Robust hedging of barrier options

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Abstract

This article considers the pricing and hedging of barrier options in a market in which call options are liquidly traded and can be used as hedging instruments. This use of call options means that market preferences and beliefs about the future behaviour of the underlying are in some sense incorporated into the hedge and do not need to be specified exogenously. Thus we are able to find prices for exotic derivatives which are independent of any model for the underlying asset. For example we do not need to assume that the underlying follows an exponential Brownian motion.

We find model-independent upper and lower bounds on the prices of knock-in and knock-out puts and calls. If the market prices the barrier options outside these limits then we give simple strategies for generating profits at zero risk. Examples illustrate that the bounds we give can be fairly tight.

1 Super-replication and robust hedging

The focus of this article is an analysis of the hedging of barrier options. The standard approach to such a problem is to postulate a model for the behaviour of the underlying asset, and from this model to derive a fair price, and associated hedging strategy. The hedge will replicate the option payout but only if the model precisely describes the real world. Clearly this last assumption is unlikely to be satisfied. This motivates attempts to find hedging strategies for options which are robust to a mis-specification of the model for the underlying asset. Here we take an extreme philosophy and consider hedging strategies which are certain to work whatever the behaviour of the underlying asset. This article describes how to use call options in an optimal fashion to hedge barrier options in a manner which is robust to model mis-specification.

The standard stock model for option pricing, namely the Black-Scholes exponential Brownian motion model, is a parametric family of models, within unknown parameter volatility. The model is often

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calibrated using the observed price of a call or other derivative. This provides a form of consistency but leads to an immediate problem; what if the prices of two traded calls correspond to different implied volatilities? Which volatility should be used for option pricing and how can the option writers be sure that they are not creating arbitrage opportunities? Our response is to incorporate the prices of liquidly traded options into the model.

In a similar vein Avellaneda and Parás [1] suppose that there is a family of traded calls or other derivative securites which, can be used, together with the underlying, as hedging instruments. However they also suppose that the volatility of the underlying asset is known to lie in some interval. They deduce a price for an exotic option not included in the original family. This involves minimising an associated hedging cost over the space of static hedging portfolios where the hedging portfolio is built up from the family of calls. This cost is the price of the hedging portfolio plus the worst case price for the unhedged component of the exotic payoff.

We use a similar philosophy to analyse the pricing of barrier options. We assume that the family of derivative securities which can be used for hedging consists of calls and puts of all possible strikes. This is consistent with a derivatives market in which puts and calls are traded liquidly. Unlike Avellaneda and Parás we make no assumptions on the volatility of the underlying asset. Remarkably knowledge of call and put prices is by itself sufficient to place often tight bounds on the possible prices of the barrier options. These bounds hold without the need for assumptions on the behaviour of the underlying asset. Further, implicit in the calculation of these bounds is a simple hedging strategy. If the barrier option trades outside these bounds then the hedging strategy can be used to generate riskless profits.

Our results, a taster for which is to be found later in this introduction, are most powerful in markets with zero interest rates or equivalently in forward markets.

The power of our method comes from the representation of financial price processes as martingales, and from theorems on the possible laws of the maximum of a martingale. The most relevant of these theorems, which are due to Blackwell and Dubins [3] and Dubins and Gilat [6] determine the stochastically largest possible law of the maximum of a martingale whose terminal distribution has been specified. Perkins [16] determines the corresponding stochastically smallest possible law for the maximum of a continuous martingale whose distribution is fixed. These were results were used in Hobson [12] in an analysis of the possible prices of lookback options.

The conclusions that we draw, that there is a range of possible prices for contingent claims, and that associated with the largest of these prices there is a super-replicating strategy, are similar to the results of El Karoui and Quenez [10]. See also Kramkov [14] for a more abstract treatment. These authors however are addressing a slightly different problem and are concerned with model incompleteness rather than model robustness. In particular they suppose that the option writer knows that the law of the price process lies in a family of equivalent martingale measures. We make no such supposition, instead we operate with a family of candidate models which are restricted to be consistent with call option prices.

The problem of producing static hedges for barrier options was also considered by Carr *et al* [5]. Their proposed strategies are based upon a put-call parity result, and in a similar fashion to the analysis here, the associated hedging strategies involve purchase of a portfolio of puts and calls. The results in [5] include exact unique prices for barrier options in terms of call option prices; however the hedging strategies are only guaranteed to work if the volatility of the underlying asset satisfies a symmetry relation, and if this relation is preserved through time. The prices and strategies are not robust to arbitrary mis-specification of the underlying model, unlike the prices we give below.

The remainder of this article is structured as follows. In the next section we review the martingale inequalities mentioned above and give an application to the pricing of digital options. The main body of results on the bounds on the prices of barrier options, including knock-in and knock-out puts and calls with barrier above the initial price, are given in Section 3. (The corresponding results for barriers below the strike are presented in an appendix.) Then we review some numerical results which show that a naive option writer can easily propose model-based prices for barrier options which are inconsistent with no-arbitrage. In all this work we make the idealistic assumption that calls are traded with all possible strikes; in a final section we discuss the discrete situation when only a finite number of puts and calls are traded. First however we present a simple example which illustrates our ideas.

1.1 An up-and-in put with barrier at the strike

Let the price process of an asset be given by S_t so that the payoff of the put knocked in at K is given by $(K - S_T)^+$ (with $K > S_0$) provided that for some $t \leq T$ we have $S_t \geq K$; otherwise the barrier option payoff is zero. Suppose that the price process for the underlying asset is continuous, so that in particular at the moment that the barrier is first reached, the asset price is exactly on the barrier.

Consider the following hedging strategy;

> buy a call with strike K and at the instant (if ever) the underlying is at K or above, sell forward one unit of the underlying.

By our continuity assumption, the return from entering into the forward transaction is $(K - S_T)$. If the underlying never reaches K, then the call matures worthless, and the forward transaction is not entered into, so the hedging strategy yields the same payoff as the knock-in put. Conversely if the underlying reaches K then the hedging strategy yields from call and forward respectively

$$(S_T - K)^+ + (K - S_T) \equiv (K - S_T)^+.$$

Using the fact that two strategies which yield the same income stream must have the same value it follows that the unique fair price for this up-and-in put is the price of a call option with strike K.

If the barrier option trades at any other price then there are arbitrage opportunities. For example if the option writer has used the formula for a knock-in put derived from a Black-Scholes model then they will in general have mispriced the barrier option unless their model was calibrated using the call with strike K. (For instance a model calibrated using an at-the-money call, or which uses the instantaneous volatility will generally lead to mispricing). Further the δ-hedging strategy proposed by the Black-Scholes model is infinitely more complex than the one or two transaction hedge described here.

This example is atypical in the sense that we can exhibit a hedging strategy involving investment in calls which exactly replicates the knock-in option. In general the best we can do is to propose a pair of strategies one of which super-replicates and one of which sub-replicates the barrier option payoff, and which together produce upper and lower bounds on the barrier price. However the example is typical in the way that it suggests simple strategies for hedging barrier options which are independent of the dynamics of the underlying asset price process. The simplicity and generality of these strategies makes them very appealing as an alternative to δ-hedging even when the option-writer has confidence in a particular model.

2 Martingale inequalities and digital options

The purpose of this section is to recall how options prices imply a marginal law for the price process, and to rederive some classical martingale inequalities concerning the distribution of the maximum of a martingale. We use these inequalities to derive arbitrage-free bounds on the prices of digital options.

The standard approach to option pricing is to propose a model for the dynamics of the underlying price process and to use this model (and risk-neutrality arguments) to derive prices for call and other options. With judicious choice of model parameters one can hope that these derived prices match, or at least approximate the prices at which the derivative securities actually trade. Here, following Ross [17], Breeden and Litzenberger [4] and Dupire [8, 9], we reverse the analysis; we use traded call prices to determine a law for the asset price. By definition this is the law used to price calls; it is natural to ask if it can also be used to price other options. Indeed this is so; any option whose payoff is contingent upon only the value of the underlying at expiry can be uniquely priced and exactly replicated using call options with that same maturity. Arbitrage considerations mean that this (unique) price is the only price at which the option can trade. For simplicity we present our analysis in terms of discounted prices, or equivalently we work in a regime of zero-interest rates.

Definition 2.1 *Suppose the family of call options with strikes K and common maturity T (with $T = 1$ say) have prices $C(K)$. Then the implied law μ of S, the law used for pricing options, is the solution to*

$$C(K) = \int (x-K)^+ \mu(dx) \equiv \mathbb{E}^{\mu}(S_1 - K)^+.$$

For example, if C is twice differentiable, then $\mu(dx) = C''(x)dx$.

Corollary 2.2 (Breeden and Litzenberger [4]) *An option with payoff $f \equiv f(S_1)$, with f twice differentiable, has unique, arbitrage-free price*

$$\mathbb{E}^{\mu}(f(S_1)) \equiv f(0) + f'(0)S_0 + \int_{x>0} f''(x)C(x)dx \tag{1}$$

This corollary is just an exercise in integration by parts. Most importantly, to quote from [4, p627-628], 'no assumptions have been made as to the stochastic process governing the movement of the underlying security's price' and, 'individuals preferences and beliefs have not been restricted as they will be reflected in the call option prices'.

Indeed (1) also suggests a model-independent hedge for the option with payoff f; keep $f(0)$ as cash, buy $f'(0)$ units of the underlying security, and purchase a portfolio of call options with $f''(x)dx$ options with strike in $[x, x+dx)$. This static hedge replicates the option payoff.

Corollary 2.2 can be extended to more general payoffs, by approximation if necessary. For example, a binary option which we define to have payoff $I_{\{S_1 \geq B\}}$, has price

$$\mu([B,\infty)) \equiv \mathbb{P}^{\mu}(S_1 \geq B).$$

If the derivative C' is well defined at B then we also have $\mu([B,\infty)) = -C'(B)$.

In this article we are concerned with path-dependent options. In particular we are interested in barrier options where the payoff is contingent upon the maximum price attained by the underlying. If call options are traded then we can infer the time 1 law of the underlying (under the pricing measure) but this is in general not sufficient to uniquely characterise the price process. As a consequence call prices will only specify a range of possible prices for the barrier option. The barrier option cannot trade at a price outside this range without creating arbitrage opportunities. As well as using call options to determine a range of feasible prices, we also use them to construct hedges. These hedges are independent of any modelling assumptions on the underlying, and any preferences of the investor.

The only assumption that we make on the price process is that it is right-continuous. This is a standard assumption which is needed to define the gains from trade process as a stochastic integral of the dynamic hedging portfolio against the price process. (In the sequel we may sometimes additionally assume that S is continuous, but these occaisions will be clearly highlighted.) We suppose that for the maturity date $T = 1$ a continuum of calls is traded at all possible strikes, (see Section 5 for a relaxation of this condition) and that these calls can be bought and sold in arbitrary quantities. Investment in the forward market on the underlying is also possible, there are no transaction costs and trades can be executed instantaneously.

As mentioned above we work throughout using a model with zero-interest rates. The notion of discounted prices means that most problems in mathematical finance can be reduced to this case at the expense of extra notation. However in this problem the translation to zero-interest rates is non-trivial since a constant barrier in the undiscounted world becomes a moving barrier with respect to the discounted variables. The exception to this rule is when the underlying asset is a forward price. The results we obtain are most striking when the barriers are constant and so we present them in that context. It is possible to apply the superhedging arguments we suggest to undiscounted markets, but then the relative timings of stock volatility and bond price movements becomes crucial.

Our aim is to find model-independent and arbitrage-free prices for exotic options, taking the prices of calls as given. The prices are arbitrage free in the sense that if the option is traded at a higher price, then necessarily there is an arbitrage opportunity, and we give a simple strategy for capturing these riskless profits. The price is model-independent in the sense that the success of this strategy does not depend on an assumption that a proposed model correctly describes the real world. There is no cheaper strategy with the arbitrage-free model-independent super-replication property unless some additional assumptions are made about the behaviour of the underlying security.

The first results we prove concern digital options which are defined to have a unit payoff if the underlying crosses a barrier B before time $T = 1$. In mathematical terms the payoff is $I_{\{H_B \leq 1\}}$, where $H_B = \inf\{t : S_t \geq B\}$. Our proof is based on a new financial interpretation and proof of a martingale inequality of Blackwell and Dubins [3] and Dubins and Gilat [6].

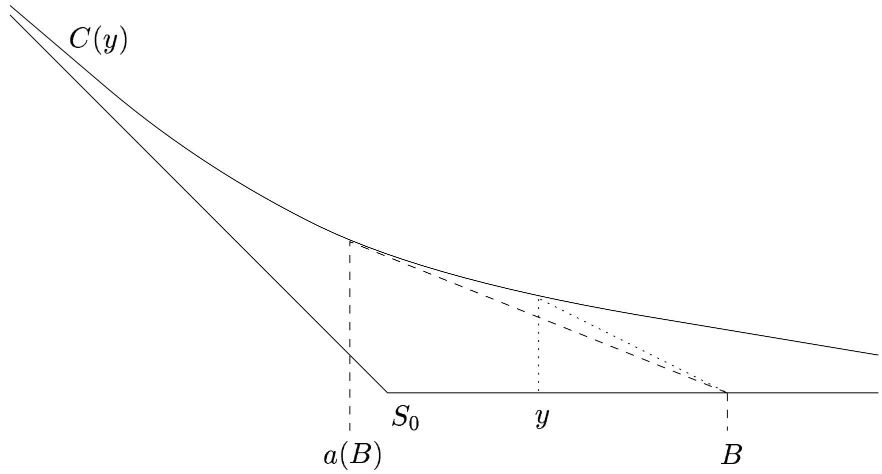

Figure 1: The call price function $C(y)$, note that $C(y) \geq (S_0 - y)^+$, and the optimal choice $a = a(\mu, B)$.

Lemma 2.3 *Let S be any martingale with S_0 given, and such that S_1 has law μ. Let $B > S_0$. Then*

$$\mathbb{P}(H_B \leq 1) \leq \inf_{y \leq B} \frac{C(y)}{B - y}.$$

Let $a = a(\mu, B)$ be the point where the infimum is attained. Then there is a martingale $S^\dagger$ such that, with $H_B^\dagger = \inf\{t : S_t^\dagger \geq B\}$,

$$\mathbb{P}(H_B^\dagger \leq 1) = \frac{C(a)}{B - a}.$$

Proof:
For any $y < B$ we have

$$I_{\{H_B \leq 1\}} \leq \frac{(S_1 - y)^+}{(B - y)} + \frac{(B - S_1)}{(B - y)} I_{\{H_B \leq 1\}}. \tag{2}$$

(This inequality is represented pictorially in Figures 2 and 3). Taking expectations, and using the martingale property, this last term has non-positive expectation and

$$\mathbb{P}(H_B \leq 1) \leq \frac{C(y)}{(B - y)}.$$

The remainder of the proof, on the existence and form of the martingale which attains the bound, is relegated to Appendix A. □

Remark 2.4 If we let $\overline{S}_1 = \sup\{S_t; t \leq 1\}$, and if the sequence B_n increases to B then $(\overline{S}_1 \geq B) = \cap_n (H_{B_n} \leq 1)$ and we deduce

$$\mathbb{P}(\overline{S}_1 \geq B) = \lim_{n \to \infty} \mathbb{P}(H_{B_n} \leq 1) \leq \liminf \frac{C(y)}{B_n - y} = \inf_{y \leq B} \frac{C(y)}{B - y}.$$

In fact all our results have identical analogues with $(\overline{S}_1 \geq B)$ replacing $(H_B \leq 1)$.

Corollary 2.5 *The model-free upper bound on the price of a digital option is*

$$\frac{C(a)}{B - a}.$$

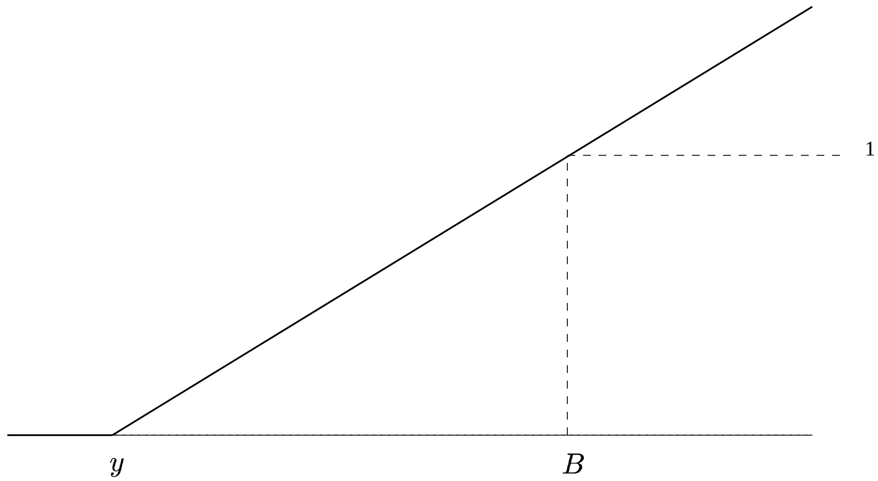

Figure 2: Prior to the first time S crosses the level B: the term $(S_1 - y)^+/(B - y)$ (thick line) dominates the indicator function (dashed line).

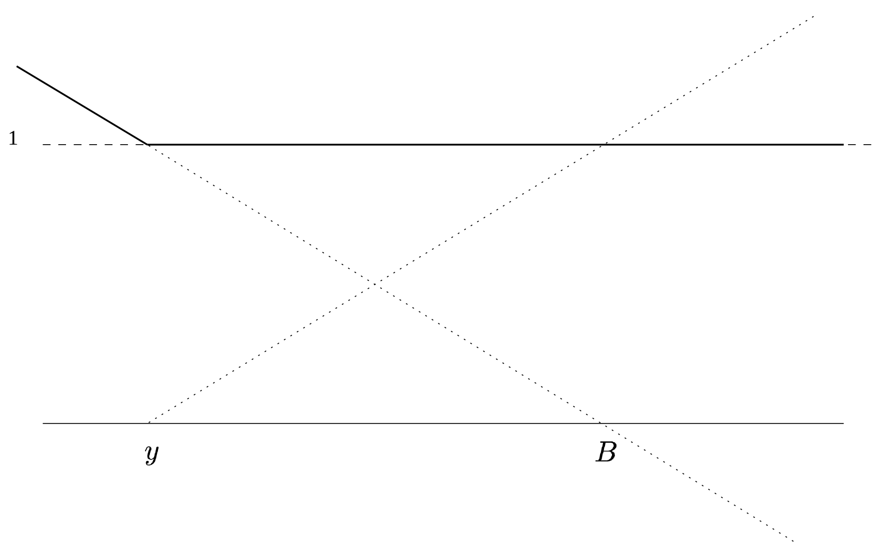

Figure 3: After H_B: the payoff structure (thick line), the sum of $(S_1 - y)^+/(B - y)$ and $(B - S_1)/(B - y)$ (dotted lines), exceeds 1 uniformly in the final value of S.

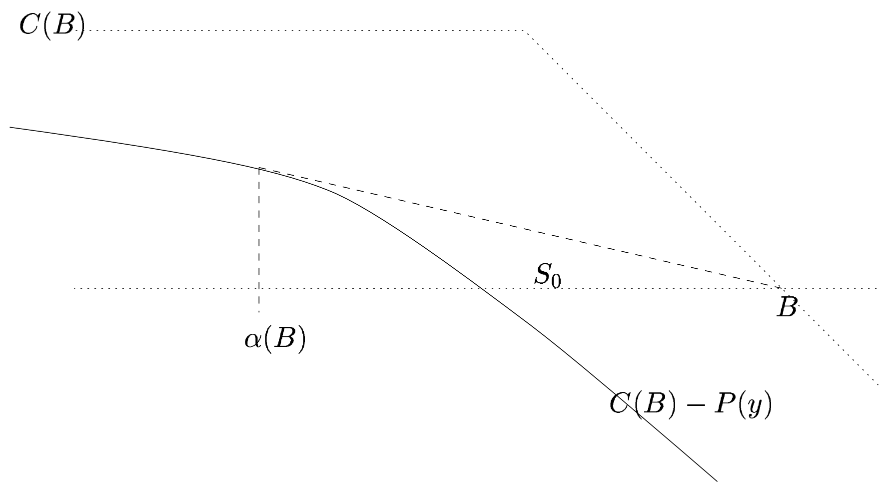

Figure 4: The maximisation in Lemma 2.6 involves finding the tangent to $C(B) - P(y)$ which crosses the x-axis at B.

Proof:
From (2) a super-replicating strategy for the digital option is to purchase $1/(B-y)$ calls with strike y, and at the moment H_B when the underlying first crosses the barrier B to sell forward $1/(B-y)$ units of the underlying. (If $H_B > 1$ then no forward transaction is made.) The forward transaction raises $(S_{H_B} - S_1)/(B-y)$ and by the right continuity assumption on the price process which guarantees that $S_{H_B} \geq B$ this gives at least the last term in (2).

The cheapest super-replicating strategy of this form uses $y = a(\mu, B)$, and since there is a model for the price process, namely $S^\dagger$, for which the expected payoff of the option equals the cost of the hedging strategy, there can be no cheaper model-independent hedging strategy.

$\square$

Lemma 2.6 *Let S be any martingale with S_0 given, and such that S_1 has law μ. Let $B > S_0$. Then the trivial inequality*

$$\mathbb{P}(H_B \leq 1) \geq \mathbb{P}(S_1 \geq B) \equiv \mu([B, \infty)) \tag{3}$$

holds, and there is a martingale S^J for which equality holds in (3).

If S is known to be a continuous martingale then

$$\mathbb{P}(H_B \leq 1) \geq \mu([B, \infty)) + \sup_{y<B} \frac{C(B) - P(y)}{B-y}.$$

Let $\alpha = \alpha(\mu, B)$ be the point at which this supremum is attained. Then $\alpha < S_0$ and there is a martingale $S^\#$ for which

$$\mathbb{P}(H_B^\# \leq 1) = \mu([B, \infty)) + \frac{C(B) - P(\alpha)}{B-\alpha}.$$

Proof:
The first part follows from the set inequality

$$(H_B \leq 1) \supseteq (S_1 \geq B).$$

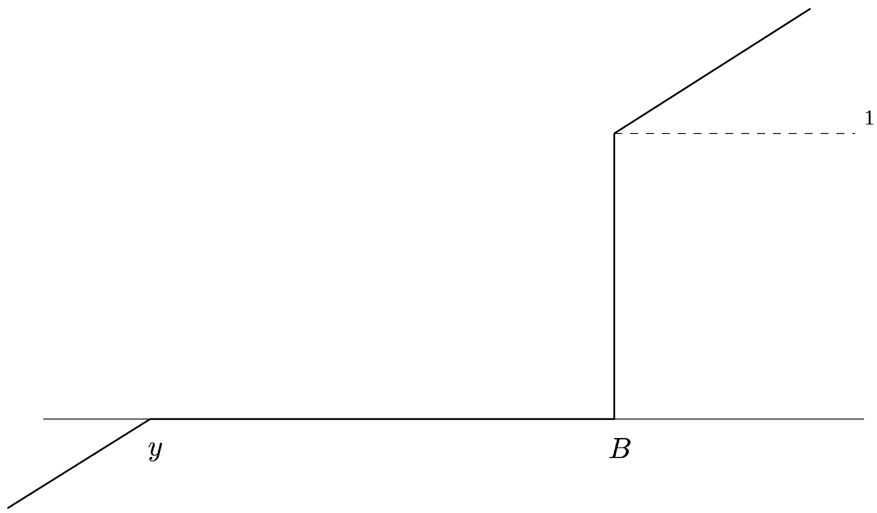

Figure 5: The inequality (4) in the case ($H_B > 1$), which states that the thick line, representing the RHS, is non-positive at least for $S_1 < B$.

Let S^J be the pure jump martingale with $S^J_t = S_0$ for $t < 1$ and with a jump at time 1 such that S^J_1 has law μ; then equality holds in (3).

Now we specialise to continuous martingales. For any $y < B$ we have

$$I_{\{H_B \leq 1\}} \geq I_{\{S_1 \geq B\}} + \frac{(S_1 - B)^+}{(B-y)} - \frac{(y - S_1)^+}{(B-y)} + \frac{(B - S_1)}{(B-y)} I_{\{H_B \leq 1\}}. \tag{4}$$

Taking expectations, and using the martingale property

$$\mathbb{P}(H_B \leq 1) \geq \mu([B, \infty)) + \frac{C(B) - P(y)}{(B-y)}.$$

The proof that there is a martingale for which this bound is attained is again relegated to the appendix.

□

Corollary 2.7 *The model-free lower bound on the price of a digital option is $\mathbb{P}(S_1 \geq B) \equiv \mu([B, \infty))$. If the underlying can be assumed to be continuous then the model-free lower bound rises to*

$$\mu([B, \infty)) + \frac{C(B) - P(\alpha)}{B - \alpha}. \tag{5}$$

Proof:

The result is immediate when jumps are not prohibited. In the continuous case suppose the option-hedger has bought the digital option. Motivated by (4) suppose they hedge the digital option by fixing some $y < B$, and by selling binary option with strike B and $1/(B-y)$ calls with strike B, and purchasing $1/(B-y)$ puts with strike y. Further at the moment the underlying first crosses the barrier B, they buy forward $1/(B-y)$ units of the underlying. By the continuity assumption this last transaction, if it takes place, raises $(S_1 - B)/(B-y)$. From (4) combination of the payoff from the digital option and the other constituent parts of this strategy is certain to be non-negative. As long as the option hedger has paid

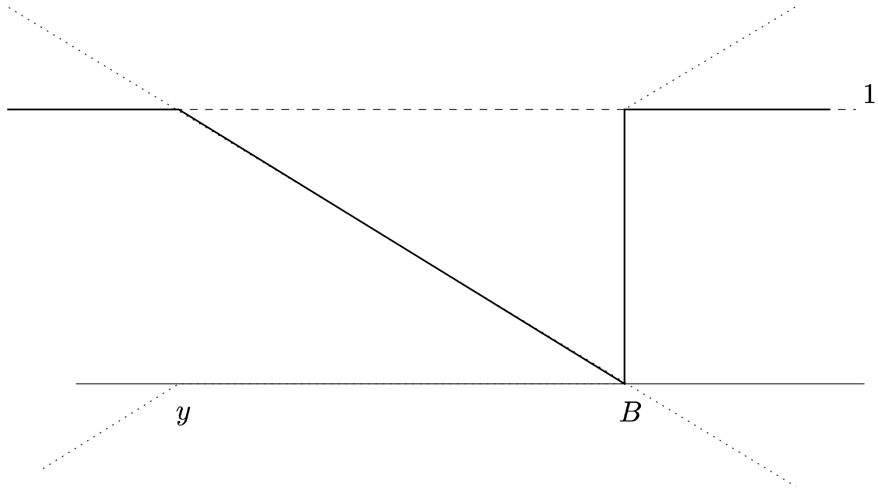

Figure 6: The inequality (4) in the case $(H_B \leq 1)$.

no more than the bound in (5) he can afford the hedge portfolio of calls and puts (using $y = \alpha$) and can guarantee not to make a loss.

Finally, we see from the example of the martingale $S^{\#}$, that $\mu([B, \infty)) + (C(B) - P(\alpha))/(B - \alpha)$ is the highest price that the option hedger can pay for the digital option without exposure to model risk, provided the continuity assumption holds. □

3 Barrier Options

The options we consider in this section are single barrier puts and calls, both of knock-in and knock-out type. In each case we find a family of super-replicating strategies and an associated upper bound on the price of the barrier option. By giving an example of a martingale price process for which the proposed bound is the true fair price for the option we show further that the upper bound is a least upper bound; no cheaper strategy exists which is guaranteed to super-replicate the barrier option whatever the behaviour of the underlying.

Let the initial price of the underlying asset be S_0, and consider barrier options with strike K, and barrier at $B > S_0$; specifically we are concentrating on *up-and-in* and *up-and-out* options. Results for down-and-in and down-and-out options can be deduced from these results by simple transformations, see Appendix B. Again we present only upper bounds on the option price in the subsequent analysis, but we can deduce lower bounds from the relationship

$$\text{Call} - \text{Knock-in-Call} = \text{Knock-out-Call}. \tag{6}$$

3.1 Up-and-in Call

The payoff of a up-and-in call is $(S_1 - K)^+ I_{\{H_B \leq 1\}}$.

If $B \leq K$ then introduction of the barrier has no effect on the payoff, and the knock-in call has the same price as a standard call. Recall the definition of $a = a(\mu, B)$ from Lemma 2.3 and Figure 1.

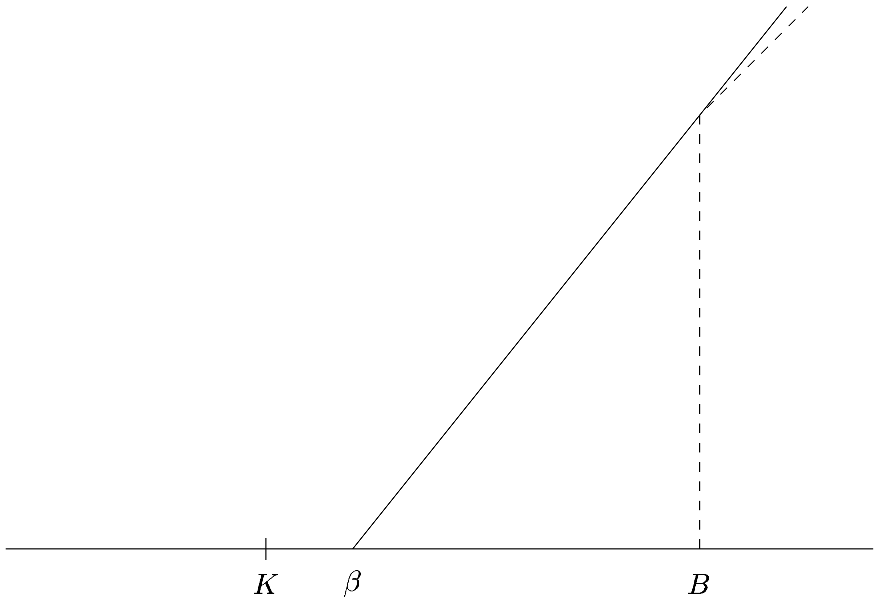

Figure 7: A pictorial representation of inequality (7), on $(H_B > 1)$.

Proposition 3.1 *For $K < B$ the arbitrage-free price of a knock-in call is $C(K)$ if $a \equiv a(\mu, B) \leq K$ and*

$$\frac{(B-K)}{(B-a)} C(a)$$

otherwise.

Proof:
For any $\beta \in (K, B)$, we have

$$(S_1 - K)^+ I_{\{H_B \leq 1\}} \leq \frac{(B-K)}{(B-\beta)}(S_1 - \beta)^+ + \frac{(\beta - K)}{(B-\beta)}(B - S_1) I_{\{H_B \leq 1\}}. \tag{7}$$

To see this note that if $(S_1 \leq K)$ or $(H_B > 1)$ then the left-hand-side is zero and the right-hand-side contains only non-negative terms. Conversely if $(S_1 > K)$ and $(H_B \leq 1)$ then there is equality in (7) for $(S_1 \geq \beta)$, and strict inequality for $(K < S_1 < \beta)$.

The right-hand-side of (7) has an interpretation in terms of a super-replicating strategy. This strategy is to purchase $(B-K)/(B-\beta)$ calls with strike β, and, if the underlying ever passes through the barrier, to sell forward $(\beta - K)/(B - \beta)$ units of the underlying. Since $S_{H_B} \geq B$ each unit of the forward transaction will raise at least $(B - S_1)$.

On taking expectations we get that, irrespective of the model for the underlying price process, the price of the knock-in call is bounded above by $(B-K)C(\beta)/(B-\beta)$. If we minimise over $\beta \in (K, B)$ we find that the optimum choice of β is the value $\beta \equiv a(\mu, B) \vee K$.

Finally we show that there is a price process for which the expected payoff of the option equals this proposed bound. Let $S^\dagger$ be the martingale of Lemma 2.3, so that

$$(S_1^\dagger > a) \subseteq (H_B^\dagger \leq 1) \subseteq (S_1^\dagger \geq a)$$

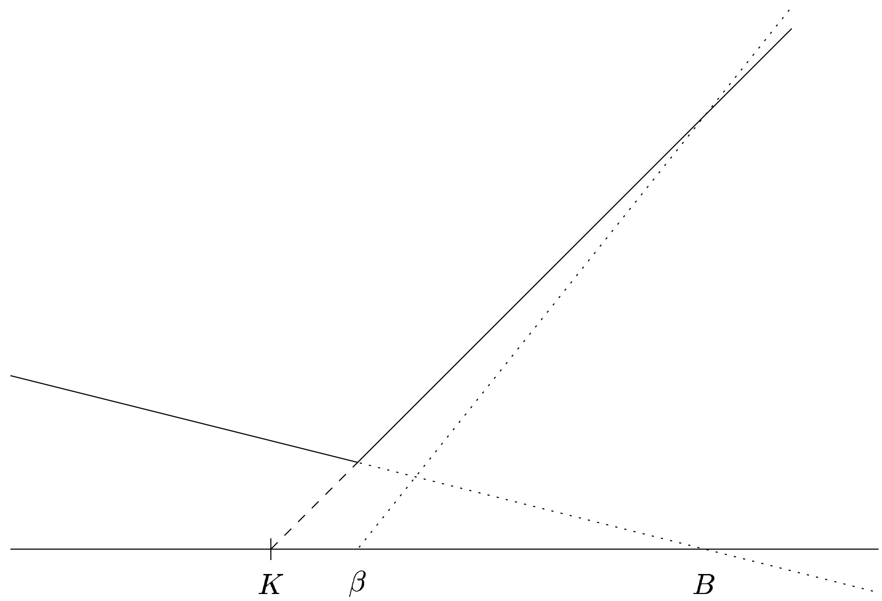

Figure 8: The same inequality, (7), on $(H_B \leq 1)$. The RHS of (7) represented by the thick line, is the sum of two components (dotted lines).

see Lemma A.1. If $a < K$ then $(S_1^\dagger - K)^+ I_{\{H_B^\dagger \leq 1\}} = (S_1^\dagger - K)^+$ and equality holds in (7). Conversely if $a \geq K$ then $(S_1^\dagger - a)^+ = 0$ on $(H_B^\dagger > 1)$ and

$$\frac{(B-K)}{(B-a)}(S_1^\dagger - a)^+ + \frac{(a-K)}{(B-a)}(B - S_1^\dagger) = (S_1^\dagger - K)$$

on $(H_B^\dagger \leq 1)$, so that for $\beta = a$ there is equality, for all realisations of $S_1^\dagger$, in (7). □

3.2 Up-and-out Call

The payoff of an up-and-out call is $(S_1 - K)^+ I_{\{H_B > 1\}}$.

If $B \leq K$ then the call is always knocked-out if there is to be a positive payoff, so the option is worthless.

Proposition 3.2 *For $K < B$ the arbitrage-free price of an up-and-out call is $C(K) - C(B) - (B - K)\mu([B,\infty))$. If the underlying price process can be assumed to be continuous, and if $\alpha \equiv \alpha(\mu, B) > K$, then the price is*

$$\frac{(\alpha - K)(B - S_0)}{B - \alpha} - \frac{B-K}{B-\alpha}(C(B) - C(\alpha)) - (B-K)\mu([B,\infty)),$$

else the continuity assumption has no affect and the arbitrage-free price is unchanged.

Proof:

If no continuity assumption is made then the upper bound on the price follows from taking expectations in the trivial inequality

$$(S_1 - K)^+ I_{\{H_B > 1\}} \leq (S_1 - K)^+ I_{\{S_1 < B\}}$$

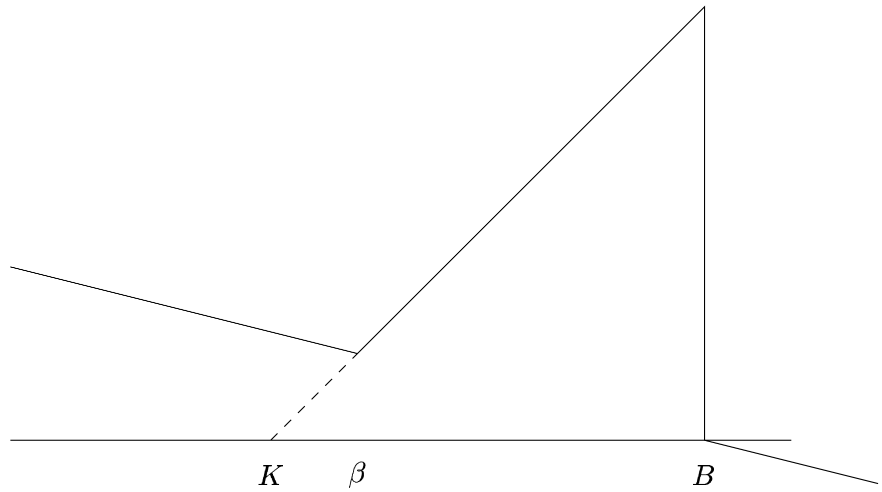

Figure 9: The case $(H_B > 1)$; the RHS of (8) (thick lines) dominates the LHS (dashed line), at least on $S_1 < B$.

Moreover, for the pure-jump martingale S^J from Lemma 2.6, this bound is attained.

For price-processes which are known to be continuous this bound can be refined. For any $\beta \in (K, B)$, we have

$$
\begin{aligned}
(S_1 - K)^+ I_{\{H_B>1\}} \quad \leq \quad & \frac{(\beta - K)(B - S_0)}{(B-\beta)} + \frac{(\beta-K)}{(B-\beta)}(S_0 - S_1) \\
& + \frac{(B-K)}{(B-\beta)}\{(S_1-\beta)^+ - (S_1 - B)^+\} - (B-K)I_{(S_1 \geq B)} \\
& + \frac{(\beta-K)}{(B-\beta)}(S_1 - B)I_{\{H_B \leq 1\}}.
\end{aligned} \tag{8}
$$

The key observation is that for a continuous martingale this last term has zero expectation. On both $(H_B > 1)$ and $(H_B \leq 1)$ each side of this expression is piecewise linear. Thus to verify that (8) holds it is sufficient to check the inequality at the points K, β, B and to consider the gradients near these points; alternatively see Figures 9 and 10.

Again the right-hand-side of the expression has an interpretation in terms of a super-replicating strategy. This strategy involves an initial cash holding, sale of the underlying, purchase of a portfolio of calls and a binary, and a dynamic component involving buying forward $(\beta - K)/(B - \beta)$ units of the underlying, if and when the underlying first reaches the barrier B. By the continuity assumption this last transaction is made at the price B, and on the forward market is costless; the super-replication argument fails if the underlying can jump over the level B.

On taking expectations we get that for each $\beta \in (K, B)$ the price of the knock-out call is bounded above by

$$\frac{(\beta-K)(B-S_0)}{(B-\beta)} + \frac{(B-K)}{(B-\beta)}\{C(\beta) - C(B)\} - (B-K)\mu([B,\infty)).$$

If we minimise over $\beta \leq B$ we find that the optimum choice of β is the value $\beta \equiv \alpha(\mu, B) \vee K$.

Finally we show that there is a continuous price process for which the expected payoff of the option

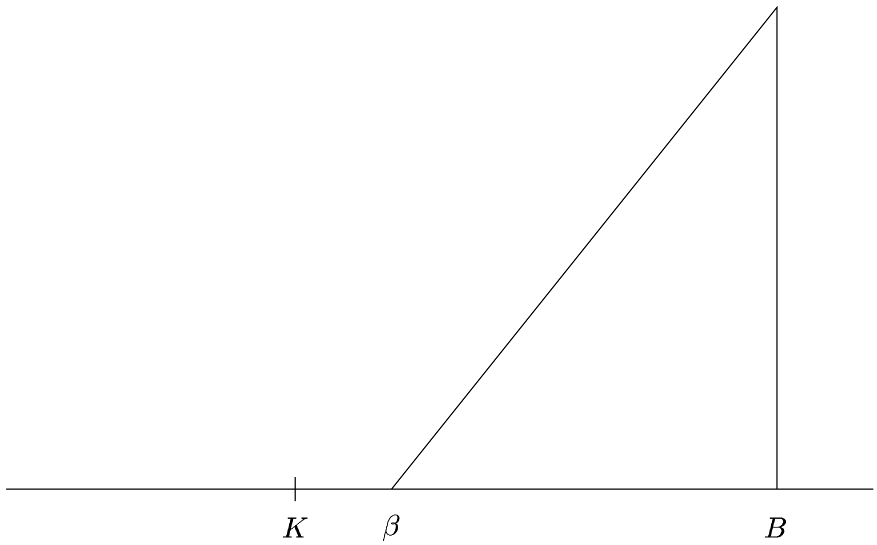

Figure 10: The case $(H_B > 1)$, the RHS of (8) is now non-negative.

equals this proposed bound. Let $S^{\#}$ be the martingale of Lemma 2.6, so that

$$(\alpha < S_1^{\#} < B) \subseteq (H_B^{\#} > 1) \subseteq (\alpha \leq S_1^{\#} < B),$$

see Lemma A.4. If $\alpha \leq K$ then $(S_1^{\#} - K)^+ I_{\{H_B^{\#} > 1\}} = (S_1^{\#} - K)^+ I_{\{S_1^{\#} < B\}}$. Conversely, in the interesting case where $\alpha > K$, then $(S_1^{\#} - \alpha)^+ = (S^{\#} - \alpha)$ on $(H_B^{\#} > 1)$ so that on this set

$$\frac{(\alpha - K)}{(B - \alpha)}(B - S_1^{\#}) + \frac{(B - K)}{(B - \alpha)}(S_1^{\#} - \alpha) = (S_1^{\#} - K).$$

Further on $(H_B^{\#} \leq 1)$ both

$$\begin{aligned} \frac{(\alpha - K)(B - S_0^{\#})}{(B - \alpha)} + \frac{(\alpha - K)}{(B - \alpha)}(S_0^{\#} - S_1^{\#}) + \frac{(\alpha - K)}{(B - \alpha)}(S_1^{\#} - B) &= 0\,; \\ \frac{(B - K)}{(B - \alpha)}\{(S_1^{\#} - \alpha)^+ - (S_1^{\#} - B)^+\} - (B - K) I_{(S_1^{\#} \geq B)} &= 0. \end{aligned}$$

Thus equality holds in (8), always, for the martingale $S^{\#}$. □

Remark 3.3 Using put-call parity we can rewrite the price of a up-and-out call as

$$(\alpha - K) + C(\alpha) + \frac{(\alpha - K)}{(B - \alpha)}P(\alpha) - \frac{(B - K)}{(B - \alpha)}C(B) - (B - K)\mu([B, \infty)).$$

3.3 Up-and-in Put

The payoff of a up-and-in put is $(K - S_1)^+ I_{\{H_B \leq 1\}}$.

Proposition 3.4 *For $K < B$ the arbitrage-free price of an up-and-in put is $P(K)$ if $\alpha \equiv \alpha(\mu, B) > K$ and*

$$\frac{(K - \alpha)}{(B - \alpha)}C(B) + \frac{(B - K)}{(B - \alpha)}P(\alpha)$$

otherwise. For $B \leq K$ the arbitrage-free price of an up-and-in put is

$$C(K) + \frac{(K-B)}{(B-a)} C(a).$$

Proof:
For $K < B$ the proof is based on the following inequality which holds for any $\beta \leq K$:

$$(K - S_1)^+ I_{\{H_B \leq 1\}} \leq \frac{(K-\beta)}{(B-\beta)}(S_1 - B)^+ + \frac{(B-K)}{(B-\beta)}(\beta - S_1)^+ + \frac{(K-\beta)}{(B-\beta)}(B - S_1) I_{\{H_B \leq 1\}}. \tag{9}$$

The right-hand-side of this inequality describes the payoff of a super-replicating strategy. The last term is interpreted as the balance from selling forward the underlying when the price process first crosses the barrier at time τ. If the price process jumps over the barrier and $S_\tau > B$ then the hedging strategy makes additional profits of $(K-\beta)(S_\tau - B)/(B-\beta)$. For the martingale $S^{\#}$, and the choice $\beta = \alpha \vee K$, there is equality in (9).

For $B \leq K$ the super-replication strategy follows from the inequality (valid for any $\beta \leq B$)

$$(K - S_1)^+ I_{\{H_B \leq 1\}} \leq (S_1 - K)^+ + \frac{(K-B)}{(B-\beta)}(S_1 - \beta)^+ + \frac{(K-\beta)}{(B-\beta)}(B - S_1) I_{\{H_B \leq 1\}}. \tag{10}$$

The optimal choice of β is $\beta \equiv a$. For the martingale $S^{\dagger}$ there is equality in (10) for $\beta = a(\mu, B)$. $\square$

3.4 Up-and-out Put

The payoff of an up-and-out put is $(K - S_1)^+ I_{\{H_B > 1\}}$.

Proposition 3.5 *For $K < B$ the arbitrage-free price of an up-and-out put is $P(K)$. If the underlying price process can be assumed to be continuous then the bound becomes*

$$\frac{(K - a \wedge K)}{(B - a \wedge K)}(B - S_0) + \frac{(B-K)}{(B - a \wedge K)} P(a \wedge K) \equiv (K - S_0) + \frac{(B-K)}{(B - a \wedge K)} C(a \wedge K).$$

For $B \leq K$ the arbitrage-free price of an up-and-out put is $P(B) + (K-B)\mu((-\infty, B))$. If the underlying price process can be assumed to be continuous then the bound becomes

$$(K-B)\mu((-\infty, B)) + P(B) + \frac{(K-B)}{(B-\alpha)} P(\alpha) - \frac{(K-\alpha)}{(B-\alpha)} C(B).$$

Remark 3.6 In the financial case, with non-negative price processes it must be true that $\mu((-\infty, B)) = \mu([0, B))$.

Proof:
If no continuity assumption is made then the upper bounds on the price follow from taking expectations in the trivial inequality

$$(K - S_1)^+ I_{\{H_B > 1\}} \leq (K - S_1)^+ I_{\{S_1 < B\}}$$

Moreover, for the pure-jump martingale S^J from Lemma 2.6, the bounds are attained.

For price-processes which are known to be continuous the bounds can be refined. Consider first the case where $K < B$. Then, for $\beta < K$,

$$
\begin{aligned}
(K-S_1)^+ I_{\{H_B>1\}} \;\leq\; & \frac{(K-\beta)}{(B-\beta)}(B-S_1) \\
& + \frac{(B-K)}{(B-\beta)}(\beta-S_1)^+ + \frac{(K-\beta)}{(B-\beta)}(S_1-B)I_{\{H_B\leq 1\}}.
\end{aligned} \tag{11}
$$

(If ($H_B \leq 1$) the outer terms on the right-hand-side cancel and the middle term is non-negative. Conversely, if ($H_B > 1$) then the two-sides agree for ($S_1 \leq \beta$), and there is strict inequality in (11) for ($\beta < S_1 < B$).)

The cost of the super-replicating strategy is

$$\frac{(K-\beta)}{(B-\beta)}(B-S_0) + \frac{(B-K)}{(B-\beta)}P(\beta)$$

which, using put-call parity becomes

$$(K-S_0) + \frac{(B-K)}{(B-\beta)}C(\beta).$$

This is minimised over $\beta \leq K$ at $\beta = a(\mu, B) \wedge K$. Moreover, for the martingale $S^\dagger$ there is equality in (11) for $\beta = a \wedge K$.

For $B \leq K$ the optimal bound is based on the inequality

$$
\begin{aligned}
(K-S_1)^+ I_{\{H_B>1\}} \;\leq\; & (K-B)I_{\{S_1<B\}} + (B-S_1)^+ \\
& + \frac{(K-B)}{(B-\beta)}(\beta-S_1)^+ - \frac{(K-\beta)}{(B-\beta)}(S_1-B)^+ \\
& + \frac{(K-\beta)}{(B-\beta)}(S_1-B)I_{\{H_B\geq 1\}}.
\end{aligned} \tag{12}
$$

Taking expectations we find that the optimal value of β minimises $[(K-B)P(\beta)-(K-\beta)C(B)]/(B-\beta)$ and thus the optimal β is $\beta = \alpha(\mu, B)$. For this choice of β there is equality in (12) for all realisations of the martingale price process $S^\#$. □

3.5 Lower bounds

In the above analysis we have concentrated on finding upper bounds on the prices of barrier options. If the barrier option is traded for a price which exceeds the upper bound then arbitrage opportunities are created.

Here we give the corresponding lower bound results, which follow from (6).

Proposition 3.7 *(i) The lower bound on the price of an up-and-in call with payoff $(S_1-K)^+ I_{\{H_B\leq 1\}}$ is $C(K)$ if $B \leq K$. If $K < B$ then the lower bound is $C(B) + (B-K)\mu([B,\infty))$. If further the underlying price process can be assumed to be continuous, and if $K < \alpha$ then the lower bound rises to*

$$C(K) - \frac{(\alpha-K)(B-S_0)}{B-\alpha} + \frac{B-K}{B-\alpha}(C(B)-C(\alpha)) + (B-K)\mu([B,\infty)).$$

If $\alpha \le K$ the continuity assumption has no effect.

(ii) The lower bound on the price of an up-and-out call with payoff $(S_1 - K)^+ I_{\{H_B > 1\}}$ is 0 if $B \le K$ or if $K < B$ and $a \le K$. If $B > a > K$ then the lower bound is

$$C(K) - \frac{(B-K)}{(B-a)} C(a).$$

These bounds on the up-and-out call are not improved if the additional assumption that the price process is continuous is introduced.

(iii) The lower bound on the price of an up-and-in put with payoff $(K - S_1)^+ I_{\{H_B \le 1\}}$ depends on whether $K < B$ or $B \le K$. If $K < B$ the lower bound is 0. If the price process can be assumed to be continuous, and if $a < K$ then the lower bound rises to

$$P(K) - \frac{(K-a)}{(B-a)}(B - S_0) - \frac{(B-K)}{(B-a)} P(a) \equiv P(K) - (K - S_0) - \frac{(B-K)}{(B-a)} C(a).$$

If $B \le K$ then the lower bound is $P(K) - P(B) - (K - B)\mu((-\infty, B))$. If the price process can be assumed to be continuous the lower bound rises to

$$P(K) - (K-B)\mu((-\infty, B)) - P(B) - \frac{(K-B)}{(B-\alpha)} P(\alpha) + \frac{(K-\alpha)}{(B-\alpha)} C(B).$$

(iv) The lower bound on the price of an up-and-out put with payoff $(K - S_1)^+ I_{\{H_B > 1\}}$ depends on whether $K < B$ or $B < K$. If $K < B$ the lower bound is 0 if $K \le \alpha$ and

$$P(K) - \frac{(K-\alpha)}{(B-\alpha)} C(B) - \frac{(B-K)}{(B-\alpha)} P(\alpha)$$

otherwise. For $B \le K$ the lower bound is

$$K - S_0 - \frac{(K-B)}{(B-a)} C(a).$$

These bounds on the up-and-out put are not improved if the additional assumption that the price process is continuous is introduced.

4 Numerical Examples

In this section we review some numerical results which illustrate the conclusions of the previous sections. In particular we generate call and put prices from a constant elasticity of variance model (Cox and Ross [7]) and use these prices to calculate upper and lower bounds on the prices of knock-in options, and, for comparison purposes, prices produced by the Black-Scholes formula.

Call prices were generated assuming that they had arisen from a price process which was known to satisfy the stochastic differential equation $dS = \sigma\sqrt{S}dB$. (We assumed an initial asset value of 1, an option expiry at time $T = 1$, and a barrier at 1.04.) Since the source model was known it was also possible to calculate (numerically) the prices of knock-in puts and knock-in calls, to give 'true prices' for barrier derivatives. It should be noted however that there are many models consistent with the observed put and call prices, and the 'true price' is only correctly titled if the underlying process follows the CEV price.

Using these vanilla option prices it was possible to calculate, for each of a range of possible strikes, four estimates of the barrier option price for both up-and-in puts and calls. The first two were the upper and lower bound (using the assumption that the underlying price process was continuous where necessary), in accordance with the analysis in the previous section. The remaining two were calculated by inputting a volatility into the Black-Scholes (exponential Brownian motion) price for the barrier option. The first volatility (the associated option price is labelled 'BS price (1)' on the plots below), was the implied volatility of the call with the same strike as the knock-in call that was being priced. The second inputted volatility was the volatility of an at-the-money call.

Figures 11 and 12 present the results for knock-in puts and calls respectively.

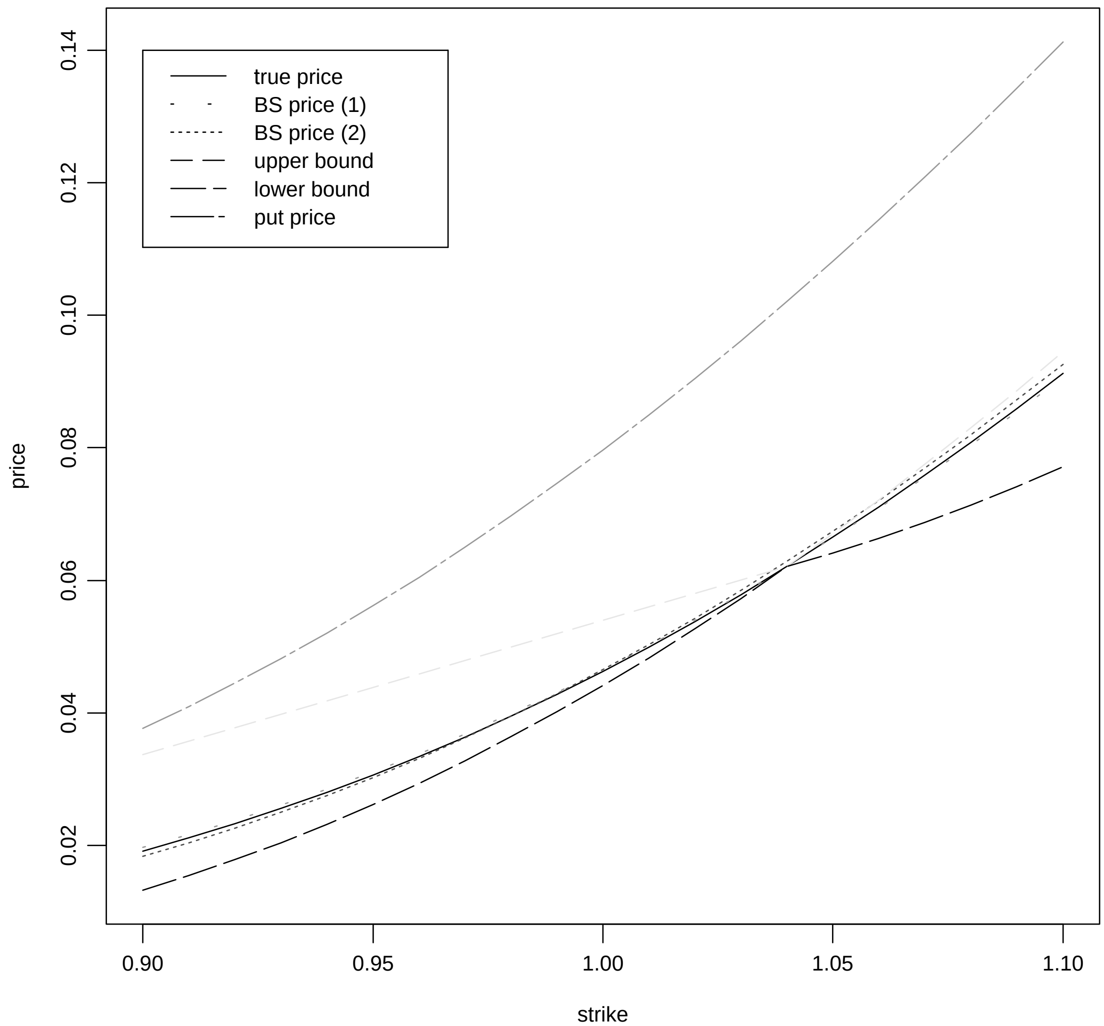

Figure 11: Prices for up-and-in puts as a function of strike from the constant elasticity of variance model $dS = 0.2\sqrt{S}dB$ with $S_0 = 1$. The knock-in barrier is at 1.04. The highest line represents the price of a put without the knock-in feature.

In Figure 11 the highest price corresponds to the put prices and the knock-in puts are considerably cheaper. For strikes near the the barrier the upper and lower bounds on the barrier option are very close, and when the strike equals the barrier they are equal (see the example in the introduction). The Black-Scholes price calculated using the implied volatility of a call with the same strike as the barrier option is almost indistinguishable from the CEV price (though it tends to overestimate knock-in put prices with strikes below the barrier). In contrast the Black-Scholes price calculated using the volatility of an at-the-money call tends to have the reverse bias when compared with the CEV price, and to differ from the CEV price by a larger amount. These biases are a function of the fact that the instantaneous volatility of the true price process is a decreasing function of the price, and this results, in turn, in the implied volatility of vanilla options being a decreasing function of strike. Moreover, for a CEV model with co-efficient greater than one these biases are reversed.

Most importantly, for a range of strikes, the Black-Scholes prices calculated using an at-the-money implied volatility lie above the no-arbitrage upper bound. Thus, the options writer who naively calibrates a Black-Scholes model with an at-the-money option is pricing derivatives in a fashion which leads to arbitrage opportunities.

It appears that the upper bound for strikes above the barrier, and the lower bound for strikes below the barrier, are the tighter of the two theoretical bounds. These are the bounds which correspond to the maximal maximum inequality (Blackwell and Dubins [3], Lemma 2.3) rather than the minimal maximum inequality (Perkins [16], Lemma 2.6).

From Figure 12 we see that for all the strikes in the range 0.9 to 1.1, the upper bound on the price of a knock-in call is identical to the price of a call without the knock-in feature, and the associated super-replicating strategy is simply to hold one unit of the underlying call. Further, for strikes above the barrier, the knock-in call is equivalent to the vanilla call, and the upper and lower bounds are equal; note also that the 'true price', and the first Black-Scholes price also give the same price. For this example the barrier feature appears to make little difference and all the estimates of the knock-in call price lie close to the price of the associated vanilla call. However, again naive use of the Black-Scholes model and the implied volatility of an at-the-money call leads to mis-pricing of barrier options and arbitrage profits.

Under certain circumstances, the options writer who is over-reliant on the exponential Brownian motion model, and who calibrates their model using an at-the-money option creates arbitrage opportunities. In these circumstances these opportunities can be used to create riskless and model-free profits for the trader who follows the strategies outlined in Section 3. A more sophisticated trader will be aware of the non-constant nature of (implied) volatility and adjust their models to account for this feature, thus preventing arbitrage. However, this complicates the hedging strategies they must then use. Thus, even when barrier options are trading within the theoretical bounds, the simplicity (and therefore the reduced transaction costs) associated with the strategies of Section 3, makes them effective, practical and attractive.

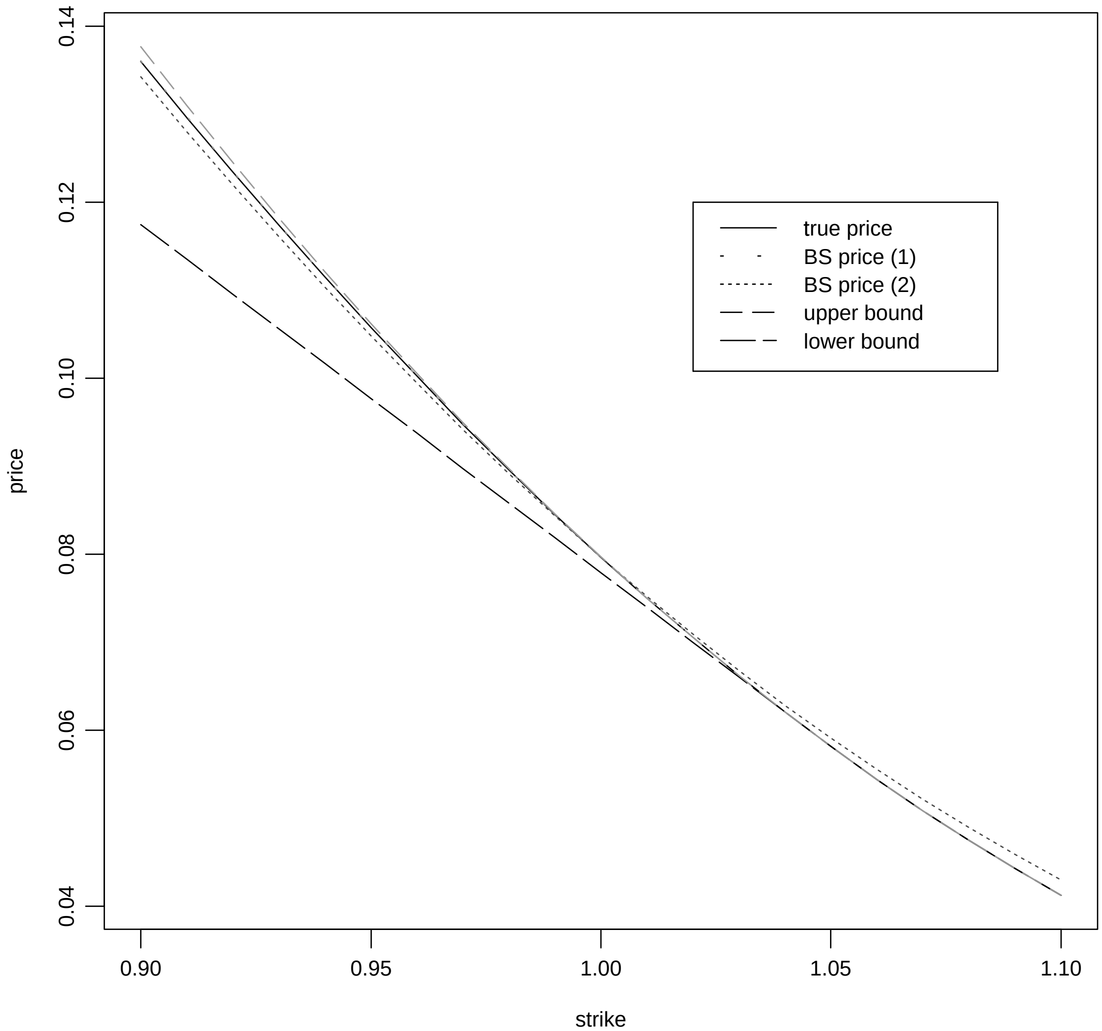

Figure 12: Prices for up-and-in calls as a function of strike from the constant elasticity of variance model $dS = 0.2\sqrt{S}dB$, with $S_0 = 1$. The knock-in barrier is at 1.04. The price of a call without the knock-in feature is the same as the upper bound on the knock-in call, at least for the range of strikes shown in this diagram.

5 The finite basis situation

To date we have considered the idealistic and unrealistic situation in which the space of liquidly traded instruments which can be used as hedging instruments includes calls of all possible strikes. In contrast in this section we suppose that only a finite number of calls are traded. This has two immediate consequences for our previous analysis: firstly the implied pricing distribution μ is not uniquely specified; and secondly calls of arbitrary strike are not necessarily available as hedging instruments. It is still possible however to determine super-replicating hedging strategies (and the associated cost of the initial portfolio detemines an upper bound on the price of the option). These strategies remain robust to model mis-specification.

We assume that calls with maturity 1 and strikes $k_1 < \cdots < k_n$ are traded. Since the option hedger always has the opportunity to hedge using the underlying instead of the traded calls, and since the underlying asset is equivalent to a call with zero strike, it is convenient to suppose that there is an additional call in our family of traded assets with strike $k_0 = 0$ and price $C(0) = S_0$.

The main ideas and techniques are the same as in the continuum case. We provide an example of an upper bound for a digital option; the case for other barrier options being similar, if more complicated.

Lemma 5.1 *The upper bound on the price of a digital option with payoff $I_{\{H_B \le 1\}}$ is*

$$\min_{k_l < B} \frac{C(k_l)}{B - k_l}. \tag{13}$$

There is a model consistent with the observed option prices for which the true call price is as given in (13).

Proof:
For each $k_l < B$ the strategy outlined in Lemma 2.3 and Corollary 2.5 super-replicates and yields a bound on the digital option price. The smallest of these upper bounds is the expression (13).

For the second part of the result we exhibit a model consistent with the observed call prices, for which the results of Section 2 show that the value in (13) is tight.

Define

$$k_{n+1} = (k_n C(k_{n-1}) - k_{n-1} C(k_n))/(C(k_{n-1}) - C(k_n))$$

and let $k^B_{n+1} = \max\{k_{n+1}, B+1\}$. Now define $C^\dagger$ for all $k \ge 0$ by linear interpolation between the points $(k_i, C(k_i))_{0 \le i \le n}$ and $(k^B_{n+1}, 0)$, and by $C^\dagger(k) = 0$ for $k \ge k^B_{n+1}$. This call price function is consistent with the observed call prices $\{C(k_l)\}_{l=0,1,\dots,n}$. Moreover for this C, the infimum of $C^\dagger(k)/(B-k)$ must be attained at one of the values $k_0, \dots, k_n$, see Figure 5. Hence the results in Section 2 show that the bound in (13) is tight for this call price function, and hence cannot be refined for the general problem. □

A Appendix

We need to prove Lemmas 2.3 and 2.6. We make repeated use of the fact that given a target distribution ν with mean x, there is a continuous martingale M, with $M_0 = x$, such that $M_1 \sim \nu$. One way to construct such a martingale is to take a solution of the Skorokhod problem (which concerns embedding distributions in Brownian motion, see [2]), and to time-change the resulting process.

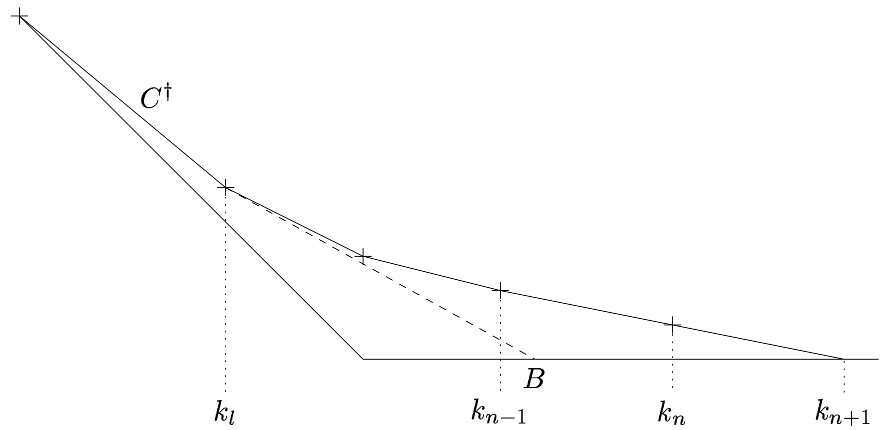

Figure 13: The call price function which shows that the bound in (13) is tight. The crosses denote the points $(k_i, C(k_i))$ which represent the prices of traded calls.

Lemma A.1 (Lemma 2.3) *Let S be any martingale with S_0 given, and such that S_1 has law μ. Let $B > S_0$. Then*

$$\mathbb{P}(H_B \leq 1) \leq \inf_{y \leq B} \frac{C(y)}{B-y}. \tag{14}$$

The infimum is attained, at $a = a(\mu, B)$ say, and there is a martingale $S^\dagger$ such that if $H_B^\dagger = \inf\{u : S_u^\dagger \geq B\}$ then

$$\mathbb{P}(H_B^\dagger \leq 1) = \frac{C(a)}{B-a}. \tag{15}$$

Moreover for this martingale

$$(S_1^\dagger > a) \subseteq (H_B^\dagger \leq 1) \subseteq (S_1^\dagger \geq a). \tag{16}$$

Proof:
Recall that we proved (14) in Section 2.

Throughout $B > S_0$ is fixed. In the financial context μ has support contained in the positive reals, but we make no such restriction.

Since $\lim_{y \downarrow -\infty} \mathbb{E}(y - S_1)^+ = 0$ it follows that $\mathbb{E}(S_1 - y)^+ + y < B$ for sufficiently small y. In particular $\inf_{y<B} C(y)/(B-y) < 1$, and $\lim_{y \downarrow -\infty} C(y)/(B-y) = 1$. Further $C(y)/(B-y)$ is continuous in y so that the infimum is attained.

Let this infimum be attained at $y = a = a(\mu, B)$. Note that a need not be unique. By differentiation we see that

$$\mathbb{P}(S_1 > a) \leq \frac{C(a)}{(B-a)} \leq \mathbb{P}(S_1 \geq a).$$

Now let γ solve

$$\frac{S_0 - \gamma}{B - \gamma} = \frac{C(a)}{(B-a)}.$$

In fact γ is the x-co-ordinate of the point where the tangent to C which passes through the x-axis at B, meets the line $y = S_0 - x$. In particular $\gamma < S_0$.

Let A_B be any set such that $(S_1 > a) \subseteq A_B \subseteq (S_1 \geq a)$ and such that $\mathbb{P}(A_B) = C(a)/(B-a)$. Then

$$\mathbb{E}(S_1; A_B) = C(a) + a\mathbb{P}(A_B) = \frac{BC(a)}{B-a} = B\mathbb{P}(A_B)$$

and

$$\mathbb{E}(S_1; A_B^c) = S_0 - \frac{BC(a)}{B-a} = S_0 - \frac{B(S_0-\gamma)}{B-\gamma} = \frac{\gamma(B-S_0)}{B-\gamma} = \gamma\mathbb{P}(A_B^c).$$

Now fix $\epsilon \in (0,1)$ and consider the martingale $S^\dagger$ with $S_0^\dagger = S_0$ and such that

$$S_\epsilon^\dagger = \begin{cases} B & \text{with probability } C(a)/(B-a), \\ \gamma & \text{otherwise.} \end{cases}$$

Suppose further that, on $(S_\epsilon^\dagger = B)$, $S_1^\dagger$ has the law μ restricted to A_B, and on $S_\epsilon^\dagger = \gamma$, $S_1^\dagger$ has the law μ restricted to A_B^c. Then the martingale property is preserved and $(H_B^\dagger \leq 1) \equiv A_B$. □

Remark A.2 The Azema-Yor construction produces a martingale for which (16), and therefore (15), holds uniformly in B.

Remark A.3 The identity (16) means that for the martingale price process $S^\dagger$ the super-replicating strategy in Corollary 2.5 is a replicating strategy and makes zero profits.

Lemma A.4 (Lemma 2.6) *Let S be any martingale with S_0 given, and such that S_1 has law μ. Let $B > S_0$. Then the trivial inequality*

$$\mathbb{P}(H_B \leq 1) \geq \mathbb{P}(S_1 \geq B) \equiv \mu([B,\infty)) \tag{17}$$

holds, and there is a martingale S^J for which equality holds in (17).

If S is known to be a continuous martingale then

$$\mathbb{P}(H_B \leq 1) \geq \mathbb{P}(S_1 \geq B) + \sup_{y<B} \frac{C(B) - P(y)}{B-y}.$$

This supremum is attained, at $\alpha = \alpha(\mu, B)$ say, and there is a martingale $S^\#$ for which

$$\mathbb{P}(H_B^\# \leq 1) = \mathbb{P}(S_1 \geq B) + \frac{C(B) - P(\alpha)}{B-\alpha}. \tag{18}$$

For this martingale

$$(S_1^\# \geq B) \cup (S_1^\# < \alpha) \subseteq (H_B^\# \leq 1) \subseteq (S_1^\# \geq B) \cup (S_1^\# \leq \alpha). \tag{19}$$

Proof:

Only the last part remains to be proved. Since $C(B) - P(y)$ is concave decreasing in y, with limit $C(B)$ at $y = -\infty$, and negative at $y = S_0$ we have that the supremum of $(C(B) - P(y))/(B-y)$ is attained with maximum value at $y = \alpha = \alpha(\mu, B) < S_0$. By differentiation we see that

$$\mathbb{P}(S_1 < \alpha) \leq \frac{C(B) - P(\alpha)}{(B-\alpha)} \leq \mathbb{P}(S_1 \leq \alpha).$$

Let F_B be any set such that $(S_1 < \alpha) \subseteq F_B \subseteq (S_1 \le \alpha)$ and such that $\mathbb{P}(F_B) = (C(B) - P(\alpha))/(B - \alpha)$. Then if $G_B \equiv F_B \cup (S_1 \ge B)$,

$$\begin{aligned} \mathbb{E}(S_1; G_B) &= \mathbb{E}(S_1; S_1 \ge B) + \mathbb{E}(S_1; F_B) \\ &= C(B) + B\mathbb{P}(S_1 \ge B) - P(\alpha) + \alpha\mathbb{P}(F_B) \\ &= B\mathbb{P}(S_1 \ge B) + B\mathbb{P}(F_B) \;=\; B\mathbb{P}(G_B). \end{aligned} \tag{20}$$

Let λ solve

$$\frac{(S_0 - \lambda)}{(B - \lambda)} = \frac{C(B) - P(\alpha)}{B - \alpha} + \mathbb{P}(S_1 \ge B) \equiv \mathbb{P}(G_B).$$

It follows that $\lambda \in (\alpha, S_0)$, and more importantly

$$\mathbb{E}(S_1; G_B^c) = S_0 - \frac{B(S_0 - \lambda)}{B - \lambda} = \frac{\lambda(B - S_0)}{B - \lambda} = \lambda\mathbb{P}(G_B^c).$$

Now, for $\epsilon \in (0, 1)$, consider the continuous martingale $S^{\#}$ with $S_0^{\#} = S_0$, and such that

$$S_\epsilon^{\#} = \begin{cases} B & \text{with probability } (S_0 - \lambda)/(B - \lambda), \\ \lambda & \text{otherwise.} \end{cases}$$

On the set $(S_\epsilon^{\#} = B)$ let $S_1^{\#}$ have the law μ restricted to G_B; since (20) guarantees that the first moment condition is satisfied, this can be done in such a way as to preserve the martingale property. Conversely, associate $(S_\epsilon^{\#} = \lambda)$ with G_B^c. By construction the sets $(H_B^{\#} \le 1)$ and G_B are identical.

□

Remark A.5 The construction given in Perkins [16] produces a martingale for which (19) and therefore (18) holds uniformly in $B > S_0$.

B Barriers below S_0

To date the analysis has considered digital and barrier options with $B > S_0$. It is possible to modify the above discussion to cover the case $B < S_0$, but instead we can deduce the relevant results by appropriate transforms; an approach we now outline.

Fix $B < S_0$. Let $s_t = 2S_0 - S_t$ and let $b = 2S_0 - B$. If also $k = 2S_0 - K$ then $(s_1 - k)^+ = (K - S_1)^+$ so that

$$c(k) := \mathbb{E}^\mu((s_1 - k)^+) = P(K)$$

and call prices for the 'price process' s_t are known. Further, if $H_B = \inf\{t : S_t \le B < S_0\}$, then $H_B \equiv h_b = \inf\{t : s_t \ge b > S_0\}$. Then

$$\mathbb{P}(H_B \le 1) = \mathbb{P}(h_b \le 1) \le \inf_{y<b} \frac{c(y)}{b - y} = \inf_{z>B} \frac{P(z)}{z - B}$$

and, if p gives the price of a put for s,

$$\begin{aligned} \mathbb{P}(H_B \le 1) = \mathbb{P}(h_b \le 1) &\ge \mathbb{P}(s_1 \ge b) + \sup_{y<b} \frac{c(b) - p(y)}{b - y} \\ &= \mathbb{P}(S_1 \le B) + \sup_{z>B} \frac{P(B) - C(z)}{z - B} \end{aligned}$$

In summary:

Lemma B.1 *For $B < S_0$,*

$$\mathbb{P}(H_B \leq 1) \leq \frac{P(d)}{d-B}$$

where $d = d(\mu, B)$ is chosen to minimise $P(z)/(z-B)$, and,

$$\mathbb{P}(H_B \leq 1) \geq \mathbb{P}(S_1 \leq B) + \frac{P(B) - C(\delta)}{\delta - B}$$

where $\delta = \delta(\mu, B)$ is chosen to minimise $(P(B) - C(z))/(z-B)$.

Corollary B.2 *For $B < S_0$ the price of a digital option with payoff $I_{\{H_B \leq 1\}}$ must lie in the range*

$$\left[\mathbb{P}(S_1 \leq B), \frac{P(d)}{d-B}\right]$$

else arbitrage opportunities exist. If the price process is known to be continuous then the feasible range becomes

$$\left[\mathbb{P}(S_1 \leq B) + \frac{P(B) - C(\delta)}{\delta - B}, \frac{P(d)}{d-B}\right].$$

We now consider down-and-in and down-and-out puts and calls. Results for these options can be deduced from the bounds for up-and-in and up-and-out by transformations similar to those used before Lemma B.1. For $B < S_0$ let $H_B = \inf\{t : S_t \leq B < S_0\}$.

Proposition B.3 *(i) Down-and-in call with payoff $(S_1 - K)^+ I_{\{H_B \leq 1\}}$.*
For $K \leq B$ the price of the down-and-in call must lie in the range

$$\left[C(K) - C(B) - (B-K)\mu((B,\infty)), P(K) + \frac{(B-K)}{(d-B)}P(d)\right].$$

If also the price process is necessarily continuous then the lower bound becomes

$$C(K) - C(B) - (B-K)\mu((B,\infty)) - \frac{(B-K)}{(\delta-B)}C(\delta) + \frac{(\delta-K)}{(\delta-B)}P(B).$$

For $B < K$ the price must lie in the range

$$\left[0, \frac{(\delta \vee K - K)}{(\delta \vee K - B)}P(B) + \frac{(K-B)}{(\delta \vee K - B)}C(\delta \vee K)\right].$$

If also the price process is necessarily continuous then the lower bound becomes

$$P(K) - \frac{(K-B)}{(d \vee K - B)}P(d \vee K).$$

(ii) Down-and-out call with payoff $(S_1 - K)^+ I_{\{H_B > 1\}}$.
For $K < B$ the price must lie in the range

$$\left[S_0 - K - \frac{(B-K)}{(d-B)}P(d), C(B) + (B-K)\mu((B,\infty))\right].$$

If also the price process is necessarily continuous then the upper bound becomes

$$C(B) + (B-K)\mu((B,\infty)) + \frac{(B-K)}{(\delta-B)}C(\delta) - \frac{(\delta-K)}{(\delta-B)}P(B).$$

For $B \leq K$ the price must lie in the range

$$\left[C(K) - \frac{(\delta \vee K - K)}{(\delta \vee K - B)} P(B) - \frac{(K-B)}{(\delta \vee K - B)} C(\delta \vee K), C(K)\right].$$

If also the price process is necessarily continuous then the upper bound becomes

$$\frac{(d \vee K - K)}{(d \vee K - B)}(S_0 - B) + \frac{(K-B)}{(d \vee K - B)} C(d \vee K).$$

(iii) Down-and-in put with payoff $(K - S_1)^+ I_{\{H_B \leq 1\}}$.
For $K \leq B$ the knock-in condition is redundant and the price of the down-and-in put is $P(K)$.
For $B < K$ the price must lie in the range

$$\left[P(B) + (K-B)\mu((-\infty, B]), \frac{(K-B)}{(d \wedge K - B)} P(d \wedge K)\right].$$

If also the price process is necessarily continuous then the lower bound becomes

$$P(K) - \frac{(K - \delta \wedge K)}{(\delta \wedge K - B)}(S_0 - B) + \frac{(K-B)}{(\delta \wedge K - B)}(P(B) - P(\delta \wedge K)) + (K-B)\mu((-\infty, B]).$$

(iv) Down-and-out put with payoff $(K - S_1)^+ I_{\{H_B > 1\}}$.
For $K \leq B$ the knock-in condition is redundant and the price of the down-and-in put is 0.
For $B < K$ the price must lie in the range

$$\left[P(K) - \frac{(K-B)}{(d \wedge K - B)} P(d \wedge K), P(K) - P(B) - (K-B)\mu((-\infty, B])\right].$$

If also the price process is necessarily continuous then the upper bound becomes

$$\frac{(K - \delta \wedge K)}{(\delta \wedge K - B)}(S_0 - B) + \frac{(K-B)}{(\delta \wedge K - B)}(P(\delta \wedge K) - P(B)) - (K-B)\mu((-\infty, B]).$$

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