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The value of insight

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An investor may invest in a riskless bank account, and in a stock which is a standard Black-Scholes asset with occasional Gaussian jumps of the log price, as proposed by [16]. It is well known how to solve the standard running consumption problem for this investor, which we take as a benchmark for comparing the performance of two different insiders, one who knows in advance of each jump exactly when the jump will happen, and the other who has information in advance of each jump about the size of the jump, but no information about the time. These considerations give rise to two novel and concrete stochastic control problems. For each problem, rigorous verification proofs for optimality are presented.

Key words: Hamilton-Jacobi-Bellman equation; insider trading; jump diffusion
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We dedicate this work to our colleague, mentor, and friend, Professor Larry Shepp (1936-2013)

1. Introduction In this paper, we consider two types of partially-informed insider traders. The first type of insider knows in advance of each jump exactly when the jump will happen, but not its size. This may model one who knows the timing of a public announcement that the CEO of his firm will resign. The second type of insider we will consider has information in advance of each jump about the size of the jump, but has no information about when it will occur. Let the jump size be ξ and further assume that the insider knows η, where (ξ,η) has a joint Gaussian law. This may be a plausible model for the insider who has some knowledge about a similar firm exposed to similar influences, leading him to know η for his own firm, which in turn gives him some information about ξ.

On a macro level, the knowledge gained by the first insider may model the scenario in which there is going to be an election in a major country on a known date. The story of the second insider may be used to model the situation where we have some idea what will follow the death of a despotic leader, but no idea when he will die.

Let us now formalize the above considerations. Consider a continuous-time model of investment and consumption, where an agent may invest in a risk-free bank account, paying interest at fixed positive rate $r>0$, and in a risky asset, the stock S. The dynamics of the stock are given by

$$dS_t = S_{t-}\big[\,\sigma dW_t + \mu dt + \int (e^x-1)\, n(dt,dx)\,\big], \tag{1}$$

where W is a standard Brownian motion, $\sigma > 0$ is the (constant) volatility, μ is the (constant) drift and n is a Poisson random measure independent of W with expectation measure

$$\mathbb{E}\,[n(dt,dx)] = \lambda\,dt\,p(x)\,dx.$$

Here, $\lambda > 0$ is the rate at which events occur, and p is the density of the jump in log-price, which we assume is distributed as $N(m,v)$, where m is the mean of the normal distribution and v is the variance of the normal distribution. The standard Black-Scholes model is a special case when $m = v = 0$, but more generally this model of Black-Scholes with jumps appears to date back to [16].

The wealth of the investor at time t is denoted by w_t, and his rate of consumption at time t is denoted by $c_t \geq 0$. At time t, the investor chooses his consumption rate c_t and the quantity θ_t of his wealth to be invested in the stock[1]. His wealth therefore evolves as

$$dw_t = (rw_t - c_t)\,dt + \theta_t\Big[\,\sigma dW_t + (\mu - r)dt + \int (e^x - 1)\,n(dt,dx)\,\Big], \tag{2}$$

with initial wealth w; for an explanation, see, for example, Section 1.1 of [20]. Controls (c,θ) will be said to be admissible if they are previsible and if $w_t \geq \theta_t \geq 0, c_t \geq 0$ for all t; the set of admissible controls for initial wealth w will be denoted $\mathcal{A}(w)$. The objective of the agent is to choose the controls (c,θ) in such a way as to achieve

$$V(w) \equiv \sup_{(c,\theta)\in\mathcal{A}(w)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \mid w_0 = w\right], \tag{3}$$

where ρ is the time discounting factor. We assume that the utility U has CRRA form

$$U(x) = \frac{x^{1-R}}{1-R}$$

for some $R > 0$ different from 1. As usual, this reduces the dimensionality of the problem by one, and renders it more amenable.

We shall treat three problems, which differ in the information available to the agent.

1. The agent has no prior knowledge about when the jumps occur, nor of their magnitudes;
2. The agent knows precisely the *time* of the next jump, but not the magnitude;
3. The agent knows nothing about the time of the next jump, but sees the signal

$$\eta = \xi + \varepsilon$$

where ξ is the jump in the log-price at the next jump, and ε is an independent $N(0,v_\varepsilon)$ random variable.

Of course, the optimal value for Problems 2 and 3 must be at least as big as the optimal value for Problem 1, as we shall see, but there is no reason to suppose that Problem 2 will be better than Problem 3, or *vice versa*. In Section 3, we indeed show that for realistic parameter values that knowing the magnitude of each jump, albeit with noise, is far more valuable than knowing the time of the jump. In particular, Problems 2 and 3 are *concrete, challenging, and are not amenable to direct application of general theory* — which is why they would have appealed to Larry Shepp.

We attack the above problems as stochastic optimal control problems. This being said, we wish to note that these problems can of course (but need not) be considered as an example of *grossissement* — the enlargement of filtrations. This theory was developed from the late 1970's on, starting with the works of [3], Jeulin & Yor ([11, 12, 13]) and further developed by others including [22], and by

[1] The processes c and θ are measurable with respect to the previsible σ-field $\mathcal{P}$ of (W,n); see, for example, [21].

Itô's extension of the stochastic integral (see [9]). The topic has an intimate connection with insider trading, which is well addressed by the recent book of [2], and the over 100 references therein. Observing that Problems 2 and 3 can be regarded as instances of *grossissement* does not appear to make solving them any easier.

We conclude the section with a brief literature overview. Our Problem 1 has been extensively considered; it is a special case of the work of [1] and [7]; the former analyzes a multi-dimensional version of our Problem 1 with a more general jump diffusion, while the latter considers a jump diffusion market with proportional transaction costs. [18] considers a version of our Problem 1 allowing for infinite variation price jumps, constraints, etc and [17] provides general verification results for semimartingale price processes, albeit these results cannot be applied directly for the verification proofs of Problems 2 and 3. We also acknowledge the work of [8], which looks at a very similar problem to our Problem 1. Other works related to our Problem 1 include those of [5], [6], [14] (in particular, Section 10). Other related works to the paper are [19], which considers a market where the price process is fixed. To the best of our knowledge, there is nothing in the literature remotely close to Problems 2 and 3. For each problem, rigorous verification proofs for optimality are presented.

The remainder of this manuscript is organized as follows. Section 2 formulates and solves the Hamilton-Jacobi-Bellman (HJB) equations for Problems 1-3. Section 3 offers numerical results. As is often the case with concrete stochastic control problems, the most challenging technical difficulties lie in the verification proofs, which are provided in the Appendix.

2. Main Results

Throughout this section, we will need the function

$$q \mapsto g(q) \equiv \int \{1+q(e^x-1)\}^{1-R}\; p(x)\, dx, \tag{4}$$

which is plainly a smooth monotone function defined for $0 \le q \le 1$, concave for $0 < R < 1$ and convex for $R > 1$.

2.1. Solution of Problem 1.

The solution to Problem 1 is a simple modification of the solution to the standard Merton problem without jumps. By the Davis-Varaiya Martingale Principle of Optimal Control (MPOC) - see, for example, Theorem 1.1 in [20] - if we can find a function V such that

$$Y_t \equiv \int_0^t e^{-\rho s} U(c_s)\, ds + e^{-\rho t} V(w_t) \text{ is a supermartingale for all } (c,\theta) \in \mathcal{A}(w_0)$$

and a martingale for some $(\bar c, \bar\theta) \in \mathcal{A}$, then V is the value function defined in (3) and $(\bar c, \bar\theta)$ is optimal. Using the symbol $\doteq$ to signify that the two sides of the expression differ by a local martingale, we apply Itô's formula to Y to give[2]

$$\begin{aligned} e^{\rho t} dY &\doteq \{\, U(c) - \rho V(w) + (rw - c + \theta(\mu - r))V'(w) + \tfrac{1}{2}\sigma^2\theta^2 V''(w)\, \}dt \\ &\qquad + \int \{V(w_{t-} + \theta_t(e^x - 1)) - V(w_{t-})\}\, n(dt, dx) \\ &\doteq \{\, U(c) - \rho V(w) + (rw - c + \theta(\mu - r))V'(w) + \tfrac{1}{2}\sigma^2\theta^2 V''(w) \\ &\qquad + \lambda \int \{V(w + \theta(e^x - 1)) - V(w)\}\, p(x)\, dx\, \}\, dt. \end{aligned} \tag{5}$$

We want Y to be a supermartingale whatever the control, and a martingale under some (optimal) control, so if we ignore the distinction between martingales and local martingales the condition would be that the supremum over (c,θ) of the drift term in (5) should be zero:

$$0 = \sup_{c,\theta} \big[\, U(c) - \rho V(w) + (rw - c + \theta(\mu - r))V'(w) + \tfrac{1}{2}\sigma^2\theta^2 V''(w)$$

[2] We omit uninformative appearances of the time index.

$$+\lambda \int \{V(w+\theta(e^x-1)) - V(w)\}\, p(x)\, dx\, \Big], \tag{6}$$

and this is the Hamilton-Jacobi-Bellman (HJB) equation for this problem.

A familiar scaling argument (see, for example, Proposition 1.2 in [20]) shows that the value function V should be a multiple of U:

$$V(w) = A_1 U(w) \tag{7}$$

for some constant A_1, and this gives us a way of solving the HJB equation (6). Assuming the scaling form (7) of the solution, optimizing over c in (6) gives

$$\bar{c} = A_1^{-1/R} w.$$

Writing $\theta = qw$ and substituting this value of c, straightforward algebra transforms the HJB equation (6) to

$$\begin{aligned} 0 = \sup_q A_1 U(w) \big[\, & R A_1^{-1/R} - \rho - (R-1)(r + q(\mu - r)) + \tfrac{1}{2}\sigma^2 q^2 R(R-1) \\ & + \lambda\{g(q) - 1\}\,\big]. \end{aligned}$$

This is optimized over q by choosing q to maximize

$$q \mapsto g_1(q) \equiv r + q(\mu - r) - \tfrac{1}{2}\sigma^2 R q^2 + \lambda\{g(q) - 1\}/(1-R). \tag{8}$$

The function g_1 is concave in q, so maximizing it (numerically) is quite easy. We need to be aware that q is constrained to lie in $[0,1]$, so the optimizing value could be an endpoint of the interval. Denoting the optimal q by $\bar{q}_1$, we have finally the equation

$$0 = R A_1^{-1/R} - [\, \rho + (R-1)\{r + \bar{q}_1(\mu - r)) - \tfrac{1}{2}\sigma^2 (\bar{q}_1)^2 R\}\,] + \lambda\{g(\bar{q}_1) - 1\},$$

which determines the value of A_1:

$$R A_1^{-1/R} = \rho + (R-1) g_1(\bar{q}_1). \tag{9}$$

In the special case $\lambda = 0$, we are back with the original Merton problem, where the q which optimizes (8) is the Merton proportion

$$\pi_M \equiv \frac{\mu - r}{\sigma^2 R} \tag{10}$$

and the expression for A_1 agrees with the known solution for that special case - see (1.30) in [20]. It is possible for the standard Merton problem to have unbounded payoff in the case $0 < R < 1$, which corresponds to the right-hand side of (9) being non-positive, and the same can happen here.

To summarize, then, we have the following result.

THEOREM 1. *Provided $\rho + (R-1) g_1(\bar{q}_1) > 0$, Problem 1 is well posed and the optimal solution is to use controls*

$$\bar{c}_t = A_1^{-1/R}\, w_t, \quad \bar{\theta}_t = \bar{q}_1\, w_t$$

where $\bar{q}_1$ maximizes the function g_1 given at (8) over $[0,1]$, and A_1 is given by (9). The value function is

$$V(w) = A_1\, U(w).$$

Proof By solving the HJB equation, we have identified what we believe is the optimal control for this problem. What remains is to verify that this is indeed the optimal solution. We do this in the Appendix A.1. □

2.2. Solution of Problem 2. Problem 2 considers the first type of insider agent, who knows the time T_1 of the first jump of the stock, and immediately each jump happens he is told the time of the next jump. Since the problem is time inhomogeneous, the value function should be parameterized by w and t. Immediately after T_1, the process restarts afresh; the time T_2 of the next jump is revealed and the difference $T_2 - T_1$ is an $\exp(\lambda)$ random variable independent of the past. Hence, it suffices to consider the value function

$$V(t,w;T_1) \equiv \sup_{(c,\theta)\in\mathcal{A}(w)} \mathbb{E}' \left[\int_t^\infty e^{-\rho(s-t)} U(c_s)\, ds \mid T_1, w_t = w \right], \quad t \in [0,T_1). \tag{11}$$

where $\mathbb{E}'$ denotes the expected value for the first type insider. Again a scaling argument shows that the value function up to T_1 has the form

$$V(t,w;T_1) = f(T_1 - t)\, U(w), \quad t \in [0,T_1). \tag{12}$$

Since $T_1 \sim \exp(\lambda)$, the unconditional expected value of $V(0,w)$ would be

$$V(0,w) = \int \lambda e^{-\lambda s} f(s)\, ds\, U(w) \equiv A_2\, U(w).$$

The behaviour of the investor has two elements. He is able to choose the fraction a of his wealth to hold in stock from the time T_1- to T_1, during which infinitesimal interval the log-price jumps by X, say. His wealth will jump from w_{T_1-} to $w_{T_1-}(1+a(e^X-1))$; the utility of his wealth will get scaled by a factor $\{1+a(e^X-1)\}^{1-R}$, so he will therefore choose

$$a^* \equiv \underset{0\le a\le 1}{\arg\max}\; g(a)/(1-R) \tag{13}$$

to maximize his expected gain in utility at the time of the jump, and thus

$$\mathbb{E}' \int_t^\infty e^{-\rho(s-t)} U(c_s)\, ds = \mathbb{E}' \left[\int_t^{T_1} e^{-\rho(s-t)} U(c_s)\, ds + e^{-\rho(T_1-t)} g(a^*) A_2 U(w_{T_1-}) \right].$$

By (11) and (12), the above equality further implies

$$f(0+) = g(a^*) A_2 = g(a^*) \int_0^\infty \lambda e^{-\lambda s} f(s)\, ds. \tag{14}$$

This relation will be needed to characterize the solution. Prior to T_1, we invoke the MPOC to assert that

$$Y_t \equiv \textstyle\int_0^t e^{-\rho s} U(c_s)\, ds + e^{-\rho t} V(t,w_t) \text{ is a supermartingale,}$$

and a martingale under optimal control. As previously, an application of Itô's formula leads to the HJB equation for this problem:

$$\begin{aligned} 0 = \sup_{c,\theta} \big[\, & U(c) - \rho V(t,w) + (rw - c + \theta(\mu - r)) V'(t,w) \\ & + \tfrac{1}{2}\sigma^2\theta^2 V''(t,w) + \dot V(t,w) \,\big], \end{aligned} \tag{15}$$

where V', V'' denote the derivatives with respect to w and $\dot V$ denotes the derivative with respect to t. The explicit form (12) of the value simplifies the HJB equation substantially; the optimal choices are

$$\bar c_t = f(T_1 - t)^{-1/R}\, w_t, \qquad \bar\theta_t = \pi_M\, w_t.$$

Substituting this back into (15) simplifies the equation to

$$0 = Rf^{1-1/R} - R\gamma_M f - \dot{f}, \tag{16}$$

where

$$\gamma_M \equiv \{\rho + (R-1)(r + (\mu - r)^2/2R\sigma^2)\}/R \tag{17}$$

is the optimal consumption rate in the standard Merton problem. Writing $f(t) = \psi(t)^R$ linearizes (16) to

$$0 = 1 - \gamma_M \psi - \dot{\psi}$$

which can be solved explicitly. The solution is

$$f(t) = \left(\frac{1 - e^{-\gamma_M t}}{\gamma_M} + e^{-\gamma_M t} f(0)^{1/R} \right)^R. \tag{18}$$

The initial value $f(0)$ is not determined by the differential equation (16), but the boundary condition (14) determines it when there is a solution. To explain in more detail, define the function

$$x \mapsto \varphi(x) \equiv g(a^*) \int_0^\infty \lambda e^{-\lambda s} \left(\frac{1 - e^{-\gamma_M s}}{\gamma_M} + e^{-\gamma_M s} x^{1/R} \right)^R ds, \tag{19}$$

so that $f(0)$ is a fixed point of φ. Two cases now need to be considered.

Case 1: $0 < R < 1$**.** In this case, φ is convex positive increasing, and $g(a^*) > 1$. We also see that $\varphi(x)/x$ decreases from $+\infty$ to limit $\lambda g(a^*)/(\lambda + R\gamma_M)$, so there will be a fixed point of φ if and only if

$$\frac{\lambda g(a^*)}{\lambda + R\gamma_M} < 1. \tag{20}$$

One way to interpret condition (20) is to say that λ must not be too big. If λ is too big, we get frequent opportunities to benefit from the jumps, and if these opportunities come too often then the boost to the objective will defeat the discounting and we can obtain unbounded objective. It is not surprising that the problem can be ill posed when $R \in (0,1)$, as the original Merton problem can also be ill posed in this regime - see Proposition 1.3 in [20].

Case 2: $R > 1$**.** In this situation, $g(a^*) < 1$ and $\varphi(x)/x$ is once again decreasing from $+\infty$ to $\lambda g(a^*)/(\lambda + R\gamma_M)$. A unique fixed point of φ therefore exists provided (20) holds - but this is guaranteed since $g(a^*) < 1$.

We summarize this in the following result.

THEOREM 2. *Let T_1 denote the time of the first jump of the stock, known to the investor from time $t = 0$. Problem 2 is ill posed if $0 < R < 1$ and*

$$\frac{\lambda g(a^*)}{\lambda + R\gamma_M} \geq 1.$$

Otherwise, the optimal controls are characterized as follows.
- $\bar{\theta}_t = \pi_M w_t$ *for* $0 \leq t < T_1$*;*
- $\bar{\theta}_{T_1} = a^* w_{T_1-}$ *;*
- $\bar{c}_t = f(T_1 - t)^{-1/R} w_t$ *for* $0 \leq t < T_1$,

where π_M is the Merton proportion (10), a^* is the maximizer in (13), f is the function given by (18), and $f(0)$ is the unique fixed point of the function φ given by (19).

The time T_1 is a renewal time, and the solution after T_1 conforms with the solution stated above for $[0, T_1]$ throughout the interval $[T_1, T_2]$, and recursively thereafter.

Proof By solving the HJB equation, we have identified what we believe is the optimal control for this problem. We verify that this is an optimal solution in the Appendix A.2. □

2.3. Solution of Problem 3. In the third version of the problem, the investor has some advance information about the magnitude ξ of the first jump of the Poisson random measure n, which comes at time T_1 about which he knows only the distribution. Specifically, the investor receives the signal

$$\eta = \xi + \varepsilon$$

where $\varepsilon \sim N(0, v_\varepsilon)$ independent of everything else. Note that for $\epsilon = 0$ there would be arbitrage. It is a routine calculation to derive the distribution of ξ given η:

$$(\xi|\eta) \sim N\left(\frac{v\eta + v_\varepsilon m}{v + v_\varepsilon}, \frac{v v_\varepsilon}{v + v_\varepsilon} \right). \tag{21}$$

Let $\mathbb{E}''$ denote the expected value for the second type of insider trader considered. The standard scaling argument tells us that the value function given the signal η will separate as

$$V(w;\eta) \equiv \sup_{(c,\theta)\in\mathcal{A}(w)} \mathbb{E}'' \left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \mid \eta, w_0 = w \right] = h(\eta)\, U(w),$$

for some function h to be determined. After the n^{th} jump at time T_n, the investor receives a new signal, independent of the past, about the magnitude of the $(n+1)^{\text{th}}$ jump at time T_{n+1}, and the process continues.

Consider what happens at time T_1. Let w denote the wealth of the investor just before T_1, and q the fraction of his wealth invested in the stock just before time T_1. The jump ξ in log-price causes the wealth of the investor to change to $w(1 + q(e^\xi - 1))$, so the value to the investor changes at time T_1+ from $h(\eta)U(w)$ to

$$(1 + q(e^\xi - 1))^{1-R}\, U(w)\, h(\eta'),$$

where η' is the new signal about the magnitude of the jump at time T_2. Thus the expected value to the investor at time T_1+ will be

$$U(w) \int (1 + q(e^x - 1))^{1-R}\, \mathbb{P}(dx|\eta) \int h(y)\mathbb{P}_0(dy), \tag{22}$$

where $\mathbb{P}$ denotes the distribution in (21) and $\mathbb{P}_0$ denotes the $N(m, v + v_\varepsilon)$ distribution. Introducing the notation

$$A_3 = \int h(y)\mathbb{P}_0(dy),$$

the value to the investor unconditional on η is $V(w) = A_3 U(w)$, and the expected value at time T_1+ is expressed as

$$A_3\, U(w) \int (1 + q(e^x - 1))^{1-R}\, \mathbb{P}(dx|\eta)\,;$$

of course, determining the value of A_3 is a major part of solving the problem. Analogously to (4), we define

$$g(q,\eta) \equiv \int (1 + q(e^x - 1))^{1-R}\, \mathbb{P}(dx|\eta),$$

so that the expected change in value at time T_1 can be more compactly expressed as

$$U(w)\{\, A_3\, g(q,\eta) - h(\eta)\, \}.$$

Using the MPOC once again, we derive similarly the HJB equation for this problem up until the time T_1:

$$\begin{aligned} 0 = \sup_{c,q} \big[\, & U(c) - \rho V + (rw - c + qw(\mu - r))V' + \tfrac{1}{2}\sigma^2 w^2 q^2 V'' \\ & + \lambda\, U(w)\{\, A_3\, g(q,\eta) - h(\eta)\, \}\,\big]. \end{aligned} \tag{23}$$

The optimization over c goes as before; we obtain

$$\bar{c} = h(\eta)^{-1/R}\, w.$$

Returning this to (23) leads to the equation

$$0 = \sup_q U(w)\Big[\, Rh(\eta)^{1-1/R} - h(\eta)\big(\, \rho + (R-1)\{r + q(\mu - r) - \\ - \tfrac{1}{2}\,\sigma^2 q^2 R\}\,\big) + \lambda\{\, A_3\, g(q,\eta) - h(\eta)\,\}\,\Big]. \tag{24}$$

At first sight, this looks hard to tackle, because when we attempt to optimize over q we do not know $h(\eta)$, or A_3. We shall deal with this using *value improvement*, by successively solving the optimization problem where we obtain a signal only for the first n jumps, $n = 0, 1, \ldots$. The usual scaling applies, and the value for the problem when we receive n signals in total will be of the form

$$V_n(w;\eta) = h_n(\eta)\, U(w).$$

Of course, when $n = 0$, we already have the solution - this is Problem 1, so we know that $h_0(\eta) \equiv A_1$, as given at (9). Clearly the more signals we get, the better we will be able to do, so for each η the sequence $h_n(\eta)$ is monotone, and therefore a limit $h_\infty(\eta) \equiv \lim_n h_n(\eta)$ exists. In the case $R \in (0,1)$, this limit might be infinite, in which case the problem is ill posed, but in the case $R > 1$ the value is non-positive, so the finite limit does exist.

The value-improvement solution is expressed in terms of a solution to a simpler problem, which we now present and solve. For simplicity of exposition, we assume from now on that

$$R > 1, \qquad \pi_M \in (0,1). \tag{25}$$

This avoids the need to discuss technical details; likely, the assumptions (25) are not needed, but realistic values of R are in any case above 1.5 [10, 15] so any interesting case is covered by (25).

A simpler problem. Suppose that wealth evolves as

$$dw_t = rw_t\, dt + q_t w_t(\sigma dW_t + (\mu - r)dt) - c_t dt \tag{26}$$

up until random time $T_1 \sim \exp(\lambda)$. At T_1, the log-price jumps by $\xi \sim N(m', v')$, so that

$$w_{T_1} = w_{T_1-}\{\, 1 + q(e^\xi - 1)\,\}, \tag{27}$$

where $q = q_{T_1-}$ is the fraction of wealth in the risky asset just before the jump. After T_1, everything stops, and the agent receives utility $BU(w_{T_1})$ for some constant $B > 0$. The first thing to do is to characterize the value to an ordinary trader with no information about ξ,

$$\tilde{V}(w) \equiv \tilde{V}(w; B, m', v') = \sup_{c,q} \mathbb{E}\left[\int_0^{T_1} e^{-\rho s} U(c_s)\, ds + e^{-\rho T_1} BU(w_{T_1}) \,\middle|\, w_0 = w \right], \tag{28}$$

and to prove that this is the value. The usual scaling argument will give us that

$$\tilde{V}(w; B, m', v') = H(B, m', v')\, U(w)$$

for some function H. The HJB equation for this problem is derived exactly as the HJB equation for Problem 1; we find that

$$0 = \sup_q U(w)\big[\, RH^{1-1/R} - (\rho + \lambda)H + (1-R)H\{r + q(\mu - r)\} - \\ - \tfrac{1}{2}\, R(1-R)H\sigma^2 q^2 + \lambda B \int (1 + q(e^x - 1))^{1-R}\, \mathbb{P}(dx|m', v')\,\big], \tag{29}$$

where $\mathbb{P}(dx|m',v')$ denotes the distribution $N(m',v')$. We express this more compactly as

$$0 = \sup_{0\le q\le 1}\left[\frac{RH^{1-1/R}}{1-R} + H\psi_1(q) + \lambda B\psi_2(q;m',v')\right], \tag{30}$$

where we define the functions

$$\psi_1(q) = r + q(\mu - r) - \tfrac{1}{2}\sigma^2 q^2 R - \frac{\rho+\lambda}{1-R},$$
$$\psi_2(q;m',v') = \int U(1+q(e^x-1))\,\mathbb{P}(dx|m',v'),$$

both concave functions of q. Notice the bounds

$$\psi_1(q) \le \alpha_1 \equiv \psi_1(\pi_M) > 0 \tag{31}$$
$$\psi_2(q;m',v') \le \alpha_2(m',v') \equiv \sup_{0\le y\le 1}\psi_2(y;m',v') < 0. \tag{32}$$

Stated equivalently, the HJB equation (30) to be solved is

$$\frac{H^{1-1/R}}{1-1/R} = \sup_{0\le q\le 1}\left[H\psi_1(q) + \lambda B\psi_2(q;m',v')\right]. \tag{33}$$

The left-hand side of (33) is a concave function of H, increasing from 0 to ∞ with first derivative approaching 0. The right-hand side is a convex function of H (the supremum of a family of linear functions) which is negative for H in a neighbourhood of zero because of (31), (32) and for large H grows at least as fast as $\alpha_1 H$, which we would get by taking $q = \pi_M$. Note that by assumption (25) that this is always permitted. The conclusion is that (33) has a solution, and moreover it is unique. We summarize the solution to this simpler problem in the following lemma.

Lemma 1. *Assume that $R>1$, $\pi_M \in (0,1)$ and a jump in log-price with size $N(m',v')$ occurs at time $T_1 \sim \exp(\lambda)$. The solution to the finite-horizon optimal control problem with associated value function (28) is*

$$\bar{c}_t = \bar{H}^{-1/R} w_t, \qquad \bar{\theta}_t = \bar{q} w_t$$

where the pair $\bar{q} = \bar{q}(B,m',v')$ and $\bar{H} = \bar{H}(B,m',v')$ is the solution to (33). The value function is then expressed as $\tilde{V}(w) = \bar{H}U(w)$.

Proof According to our previous argument, there is a unique solution $(\bar{q},\bar{H})$ to (33), and thus a unique solution to the problem (28). Of course, it remains to prove that the unique solution to the HJB equation actually gives the optimal policy and the value function for this simpler problem; we leave this routine but technical verification to the Appendix A.3. □

Back to Problem 3.

Let us consider what happens up until the time T_1 of the first jump when we are given $n+1$ signals. As we argued for (22), conditional on the past, the expected value to the investor at time T_1+ will be

$$U(w_{T_1-})\int (1+q(e^x-1))^{1-R}\,\mathbb{P}(dx|\eta)\int h_n(y)\mathbb{P}_0(dy),$$

where q is the fraction of wealth held in the risky asset at T_1-. Given the signal η about the jump ξ in log-price at time T_1, the distribution of ξ will be

$$\xi \sim N\left(\frac{v_0\eta + v_\varepsilon m_0}{v_0+v_\varepsilon}, \frac{v_0 v_\varepsilon}{v_0+v+\varepsilon}\right) \equiv N(m'(\eta),v').$$

So if we define

$$A_3^{(n)} \equiv \int h_n(y)\mathbb{P}_0(dy),$$

then the problem with $n+1$ signals is exactly the simpler problem when we have time-T_1 bequest $A_3^{(n)}U(w)$, and jump distribution $N(m'(\eta), v')$. Accordingly, the h_n are generated recursively by the relations

$$\frac{h_{n+1}(\eta)^{1-1/R}}{1-1/R} = \sup_{0\le q\le 1}\Big[h_{n+1}(\eta)\psi_1(q) + \lambda A_3^{(n)}\psi_2(q; m'(\eta), v')\Big], \tag{34}$$

$$A_3^{(n)} = \int h_n(y)\mathbb{P}_0(dy); \tag{35}$$

see (33). Since the value improves as we get more signals, it follows that $h_n(\eta) \ge h_{n+1}(\eta)$. Therefore the $A_3^{(n)}$ are also decreasing.

We summarize this as the following result.

THEOREM 3. *Assume that $R > 1$ and $\pi_M \in (0,1)$. The optimal control for Problem 3 has the form*

$$\bar{c}_t = (h(\eta_t))^{-1/R}\, w_t, \qquad \bar{\theta}_t = \bar{q}(\eta_t)\, w_t$$

where η_t is the signal known at time t about the next jump after time t, $\bar{q}(\eta)$ is the value of q which maximizes the function

$$q \mapsto h(\eta)\psi_1(q) + \lambda A_3\psi_2(q; m'(\eta), v')$$

and (h, A_3) are the maximal solution to the equation system

$$\frac{h(\eta)^{1-1/R}}{1-1/R} = \sup_{0\le q\le 1}\Big[h(\eta)\psi_1(q) + \lambda A_3\psi_2(q; m'(\eta), v')\Big], \tag{36}$$

$$A_3 = \int h(y)\mathbb{P}_0(dy) \tag{37}$$

subject to the condition that $A_3 \le A_1$. The value function is

$$V(w;\eta) = h(\eta)\, U(w).$$

Proof The equation system (36)-(37) has the trivial solution $h(\eta) \equiv 0$, $A_3 = 0$, so we do not have a unique solution. However, the solution we obtain by value improvement, the value of the problem, is maximal as claimed, by the following argument.

Suppose that $(\tilde{h}, \tilde{A}_3)$ solves (36)-(37), and that $\tilde{A}_3 \le A_1 \equiv A_3^{(0)}$. From (34) we discover that $\tilde{h}(\eta) \le h_1(\eta)$, which implies that $\tilde{A}_3 \le A_3^{(1)}$, from (34) and (36). Repeating the argument, we learn that $\tilde{h}(\eta) \le h_2(\eta)$ and $\tilde{A}_3 \le A_3^{(2)}$. In the limit, we have that $\tilde{h}(\eta) \le h(\eta)$ and $\tilde{A}_3 \le A_3$.

All that remains is to give the verification for Lemma 1, which is done in Appendix A.3. □

REMARK 1. One may consider the story where the agent knows the time and has a signal on the jump (i.e., a combination of Problems 2 and 3). However, this story would be very much like our Problem 2; at the instant of the jump one chooses the best portfolio for the jump, at other times the continuous investment/consumption story is employed.

3. Numerical results. In this Section, we compare the effects of different types of insider information. There are so many parameters at our disposal that we have to try to make restrictions which give us comparisons that can be interpreted; so throughout this Section we shall fix

$$r = 0.05, \qquad \rho = 0.15, \qquad R = 2. \tag{38}$$

We shall also suppose that whatever the dynamics considered, there is a common annualized growth rate $\bar{\mu}$ and a common annualized volatility $\bar{\sigma}$. Thus if we were considering the basic Black-Scholes model with no jumps, the stock would evolve as

$$dS_t = S_t\,(\,\bar{\sigma} dW_t + \bar{\mu} dt\,).$$

For this baseline situation, the value is known explicitly [20](equation (1.30)):

$$V_M(w) = A_M\, U(w),$$

where

$$A_M^{-1/R} = \gamma_M \equiv \left\{\, \rho + (R-1)(\,r + (\bar{\mu}-r)^2/2\bar{\sigma}^2 R\,)\,\right\}/R.$$

If we force the investor only to invest in the bank account by letting $\bar{\sigma} \to \infty$, then the limiting value of A_M, which we denote by A_r, turns out to be $A_r = 100$ if we use the values given in (38), as the reader may quickly verify.

What happens when we include the jumps? We will suppose that a fraction $\beta \in [0,1]$ of the annualized return and volatility comes from the jumps, which arrive at intensity λ. The parameters of the model (1) are therefore

$$\mu = (1-\beta)\bar{\mu}, \qquad \sigma = \bar{\sigma}\sqrt{1-\beta}, \qquad m = \beta\bar{\mu}/\lambda, \qquad v = \beta\bar{\sigma}^2/\lambda.$$

By definition, for the problems considered in this work, the value at time 0 with initial wealth w is given by $A_k U(w)$ for $k = 1,2,3$. For the two types of insider traders, this value is the maximum expected value of $\int_0^\infty e^{-\rho t} U(c_t) dt$ before the insiders receive information about the first jump. Since $AU(w) = U(A^{1/(1-R)} w)$, we report the normalized values

$$\omega_k \equiv \left(\frac{A_k}{A_M}\right)^{1/(1-R)}$$

instead of A_k for $k = 1,2,3$. ω_k is the value of a dollar in the k-th problem. For $R = 2$, $\omega_k = A_M/A_k$. For Problem 3, we always use $v_\epsilon = 0.5v$. A_M, A_1 are obtained by direct calculations and A_2, A_3 are computed by value improvement.

The numerical results are presented in Table 1. In each block, the annualized return and volatility are fixed. A_M and A_1 represent two situations with no insider information but different stochastic structures. Hence ω_1 tends to be close to 1 but it may or may not be greater than 1. ω_2 and ω_3 are always greater than ω_1 since they represent two insider situations with the same stochastic structure as Problem 1, i.e., insiders can convert one dollar to more "utility" than ordinary investors. With all the other parameters fixed, ω_2 (or ω_3) is greater for $\beta = 0.5$ than for $\beta = 0.1$. An interesting observation is that ω_3 seems significantly greater than ω_2. One plausible reason is that the jump, on the average, tends to be neutral (recall $m = \beta\bar{\mu}/\lambda$). Therefore, the insider information for the time of the jump does not generate a large profit. On the contrary, knowing the magnitude of each jump, albeit with noise, is far more valuable. There may indeed be some large jumps from which insiders can make a big profit. Moreover, we notice that ω_3 grows quickly with λ while ω_2 does not. Intuitively, it is difficult to predict whether a larger λ benefits the insiders, because though the insider information comes more frequently, both m and v become closer to 0. A plausible reason for this observation is that, for the second type of insiders, as λ grows larger, their ignorance of the arrival time of the jump hurts them much less. For the first type of insiders, as m decreases, their insider information does not constitute much of an advantage over ordinary investors.

$\bar{\mu}$	$\bar{\sigma}^2$	A_M	β	λ	ω_1	ω_2	ω_3
			0.1	1	1.005	1.034	1.123
			0.1	2	1.005	1.034	1.211
0.07	0.05	98.030	0.1	4	1.005	1.034	1.351
			0.5	1	1.033	1.236	1.448
			0.5	2	1.033	1.237	1.712
			0.5	4	1.033	1.238	2.124
			0.1	1	1.005	1.033	1.107
			0.1	2	1.005	1.033	1.168
0.07	0.02	95.181	0.1	4	1.005	1.033	1.258
			0.5	1	1.026	1.316	1.309
			0.5	2	1.027	1.319	1.466
			0.5	4	1.027	1.320	1.702
			0.1	1	1.012	1.037	1.138
			0.1	2	1.012	1.037	1.229
0.1	0.05	88.581	0.1	4	1.012	1.037	1.368
			0.5	1	1.064	1.242	1.467
			0.5	2	1.066	1.246	1.722
			0.5	4	1.067	1.247	2.116
			0.1	1	0.998	0.999	1.036
			0.1	2	0.998	1.000	1.077
0.1	0.02	74.799	0.1	4	0.998	1.000	1.145
			0.5	1	1.024	1.114	1.187
			0.5	2	1.028	1.118	1.313
			0.5	4	1.030	1.120	1.505

TABLE 1. The numerical values for $\omega_1, \omega_2, \omega_3$ in different settings.

APPENDIX

A.1. Problem 1 Verifications The verification proofs for Problem 1 can be found in [7]. Nonetheless, we provide them here in our notation for use in the verifications of Problems 2 and 3. By letting $\theta_t = q_t w_t$, we rewrite (2) as

$$dw_t = [(r + q_t(\mu - r))w_t - c_t]\,dt + \sigma q_t w_t\,dW_t + q_t w_{t-} \int (e^x - 1) n(dt, dx).$$

Recall that $\bar{c}_t = f(T_1 - t)^{-1/R}\, w_t$ for $0 \le t < T_1$. When $q_t = \bar{q}_1$ and $c_t = \bar{c}_t$, the solution is given by

$$\bar{w}_t = w_0 \exp[\,\bar{q}_1 \sigma\, W_t + (r + \bar{q}_1(\mu - r) - \tfrac{1}{2}\,\bar{q}_1^2 \sigma^2 - A_1^{-1/R})\,t\,] \prod_{i=1}^{J_t} [1 + \bar{q}_1(e^{X_i} - 1)].$$

where J_t denotes the the number of jumps up to time t and the X_i are the jumps in the log-price. Guided by [4], we study the state-price density process ζ, which is the marginal utility of optimal consumption:

$$\zeta_t = e^{-\rho t} U'(\bar{c}_t) = A_1 e^{-\rho t} \bar{w}_t^{-R}.$$

We shall confirm the following properties:

(P1.1) for any admissible control (c_t, q_t), the stochastic process

$$Z_t = \zeta_t w_t + \int_0^t \zeta_s c_s\, ds$$

is a nonnegative supermartingale;

(P1.2) $\mathbb{E}[\int_0^\infty \zeta_t \bar{c}_t\, dt] = \zeta_0 w_0$.
Given these, we have

$$\mathbb{E}\left[\int_0^\infty \zeta_t \bar{c}_t\, dt\right] = \zeta_0 w_0 = Z_0 \geq \mathbb{E}[Z_\infty] \geq \mathbb{E}\left[\int_0^\infty \zeta_t c_t\, dt\right]. \tag{A.1}$$

By the concavity of U,

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt\right] &\leq \mathbb{E}\left[\int_0^\infty e^{-\rho t}\{U(\bar{c}_t) + (c_t - \bar{c}_t)U'(\bar{c}_t)\}\, dt\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(\bar{c}_t)\, dt\right] + \mathbb{E}\left[\int_0^\infty \zeta_t (c_t - \bar{c}_t)\, dt\right].\end{aligned}$$

From (A.1) we conclude that

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt\right] \leq \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(\bar{c}_t)\, dt\right],$$

proving the optimality of $(\bar{c}, \bar{q})$.

Now we must verify (P1.1) and (P1.2). By routine calculations, the dynamics of ζ_t are given by

$$\begin{aligned} d\zeta_t/\zeta_{t-} = &- \rho\, dt + R\{-r - \bar{q}_1(\mu - r) + A_1^{-1/R} + \tfrac{1}{2}(R+1)\sigma^2 \bar{q}_1^2\}dt - \\ &- R\bar{q}_1 \sigma\, dW_t + \int \{(1 + \bar{q}_1(e^x - 1))^{-R} - 1\} n(dt, dx).\end{aligned}$$

By Itô's lemma,

$$\begin{aligned} dZ_t &= w_{t-} d\zeta_t + \zeta_{t-} dw_t + d\zeta_t dw_t + \zeta_t c_t dt \\ &\doteq \zeta_t w_t \big[\, (1-R)r - \rho + RA_1^{-1/R} + (q_t - R\bar{q}_1)(\mu - r) + \tfrac{1}{2} R(1+R)\sigma^2 {\bar{q}_1}^2 \\ &\quad - \sigma^2 R q_t \bar{q}_1 + \lambda \int \big(\{1 + \bar{q}_1(e^x - 1)\}^{-R}\{1 + q_t(e^x - 1)\} - 1\big)\, p(x)dx \,\big]\, dt \\ &= \zeta_t w_t \big[\, RA_1^{-1/R} + (1-R)\{r + \bar{q}_1(\mu - r) - \tfrac{1}{2}\sigma^2 R \bar{q}_1^2\} - \rho + \lambda(g(\bar{q}_1) - 1) \\ &\quad + (q_t - \bar{q}_1)\big\{\, \mu - r - \sigma^2 R \bar{q}_1 + \lambda \int (1 + \bar{q}_1(e^x - 1))^{-R}(e^x - 1)\, p(x)dx \,\big\}\big] dt \\ &= \zeta_t w_t (q_t - \bar{q}_1)\big[\, \mu - r - \sigma^2 R \bar{q}_1 + \lambda \int (1 + \bar{q}_1(e^x - 1))^{-R}(e^x - 1)\, p(x)dx \,\big] dt.\end{aligned}$$

The second last equality can be derived using $q_t = \bar{q}_1 + (q_t - \bar{q}_1)$ and the last equality follows from (9). Using the function g_1 defined in (8), the drift term can be further simplified to

$$dZ_t \doteq \zeta_t w_t (q_t - \bar{q}_1) g_1'(\bar{q}_1) dt.$$

Recall that $\bar{q}_1$ is the maximizer of g_1, so if $\bar{q}_1 \in (0,1)$ the derivative $g_1'(\bar{q}_1) = 0$. If $\bar{q}_1$ is an endpoint, it is not difficult to see that $(q_t - \bar{q}_1) g_1'(\bar{q}_1) \leq 0$. Hence the drift must be nonpositive, which implies that Z_t is a supermartingale.

Lastly, we verify $\mathbb{E}[\int_0^\infty \zeta_t \bar{c}_t dt] = \zeta_0 w_0$. Direct calculation gives us

$$\begin{aligned}\mathbb{E}[\zeta_t \bar{w}_t] =& A_1 w_0^{1-R} \exp\left[\frac{1-R}{R}\{r + (\mu - r)\bar{q}_1 - \frac{1}{1-R}\rho - \frac{1}{2} R\sigma^2 \bar{q}_1^2 + \frac{\lambda}{1-R}(g(\bar{q}_1) - 1)\} t\right] \\ =& \zeta_0 w_0 \exp(-A_1^{-1/R} t).\end{aligned}$$

Then, using $\bar{c}_t = A_1^{-1/R} \bar{w}_t$, we obtain

$$\mathbb{E}\left[\int_0^\infty \zeta_t \bar{c}_t dt\right] = \int_0^\infty \mathbb{E}[\zeta_t \bar{c}_t] dt = \zeta_0 w_0 \int_0^\infty A_1^{-1/R} \exp(-A_1^{-1/R} t) dt = \zeta_0 w_0.$$

A.2. Problem 2 Verifications Recall that in Problem 2, the jumps come at times $T_1, T_2, \ldots$ which are the times of a Poisson process of rate λ ($T_0 = 0$ by convention). The wealth evolves as:

$$dw_t = w_{t-}\left[rdt + q_t(\sigma dW_t + (\mu - r)dt) + a_t \int (e^x - 1)n(dt, dx)\right] - c_t dt, \tag{A.2}$$

where (q, c) is the control, and (a_t) is the choice of portfolio at the time of the jump. We conjecture that $\bar{q}_t = \pi_M$ for all t,

$$\bar{c}_t = \bar{w}_t f(T_j - t)^{-1/R} \quad \text{for} \quad T_{j-1} \le t < T_j, \tag{A.3}$$

$$a_t = a^* \equiv \operatorname{argmax} \frac{g(a)}{(1-R)}, \tag{A.4}$$

where

$$g(a) \equiv \int \{1 + a(e^x - 1)\}^{1-R} p(x)dx \tag{A.5}$$

and $\mathbb{E}[n(dt, dx)] = \lambda dt p(x)dx$. We conjecture that the value function at time t will be

$$V(t, w_t; T_j) = f(T_j - t)U(w_t) \quad (T_{j-1} \le t < T_j),$$

where f solves

$$0 = Rf^{1-1/R} - \gamma_M R f - \dot{f} \tag{A.6}$$

with the boundary condition

$$f(0) = g(a^*) \int_0^\infty \lambda e^{-\lambda s} f(s) ds.$$

This leads us to believe that the marginal utility $V_w(t, \bar{w}_t) \equiv \zeta_t$ will be the required state-price density,

$$\zeta_t = \sum_{j \ge 1} I_{\{T_{j-1} \le t < T_j\}} f(T_j - t)(\bar{w}_t)^{-R} e^{-\rho t}. \tag{A.7}$$

From the definition, it is immediate that:

$$e^{-\rho t} U'(\bar{c}_t) = \zeta_t.$$

We may now proceed with the verification proof, which requires two propositions.

PROPOSITION A.1. *With ζ defined as (A.7), for any feasible control we have:*

$$\zeta_t w_t + \int_0^t \zeta_s c_s ds$$

is a non-negative supermartingale.

Proof The processes ζ and $\bar{w}$ will be continuous except at the jump times T_j. Without loss of generality, we look at $(0,T_1)$, where we have:

$$\frac{d\bar{w}_t}{\bar{w}_t} = r\,dt + \pi_M(\sigma dW + (\mu - r)dt) - f(T_1 - t)^{-1/R}dt.$$

Hence the evolution of $\zeta_t \equiv e^{-\rho t} f(T_1 - t)\bar{w}_t^{-R}$ is given by

$$\begin{aligned}\frac{d\zeta_t}{\zeta_t} &= \Big[-\rho - \frac{\dot{f}(T_1-t)}{f(T_1-t)} - R(r + \pi_M(\mu - r) - f(T_1-t)^{-1/R}) + \\ &\qquad + \tfrac{1}{2} R(R+1)\sigma^2\pi_M^2 \Big]\, dt + \pi_M \sigma\, dW_t \\ &= -r\, dt - R\sigma\pi_M\, dW_t \end{aligned} \tag{A.8}$$

where we pass to the final line by using (A.6) and the explicit expression (17) for γ_M. In terms of the Sharpe ratio $\kappa \equiv (\mu - r)/\sigma$, this is more cleanly expressed as

$$d\zeta_t/\zeta_t = -r\, dt - \kappa\, dW_t$$

between jump times. At time T_j, the log price jumps by X_j, and so

$$\begin{aligned}\bar{w}(T_j) &= \bar{w}(T_j-)\left\{1 + a^*(e^{X_j} - 1)\right\} \\ \zeta(T_j) &= \zeta(T_j-)\left\{1 + a^*(e^{X_j} - 1)\right\}^{-R} f(T_{j+1} - T_j)/f(0).\end{aligned} \tag{A.9}$$

If we were following some other feasible rule that chose a fraction $a \in [0,1]$ of wealth to invest in the stock at time T_j, then

$$\begin{aligned}&\mathbb{E}\left[(\zeta w)_{T_j} \,|\, \mathcal{F}_{T_j-}\right] \\ &= (\zeta w)_{T_j-} \int (1 + a^*(e^x - 1))^{-R}(1 + a(e^x-1))p(x)dx \int_0^\infty \lambda e^{-\lambda t} f(t)dt/f(0) \\ &= \frac{(\zeta w)_{T_j-}}{g(a^*)} \int (1 + a^*(e^x - 1))^{-R}(1 + a(e^x-1))p(x)dx.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathbb{E}\left[(\zeta w)_{T_j} - (\zeta w)_{T_j-} \,|\, \mathcal{F}_{T_j-}\right]}{(\zeta w)_{T_j-}} &= \frac{a - a^*}{g(a^*)} \int (1 + a^*(e^x - 1))^{-R}(e^x - 1)p(x)dx \\ &= \frac{(a - a^*)g'(a^*)}{g(a^*)(1-R)} \le 0,\end{aligned}$$

since a^* maximizes $g(a)/(1-R)$ on $[0,1]$ from formulas (A.4) and (A.5). Hence the process

$$\zeta_t w_t + \int_0^t \zeta_s c_s ds$$

is a local supermartingale for any feasible (c,w). Away from the jump times, this can be verified by the usual Itô verification as we did in Appendix A.1; at the jump times we have the expected change is less than or equal to 0, by what we have proven above. □

Proposition A.2. *With the optimal consumption process $\bar{c}_t$ and the optimal wealth process $\bar{w}$ defined at (A.2), (A.3), (A.4), and ζ as at (A.7),*

$$M_t^* \equiv \zeta_t \bar{w}_t + \int_0^t \zeta_s \bar{c}_s\, ds,$$

is a non-negative martingale, with $\zeta_0 w_0 = \mathbb{E}\left[\int_0^\infty \zeta_s \bar{c}_s ds\right]$.

Proof Notice that the f used in Proposition A.1 is bounded away from zero; for some $A > 0$,

$$A^{-1} \leq f(t) \leq A \quad \text{for all } t.$$

We have (see (A.9)) that

$$\frac{(\zeta \bar{w})_{T_j}}{(\zeta \bar{w})_{T_j -}} = \left\{1 + a^*(e^{X_j} - 1)\right\}^{1-R} \frac{f(T_{j+1} - T_j)}{f(0)}$$

and between jumps

$$d(\zeta_t \bar{w}_t) = -\zeta_t \bar{c}_t dt + \zeta_t \bar{w}_t \, b \, dW_t, \qquad T_j < t < T_{j+1},$$

where $b = \kappa(1-R)/R$. Now if Z_t solves

$$dZ_t = bZ_t \, dW_t, \;\; Z_0 = 1,$$

we find that

$$d\left(\frac{\zeta_t \bar{w}_t}{Z_t}\right) = \frac{\zeta_t \bar{w}_t}{Z_t}\left(-\frac{\bar{c}_t}{\bar{w}_t}\right) dt,$$

from which we conclude that

$$\frac{\zeta_t \bar{w}_t}{Z_t} = \zeta_0 w_0^* \exp\left(-\int_0^t \frac{\bar{c}_s}{\bar{w}_s} ds\right) \prod_{T_j \leq t} \left\{1 + a^*(e^{X_j} - 1)\right\}^{1-R} \frac{f(T_{j+1} - T_j)}{f(0)}. \tag{A.10}$$

The aim is to prove that for any t,

$$\sup_{0 \leq s \leq t} \zeta_s \bar{w}_s \in L^1,$$

which will guarantee that the process $M_t^* \equiv \zeta_t \bar{w}_t + \int_0^t \zeta_s \bar{c}_s ds$ is not just a local martingale but also a martingale. This is straightforward: $\bar{c}_s / \bar{w}_s$ is $f(T_j - s)^{-1/R}$ and so it is bounded both above and below. The process Z is an exponential Brownian motion and thus $\sup_{0 \leq s \leq t} Z_s \in L^1$. This just leaves the product term in (A.10), which is bounded above by

$$\tilde{\pi}_t \equiv \prod_{T_j \leq t} \max\{1, \left(1 + a^*(e^{X_j} - 1)\right)^{1-R}\} K,$$

where $K = \sup\{f(t)/f(0)\} \in (1, \infty)$. Every time a jump occurs, the product $\tilde{\pi}_t$ gets an additional independent factor with finite expectation

$$\Delta \equiv \mathbb{E}\left[\max\left\{1, \left(1 + a^*(e^{X_j} - 1)\right)^{1-R}\right\} K\right],$$

so

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} \tilde{\pi}_s\right] = \mathbb{E}\left[\tilde{\pi}_t\right] = \sum_{n > 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \Delta^n = \exp\left(\lambda t(\Delta - 1)\right),$$

which is finite.

The final assertion to prove is that $\mathbb{E}\left[\zeta_t \bar{w}_t\right] \to 0$ as $t \to \infty$. We have $\mathbb{E}\left[Z_t\right] = 1$ for all t, and Z_t is independent of

$$Y_t \equiv \exp\left[-\int_0^t \frac{\bar{c}_s}{\bar{w}_s} ds\right] \prod_{T_j \le t} \left\{1 + a^*(e^{X_j} - 1)\right\}^{1-R} \frac{f(T_{j+1} - T_j)}{f(0)}.$$

We observe that

$$\begin{aligned}
\int_{T_{j-1}}^{T_j} \frac{\bar{c}_s}{\bar{w}_s} ds &= \int_{T_{j-1}}^{T_j} f(T_j - s)^{-1/R} ds && \text{(from (A.3))} \\
&= \int_{T_{j-1}}^{T_j} \left\{\gamma_M + \frac{\dot{f}(T_j - s)}{Rf(T_j - s)}\right\} ds && \text{(from (A.6))} \\
&= \gamma_M (T_j - T_{j-1}) + \frac{1}{R} \log \frac{f(T_j - T_{j-1})}{f(0)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}[Y(T_j)] &= \mathbb{E}\left[\exp(-\gamma_m T_j) \prod_{i=1}^{j} \left(1 + a^*(e^{X_i} - 1)\right)^{1-R} \prod_{i=1}^{j} \left(\frac{f(T_i - T_{i-1})}{f(0)}\right)^{1-1/R} \frac{f(T_{j+1} - T_j)}{f(0)}\right] \\
&= g(a^*)^j \frac{1}{g(a^*)} \left(\int_0^\infty \lambda e^{-\lambda t - \gamma_M t} \left(\frac{f(t)}{f(0)}\right)^{1-1/R} dt\right)^j.
\end{aligned}$$

Notice that

$$g(a^*) = f(0) \Big/ \int_0^\infty \lambda e^{-\lambda s} f(s)\, ds,$$

so, by (18),

$$\begin{aligned}
\beta &\equiv g(a^*) \int_0^\infty \lambda \exp\{-\lambda t - \gamma_M t\} \left(f(t)/f(0)\right)^{1-1/R} dt \\
&= \frac{\int_0^\infty \lambda \exp\{-\lambda t - \gamma_M t\} f(t) \left(f(0)/f(t)\right)^{1/R} dt}{\int_0^\infty \lambda e^{-\lambda t} f(t)\, dt} \\
&= \frac{\int_0^\infty \lambda e^{-\lambda t} f(t) \left\{1 + \frac{1 - \exp\{-\gamma_M t\}}{\gamma_M \exp\{-\gamma_M t\} f(0)^{1/R}}\right\}^{-1} dt}{\int_0^\infty \lambda \exp\{-\lambda t\} f(t)\, dt} \\
&< 1.
\end{aligned}$$

Thus $\mathbb{E}[Y(T_j)] \to 0$.

Now observe that Y_t is decreasing on each interval $[T_j, T_{j+1}]$, so $\sup_{t \ge T_j} Y_t = \sup_{k \ge j} Y(T_k)$ and therefore:

$$\begin{aligned}
\mathbb{E}\left[\sup_{t \ge T_j} Y_t\right] &\le \mathbb{E}\left[\sup_{k \ge j} Y(T_k)\right] \\
&\le \sum_{k \ge j} \mathbb{E}[Y(T_k)] \\
&\le C\beta^j,
\end{aligned}$$

for a constant C.

We have

$$\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}[Y_t : t < T_j] + \mathbb{E}[Y_t : t \ge T_j] \\
&\le \mathbb{E}[Y_t : t < T_j] + \mathbb{E}\left[\sup_{u \ge T_j} Y_u\right],
\end{aligned}$$

so given $\epsilon > 0$ we can choose j sufficiently large such that $\mathbb{E}\left[\sup_{u\geq T_j} Y_u\right] \leq \epsilon/2$. As for the first term, we see that

$$\begin{aligned}
Y_t &= \exp\left[-\int_0^t \frac{\bar{c}_s}{\bar{w}_s} ds\right] \prod_{T_j \leq t} \left(1 + a^*(e^{X_j} - 1)\right)^{1-R} \frac{f(T_{j+1} - T_j)}{f(0)} \\
&\leq \prod_{T_j \leq t} \left(1 + a^*(e^{X_j} - 1)\right)^{1-R} \frac{f(T_{j+1} - T_j)}{f(0)}.
\end{aligned}$$

Thus, all the way until T_j, the process Y is bounded by $\prod_{i=1}^j \left(1 + a^*(e^{X_i} - 1)\right)^{1-R} K$ (since f is bounded). Since the jump sizes X_j are independent of the jump times, we have

$$\mathbb{E}\left[Y_t : t < T_j\right] \leq K \left\{\mathbb{E}\left[1 + a^*(e^{X_i} - 1)\right]^{1-R}\right\}^j \mathbb{P}\left(t < T_j\right).$$

We now may choose t sufficiently large such that the right-hand side of the above display is less than $\epsilon/2$. Hence $\mathbb{E}\left[Y_t\right] = \mathbb{E}\left[\zeta_t \bar{w}_t\right] \to 0$, and $\zeta_0 w_0 = \mathbb{E}\left[\int_0^\infty \zeta_t \bar{c}_s ds\right]$ as required. □

A.3. Problem 3 Verifications

Verification for the simpler problem Recall that we assume $R > 1$, $\pi_M \in (0,1)$. The agent chooses previsible portfolio proportions (q_t) and non-negative optional consumption process (c_t), resulting in wealth evolution

$$dw_t = w_{t-}\left[rdt + q_t\left\{\sigma dW_t + (\mu - r)dt + (e^\xi - 1)dJ_t\right\}\right] - c_t dt, \tag{A.11}$$

where $J_t = I_{\{t\geq T_1\}}$ and $T_1 \sim \exp(\lambda)$, independent of everything else, is the time at which the log-price undergoes a jump $\xi \sim N(m', v')$, The initial wealth is denoted by $w_0 = w$, and the set $\mathcal{A}(w)$ of admissible pairs (c,q) is the set of pairs for which $w_t \geq 0$ and $q_t \in [0,1]$. The objective is to obtain the value

$$\tilde{V}(w) \equiv \sup_{(c,q)\in\mathcal{A}(w)} \mathbb{E}\left[\int_0^{T_1} e^{-\rho s} U(c_s)\, ds + e^{-\rho T_1} BU(w_{T_1})\right],$$

where $B > 0$ is constant. The HJB equation to be solved is

$$\frac{H^{1-1/R}}{1 - 1/R} = \sup_{0\leq q\leq 1}\left[H\psi_1(q) + \lambda B\psi_2(q; m', v')\right], \tag{A.12}$$

where

$$\begin{aligned}
\psi_1(q) &= r + q(\mu - r) - \tfrac{1}{2}\sigma^2 q^2 R - \frac{\rho + \lambda}{1 - R}, \\
\psi_2(q; m', v') &= \int U(1 + q(e^x - 1))\, \mathbb{P}(dx|m', v'),
\end{aligned}$$

and there is a unique solution $(\bar{q}, \bar{H})$ to (A.12). We conjecture that the marginal utility

$$\begin{aligned}
\zeta_t &= e^{-\rho t}\, \bar{c}_t^{-R} I_{\{t<T_1\}} + e^{-\rho T_1} B\bar{w}_{T_1}^{-R} I_{\{t\geq T_1\}} \\
&= e^{-\rho t}\, \bar{H}\, \bar{w}_t^{-R} I_{\{t<T_1\}} + e^{-\rho T_1} B\bar{w}_{T_1}^{-R} I_{\{t\geq T_1\}}
\end{aligned}$$

will serve as a state-price density, where the optimal consumption process $\bar{c}_t = \bar{H}^{-1/R}\bar{w}_t$. The conjectured optimal wealth process evolves as

$$d\bar{w}_t = \bar{w}_{t-}\left[\,(r + \bar{q}(\mu - r) - \bar{H}^{-1/R})\, dt - \sigma\bar{q}\, dW_t + \bar{q}(e^\xi - 1)dJ_t\,\right]. \tag{A.13}$$

Routine calculations give us

$$\frac{d\zeta_t}{\zeta_{t-}} = -\rho dt - R(r+\bar{q}(\mu-r) - \bar{H}^{-1/R})dt + \tfrac{1}{2}R(1+R)\sigma^2\bar{q}^2 dt - \\ -R\sigma\bar{q}\,dW + \left\{\,\bar{H}^{-1}B(1+\bar{q}(e^{\xi}-1))^{-R} - 1\,\right\} dJ_t.$$

As before, we aim to show that for any admissible (c,q)

$$Z_t = \zeta_t w_t + \textstyle\int_0^t \zeta_s c_s\, ds \text{ is a supermartingale}$$

and is a martingale under control $(\bar{c},\bar{q})$. Itô's formula and routine simplifications give

$$\begin{aligned} dZ_t &= \zeta_{t-}dw_t + w_{t-}d\zeta_t + dw_t d\zeta_t + \zeta_t c_t dt \\ &\doteq \zeta_t w_t\Big[\,(q_t-\bar{q})\big\{(\lambda B/\bar{H})\int (1+\bar{q}(e^x-1))^{-R}(e^x-1)\,\mathbb{P}(dx|m',v') + \\ &\qquad +\mu - r - \sigma^2 R\bar{q}\,\big\} + (1-R)(r+\bar{q}(\mu-r) - \tfrac{1}{2}R\sigma^2\bar{q}^2) - (\rho+\lambda) + \\ &\qquad\quad + R\bar{H}^{-1/R} + (\lambda B/\bar{H})\int (1+\bar{q}(e^x-1))^{1-R}\,\mathbb{P}(dx|m',v')\,\Big]\,dt \\ &= \zeta_t w_t(q_t-\bar{q})\Big[\,(\lambda B/\bar{H})\int (1+\bar{q}(e^x-1))^{-R}(e^x-1)\,\mathbb{P}(dx|m',v') + \\ &\qquad +\mu - r - \sigma^2 R\bar{q}\,\Big]\,dt \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.14)} \\ &= \zeta_t w_t\, dt\,(q_t-\bar{q})\big\{\,\bar{H}\psi_1'(\bar{q}) + \lambda B\psi_2'(\bar{q};m',v')\,\big\}/\bar{H} \qquad\qquad\qquad\qquad\qquad\qquad \text{(A.15)} \end{aligned}$$

where many terms disappear in the step to (A.14) because of the HJB equation (A.12). The final form of the drift in (A.15) is non-positive because $\bar{q}$ is the maximiser for (A.12). If we use $q_t = \bar{q}$, the drift is clearly zero, so Z is then a local martingale. To finish, we need to show that for any $t > 0$

$$\sup_{0\le s\le t} Z_s \in L^1.$$

The argument is similar to that used for Problem 2, but easier, so we leave the details to the reader.

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