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# OPTION PRICING WITH MARKOV-MODULATED DYNAMICS*

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**Abstract.** Markov-modulated models for equity prices have recently been extensively studied in the literature. In this paper, we apply some old results on the Wiener–Hopf factorization of Markov processes to a range of option-pricing problems for such models. The first example is the perpetual American put, where the exact (numerical) solution is obtained without discretizing any PDE. We then show how the methodology of Rogers and Stapleton [*Finance Stoch.*, 2 (1997), pp. 3–17] can be used to tackle finite-horizon problems and illustrate the methodology by pricing European, American, single barrier, and double barrier options under Markov-modulated dynamics.

**Key words.** Markov-modulated, Markov-chain, option, Black–Scholes model, Wiener–Hopf factorization, risk-sensitive control, regime switching

**AMS subject classifications.** 15A23, 15A24, 60J27, 60J70, 93E20

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## 1. Introduction.

Though outstandingly successful as a leading-order model for an asset price, the familiar log-Brownian paradigm fails in various ways, such as the fact that implied volatility is not constant. Among the many attempted variations and extensions, one of the most natural is to allow the dynamics of the underlying process to be a log-Brownian motion whose volatility and rate of return are stochastic in some way. Allowing the volatility to be stochastic is the central theme of the extensive literature on *stochastic volatility modeling*, of which [24, 15, 13, 2, 14] make up a small sample. Allowing the rate of return to be stochastic is of relevance to portfolio optimization, but not to asset pricing,[1] and the literature on *risk-sensitive optimal control* develops this theme in various ways; see, for example [5, 6, 3, 20].

Perhaps the simplest way to introduce additional randomness into the standard log-Brownian model is to let the volatility and rate of return be functions of a finite-state Markov chain; we can imagine that such a model might describe regime-switching behavior of some kind, perhaps related to the business cycle, or other economic indicators. The terms *regime-switching* and *Markov-modulated dynamics* are used to describe such models, and there are already interesting contributions here, such as applications to option pricing [12, 11, 10, 9, 8, 4, 26], portfolio optimization [27, 25], and optimal trading strategies [28]. In applications, it is likely that the number of states of the Markov chain will be small (otherwise estimation becomes a problem), and it is then natural to think of such a model as "nearly" a log-Brownian motion, with occasional parameter shifts. Some explicit solutions can be found for a two-state Markov chain, but as the problems get harder we are soon led into PDE-related numerical methods (smoothed approximation of boundary conditions [26], two-point boundary value problems [28], discretization of associated dynamic programming equations [12]). The coupled PDEs which arise in these models will rarely be soluble in closed form, though finite-difference methods are still quite competitive.

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[1] Of course, for asset pricing we change measure so that the rate of return becomes the riskless rate.

This paper approaches such models from a different direction: by viewing the Markov-modulated asset as nearly a Markov chain. This approach relies on some old work on Wiener–Hopf factorization of Markov processes (and in particular, Markov chains) dating back to the paper of Barlow, Rogers, and Williams [1] from 1980; the focus is on level-crossings of the asset price process. We find a quite different toolkit applies in this approach, namely, linear algebra; this leads to numerical schemes that are very efficient and quite able to handle moderate-sized problems, which we will illustrate by pricing a perpetual American put[2] on such an asset. Let us emphasize immediately a key difference between the present approach and the traditional PDE approach: here we shall be obtaining (numerically) *exact* solutions to the problem, *not* just approximations. There is no need to solve any dynamic programming equation or discretize any PDE.

Next, we move to a pricing framework with finite time horizon and Markov regimes. The approach here extends the methodology developed by Rogers and Stapleton [22] for the standard log-Brownian model and is illustrated by pricing the European call, the American put, and finally double barrier options.

**2. General setup and noisy Wiener–Hopf factorization.** The stock price is modeled as

$$dS_t = S_t[\sigma(\xi_t)dW_t + r(\xi_t)dt], \tag{1}$$

where $W_t$ is a standard Brownian motion, $r$ denotes as usual the risk-free interest rate, $\sigma$ denotes the Markov-modulated volatility of the stock, and $\xi$ is an irreducible Markov chain with values in the finite set $I$, $|I| = d$. Notice that the riskless rate may vary with the underlying Markov chain. The log price $X_t = \log(S_t)$ then satisfies

$$dX_t = \sigma(\xi_t)dW_t + \left[r(\xi_t) - \frac{1}{2}\sigma(\xi_t)^2\right]dt, \tag{2}$$

which can be rewritten as, say,

$$dX_t = \sigma(\xi_t)dW_t + v(\xi_t)dt. \tag{3}$$

The idea of the Wiener–Hopf factorization approach is to study the crossings back and forth over levels of $X$. To help in this, define for $t \geq 0$[3]

$$\tau_t^{\pm} \equiv \inf\{u : \pm X_u > t\}. \tag{4}$$

We aim to characterize the distribution of the times $\tau_t^{\pm}$ and the law of the chain at these times, and to do this we will seek martingales $M_t^f$ of the following form:

$$M_t^f = \exp\left(-\int_0^t r(\xi_u)du\right) f(\xi_t, X_t) \tag{5}$$

for some function $f$. Itô's formula gives, up to a local martingale part,

$$dM_t^f \doteq \exp\left(-\int_0^t r(\xi_u)du\right)\left[(Q-R)f + \frac{1}{2}\Sigma f_{XX} + V f_X\right]dt, \tag{6}$$

[2] This problem was solved for a two-state chain by Guo and Zhang [12].

[3] With the usual convention that $\inf(\emptyset) = \infty$.

where $R$ is the diagonal matrix whose $i$th diagonal element is equal to $r(i)$, $\Sigma$ is the diagonal matrix whose $i$th diagonal element is equal to $\sigma(i)^2$, and $V = R - \frac{1}{2}\Sigma$. We therefore require

$$(Q - R)f + \frac{1}{2}\Sigma f_{XX} + V f_X = 0. \tag{7}$$

Seeking separable $f$ of the form $f(\xi_t, X_t) = g(\xi_t)\exp(-\lambda X_t)$ gives rise to the following equation to be solved in $\lambda$ and $g$:

$$(Q - R)g + \frac{1}{2}\lambda^2 \Sigma g - \lambda V g = 0, \tag{8}$$

Now this is just the "quadratic eigenvalue" problem considered by Kennedy and Williams [17], which can be reduced to a standard eigenvalue problem as follows. Premultiplying the above equation by $2\Sigma^{-1}$ gives

$$2\Sigma^{-1}(Q - R)g + \lambda^2 g - 2\lambda\Sigma^{-1}V g = 0. \tag{9}$$

This can be reformulated as a system of equations

$$\begin{cases} \lambda g &= h, \\ \lambda h &= 2\Sigma^{-1}V h - 2\Sigma^{-1}(Q - R)g, \end{cases} \tag{10}$$

which can be rewritten as the following (standard) eigenvalue problem:

$$A \begin{pmatrix} g \\ h \end{pmatrix} \equiv \begin{pmatrix} 0 & I \\ -2\Sigma^{-1}(Q - R) & 2\Sigma^{-1}R - I \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} = \lambda \begin{pmatrix} g \\ h \end{pmatrix}. \tag{11}$$

If $(g, \lambda)$ solve (11), then

$$M_t^f = \exp\left(-\int_0^t r(\xi_u)du - \lambda X_t\right) g(\xi_t) \tag{12}$$

is a martingale. The argument given in [1] serves to show that there are exactly $d$ eigenvalues of $A$ in the left open half plane, and $d$ in the right open half plane, a fact that will be needed later.

**3. Markov-modulated perpetual American put.** Our goal in this section is to compute the value

$$v(j, x) \equiv \sup_\tau E\left[ \exp\left(-\int_0^\tau r(\xi_s)ds\right) (K - e^{X_\tau})^+ \mid \xi_0 = j, X_0 = x \right] \tag{13}$$

of the perpetual American put with Markov-modulated dynamics. The special case of the problem where there is no Markov modulation (that is, $|I| = 1$) is well known:[4] the optimal rule is to wait until the price of the asset falls below some critical boundary value $L^*$, and then immediately exercise. Standard first passage time calculations for Brownian motion lead to the following closed-form expression for the perpetual American put:

$$v(x) = \begin{cases} K - \exp(x) & \text{if } x \le \log(L^*), \\ (K - L^*)(L^*)^\gamma \exp(-\gamma x) & \text{if } x > \log(L^*), \end{cases} \tag{14}$$

where $\gamma = 2r/\sigma^2$, $L^* = \gamma K/(\gamma + 1)$.

[4]See the original solution of McKean [19] and Karatzas [16] for a discussion in a more general setting.

When $|I| > 1$, the optimal rule is to exercise when the price of the asset falls below some critical level, which depends on the current state of the modulating Markov chain $\xi$. This intuitively obvious form of the solution follows immediately from the next simple result.

PROPOSITION 1. *If $\varphi(x) \equiv (K - e^x)^+$, then for each $j \in I$ the function*

$$x \mapsto v(j,x) - \varphi(x)$$

*is nondecreasing in* $(0, \log(K))$.

*Proof.* Pick $0 < x < x + \delta < \log(K)$ and let $\tau^*$ denote the optimal stopping time to be used if $X_0 = x$. We consider instead what would happen if we were to use the stopping rule $\tau^*$ but with initial log-price $x+\delta$. Using the elementary inequality $(a-b)^+ \geq a^+ - b^+$, we get[5]

$$\begin{aligned}
v(j, x+\delta) &\equiv \sup_\tau E[e^{-R(\tau)}\varphi(X_\tau)|\, \xi_0 = j, X_0 = x+\delta] \\
&\geq E[e^{-R(\tau^*)}\varphi(X_{\tau^*})|\, \xi_0 = j, X_0 = x+\delta] \\
&= E[e^{-R(\tau^*)}\varphi(X_{\tau^*}+\delta)|\, \xi_0 = j, X_0 = x] \\
&= E[e^{-R(\tau^*)}(K - e^{X(\tau^*)+\delta})^+|\, \xi_0 = j, X_0 = x] \\
&= E[e^{-R(\tau^*)}(K - e^{X(\tau^*)} - (e^\delta - 1)e^{X(\tau^*)})^+|\, \xi_0 = j, X_0 = x] \\
&\geq E[e^{-R(\tau^*)}\{\, (K - e^{X(\tau^*)})^+ - (e^\delta - 1)e^{X(\tau^*)}\,\}|\, \xi_0 = j, X_0 = x] \\
&= v(j,x) - (e^\delta - 1)E[e^{-R(\tau^*)}e^{X(\tau^*)}|\, \xi_0 = j, X_0 = x] \\
&\geq v(j,x) - (e^\delta - 1)e^x \\
&= v(j,x) - \varphi(x) + \varphi(x+\delta),
\end{aligned}$$

using the fact that $e^{-R(t)+X(t)}$ is a martingale, and therefore a supermartingale. □

Immediately from Proposition 1, the optimal stopping time is of the form

$$\tau = \inf\{t : X_t < b(\xi_t)\}, \tag{15}$$

where the constants $(b_i)_{i\in I}$ must be found.

This problem was solved by Guo and Zhang [12] in the simple case of two states, where a closed-form expression can be derived for the price. Note that $-1$ is always an eigenvalue of $A$, which is a key observation that makes the two-state problem tractable. However, the current methodology will work for any number of states. The time-0 value of the stopping rule (15) defined by the levels $(b_i)_{i\in I}$ is

$$v(j,x) = \mathbf{E}\left[\exp\left(-\int_0^\tau r(\xi_t)dt\right)(K - \exp(b(\xi_\tau)))^+ | S_0 = \exp(x); \xi_0 = j\right] \tag{16}$$

There are thus two problems:

(1) Given some thresholds $b_i$, derive the value function;

(2) find the optimal $b_i$.

PROBLEM 1. *Let us suppose given $(b_i)_{i\in I}$, where without loss of generality*[6] *$b_1 > b_2 > \cdots > b_d$; our goal is to compute the value function associated with this set of threshold levels.*

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[5]We use the abbreviation $R(t) \equiv \int_0^t r(\xi_s)ds$.

[6]This assumption amounts to an inessential relabeling of the states and is merely for convenient discussion. When it comes in practice to identifying the thresholds, no assumption is made on the ordering, and all possible orderings are considered. We show in Proposition 2 that there is a unique solution for the thresholds, whose ordering is determined by the parameters of the problem.

Let us start with $x$ in the interval $[b_1, \infty)$. Here, the value function is larger than the payoff function whatever the initial state. Recall that we are looking for a martingale $M_t^f$ of the form of (5) for some function $f$ which satisfies (7) and which will be represented as a weighted sum

$$f(\xi, x) = \Sigma_{i=1}^d w_i g_i(\xi) \exp(-\lambda_i x), \tag{17}$$

where for each $i$, $(\lambda_i, g_i)$ satisfies (8), with $\lambda_i > 0$. We restrict our attention to the $d$ eigenvalues with positive real part because this means that the martingale

$$M_t \equiv \exp\left(-\int_0^t r(\xi_u)du\right) \sum_i w_i g_i(\xi_t) \exp(-\lambda_i X_t)$$

is bounded on $[0, \tau_1]$, where $\tau_1 \equiv \inf\{t : X_t < b_1\}$. Therefore we may apply the optional sampling theorem to obtain

$$\begin{aligned} &\mathbf{E}\left[\exp\left(-\int_0^{\tau_1} r(\xi_u)du\right) \sum_i w_i g_i(\xi_{\tau_1}) \exp(-\lambda_i X_{\tau_1}) \,\middle|\, X_0 = x, \xi_0 = j\right] \\ &\quad = \sum_i w_i g_i(j) \exp(-\lambda_i x). \end{aligned} \tag{18}$$

This is the expression for the value function over the interval $[b_1, \infty)$. In particular, for $j = 1$, this completes the determination of the time-0 price when the underlying chain is initially in state 1, provided we impose

$$(K - \exp(b_1))^+ = \sum_i w_i g_i(1) \exp(-\lambda_i b_1). \tag{19}$$

This gives us a first equation satisfied by the $d$ unknown weights $w$, and

$$v(1, x) = \begin{cases} K - \exp(x) & \text{if } x \le b_1, \\ \sum_i w_i g_i(1) \exp(-\lambda_i x) & \text{if } x \ge b_1, \end{cases} \tag{20}$$

where $w$ still needs to be determined. Continuity at $b_1$ in (19) restricts $w$ to a $(d-1)$-dimensional subspace; to go further, we must look at the next interval $I_2 = [b_2, b_1]$ and match values and slopes of $V$ across $b_1$.

When $x \in I_2$, $\xi$ can jump to state 1, causing exercise to happen. So we now need to modify slightly the Wiener–Hopf argument and the equation for $f$. Let $\tilde{\Sigma}$, $\tilde{R}$, $\tilde{V}$ be the diagonal matrices defined in the following way: for every $i = 2, \ldots, d$, $\tilde{\Sigma}(i, i) = \sigma(i)^2$, $\tilde{R}(i, i) = r(i)$ and $\tilde{V} = \tilde{R} - \frac{1}{2}\tilde{\Sigma}$. Let $\tilde{Q}$ be the submatrix derived from $Q$ by removing its first row and first column.

We still seek a martingale $M_t^f$ of the form of (5) for some function $f$, which now satisfies the following modified equation:

$$(\tilde{Q} - \tilde{R})f + \frac{1}{2}\tilde{\Sigma} f_{XX} + \tilde{V} f_X + \tilde{K} = 0, \tag{21}$$

where $\tilde{K}$ is defined so that it accounts for jumps to the payoff function in state 1: $\tilde{K} = \tilde{q}(K - \exp(x))$, where $\tilde{q}$ denotes a $(d-1)$-dimensional vector, such that $\tilde{q}(i) = q_{i1}$ for every $i = 2, \ldots, d$. The value function over the interval $[b_2, b_1]$ is characterized by (21).

A particular solution to (21) is easily obtained and is of the form $B + C\exp(x)$. The homogeneous equation

$$(\tilde{Q} - \tilde{R})f + \frac{1}{2}\tilde{\Sigma} f_{XX} + \tilde{V} f_X = 0 \tag{22}$$

is structurally similar to (7) and is solved similarly. Let $\tilde{\lambda}_i$ and $\tilde{g}_i$ denote the $2(d-1)$ corresponding eigenvalues and eigenvectors for this new problem. For any scalars $\tilde{w}_i$,

$$\exp\left(-\int_0^t r(\xi_u)\,du\right)\left(\Sigma_{i=1}^{2(d-1)} \tilde{w}_i \exp(-\tilde{\lambda}_i X_t)\tilde{g}_i(\xi_t) + B + C\exp(X_t)\right) \tag{23}$$

is a martingale, at least if we stop at first exit from $I_2$, and is bounded up to that time. Provided that we ensure that $v(j,\cdot)$ joins in a $C^1$ fashion across $b_1$, for $j = 2, \dots, d$, we therefore have for any $x \in I_2$, and any $j = 2 \dots d$,

(24)

$$\begin{aligned}\mathbf{E}^{x,j}&\left[\exp\left(-\int_0^{\tau_2} r(\xi_u)\,du\right)\left(\sum_i \tilde{w}_i\tilde{g}_i(\xi_{\tau_2})\exp(-\tilde{\lambda}_i X_{\tau_2}) + B_j + C_j\exp(X_{\tau_2})\right)\right]\\ &\qquad = \sum_i \tilde{w}_i\tilde{g}_i(j)\exp(-\tilde{\lambda}_i x) + B_j + C_j\exp(x),\end{aligned}$$

where $\tau_2 = \min\{t : X_t \le b_2\}$, and $\mathbf{E}^{x,j}$ denotes the usual expectation conditional upon $X_0 = x$, $\xi_0 = j$ . This is the expression for the value function in the interval $I_2$; in particular, for $j = 2$, this completes the determination of the time-0 price when the underlying chain is initially in state 2, provided we impose continuity at $b_2$.

Notice that at the end of the first step, we were left with $d-1$ degrees of freedom. Once we have solved the problem over the interval $[b_2, b_1]$, matching the values and the slopes of $v(j,\cdot)$, $j = 2, \dots, d$ across $b_1$, we have $2 \times (d-1)$ new linear equations, for the $2(d-1)$ new unknowns $\tilde{w}_i$. Continuity across $b_2$ of $v(2,\cdot)$ provides us with another equation so that at the end of our second step, we are left with $d-2$ degrees of freedom.

From the above, it is now clear that we can proceed recursively, from $b_1$ to $b_d$, by solving $d$ successive problems of this type and considering the standard eigenvalue problem associated with our modified setup and our updated generator for the underlying Markov chain. At the end of the $d$th problem over the interval $[b_d, b_{d-1}]$, we no longer have any degrees of freedom, once we have imposed the continuity of $v(d,x)$ across $b_d$. Finally, over $[0, b_d]$, we have: $v(1,x) = \cdots = v(d,x) = K - \exp(x)$. This deals with the first problem, namely, given thresholds to compute the value function.[7]

PROBLEM 2. *The method just presented shows how for any given sequence of threshold values we may compute the value. For optimality, we need to make $v(j,\cdot)$ be $C^1$ at $b_j$ for $j = 1, \dots, d$. This gives us $d$ nonlinear equations to be solved in $d$ unknowns, which can be solved by standard numerical techniques; we used sequential quadratic programming. The latter optimization routine is converging efficiently toward a set of $b$ values which make $v$ to be $C^1$. It remains to check that*

$$(Q - R)v + \frac{1}{2}\Sigma v_{XX} + V v_X \le 0$$

[7]The above procedure leaves us in fact with a linear system to solve in order to determine the unknown weights on every subinterval: $d$ weights on $[b_1, \infty)$, $2 \times (d-1)$ weights on $[b_2, b_1]$, $2 \times (d-2)$ weights on $[b_3, b_2]$, ... and finally 2 weights on $[b_d, b_{d-1}]$. This gives rise to a linear system with $d^2$ unknowns and $d^2$ equations.

*everywhere; from this, it follows that the solution $v$ found is in fact optimal. The following results deals with this point.*

PROPOSITION 2. *Suppose that thresholds $(b_j) < \log K$ have been found such that the (unique) bounded solution $f$ to the coupled system of ODEs*

$$\frac{1}{2}\sigma_i^2 f_{XX}(i,X) + V_i f_X(i,X) - r_i f(i,X) + \sum_j q_{ij} f(j,X) = 0 \qquad (X > b_i), \tag{25}$$

$$f(i,X) = \varphi(X) \qquad (X \le b_i) \tag{26}$$

*is $C^1$ in $X$ at each point $(j, b_j)$. Then the $(b_j)$ are uniquely determined, and $f$ is the value of the problem.*

*Proof.* The proof proceeds in a number of steps. Given the thresholds $(b_j)$, we set $\tau^* = \inf\{t : X_t \le b(\xi_t)\}$, and we observe that

$$f(\xi_{t\wedge\tau^*}, X_{t\wedge\tau^*}) \exp\{-R(t\wedge\tau^*)\} \text{ is a bounded martingale,}$$

and so in particular

$$f(j,x) = E\big[\,\varphi(X_{\tau^*})e^{-R(\tau^*)}\,\big|\,\xi_0 = j, X_0 = x\,\big].$$

Since $\varphi \ge 0$, it follows that $f > 0$.

(i) We claim that $f(j,x) > \varphi(x)$ whenever $x > b_j$. To see why, let $\tau_0 \equiv \inf\{t : f(\xi_t, X_t) \le \varphi(X_t)\} \le \tau^*$, and observe that

$$\begin{aligned} f(j,x) &= E\big[\,\varphi(X_{\tau^*})e^{-R(\tau^*)}\,\big|\,\xi_0 = j, X_0 = x\,\big] \\ &= E\big[\,(K - e^{X(\tau^*)})e^{-R(\tau^*)}\,\big|\,\xi_0 = j, X_0 = x\,\big] \\ &= E\big[\,\varphi(X_{\tau_0})e^{-R(\tau_0)}\,\big|\,\xi_0 = j, X_0 = x\,\big] \\ &= E\big[\,(K - e^{X(\tau_0)})e^{-R(\tau_0)}\,\big|\,\xi_0 = j, X_0 = x\,\big]. \end{aligned}$$

The fact that $\exp(X_t - R_t)$ is a martingale[8] tells us that

$$E\big[\,Ke^{-R(\tau^*)}\,\big|\,\xi_0 = j, X_0 = x\,\big] = E\big[\,Ke^{-R(\tau_0)}\,\big|\,\xi_0 = j, X_0 = x\,\big],$$

whence immediately $\tau^* = \tau_0$, and the claim is proved.

(ii) We claim next that $f(j,\cdot) - \varphi(\cdot)$ is nondecreasing in $(0, \log(K))$. The proof of this is in effect a reprise of the proof of Proposition 1. As there, we take two starting points $x, x+\delta \in (0, \log(K))$, and let $\tau$ denote the stopping time that would be used if we started from $x$. Using the fact that $f \ge \varphi$, we have

$$\begin{aligned} f(j,x+\delta) &= E[e^{-R(\tau)} f(\xi_\tau, X_\tau)|\,\xi_0 = j, X_0 = x+\delta] \\ &\ge E[e^{-R(\tau)}\varphi(X_\tau)|\,\xi_0 = j, X_0 = x+\delta] \\ &= E[e^{-R(\tau)}\varphi(X_\tau + \delta)|\,\xi_0 = j, X_0 = x] \\ &= E[e^{-R(\tau)}(K - e^{X(\tau)+\delta})^+|\,\xi_0 = j, X_0 = x] \\ &= E[e^{-R(\tau)}(K - e^{X(\tau)} - (e^\delta - 1)e^{X(\tau)})^+|\,\xi_0 = j, X_0 = x] \\ &\ge E[e^{-R(\tau)}\big\{\,(K - e^{X(\tau)})^+ - (e^\delta - 1)e^{X(\tau)}\,\big\}|\,\xi_0 = j, X_0 = x] \\ &= f(j,x) - (e^\delta - 1)E[e^{-R(\tau)}e^{X(\tau)}|\,\xi_0 = j, X_0 = x] \\ &\ge f(j,x) - (e^\delta - 1)e^x \\ &= f(j,x) - \varphi(x) + \varphi(x+\delta). \end{aligned}$$

[8]It is in fact the discounted stock price.

(iii) The final step is to prove that

$$\Phi(i,x) \equiv \frac{1}{2}\sigma_i^2 f_{XX}(i,X) + V_i f_X(i,X) - r_i f(i,X) + \sum_j q_{ij} f(j,X) \le 0 \tag{27}$$

in $X \le b_i$. From (26), we have that in fact

$$\Phi(i,x) = -r_i K + \sum_{j \ne i} (f(j,X) - \varphi(X)),$$

which is seen to be nondecreasing in $[0, b_i]$, in view of point (ii) proved above. It is therefore sufficient to prove that $\Phi(i, b_i-) \le 0$ to establish (27). By the $C^1$ property of the solution $f$, we note that all of the terms in $\Phi(i,\cdot)$ are continuous across $b_i$ except perhaps the second derivative term; thus any discontinuity in $\Phi$ is entirely accounted for by the jump in this. But now consider the function $f(i,\cdot) - \varphi(\cdot)$. Its second derivative at $b_i-$ is zero, and yet its second derivative at $b_i+$ must be nonnegative, since the function is nonnegative to the right of $b_i$, and the function and its first derivative both vanish there. We deduce that the change in the second derivative of $f(i,\cdot)$ at $b_i$ is nonnegative, and the conclusion (27) follows.

(iv) The standard verification argument for optimal control now shows that stopping at $\tau^*$ is optimal and that $f$ is the value function of the problem. □

As a check, take the case $d = 1$, which is the standard perpetual American put problem mentioned earlier; we have $R = r$, $\Sigma = \sigma^2$, $V = r - \frac{1}{2}\sigma^2$, $Q = 0$ and set $\gamma = 2r/\sigma^2$. Now the matrix $A$ defined at (11) is simply

$$A = \begin{pmatrix} 0 & 1 \\ \gamma & \gamma - 1 \end{pmatrix}$$

with eigenvalues $\gamma$ and $-1$, so the solution is of the form

$$v(x) = \begin{cases} K - \exp(x) & \text{if } x \le b, \\ w \exp(-\gamma x) & \text{if } x > b, \end{cases} \tag{28}$$

where the critical level $b$ and the weight $w$ are to be determined. The $C^1$ condition for $v$ at $b$ becomes

$$\begin{cases} K - \exp(b) & = w \exp(-\gamma b), \\ \exp(b) & = \gamma w \exp(-\gamma b), \end{cases}$$

from which we easily deduce the form given in (14).

### 3.1. Numerical results.

**3.1.1. Two states.** First, we check that we recover the results of Guo and Zhang [12] for the simple case of two states. The strike $K$ is taken to be equal to 5,

$$R = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.03 \end{pmatrix},$$

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.81 \end{pmatrix}.$$

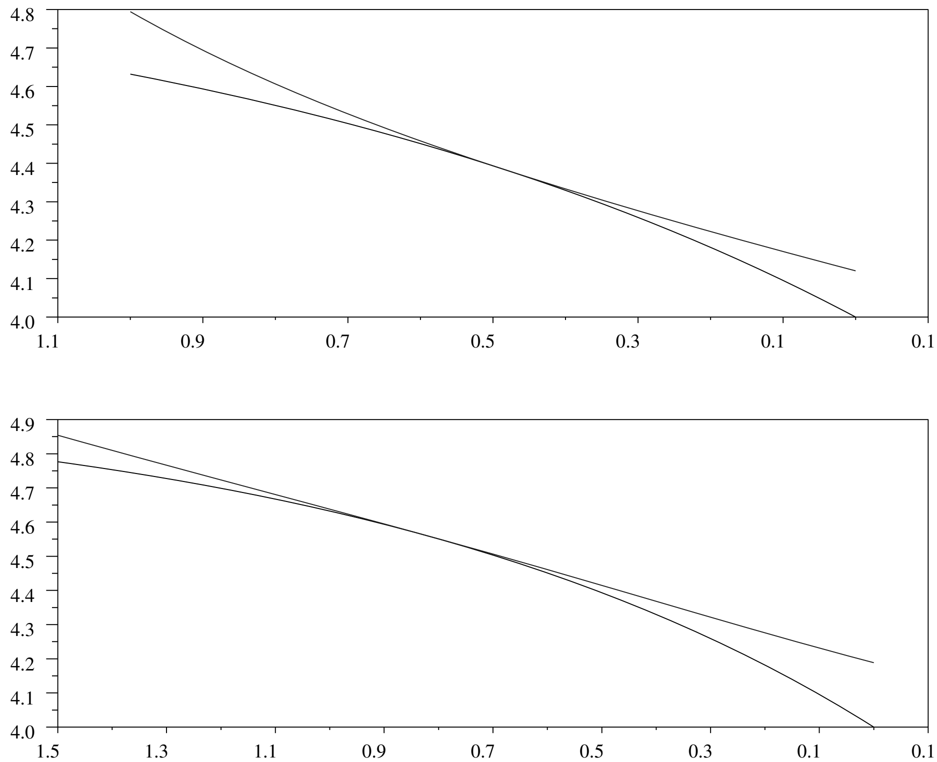


FIG. 1. *Perpetual American put with two states (value function against log price).*

This gives optimal thresholds: $\exp(b) = (0.612, 0.441)$, which is close to Guo and Zhang's solution $(0.614, 0.441)$. Figure 1 plots the price of the Markov-modulated perpetual American put in every state of the chain and enables us to visualize the corresponding smooth pasting conditions. Above the optimal thresholds, where smooth pasting occurs, the upper curve is the value function for the perpetual American put and the lower curve is the reward function. Below the optimal thresholds, the value function is equal to the reward function represented by the lower curve. We keep drawing the upper curve below the optimal thresholds for the sole purpose of assessing the quality of smooth pasting. All the plots below are drawn using a logarithmic scale for the stock price.

Decreasing the volatility in the second state,

$$\Sigma = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.49 \end{pmatrix},$$

leads to higher optimal thresholds $\exp(b) = (0.801, 0.646)$. On the other hand, increasing the jump intensity from state 2 to state 1, where $\sigma_2^2 = 0.81$ and $\sigma_1^2 = 0.25$, decreases the average volatility and we expect therefore our optimal thresholds to be bigger, which turns out to be the case: $\exp(b) = (0.633, 0.455)$.

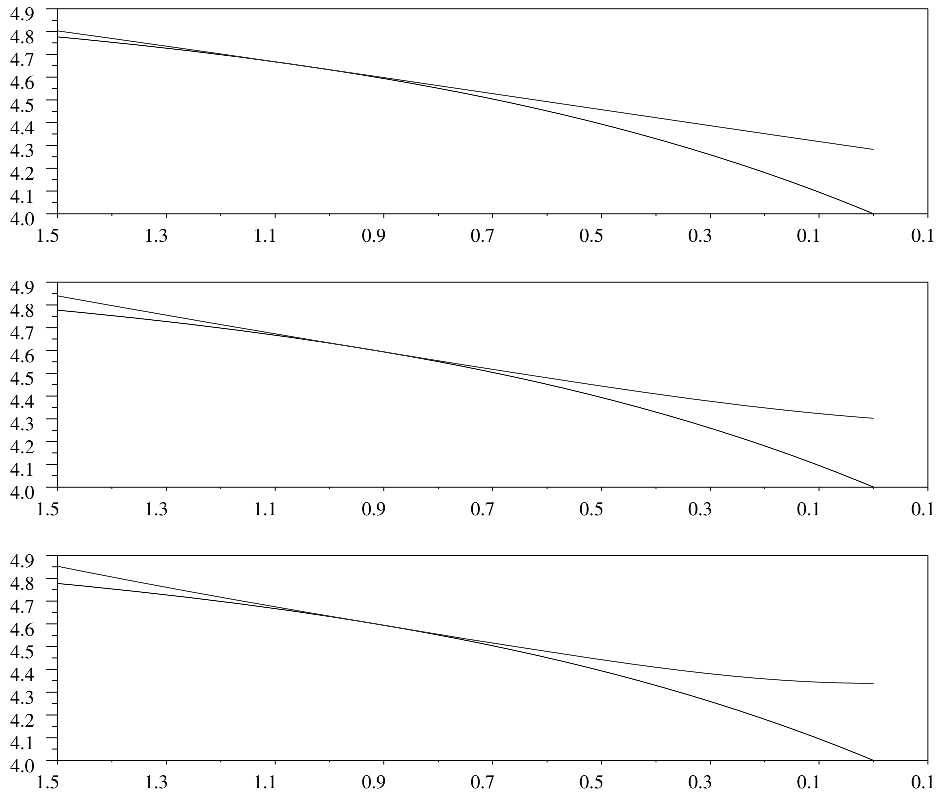


FIG. 2. *Perpetual American put with three states (value function against log price).*

**3.1.2. Three states.** Consider now the case of an underlying Markov chain with three states (high, low, and intermediate levels for the volatility):

$$R = \begin{pmatrix} 0.03 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.03 \end{pmatrix},$$

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.50 & 0 \\ 0 & 0 & 0.81 \end{pmatrix}.$$

This gives the following thresholds: $\exp(b) = (0.600, 0.544, 0.455)$. Figure 2 plots the results.

**3.1.3. More states.** The methodology specified above enables us to deal with a moderately large number of states; in this example, there are eight. In each of the states, $r$ is taken to be equal to 0.03. The jump intensities from one state to another are taken to be equal to 1, the volatility matrix is given by a diagonal matrix with diagonal entries $(0.35, 0.4, 0.6, 0.7, 0.75, 0.8, 0.85, 0.9)$, and in Figure 3, we plot

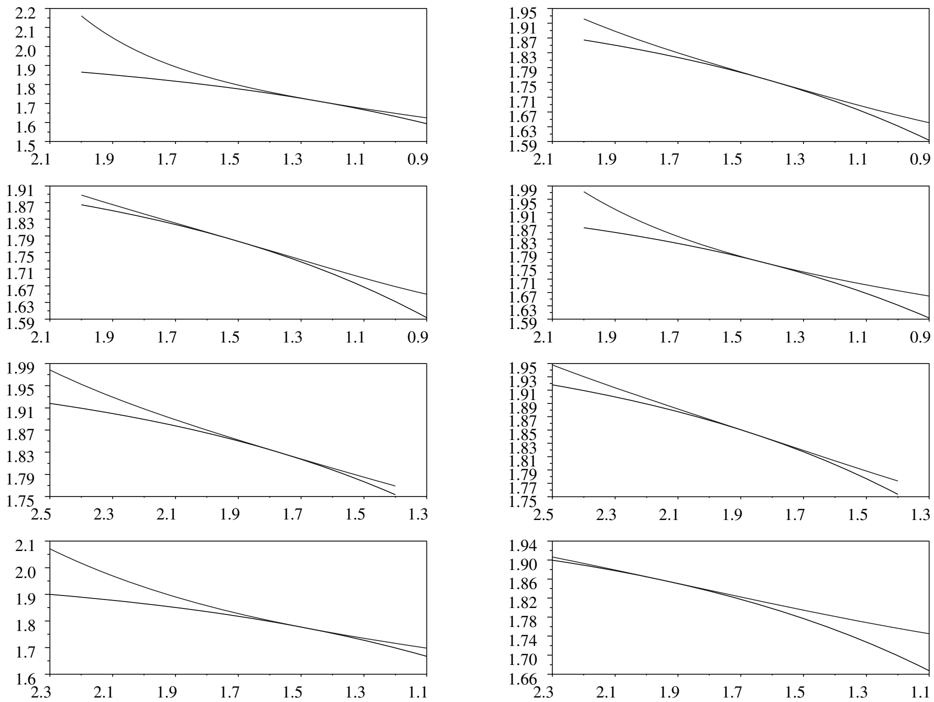

FIG. 3. *Perpetual American put with eight states (value function against log price).*

the value functions for the eight states of the chain, with the corresponding optimal thresholds,

$$\exp(b) = (0.290, 0.239, 0.238, 0.227, 0.220, 0.169, 0.155, 0.140).$$

As one would expect, the less volatile a state is, the bigger is the corresponding threshold.

**4. Binomial pricing with Markov regimes.** The standard binomial pricing method approximates the log-price process by a random walk, which jumps at the times $\Delta t, 2\Delta t, \ldots$ and, at each jump, moves either up or down. The probability of an up step and the size of the jump are chosen to match the drift and variance to the Black–Scholes asset. In this section, we will extend the alternative random walk approximation introduced by Rogers and Stapleton [22] to the case of Markov regimes.

With $X$ still denoting the Markov-modulated log-price (2), the idea of [22] was to fix some $\Delta x > 0$ and view $X$ only at the discrete set of times at which it has moved by $\Delta x$ from where we last observed it. Formally, if

$$\left\{ \begin{array}{l} \tau_0 = 0, \\ \tau_{n+1} \equiv \inf\{t > \tau_n : |X(t) - X(\tau_n)| > \Delta x\} \quad \text{if } n \geq 0, \end{array} \right. \tag{29}$$

then we take $(X(\tau_n))$ as the discrete approximation to $X$, observed for $\nu$ steps, where $\nu \equiv \sup\{n : \tau_n < T\}$ ($T$ is the expiry of the option). We need to compute the distribution of $(X(\tau_1), \xi(\tau_1))$. Take $X_0 = 0$, $\tau \equiv \tau_1$ for notational simplicity.

Let $\lambda_i$ and $g_i$ denote the eigenvalues and the eigenvectors of the eigenvalue problem (11):

$$\begin{pmatrix} 0 & I \\ -2\Sigma^{-1}(Q-R) & 2\Sigma^{-1}R - I \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} = \lambda \begin{pmatrix} g \\ h \end{pmatrix}.$$

There are $d$ negative and $d$ positive eigenvalues. For any scalars $w_i$,

$$\exp\left(-\int_0^t r(\xi_u)\,du\right) \Sigma_{i=1}^{2d} w_i g_i(\xi_t) \exp(-\lambda_i X_t)$$

is a martingale, so by the optional sampling theorem,

$$\mathbf{E}\left[\exp\left(-\int_0^\tau r(\xi_u)\,du\right) \sum_i w_i g_i(\xi_\tau) \exp(-\lambda_i X_\tau) | X_0 = 0, \xi_0 = j\right] = \sum_i w_i g_i(j). \tag{30}$$

The discounted probability of an upwards step from state $j$ to state $k$ is given by

$$P_{j,k}^+ = \mathbf{E}\left[\exp\left(-\int_0^\tau r(\xi_u)\,du\right) \mathbf{I}\{X_\tau = \Delta x, \xi_\tau = k\} | \xi_0 = j\right] \tag{31}$$

where $\mathbf{I}$ denotes the indicator function. Therefore, in order to find $P_{j,k}^+$ we need to solve the following system:

$$\begin{cases} \sum_i w_i g_i(\xi) \exp(-\lambda_i \Delta x) = \mathbf{I}\{\xi = k\} & \forall \xi = 1, \dots, d, \\ \sum_i w_i g_i(\xi) \exp(+\lambda_i \Delta x) = 0 & \forall \xi = 1, \dots, d. \end{cases} \tag{32}$$

This leaves us with $2d$ equations for the $2d$ unknown weights, from which we calculate the discounted probability of an upwards step. Similarly, we can compute the probability of a downwards step from state $j$ to state $k$ by replacing in the above system $\Delta x$ with $-\Delta x$. When the initial logarithmic price is equal to $x$, the price of a standard European call option in this Markov-modulated framework is now computed using the following dynamic programming equation, written using vector notation:

$$\begin{cases} V_0(x) = (\exp(x) - K)^+, \\ V_{n+1}(x) = P^+ V_n(x + \Delta x) + P^- V_n(x - \Delta x), \end{cases} \tag{33}$$

where $n$ is the number of time steps to go before expiry $T$. The matrices $P^+$ and $P^-$ are defined above and denote, respectively, the up and down transition matrices for the underlying Markov chain.

Observe that (as in Rogers and Stapleton [22]) this approximation is well suited to pricing barrier options; we merely change appropriately the matrices $P^\pm$ at the vertices adjacent to the barrier(s).

Once we have computed the discounted probabilities of an upwards and a downwards step, it remains for us to deal with the fact that the number $\nu$ of time steps is random. One solution to this problem is to match bond prices so that

$$\mathbf{E}\left[\exp\left(-\int_0^T r(\xi_u)\,du\right) 1\right] \simeq \mathbf{E}\left[\exp\left(-\int_0^{\tau_\nu} r(\xi_u)\,du\right) 1\right]. \tag{34}$$

Let $\overline{P} = P^+ + P^-$. From the above, it is enough to find $\nu$ so that

$$\pi \overline{P}^{\nu} 1 = \pi \exp\left[T(Q - R)1\right], \tag{35}$$

where $\pi$ denotes the invariant distribution of the underlying Markov chain. This simple approximation turns out to give very satisfactory results for the Markov-modulated setup.

Notice finally that for the case of the American put, the dynamic programming equation for the value function now becomes

$$\begin{cases} V_0(x) = (\exp(x) - K)^+, \\ V_{n+1}(x) = \max\{(K - \exp(x))^+, P^+ V_n(x + \Delta x) + P^- V_n(x - \Delta x)\}. \end{cases} \tag{36}$$

The case of the finite expiry Markov-modulated American put was tackled by Buffington and Elliott [4] but only in the case of a two-state Markov chain and by extending the Barone-Adesi–Whaley analytic approximation.

### 4.1. Numerical results.

**4.1.1. Markov-modulated European call.** One way of checking our results is to consider the case when the chain switches between two identical states for the volatility. The price in each state should then be equal approximately to the Black–Scholes price for this given volatility. The expiry time is taken to be equal to one year; the initial stock price is $S_0 = 95$. The strike is $K = 100$. Finally, the size of the space grid is taken to be $\Delta x = 0.022$. We take

$$R = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.03 \end{pmatrix},$$

$$Q = \begin{pmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}.$$

The price in each state of the chain is found to be equal to 17.9667, compared to the Black–Scholes value of 17.9506 (relative error: 0.0009).

**4.1.2. Markov-modulated American put.** Here we compare with the prices tabulated in [21].

(i) Let us first consider the case of two identical states, with $T = 0.5$, $\Delta x = 0.022$, $X_0 = \log(85)$, $K = 100$:

$$R = \begin{pmatrix} 0.06 & 0 \\ 0 & 0.06 \end{pmatrix},$$

$$Q = \begin{pmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 0.16 & 0 \\ 0 & 0.16 \end{pmatrix}.$$

The price in each of the two states is found to be equal to 18.0285, which needs to be compared with the value 18.0374 found by Rogers [21] (relative error: 0.0005).

(ii) Let us now decrease the volatility in the second state:

$$\Sigma = \begin{pmatrix} 0.16 & 0 \\ 0 & 0.10 \end{pmatrix}.$$

The prices in each of the two states are now found to be $(17.3070, 16.7677)$. Decreasing the volatility decreases the price in the two states, as expected. Correspondingly, increasing the volatility in one of the states increases the price in the two states, as shown by

$$\Sigma = \begin{pmatrix} 0.40 & 0 \\ 0 & 0.16 \end{pmatrix},$$

where the price is now given by $(20.3454, 19.1986)$.

(iii) Taking example (i) and changing just the start value $X_0$ to $\log(100)$ allows us to compare our values with other values in [22]. We find the price is $(9.9279, 9.9279)$, to be compared with 9.9458 (relative error: 0.00192). Taking $X_0 = \log(115)$, the price is $(5.1109, 5.1109)$, to be compared with 5.1265 (relative error: 0.00304).

It therefore turns out that the random walk approximation provides an accurate and very quick method for Markov-modulated asset dynamics.

The solution for the finite expiry American put should provide a way of checking the results of the preceding section for the perpetual American put, by letting $T$ tend to $\infty$. For the example of Guo and Zhang [12], where $\exp(b) = (0.616, 0.441)$, the time-0 price of the Markov-modulated perpetual American putm, $(4.2239, 4.2758)$, compares well with our results for the finite expiry American put when $T = 40$: $(4.2180, 4.2692)$ (relative errors: 0.00139, 0.00155).

A similar check can be made for the perpetual American put example with three states, where the prices are $(4.2278, 4.2486, 4.2706)$, to be compared with $(4.2244, 4.2439, 4.2631)$ for the finite expiry case, where $T = 40$ (relative errors: 0.0008, 0.0011, 0.0017).

**4.1.3. Markov-modulated barrier options.** In this section, we price a number of double knockout barrier options in a Markov-modulated setup.

(i) Let us first consider the case of constant barriers, where we compare our results with those of Geman and Yor [7]. With two identical states, taking $T = 1$, $X_0 = \log(100)$, $K = 100$, $\Delta x = 0.022$, $b^* = \log(150)$, and $b_* = \log(75)$, and

$$R = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix},$$

$$Q = \begin{pmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}.$$

the price of the double knockout is then found to be $(0.8994, 0.8994)$, to be compared with the value 0.89 computed by Geman and Yor [7] (relative error: 0.01061).

(ii) Changing $K$ to 87.5 and $b_*$ to $\log(50)$, the price becomes $(3.8274, 3.8274)$, to be compared with 3.8075 from Geman and Yor (relative error: 0.00519).

(iii) The next example is the same as the previous one, but now we have two different volatility levels:

$$\Sigma = \begin{pmatrix} 0.50 & 0 \\ 0 & 0.25 \end{pmatrix}.$$

The price of the double knockout is now equal to $(2.6055, 2.5882)$.

(iv) We finally consider the case of a double knockout with moving barriers, which are linear for the log-price $b^* = \log(U) + \Delta x_1 t$ and $b_* = \log(L) + \Delta x_2 t$. We compare our results with those of Kunitomo and Ikeda [18] Let us take: $T = 0.5$, $X_0 = \log(1000)$, $K = 1000$, $\Delta x_1 = 0.1$, $\Delta x_2 = -0.1$, $L = 500$, and $U = 1500$:

$$R = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix},$$

$$Q = \begin{pmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}.$$

The price of the double knockout in this Markov-modulated setup is found to be $(67.2596, 67.2596)$, to be compared with 67.78 from Kunitomo and Ikeda [18] and 67.7834 from Rogers and Zane [23] (relative error: 0.00773).

**5. Conclusions.** We have shown how to use classical results from the Wiener–Hopf factorization of Markov processes to price options on a Markov-modulated asset. Such a model can accommodate "bull" and "bear" markets, as well as changes in interest rate and volatility. This method has been applied to the optimal stopping problem of the Markov-modulated perpetual American put. It yields a very efficient and accurate numerical method, which amounts to computing the eigenvalues and eigenvectors of some particular matrices. There is no dynamic programming nor discretization of any PDE. Finally, with a finite time horizon, the approach can be used to construct a modified binomial lattice methodology, which has been applied to the European call, the American put, and double barrier options in a Markov-modulated setup. This modified binomial method turns out to provide an efficient numerical scheme for Markov-modulated option pricing.

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