A dynamic approach to the modelling of correlation credit derivatives using Markov chains

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Abstract

The modelling of credit events is in effect the modelling of the times to default of various names. The distribution of individual times to default can be calibrated from CDS quotes, but for more complicated instruments, such as CDOs, the joint law is needed. Industry practice is to model this correlation using a copula or base correlation approach, both of which suffer significant deficiencies. We present a new approach to default correlation modelling, where defaults of different names are driven by a common continuous-time Markov process. Individual default probabilities and default correlations can be calculated in closed form. As illustrations, CDO tranches with name-dependent random losses are computed using Laplace transform techniques. The model is calibrated to standard tranche spreads with encouraging results.

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1 Introduction

The current industry approach to the pricing of multi-name credit derivatives makes use of copula functions to model the dependence between issuers in a given portfolio of defaultable securities. This approach is problematic for two main reasons: there is no dynamic consistency, and there is no theoretical basis for the choice of any particular dependence structure. The root cause of the problems is bad modelling - the dependence is forced into the model at the very last stage, rather than growing organically from the modelling assumptions. An alternative industry based method is the so called *base correlation* approach. Again the main problem with the latter approach is the lack of dynamics which does not allow us to price, for example, forward starting credit products or options on tranches.

In order to overcome the deficiencies of the copula and base correlation approach a number of models have recently emerged in the credit literature. Duffie and Garleanu [DG] propose a reduced form approach based on affine processes. In particular, the default intensity of each individual obligor is assumed to be the sum of two affine processes, one common to all the names in the portfolio and the other credit specific. Individual default probabilities can be calculated explicitly, however CDO prices have to be recovered by resorting to Monte Carlo simulation. Chapovsky, Rennie and Tavares [CRT] introduce a model which is similar in spirit to [DG]. The authors however suggest a different specification of the names' stochastic intensity in order to improve tractability. In particular, default intensities are modeled as the the sum of a compensated common *random intensity driver* with tractable dynamics (e.g. CIR with jumps) and a deterministic name depended function which allows calibration to single name default curves. The model is calibrated to CDO tranche prices one maturity at the time (although only a subset of the parameter space is allowed to vary across maturities). Baxter [BX] models the *value of the firm* X_t as a sum of two Levy processes (based on the gamma process), representing idiosyncratic and systemic risk respectively. Joshi and Stacey [JS] introduce co-dependence between default times of different credit entities within the reduced form framework by time changing the intensity of default of each reference entity using a gamma process. The model calibrates to tranche spreads, however tranche pricing requires Monte Carlo simulation and no analytic or semi-analytic formulas are available. Moreover, as noted by the authors the model does not allow for dynamic credit spreads and cannot be used to price more exotic products. Hurd and Kuznetsov [HK] model the credit migration process of each obligor by a Markov chain with an absorbing state representing default. Correlation among creditors is introduced via a common time change of affine type. Albanese et al [ACDV] propose a rating transition model within the structural framework where the *distance to default* of each single obligor is represented by a Markov Chain. Correlation is introduced using non recombining trees. As far as data fitting is concerned, the model needs to be calibrated to both historical rating transition data and market prices of CDS and CDO. Brigo et al [BPT], Schoenbucher [S] and Sidenius et al [SPA] take a very different route from the approaches so far described and model directly the cumulative portfolio loss process. In particular, [BPT] assume the cumulative loss process to be the weighted sum of independent (time inhomogeneous) Poisson processes with (possibly) stochastic intensities. In Schoenbucher [S] the loss distribution of the portfolio is derived from the transition rates of an auxiliary time-inhomogeneous Markov chain which reproduces the desired transition probability distribution. Stochastic evolution of the loss distribution is obtained by equipping the transition rates with stochastic dynamics. Finally, Sidenius et al [SPA] model the dynamics of portfolio loss distributions in the absence of information about default times. This *background process* can be in principle be calibrated to liquid tranche price. They then proceed modeling the loss process itself as a Markov process conditioned on the path taken by the background process. The *top-down* approach followed by the latter authors is fairly different from ours as it does not contemplate the modeling of individual creditor default probabilities. However, we shall show that the dynamics of the loss process arising from our model can be approximated by a compound Poisson with stochastic, name dependent intensities. This allows us to recover a simple analytic expression for the loss distribution while retaining the ability to calibrate to single name default curves.

What we propose here is a new approach to the problem based on the use of a Markov process within the reduced-form framework. This completely deals with the main problems of the copula-based and base correlation approach. Default correlation is determined from market data by fitting the model to CDS and CDO data. Also the model is fully dynamic and it is suitable to price products such forward starting tranches and option on tranches.

We start by assuming there exists a process $(\xi_t)_{t\geq 0}$ which drives the common dynamics of the credits in the portfolio. We then model the survival probabilities up to time t of a given obligor, say i, conditional on the filtration generated by the process F_t^ξ as

$$P\left(\tau^i \geq t | F_t^\xi\right) = \exp\left(-C_t^i\right), \qquad (1)$$

where τ^i indicates the default time of the i^{th} reference entity, and C^i is an additive functional of the process. The simplest thing[1] for this is to take

$$C_t^i = \int_0^t \lambda^i(\xi_u) du$$

[1]... but as we shall see, not the only thing ...

where $\lambda^i(\xi)$ is a deterministic function of the chain, which we will refer to as the *(default) intensity (function)* of entity i. For simplicity, we shall limit our discussion to the case where $(\xi_t)_{t\geq 0}$ is a continuous-time finite-state Markov chain. This framework is already sufficiently flexible for practical purposes, and is simple enough to allow explicit computation using fast linear algebra routines.

Taking (1) as a starting point, it is easy to derive the individual conditional default probabilities in closed form. It is also straightforward to compute default correlations. Moreover, we show how to obtain a fast and reasonably accurate approximation to the price of CDO tranches based on a Poisson approximation. Exact solutions can be obtained by computing the Laplace transform of the portfolio loss distribution and related quantities and then resorting to numerical inversion techniques. The model calibrates closely to liquid tranche data, thus explaining the skew effect observed in CDO markets. Results of the calibration are presented in the last section.

2 Model specification and basic results.

In this section we introduce the main modelling ideas of the paper which will form the basic building blocks for the pricing of multi-name credit derivatives.

Consider a portfolio of N defaultable securities and assume that there exist a continuous-time finite-state irreducible Markov chain $(\xi_t)_{t\geq 0}$ with Q-matrix Q, generating a filtration F_t^ξ. Assume that conditional on the path of the chain, defaults of the N names will be independent, the survival probability of the i^{th} reference entity being given by

$$q_t^i = P\left(\tau^i \geq t | F_t^\xi\right) = \exp\left(-C_t^i\right), \tag{2}$$

where C_t^i is some additive functional of the chain of the form

$$C_t^i = \int_0^t \lambda^i(\xi_u)du + \sum_{j\neq k} w_{jk}^i J_{jk}(t). \tag{3}$$

Here, τ^i is the default time of the i^{th} name in the portfolio, λ^i is a deterministic function of the chain, $J_{jk}(t)$ denotes the number of jumps by time t from state j to state k, and the w_{jk}^i are non-negative weights.

In order to gain some intuition, one could think of the chain as representing the *state of health* of the economy. If the chain jumps from a state of economic growth to a state of recession, this may cause the conditional default intensity of some of the reference entities to go up, increasing the chances of observing a larger number of defaults in the portfolio. The jump itself may also trigger defaults. Note that the information about how the various credits in the portfolio are correlated is contained in the λ^i, the w^i_{jk}, and Q. An expression for the dynamic default correlation will be derived in section 3.

Throughout this paper we will also assume that the money market account takes the following form

$$B_t = \exp\left(\int_0^t r(\xi_u)du\right), \tag{4}$$

where again r is a deterministic function of the chain.

Remarks (i) Since our Markov-chain can only take a finite number of values, we shall assume without loss of generality, that $\xi_t \in I_m$, where $I_m \equiv \{1, \ldots, m\}$ for any $t \geq 0$. It follows that a function of the chain, say $g(\cdot)$, can be thought of as a N-dimensional vector whose i^{th} component is given by $g_i \equiv g(i)$. In this paper, we shall use the notation $g(i)$ or g_i to indicate component i of the vector and g without subscripts to denote the whole vector.

(ii) The vectors $\lambda^i(\cdot)$, r, the matrix w and the infinitesimal generator Q should be seen as parameters of the problem and calibrated to market data, such as CDO tranche spreads, CDS quotes and risk-free bonds. One of the nice features of the model is that the number of parameters can be adjusted, by modifying the number of chain states, to best reflect the availability of market data. As CDO markets become more liquid, a higher number of quotes are likely to become available. By increasing the number of parameters we are more likely to capture the extra information available in the market.

(iii) We shall assume that the process ξ is not observable.

In order to price derivatives on a portfolio of N defaultable securities, we need to be able to find the distributions of some non trivial random variable. For example, if $\ell_i \equiv A_i(1 - R_i)$ denotes the loss on the i^{th} name, in terms of the notional A_i and the (possibly random) recovery rate R_i, then the portfolio *cumulative loss process*

$$L_t \equiv \sum_{i=1}^{N} \ell_i I_{\{\tau_i \leq t\}} \tag{5}$$

is an object of interest. Apart from Monte Carlo, the only tools available to find the law of L_t are based on transforms. By conditioning firstly on the path of the chain, it is easy to see that the (discounted) Laplace transform of L_t is given by

$$E \exp(-\int_0^t r(\xi_s)ds - \alpha L_t) = E\left[\exp(-\int_0^t r(\xi_s)ds) \prod_{i=1}^{N} \left((1 - q^i_t)\zeta_i(\alpha) + q^i_t\right)\right], \tag{6}$$

where q_t^i is given by (2) and $\zeta_i(\alpha) = E[e^{-\alpha \ell_i}]$. This is the key relation linking our modelling approach at an abstract level to the kinds of calculation needed to price credit derivatives of various sorts. The (numerical) inversion of the Laplace transform (6) is the common first step; the method of Hosono [Ho], popularised by Abate & Whitt [AW], [AW1], is a fast and accurate solution. We discuss numerical approaches in Section 4, but before that we record the form of default correlation given by this approach.

3 Default correlation

Default and survival correlation can be easily calculated in our framework in closed form. We shall derive the expression for survival correlation; default correlation can be calculated, *mutatis mutandis*, in a similar fashion.

Routine calculations (see Appendix A), allow us to recover the survival probability of the i^{th} reference entity. In particular, we have that

$$\begin{aligned}
\tilde{q}_t^i(\xi_0) &\equiv E\left[1_{\{\tau^i \geq t\}} \mid \xi_0\right] \\
&= E\left[\exp\left(-\int_0^t \lambda^i(\xi_u)du - \sum_{j \neq k} w_{jk}^i J_{jk}(t)\right)\right] \\
&= \exp(t\tilde{Q}^i)\mathbf{1}(\xi_0), \qquad (7)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{Q}_{jk}^i &= Q_{jj} - \lambda_j^i \qquad (j = k); \qquad (8) \\
&= \exp(-w_{jk}^i)Q_{jk} \qquad (j \neq k). \qquad (9)
\end{aligned}$$

Note that survival probabilities depend on the current (unobservable) state of the chain ξ_0.

Assume for example we want to compute the joint survival probability of obligors i and j. Using the independence of default times given ξ, we obtain

$$\begin{aligned}
\tilde{q}_t^{ij}(\xi_0) &\equiv P(\tau^i \geq t, \tau^j \geq t \mid \xi_0) \\
&= \exp(t\tilde{Q}^{ij})\mathbf{1}(\xi_0),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{Q}_{kl}^{ij} &= Q_{kk} - \lambda_k^i - \lambda_k^j \qquad (k = l); \\
&= \exp(-w_{kl}^i - w_{kl}^j)Q_{kl} \qquad (k \neq l).
\end{aligned}$$

Elementary algebraic calculations allow us to recover the survival correlation $\rho_T(\xi_t)$ of i and j from the joint survival probability function and the individual survival probabilities:

$$\rho_t(\xi_0) = \frac{\tilde{q}_t^{ij}(\xi_0) - \tilde{q}_t^i(\xi_0)\tilde{q}_t^j(\xi_0)}{\sqrt{\tilde{q}_t^i(\xi_0)(1-\tilde{q}_t^i(\xi_0))}\sqrt{\tilde{q}_t^j(\xi_0)(1-\tilde{q}_t^j(\xi_0))}} \tag{10}$$

where

$$\tilde{q}_t^i(\xi_0) = \exp(t\tilde{Q}^i)\mathbf{1}(\xi_0) \tag{11}$$

as at (7).

Note that the survival correlation is obtained endogenously from the model, rather than being exogenously imposed as in the copula-based industry approach to default correlation. Also in our set-up the survival correlation becomes a stochastic process driven by ξ, making the model dynamically consistent.

4 Computational approaches

Our attention now focuses on the exact expression (6) for the discounted Laplace transform of the cumulative loss at time t. We have good techniques for inverting the transform, but first we have to be able to calculate it (or some approximation), and in this Section we discuss three possible approaches.

4.1 Exact method

The approach here is to multiply out the product on the right-hand side of (6). The individual terms are all quite easy to deal with, because each is the exponential of some additive functional of the Markov chain, and we are able to compute these expectations using fast linear algebra routines. The problem with this approach comes when the number N of names gets too big; with N names there are 2^N terms in the product when multiplied out, and each of these needs to be evaluated and inverted separately. When $N = 10$ there are 1024 such calculations, and typically we need to be able to handle values of N that are an order of magnitude bigger. Thus the 'exact' calculation method will be too cumbersome for general use.

4.2 Poisson approximation

The expression (2) for the survival probability of name i can be understood in terms of a standard Poisson process ν independent of the chain ξ. If the jump times of ν are denoted $S_1 < S_2 < \ldots$, then we may set

$$\tau^i \equiv \inf\{t : C^i_t > S_1\},$$

and then the relation (2) holds. The Poisson approximation we propose here is to allow name i to default *more than once*, at times

$$\tau^i_m \equiv \inf\{t : C^i_t > S_m\}, \qquad m = 0, 1, \ldots.$$

By doing this, we arrive at an expression $\bar{L}_t$ for the portfolio cumulative loss which overestimates L_t, because it includes (non-existent) second and subsequent losses of each of the names. The error we are committing by this is of the same order as the default probabilities themselves; typically this would be of the order of a few percent, which would be comparable to the error we could expect from a Monte Carlo approach. However, there is some simple trick we can employ to improve the approximation. The expected (discounted) number of losses for name i by time t using the Poisson method is given by $E\left[B_t^{-1}C^i\right]$ compared with a true value of $E\left[B_t^{-1}(1 - e^{-C^i})\right]$. So if we define

$$\beta^i_t \equiv \frac{E\left[B_t^{-1}(1 - e^{-C^i})\right]}{E\left[B_t^{-1}C^i\right]} \tag{12}$$

we can get a fairly good approximation for the Laplace transform of the cumulative loss by letting

$\bar{L}_t$:

$$E\exp(-\int_0^t r(\xi_s)ds - \alpha\bar{L}_t) = E\left[\exp(-\int_0^t r(\xi_s)ds + \sum_{i=1}^N \beta^i_t(\zeta_i(\alpha) - 1)C^i_t)\right]. \tag{13}$$

For each α, we are computing the mean of the exponential of an additive functional of the chain, and this is a simple and rapid calculation which can be carried out using formula (31) in Appendix A by setting

$$
\begin{aligned}
\nu &\equiv r + \sum_{i=1}^{N} \beta_t^i (1 - \zeta_i(\alpha)) \lambda^i \\
w_{jk} &\equiv \sum_{i=1}^{N} w_{jk}^i \, \beta_t^i (1 - \zeta_i(\alpha)) \\
g &\equiv \mathbf{1}
\end{aligned}
$$

4.3 Monte Carlo

Another approach to calculating (6) is to use Monte Carlo simulation to evaluate the right-hand side, and then invert the transform. The simulation algorithm is quite standard, however we highlight here a few pitfalls to be avoided.

Firstly, we do not generate paths by discretising the time interval into a large number of subintervals and then simulating the (many) individual steps of the chain; rather, we use the jump-hold construction of the Markov chain, starting from the embedded discrete-time jump chain with exponentially-distributed residence times in the states passed through. This is far more efficient, and makes the calculation of additive functionals of the path a triviality.

Secondly, inversion of the Laplace transform will require evaluation of the transform at many different values of α; we do not of course simulate a different chain for each value of α, but keep the same chain for all evaluations.

Finally although the Monte Carlo approach is relatively fast for pricing purposes we find it more efficient to use the Poisson approximation for calibrating to tranche spreads, given the higher speed of the latter method (roughly by a factor of 20).

5 Example: CDSs

In order to calibrate the model to market prices, we need to be able to compute, among other things, spreads of simple and liquid securities such CDS. This can be done in closed form in our model. For simplicity of exposition, we shall consider a CDS that pays continuously a spread S to the protection seller. We would like to stress however, allowing for discrete payments on the premium leg amounts to a simple modification of the following calculation, and can also be done in closed form.

If we set $S = 1$, the CDS **premium leg** is given by

$$\begin{aligned} PL_T &\equiv E\left[\int_0^T I_{\{\tau>u\}} B_u^{-1} du\right] = \int_0^T \exp(u\tilde{Q})\mathbf{1} du \\ &= \hat{Q}^{-1}(\exp(\hat{Q}T) - I)\mathbf{1}(\xi_0) \end{aligned}$$

where

$$\hat{Q}^i_{jk} = Q_{jj} - r_j - \lambda^i_j \qquad (j = k); \tag{14}$$

$$= \exp(-w^i_{jk})Q_{jk} \qquad (j \neq k). \tag{15}$$

Similarly we can derive the value of the **default leg**. Define $\theta_{ij} \equiv 1 - \exp(-w_{ij})$; we have that,

$$DL_T \equiv E\left[B_\tau^{-1}; \tau \leq T\right] \tag{16}$$

$$= E\left[\int_0^T \{\lambda(\xi_u) + \sum_k Q_{\xi_u k}\theta_{\xi_u k}\} B_u^{-1} \exp(-C_u) du\right] \tag{17}$$

$$= \hat{Q}^{-1}(\exp(\hat{Q}T) - I)\tilde{\lambda}(\xi_0) \tag{18}$$

where $\tilde{\lambda}_i = \lambda_i + \sum_k Q_{ik}\theta_{ik}$.

The above calculations allow us to calibrate to CDS prices and index levels.

6 Example: synthetic CDOs

We turn now our attention to the problem of pricing a CDO tranche, and find the techniques developed so far will again serve. As before, we derive first the value of the premium leg and then the value of default leg.

6.1 Premium leg

Let L^+ and L^- be the upper and lower attachment points of the tranche respectively. At each payment date, investors receive a coupon which is proportional to the notional of the tranche, net of the losses suffered by the credit portfolio up to that point. The tranche PV01 is equal to

$$PV01 = \sum_{j=1}^{M} \Delta_i E\left[\exp\left(-\int_0^t r(\xi_u)du\right)\Phi(L_{T_j})\right], \tag{19}$$

where

$$\Phi(x) = \frac{1}{L^+ - L^-}\left[\left(L^+ - x\right)^+ - \left(L^- - x\right)^+\right], \tag{20}$$

and M is the number of total payments occurring at dates $T_1, \ldots, T_M$. In order to evaluate the PV01, we need to calculate the price of a portfolio of put options on the portfolio cumulative losses at each payment date T_j. In particular, $\Phi(x)$ is the difference of two put options of the form

$$P_t(K) \equiv E\left[B_t^{-1}(K - L_t)^+\right] \tag{21}$$

with strike K equal to L^+ and L^- respectively.

In principle we could calculate the discounted density of L_t at a set of points $I = \{x_1, \ldots, x_m\}$ by inverting its Laplace transform (6) m-times and then compute (21) resorting to a one dimensional numerical integration over the set I. However, there is a more accurate and significantly faster approach to the problem. We can in fact derive explicitly the Laplace transform, say $\hat{P}_t(\cdot)$ of (21) and then recover $P_t(K)$ by a single inversion. This will save us the time consuming numerical integration step. More precisely, note that

$$\begin{aligned}
\hat{P}_t(\alpha) &\equiv \int_0^\infty e^{-\alpha x} P_t(x)dx \\
&= E\int_{L_t}^\infty e^{-\alpha x} B_t^{-1}(x - L_t)\, dx \\
&= \frac{1}{\alpha^2} E\exp(-\int_0^t r(\xi_u)du - \alpha L_t).
\end{aligned}$$

All that remains to do is to compute $P_{T_i}(L^+)$ and $P_{T_i}(L^-)$ for $1 \leq j \leq M$ by inverting the corresponding Laplace transforms $\hat{P}_{T_j}$.

6.2 Default leg

The value of the default leg of a CDO tranche is the expected present value of the tranche's losses. More precisely, define

$$\Xi(x) = \frac{1}{L^+ - L^-}\left[\left(x - L^-\right)^+ - \left(x - L^+\right)^+\right].$$

The value of the default leg of the tranche, is then given by

$$DL = E\left[\int_0^T \exp\left(-\int_0^t r(\xi_u)du\right) d\Xi(L_u)\right].$$

Integrating by parts and noting that $\Xi(x) = 1 - \Phi(x)$, we can simplify the previous expression to

$$DL = 1 - E\left[\exp\left(-\int_0^T r(\xi_u)du\right)\Phi(L_T)\right] - E\left[\int_0^T r(\xi_u)\exp\left(-\int_0^t r(\xi_u)du\right)\Phi(L_u)du\right]. \tag{22}$$

Again all the quantities in (22) can be calculated explicitly. Note that the basic elements needed to calculate the default leg are the same as the ones we derived when calculating the premium leg, with some minor modification to account for the term $r(\xi)$ appearing in the second expectation of (22). The time integral appearing in the last term of (22) can be approximated by standard quadrature methods.

The tranche spread is recovered as usual by dividing the default leg by the PV01 of the premium leg.

Remark. All calculations simplify if we assume interest rates the Markov chain ξ are independent. Then all we need to do is substitute the relevant discount factor B_t^{-1}, with the corresponding risk-less zero-coupon bond $B(0,t)$ which we can observe in the market. This assumption, albeit crude, can be useful to simplify the calibration.

7 Calibration

In this section we shall present in some detail the methodology used to fit the model to market data and present the main results. We calibrated the model to tranches on the CDX (series 7) index (mid levels) for 4 consecutive business days from November 1st to November 6th 2006.

As previously mentioned, the parameters of the model are the N conditional intensity vectors λ^i, the jump weights w^i_{jk}, where $j, k \in \{1, \ldots, M, j \neq k\}$, the infinitesimal generator of the chain Q and the initial distribution of the chain π, which is an N-dimensional vector. Given the high dimensionality of the problem, some care has to be exercised in defining the calibration strategy.

There are two main strategies we can follow in order to fit the model to market data. One can calibrate the general model as presented in the paper. It should be noted however that in its general form the dimensionality of the calibration problem is relatively high and this can be expensive from a computational point of view. Alternatively one can try to reduce the dimensionality of the problem imposing some simplifying assumption. In particular, we can assume that the credit portfolio is homogenous, i.e. the vectors λ^i, matrices w^i and individual loss at default ℓ^i are the same for all the names. We found the second strategy is comparable to the first strategy as far as the quality of fit is concerned while being faster and easier to implement. For simplicity we shall only present the results of the simplified homogeneous model.

Since the portfolio is homogeneous we do not need to match the individual CDS curves but it is enough to fit the model to the aggregate CDX curve. Also the model needs to be calibrated to tranche spreads. The parameters of the calibration routine are the vector λ, the infinitesimal generator Q, the matrix w and the initial distribution of the chain π. We also added the recovery rate R to the parameter list. We found that a number of chain states equal to 4 is sufficient. The total number of parameters is this simplified model is equal to $2M^2$, here 32.

We fitted 5,7 and 10 year tranche and corresponding index spreads *one at the time.* Note that it is crucial to calibrate simultaneously tranche and index prices across maturities in order to be able to price tranches for non standard maturities and path dependent products in a consistent, arbitrage-free fashion. Note also that super senior tranche spreads (30%-100%) were included in the data set.

In calibrating the model we minimized a combination of the following objective functions, representing the average absolute error and the average percentage error respectively:

$$\mathcal{D}^a(\theta) \equiv \frac{1}{KT} \sum_{j=1}^{T} \sum_{k=1}^{K} |\tilde{S}_{market}(t_j, k) - \tilde{S}_{model}(t_j, k, \theta)| + \frac{1}{T} \sum_{j=1}^{T} |\bar{S}_{market}(t_j) - \bar{S}_{model}(t_j, \theta)|, \tag{23}$$

and

$$\mathcal{D}^p(\theta) \equiv \frac{1}{KT}\sum_{j=1}^{T}\sum_{k=1}^{K}|\frac{\tilde{S}_{model}(t_j,k,\theta)}{\tilde{S}_{market}(t_j,k)} - 1| + \frac{1}{T}\sum_{j=1}^{T}|\frac{\bar{S}_{model}(t_j,\theta)}{\bar{S}_{market}(t_j)} - 1|, \qquad (24)$$

where $\tilde{S}(t,k)$ is the tranche spread relative to attachment point k and maturity t, θ is a vector containing the elements of Q, ℓ and π and $\bar{S}(t,\theta)$ is the CDX spread for maturity t.

The results of the calibration where quite satisfactory. In particular, in all of the four days under consideration, the absolute error was less than 4.81bp for tranches and 1.11bp for the index. Average Percentage errors were below 3.5% for both index and tranches. Note that most of the calibration error is stemming from the 7 year equity tranche, which is notoriously difficult to fit. Detailed results can be found in tables 1 to 8. Parameters stability across calibration dates was also not an issue. As Table 9 shows parameters were fairly stable over time in the horizon under consideration.

Finally figures 1, 2 and 3 show the calibrated portfolio cumulative (percentage) loss distribution on November 1st for different initial states ξ_0 of the chain and different maturities. Note that the loss distributions do vary depending on ξ_0. As one would expect, as maturity extends, loss distributions tend to move to the right to reflect higher potential losses. Also, losses in the $[30\% - 40\%]$ interval have a non negligible mass. The latter phenomenon is a consequence of positive spreads for senior and super-senior tranches. In fact, in order for those tranches to suffer losses, a large number of defaults is required.

8 Conclusion

We presented a simple dynamic model for the pricing of correlation credit derivatives. We provide semi-analytic formulas for the pricing of CDOs tranches via Markov chain and Laplace transform techniques which are both fast and easy to implement. The model calibrates to quoted tranche prices with a high degree of precision and allows to price non-standard tranches in a consistent and arbitrage free manner. The number of parameters of the model is flexible and can be adjusted to adapt to the set of market data one is calibrating to. More importantly, the model is dynamically consistent and allows to price option on tranches and other exotic path dependent products.

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A Expectations of exponential linear functionals of Markov-chains

In this section we provide the proof of some result concerning general linear functional of the chain widely used in the main text of the paper. The material included in this appendix is quite standard for people in the probability theory circle, but it is frequently less well known to practitioners. A fuller treatment of the subject can be found for example in Rogers and Williams [RW].

In order to compute survival probabilities, survival correlation and related quantities we need to be able to evaluate expression of the form,

$$V(t,T) \equiv E\left[\exp\left(-\int_t^T \nu(\xi_u)du - \sum_{j\neq k} w_{jk}\left[J_{jk}(T) - J_{jk}(t)\right]\right) g(\xi_T) \mid \xi_t = \xi\right] \tag{25}$$

where $t \leq T$ and $J_{jk}(t)$ is the number of jumps from state j to state k occurred up to time t and w_{jk} are real numbers. To this end define

$$X_t \equiv \exp\left(-\int_0^t \nu(\xi_u)du - \sum_{j\neq k} w_{jk} J_{jk}(t)\right) \tag{26}$$

and let $M_t \equiv X_t V(t, \xi_t)$. By Dynkin formula, it follows that

$$dV_t \doteq \left(\frac{\partial V}{dt} + QV\right)(t, \xi_t) \tag{27}$$

where the sign $\doteq$ indicates that the left-hand and the right-hand side differ by a local martingale term. It can also be proved that for any previsible θ_t the following equality holds

$$\theta_t I_{\{\xi_{t-}=j,\xi_t=k\}} \doteq \theta_t Q_{jk} dt. \tag{28}$$

If we now apply integration by parts to M_t, we have that,

$$\begin{aligned}
dM_t &\doteq X_{t-}\left(\frac{\partial V}{dt}+QV\right)dt+V_{t-}dX_t+\Delta V_t\Delta X_t\\
&= X_t\left(\frac{\partial V}{dt}+QV\right)dt+V_{t-}X_{t-}\{-\nu(\xi_t)dt+\sum_{j\neq k}I_{\{\xi_{t-}=j,\xi_t=k\}}(e^{-w_{jk}}-1)\}\\
&+ X_{t-}\sum_{j\neq k}I_{\{\xi_{t-}=j,\xi_t=k\}}(e^{-w_{jk}}-1)\{V(t,k)-V(t,j)\}\\
&\doteq X_t\left(\frac{\partial V}{dt}+QV-\nu V\right)dt+\sum_j I_{\{\xi_t=j\}}X_t\sum_{k\neq j}Q_{jk}V(t,k)(e^{-w_{jk}}-1)dt\\
&= X_t\left(\frac{\partial V}{dt}+\tilde{Q}V\right)(t,\xi_t)dt
\end{aligned}$$

where where

$$\begin{aligned}
\tilde{Q}^i_{jk} &= Q_{jj}-\nu_j \qquad\qquad (j=k);\\
&= \exp(-w_{jk})Q_{jk} \qquad (j\neq k).
\end{aligned}$$

Since M is a martingale we must have that

$$\frac{\partial V}{dt}+\tilde{Q}V = 0 \tag{29}$$

$$V(T,\cdot) = g \tag{30}$$

The above ODE system admits the simple solution

$$V(t,\cdot)=\exp(\tilde{Q}(T-t))g. \tag{31}$$

Table 1: Market and model spreads - November 1st

	Market			Model		
	5Y	7Y	10Y	5Y	7Y	10Y
CDX	35	45	57	36.5	46.23	56.39
$0-3\%$	2438	4044	5125	2438.4	4008.9	5125
$3-7\%$	90	209	471	86	222.4	470.8
$7-10\%$	19	46	112	19.1	45.8	99.7
$10-15\%$	7	20	53	7	20.4	53.2
$15-30\%$	3.5	5.75	14	3.5	5.0	14.0
$30-100\%$	1.73	3.12	4	1.7	2.6	3.8

Table 2: Calibration error - November 1st

	Index	Traches
Absolute Error	1.11bp	3.77bp
Percentage Error	2.70%	3.47%

Table 3: Market and model spreads - November 2nd

	Market			Model		
	5Y	7Y	10Y	5Y	7Y	10Y
CDX	34	44	56	34.51	44	53.94
$0-3\%$	2325	3938	5056	2325	3906	5056
$3-7\%$	85.5	200	460	84.6	216.8	460
$7-10\%$	18	45.5	107	18	45.5	101
$10-15\%$	6.5	19.5	50.5	6.5	19	52.2
$15-30\%$	3.25	5.25	13.5	3.3	5.3	13.5
$30-100\%$	1.67	3.04	3.64	1.7	2.4	3.6

Table 4: Calibration error - November 2nd

	Index	Traches
Absolute Error	0.86bp	3.26bp
Percentage Error	1.73%	2.68%

Table 5: Market and model spreads - November 3th

	Market			Model		
	5Y	7Y	10Y	5Y	7Y	10Y
CDX	34	44	56	34.6	44.02	53.93
$0-3\%$	2325	3931	5038	2325	3892.7	5038.5
$3-7\%$	84.5	200	458.5	84.5	215.7	458
$7-10\%$	18.5	45.00	107.5	18.4	45	98.7
$10-15\%$	6.5	19.5	51	6.5	19.1	51.2
$15-30\%$	3.25	5.25	13.5	3.2	5.2	13.5
$30-100\%$	1.61	3.06	3.76	1.6	2.4	3.8

Table 6: Calibration error - November 3th

	Index	Traches
Absolute Error	0.90bp	3.63bp
Percentage Error	1.84%	2.55%

Table 7: Market and model spreads - November 6th

	Market			Model		
	5Y	7Y	10Y	5Y	7Y	10Y
CDX	33	43	54	34.61	43.88	53.97
$0-3\%$	2256	3863	4963	2255.9	3794.3	4963.1
$3-7\%$	77	192	438	77	201.3	438
$7-10\%$	17	41	98	17	41	93.5
$10-15\%$	6	18.5	46.5	6	17.1	47
$15-30\%$	3.13	5.75	12	3.1	5.2	12.8
$30-100\%$	1.27	2.55	3.23	1.3	2	3.2

Table 8: Calibration error - November 6th

	Index	Traches
Absolute Error	0.84bp	4.81bp
Percentage Error	2.33%	3.44%

Table 9: Calibrated parameters

Parameters	Nov 1st	Nov 2nd	Nov 3th	Nov 6th
λ_1	0.0545	0.0472	0.0482	0.0482
λ_2	0.0134	0.0131	0.0131	0.0135
λ_3	0.0000	0.0000	0.0001	0.0009
λ_4	0.0007	0.0019	0.0022	0.0021
Q_{12}	0.0000	0.0013	0.0035	0.0019
Q_{13}	0.0004	0.0132	0.0104	0.0107
Q_{14}	0.0065	0.0048	0.0035	0.0041
Q_{21}	0.0179	0.0193	0.0184	0.0182
Q_{23}	0.0001	0.0001	0.0001	0.0001
Q_{24}	0.0000	0.0000	0.0001	0.0000
Q_{31}	0.0000	0.0000	0.0000	0.0000
Q_{32}	0.0000	0.0000	0.0000	0.0000
Q_{34}	0.4291	0.4368	0.4233	0.3832
Q_{41}	0.0000	0.0000	0.0000	0.0000
Q_{42}	1.2835	1.1012	1.1198	0.9557
Q_{43}	0.0014	0.0006	0.0007	0.0005
w_{12}	9.3981	9.3980	9.3982	9.3980
w_{13}	0.1277	0.3481	0.3650	0.4741
w_{14}	14.8746	14.8746	14.8746	14.8746
w_{21}	0.0000	0.0129	0.0145	0.0158
w_{23}	10.0362	10.0362	10.0361	10.0361
w_{24}	19.6856	19.6856	19.6856	19.6856
w_{31}	9.1688	9.1688	9.1688	9.1688
w_{32}	7.5897	7.5897	7.5897	7.5897
w_{34}	0.0000	0.0000	0.0000	0.0000
w_{41}	6.4959	6.4958	6.4958	6.4958
w_{42}	0.0009	0.0000	0.0000	0.0000
w_{43}	0.7407	0.9194	0.7570	0.6270
π_1	0.0019	0.0019	0.0021	0.0026
π_2	0.0000	0.0000	0.0000	0.0000
π_3	0.9981	0.9975	0.9979	0.9970
π_4	0.0000	0.0006	0.0000	0.0004
R	0.4701	0.4784	0.4776	0.4702

Figure 1: Calibrated portfolio loss density. 5Y Maturity. X axis: L_t, Y axis: density

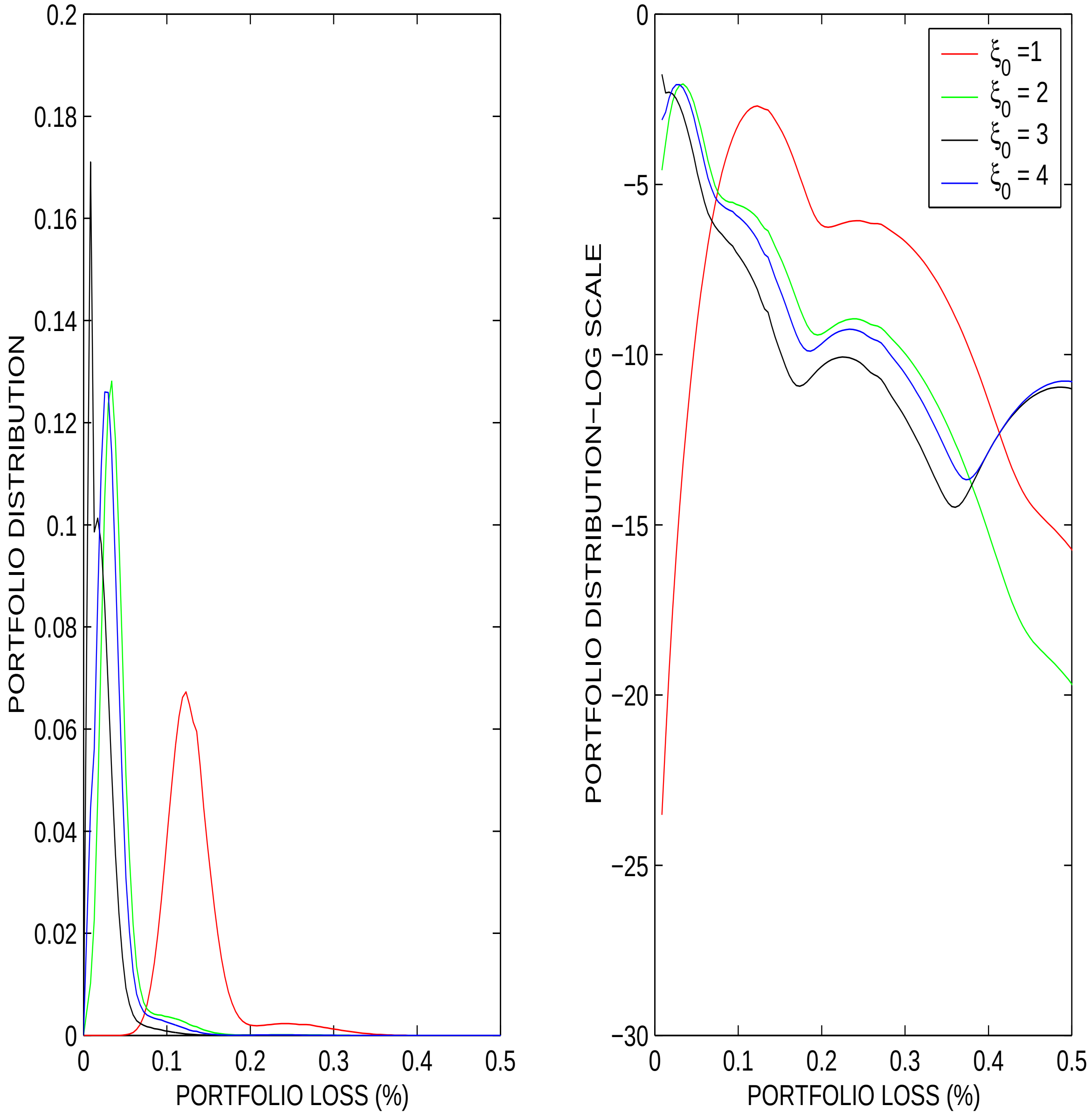

Figure 2: Calibrated portfolio loss density. 7Y Maturity. X axis: L_t, Y axis: density

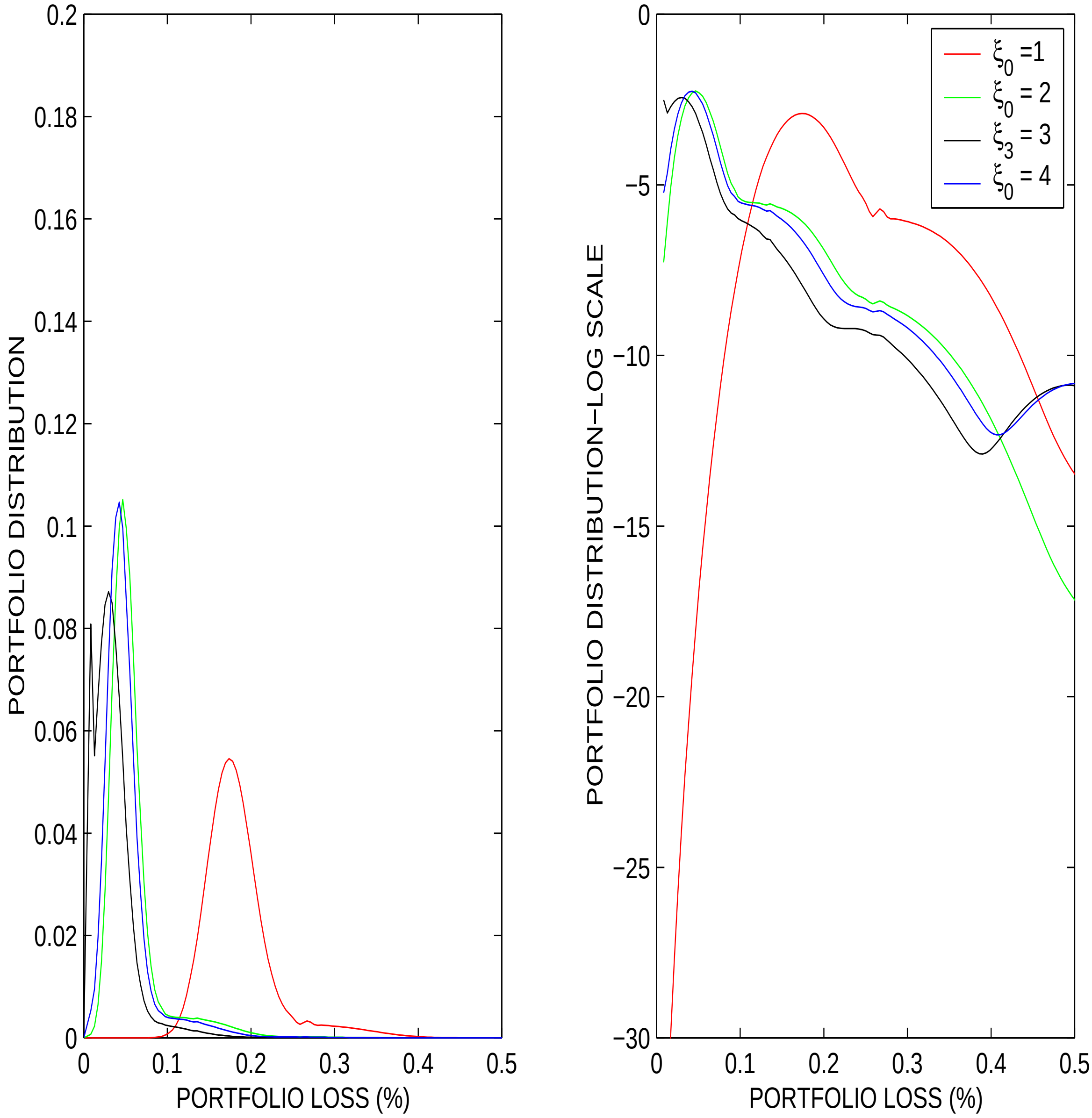

Figure 3: Calibrated portfolio loss density. 10Y Maturity. X axis: L_t, Y axis: density