact metric. Now fix some closed proper $F \subsetneq E$, $D \equiv E \backslash F$, and consider the excursions of the process away from F. The case $F = \{a\}$ is the case we have been studying up to now, and the generalisation really holds few surprises. Indeed, if F were *finite*, then we would have no hesitation in describing the excursion decomposition of X, started at $a \in F$: we would have a Poisson point process of excursions governed by some entrance law $(n_t^a)_{t>0}$, at least up until the first excursion from a which encountered some point $b \in F\backslash\{a\}$; at which instant we would switch to a Poisson point process of excursions governed by some entrance law $(n_t^b)_{t>0}$, up until the first excursion from b which met $F\backslash\{b\}$, etc.

The main problem encountered in making this precise is choosing a formulation in which this intuitive idea can be expressed. (It should be emphasised that, once the formulation of the results is complete, there remain very substantial technical elements in the proof.) Let us firstly review the familiar case $F = \{a\}$. Recalling that we regard the excursion point process Ξ sometimes as a random measure, then for any measurable $\psi: U \to \mathbb{R}$ such that $\int_U |\psi(\xi)|\, n(d\xi) < \infty$ we have that

$$M_t^\psi \equiv \Xi_t(\psi) - t \int \psi\, dn \textit{ is a martingale},$$

where $\Xi_t(\psi) \equiv \int_{(0,t]\times U} \psi(\xi)\, \Xi(ds, d\xi)$. (This is a martingale with respect to the filtration $(\tilde{\mathscr{F}}_t)$ of the excursion point process, of course.) Thus if H is any $(\tilde{\mathscr{F}}_t)$-previsible process (satisfying suitable integrability conditions),

$$\begin{aligned}(H \cdot M^\psi)_t &\equiv \int_{(0,t]} H_s\, dM_s^\psi \\ &= \int_{(0,t]\times U} H_s\, \psi(\xi)\, \Xi(ds, d\xi) - \int_0^t H_s\, ds \cdot \int \psi\, dn\end{aligned}$$

is again a martingale which will be bounded in L^1 provided $\mathbb{E}\int_0^\infty |H_s|\, ds < \infty$, and in that case

$$\mathbb{E} \int_{\mathbb{R}^+ \times U} H_s\, \psi(\xi)\, \Xi(ds, d\xi) = \int \psi\, dn \cdot \mathbb{E} \int_0^\infty H_s\, ds.$$

We can re-express this in the form

$$\mathbb{E}[\sum_{t\in G} \tilde{H}_t\, \psi(\xi_t)] = \mathbb{E}\left[\int_0^\infty \tilde{H}_s\, n(\psi)\, dL_s\right], \tag{29}$$

where G is the (countable) set of left endpoints of excursions, $\tilde{H}_t \equiv H(L_t)$ is a previsible process (with respect to the filtration of the basic Markov process X) $n(\psi)$ is short for $\int \psi(\xi)\, n(d\xi)$, and ξ_t is the excursion starting at $t \in G$:

$$\xi_t(s) = X((t+s) \wedge R_t), \quad s \geqslant 0,$$

where $R_t \equiv \inf\{U > t: X_U \in F\}$. The identity (29) can be extended to any previsible process $\tilde{H}$ (and even to optional processes), and to functionals ψ which are time-dependent (and even randomly varying in an optional fashion). Suffice it to say that for any non-negative optional process Z, and any non-negative $\psi: \mathbb{R}^+ \times U \to \mathbb{R}^+$ we have the identity

$$\mathbb{E}[\sum_{t \in G} Z_t \psi(t, \xi_t)] = \mathbb{E}\left[\int_0^\infty Z_t n(\psi(t, \cdot))\, dL_t\right]. \tag{30}$$

This identity is one which is amenable to extension to the case of general F. You can see that, in general, trying to formulate a 'Poisson process' description of the excursion process is going to be impossibly clumsy, since the characteristic measure of the process is going to be changing in time, in a way which depends on what previous excursions have done. The complete breakdown of the Poisson point process description in the general setting is the underlying explanation for the lack of any striking concrete applications of excursion theory from a general closed set.

Let us now see how to formulate the analogue of (30) in the case of general F. We make the simplifying assumption that *every point of F is regular for F*; $\mathbb{P}^x(H = 0) = 1$, where $H \equiv \inf\{t > 0 : X_t \in F\}$. This rules out the trivial but untidy possibility that certain points of F might be visited in a discrete way; for the whole story, you must read Motoo **[49]**, Dynkin **[12, 13]**, or Maisonneuve **[37, 38]**, for example, which all treat essentially the same material by various techniques. The left-hand side of (30) is still meaningful in the case of general F. As for the right-hand side, there is in general a continuous additive functional L, which grows when $X \in F$, and a kernel n from E to U such that

$$\mathbb{E}[\sum_{t \in G} Z_t \psi(t, \xi_t)] = \mathbb{E}\left[\int_0^\infty Z_t n(X(t), \psi(t, \cdot))\, dL_t\right], \tag{31}$$

where we write

$$n(x, \psi) \equiv \int_U n(x, d\xi)\, \psi(\xi).$$

If for $\lambda > 0$ and non-negative functions f on E we define

$$n_\lambda(x, f) \equiv \int_U n(x, d\xi) \left(\int_0^{\zeta(\xi)} f(\xi_s)\, e^{-\lambda s}\, ds \right),$$

then one can deduce from (31) the decomposition of the resolvent

$$R_\lambda f(x) = {}_F R_\lambda f(x) + \mathbb{E}^x\left[\int_0^\infty e^{-\lambda s} I_F(X_s)\, ds\right] + \mathbb{E}^x\left[\int_0^\infty e^{-\lambda t} n_\lambda(X_t, f)\, dL_t\right]$$

which features in Dynkin **[13]**, and (in a slightly different form) in Motoo **[49]**. It is possible also to decompose the Lévy kernel in through the Martin boundary, but we shall not go further into this here.

9. *Remarks on the literature*

With the exception of the sample-path descriptions, all the main ideas of excursion theory were known in some form or other to the pioneers of Markov chain theory. The fundamental formula (21) for the decomposition of the resolvent was known in the case where the exit boundary consisted of a single point, and also in the case where the exit boundary comprised a finite set; see Chung [**5, 6**], Dynkin [**11**], Neveu [**50, 51**], Pittenger [**55, 56**], Reuter [**57, 58**], Williams [**72**]. The expression for $R_\lambda f(a)$ in terms of the Laplace transform of an entrance law was also known to Chung, Dynkin, Neveu, Pittenger, Reuter, and Williams, and integral representations of the entrance laws were also to be found in the early work on chains. Remarkable work on additive functionals and Lévy systems in the sixties by Watanabe [**71**], Meyer [**40**], Shur [**66**] and others led to the work of Motoo [**49**], Sato [**65**], Ueno [**68**], Itô's pivotal paper [**25**] and the contemporaneous contributions of Dynkin [**12, 13**], developing the ideas of excursion theory in a general setting. After perusing [**12, 13, 25, 49**], one may well wonder what else there is to say in the general case.

Nonetheless, interest in excursion results has continued to the present, the work falling into four broad categories.

(a) *General results valid in an abstract Markovian setting.* Here we mention Getoor and Sharpe [**19**], Jacobs [**27**], Blumenthal [**4**], Rogers [**60**], Pitman [**53**], Kaspi [**30**], Mitro [**47**], Salisbury [**63, 64**]. The work of Maisonneuve [**37, 38**] is distinguished by its application of the general theory of processes, the others being more firmly rooted in Markov process theory.

(b) *Study of the local time process.* The study of regenerative sets (Hoffmann-Jørgensen [**23**], Kingman [**31**], Meyer [**41, 42**], Maisonneuve and Meyer [**44, 45**]) amounts to the study of the set of times when a Markov process is in some state a. Of course, one thereby loses all information about what the process does between visits to a, and it is surprising and pleasing that an interesting and useful theory remains. Compare this with the numerous papers which obtain local time by some limiting process, which are in effect considering the process only when at a: Chung and Durrett [**7**], Williams [**74**], and Maisonneuve [**39**], who obtain local time of Brownian motion from the limit of the number of 'small' excursions; Getoor [**18**], who gets a similar limit of downcrossings for a Lévy process which hits points; Knight [**32**], who gets local time for a reflected symmetric stable process from the occupation measure (interestingly, the local time does *not* turn out to be the occupation density in general); Smythe [**67**], who gets local time for a Markov chain by counting crossings to some finite (but growing) set; Fristedt and Taylor [**16**], who construct local time in general settings by a variety of Brownian-inspired techniques; and Greenwood and Pitman [**21**], who construct local time by an elegant and elementary martingale argument from nested arrays.

(c) *Explicit characterisation of the excursion law.* A host of such papers, using various formulae for Brownian and diffusion transitions, have appeared. The progenitor was perhaps Chung, 'Excursions in Brownian motion', *Arkiv. för Mat.* 14 (1976) 155–177, who studied the scaled Brownian excursion and the Brownian meander. Rogers [**59**] proved a very useful decomposition of the (unscaled) Brownian excursion, first observed by Williams. A recent paper of P. Biane and M. Yor, 'Quelques précisions sur le méandre Brownian', *Bull. Sci. Math.* 112 (1988) 101–109, provides simple and economical proofs of many earlier results, and would be an excellent place to start looking.

(d) *Applications.* The combination of these notions with the *Poisson* nature of the excursion process has provided beautiful and hard-hitting applications. Millar [**46**], and Monrad and Silverstein [**48**] used excursion methods on Lévy processes, Greenwood and Pitman [**22**] explained Wiener–Hopf factorisation of Lévy processes, Rogers [**59**] showed how (the Azema–Yor approach to) the classical Skorohod embedding could be tackled with excursion theory, Greenwood and Perkins [**20**], and later Barlow and Perkins [2], used excursion techniques to study slow points of Brownian motion and escape from square-root boundaries, Bass and Griffin [3] used the Ray–Knight theorem to decide that the most visited site of Brownian motion is transient, Pitman and Yor [54] used excursion theory extensively in their work on Bessel bridges and elsewhere, Rogers [61] used excursion theory and ideas of Greenwood and Pitman to investigate conditions under which a Lévy process creeps across a level, a problem also studied by Millar, and recently Barlow [1] has obtained necessary and sufficient conditions for the local time of a Lévy process to be continuous, using excursion ideas (see also 'Necessary and sufficient conditions for the continuity of local time of Lévy processes', *Ann. Probab.* 16 (1988) 1389–1427).

In a number of these papers, excursion theory only enters in a few key steps and then vanishes from the scene, and this is how it *should* be; excursion theory has begun to acquire the status of a regular tool in the probabilist's tool kit, and is used exactly when it is needed. However, I think it still needs to become more widely appreciated by probabilists, and it is largely for this reason that I was so glad to have the opportunity to write this paper. The ideas involved are not really difficult, the power of the techniques is astonishing in view of their simplicity, and their elegance can sometimes be breathtaking; certainly, whenever excursion ideas can be applied the gain in clarity is enormous. I hope that this (somewhat sketchy) survey of excursion theory has communicated some of this simplicity, power, elegance, and clarity, and that those who have not yet tried excursion theory are eager to find a nice example to apply it to; please send me a preprint of anything you discover!

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