Contracting for optimal investment with risk control

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First version: April 2008

July 4, 2008

Abstract

The theory of risk measurement has been extensively developed over the past ten years or so, but there has been comparatively little effort devoted to using this theory to inform portfolio choice. One theme of this paper is to study how an investor in a conventional log-Brownian market would invest to optimize expected utility of terminal wealth, when subjected to a bound on his risk, as measured by a coherent law-invariant risk measure. Results of Kusuoka lead to remarkably complete expressions for the solution to this problem.

The second theme of the paper is to discuss how one would actually manage (not just measure) risk. We study a principal/agent problem, where the principal is required to satisfy some risk constraint. The principal proposes a compensation package to the agent, who then optimises selfishly ignoring the risk constraint. The principal can pick a compensation package that induces the agent to select the principal's optimal choice.

Keywords: Principal, agent, contract, risk measure, optimal investment

AMS Subject Classifications:

1 Introduction

The study of risk measurement in the mathematical finance literature can be said to date from the seminal paper [2], and has since grown into one of the biggest branches of the subject. While a lot of effort has been expended on framing axioms for risk measures and

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their consequences, less attention has focused on how one might apply these notions to more practical matters. A particularly interesting question is how one might optimally invest if constrained by a bound on some risk measure. Early contributions to this literature were [8], [6], [3], which dealt with the problem of optimizing subject to a VaR constraint, among other questions. The use of VaR as a risk measure remains commonplace in the industry, though it has been comprehensively and justly discredited in the academic literature. Coherent or convex risk measures are generally preferred as ways of expressing the riskiness of a position, and we shall discuss only these. A further natural restriction to be placed on a risk measure is *law invariance*, which is to say that two contingent claims with the same law are considered equally risky[1]. Law-invariant risk measures have been characterised by [12] and [9]. In Section 3, we shall work with a standard complete log-Brownian market, and solve the problem of investing to optimize the expected utility of terminal wealth, subject to a constraint on risk expressed as a bound on the value of a general law-invariant coherent risk measure. The key observation is that the optimal terminal wealth must be a decreasing function of the state-price density[2]. Casting the problem in Lagrangian form leads to the form of the solution, which can in some cases be made reasonably explicit.

But beyond the characterisation of the optimal policy, we are interested in the contracting problem which would arise between a trader and an investment bank which employs him. The investment bank will have its own utility, which will typically be much more risk averse than that of the trader, and will in addition be subject to regulatory constraints on risk. The question we next investigate is what contract the bank might offer the trader so that *the trader acting in his own self-interest implements the bank's preferred solution*. The theory of contracts is a difficult area of economic theory; see the excellent survey of Stole [11]. We shall obviate many of the difficulties, which can arise in a situation where the agent may misreport effort or outcomes, or where the level of effort of the agent is not verifiable by the principal; the paper of Palomino & Prat [10], and further papers cited there, provide examples of the kinds of problems that are tackled in this literature. In the context of a trader in an investment bank, his every action is recorded electronically and subject to daily inspection, so misreporting is virtually impossible. Likewise, the trader has no opportunity to slack; he sits in an open-plan workspace with his boss within a few feet most of the day. Moreover, competitive pressure from other traders means that it is reasonable to assume that the trader will give maximum effort regardless. And in any case, in the story we are telling, there is no scope for cleverness or innovation on the part of the trader; whatever he may believe about his god-like insights, he is working in a complete market, and the only thing he has to do is to implement the trades that replicate his desired terminal wealth. By abstracting from issues of effort or misreporting, the construction of the contract to be offered to the agent is immediate, at least in the first case we consider in Section 2 where there are no risk constraints

[1] While initially appealing, there are reasons why this may not be desirable; a trader would not necessarily be indifferent between two contingent claims with the same law, one of which was the market portfolio, the other being negatively correlated with the market portfolio. In a similar spirit, Cherny & Grigoriev [4] point out that law invariance would not be a natural assumption when an agent has an existing position to be offset against the proposed contingent claim.

[2] This is a result of Dybvig [5].

imposed. We find that in realistic situations the overall value of the package can be quite small compared to the amounts of money in the portfolio. These ideas come together in Section 4 where we identify a contract which the principal can offer the agent under which the agent's selfish actions implement the principal's risk-responsible optimum.

2 Contracting to align objectives.

A principal employs the agent to invest on his behalf, starting with wealth w_0. The agent invests in various assets, and at some fixed horizon T, the agent is paid some fee $\varphi(w_T)$ depending on the final value of the portfolio; bearing in mind that the agent's preferences may differ from the principal's, *how should the principal set φ to achieve his own objective?* The principal aims to obtain

$$\sup E\, U_P(w_T - \varphi(w_T)),$$

whereas the agent aims to obtain

$$\sup E\, U_A(\varphi(w_T)).$$

The answer to the question is almost trivial; the principal selects constants $k > 0$ and a, and defines the wage schedule φ by

$$U_P(x - \varphi(x)) = kU_A(\varphi(x)) - a. \tag{1}$$

While this is the obvious recipe for the principal's choice of wage schedule, and ensures that the agent's objective is perfectly aligned with the principal's, there are various questions which need to be answered; how should a and k be chosen? Is φ increasing? Is φ concave? For realistic examples, what does φ look like?

It is likely that the principal is much more risk-averse than the agent; if the agent loses all the money, he walks away with nothing, but then can go on to another job - he has lost only some of his time. The principal on the other hand has lost a huge amount of capital that might have taken decades to acquire, and may result in the destruction of the business which generated the capital. For this reason, we shall suppose that there is some lower bound $\underline{x}$ that the principal will tolerate for wealth. If ever the value of the portfolio falls to $\underline{x}$, then the agent is fired immediately, no further investment takes place, and the agent is paid nothing. This gives us one condition for determining a and k, namely,

$$U_P(\underline{x}) = kU_A(0) - a. \tag{2}$$

Subject to this, there is only the choice of k, a parameter which reflects the bargaining power of the agent, the smaller k, the more the agent's bargaining power.

The properties of φ are summarised in the following result.

Proposition 1 *Assuming that U_P and U_A are strictly increasing, the function $\varphi : [\underline{x}, \infty) \to \mathbb{R}^+$ is well defined by (1) and (2). It is increasing, and $U_A \circ \varphi$ is concave.*

PROOF. For any $x > \underline{x}$, the function

$$y \mapsto U_P(x-y) - kU_A(y) + a$$

is continuous and strictly decreasing on $[0, x-\underline{x}]$, from a positive value at $y = 0$ to a negative value at $y = x - \underline{x}$. Thus there is a unique $y = \varphi(x) \in (0, x - \underline{x})$ at which the function is zero. The monotonicity of φ is obvious.

Turning to the concavity of $u(x) \equiv U_A(\varphi(x))$, suppose that concavity *fails*. Thus there exist $x_1, x_2 \geq \underline{x}$, and $p \equiv 1 - q \in (0, 1)$ such that (with $x = px_1 + qx_2$)

$$\begin{aligned} u(x) &= U_A(\varphi(x)) \\ &< pu(x_1) + qu(x_2) \\ &= pU_A(\varphi(x_1)) + qU_A(\varphi(x_2)) \\ &\leq U_A(p\varphi(x_1) + q\varphi(x_2)), \end{aligned} \tag{3}$$

and so $\varphi(x) < p\varphi(x_1) + q\varphi(x_2)$. Hence

$$\begin{aligned} u(x) = U_A(\varphi(x)) &= U_P(x - \varphi(x)) \\ &> U_P(x - p\varphi(x_1) - q\varphi(x_2)) \\ &\geq pU_P(x_1 - \varphi(x_1)) + qU_P(x_2 - \varphi(x_2)) \\ &= pU_A(\varphi(x_1)) + qU_A(\varphi(x_2)), \end{aligned}$$

contradicting (3).

□

The message of this simple result is that once the lower bound $\underline{x}$ and the bargaining power k of the agent have been fixed, *the original utility U_P is replaced by the modified utility $U_A \circ \varphi$.*

How does this modified utility look for some realistic values of the parameters? We show in Figure 1 some plots for a typical example, where the principal is a CRRA investor with coefficient of relative risk aversion $R = 2$, employing a much less risk-averse agent, again CRRA with coefficient of relative risk aversion $R = 0.005$, in effect, a risk-neutral agent. The principal has a lower bound $\underline{x} = 0.6$ for the portfolio value, and the agent's bargaining power is determined by the value $k = 20$. Notice how the wage schedule φ offered to the agent induces risk aversion; there is pronounced concavity in the wage schedule. At best, the agent receives about 3.3% of the total wealth generated, and the effect on the principal's utility is very small.

Different choices can be made, but the qualitative picture usually looks quite similar. One point to note is that if the agent's reward is never more than a few percent of the total wealth, then the principal's utility is affected very little.

3 From optimal wealth to optimal contract.

In this section, we make the common assumption of a complete market with a single risky asset[3], so that wealth dynamics take the familiar form

$$dw_t = rw_t dt + \theta_t(\sigma dW_t + (\mu - r)dt), \tag{4}$$

for some given initial value w_0 of wealth. The market price of risk

$$\kappa \equiv \sigma^{-1}(\mu - r) \tag{5}$$

and the state-price density process ζ defined by

$$d\zeta_t = \zeta_t(-rdt - \kappa \cdot dW_t), \quad \zeta_0 = 1, \tag{6}$$

play a key rôle in the solution of optimal investment problems in this context.

The objective is to generate a terminal wealth w_T which maximises the expected utility[4] $EU(w_T)$, subject to the initial wealth constraint, and subject to further constraints *which may be expressed solely in terms of the law of the terminal wealth*[5]. Such further constraints may typically be interpreted as some form of risk-management constraint. The optimization problem is therefore to select a distribution for the terminal wealth, subject to the budget constraint, so as to maximize the expected utility of terminal wealth. As Dybvig [5] proves, if the law of the terminal wealth is specified, then the cheapest way to achieve a terminal wealth with that law is to take $w_T = \psi(\zeta_T)$ for some decreasing function ψ defined by the property that w_T has the desired distribution. Thus the problem is to select a decreasing function ψ to maximise the objective $EU(\psi(\zeta_T))$ subject to the budget constraint $E\zeta_T\psi(\zeta_T) = w_0$, and whatever further constraints on the law of w_T have been imposed.

Suppose that this optimization problem is solved, and that the optimal decreasing function ψ has been found. If we define a new utility function u by

$$u'(x) = \psi(x), \tag{7}$$

then *an agent with utility u freely optimizing $Eu(w_T)$ subject to the budget constraint will choose terminal wealth $\psi(\zeta_T)$*. In terms of the agent, if the principal offers the agent a contract φ which satisfies[6] for some $k > 0$ and $a \in \mathbb{R}$

$$kU_A(\varphi(x)) - a = u(x), \tag{8}$$

[3]There would be no difficulty in extending the formulation to multiple risky assets, with suitably bounded processes for volatility and growth rate; the key assumption is that the market is complete. We take the unvariate constant coefficient case only to allow us to do explicit calculations for a number of examples.

[4]For simplicity, we shall assume that U is C^2, strictly increasing and satisfies the Inada conditions.

[5]This is quite restrictive, in that it rules out all path-dependent constraints, such as drawdown constraints, or constraints that the portfolio lie always in some convex set. Nevertheless, for some important interesting situations, this assumption applies.

[6]It may be that (8) cannot be satisfied, if u takes values not in the range of the left-hand side.

then the (unconstrained) agent will optimally choose the terminal wealth that the (constrained) principal considers optimal.

The message which comes from this is clear and simple: under the complete-markets assumption, *the principal can take all responsibility for satisfying the risk-management constraint, by offering the agent a suitably-constructed contract, and then letting the agent act in his own best interests.*

4 Law-invariant coherent risk measure constraints.

We illustrate the principles of Section 3 by solving the problem of maximizing the expected utility of terminal wealth, subject to the initial wealth constraint, and the further constraint that some coherent law-invariant risk measure does not exceed some required threshold. Similar problems with other constraints of a risk-management flavour have been solved by [8], [6], [3]; what we do here tackles a general class of problems (which does not, however, include those cited which deal with VaR constraints).

Law-invariant coherent risk measures have been characterised by [9], [1], [12], who show that any coherent risk measure takes the form

$$\rho(X) = \sup\{\rho^{\mu}(X) : \mu \in \mathcal{M}\} \tag{9}$$

where $\mathcal{M}$ is a collection of probability measures on $[0,1]$, and

$$\rho^{\mu}(X) \equiv \int \rho_a(X)\ \mu(da), \tag{10}$$

where

$$\rho_a(X) \equiv -a^{-1}E[X : X \le F_X^{-1}(a)] = -E[X|X \le F_X^{-1}(a)] = -a^{-1}\int_0^a F_X^{-1}(x)\,dx \tag{11}$$

for $a > 0$, and $\rho_0(X) \equiv -\text{essinf}(X)$.

In the assumed complete market setting, it is well known that it is possible to generate any $\mathcal{F}_T$-measurable terminal wealth X which satisfies the budget constraint

$$w_0 = E[\zeta_T X]; \tag{12}$$

see, for example, [7]. The task therefore is to maximise $EU(X)$ subject to this constraint, and the risk-measure constraint

$$\rho(X) \le -b \tag{13}$$

for some constant b. Since our objective only depends on the law of w_T, as does the risk-measure constraint, by the arguments of the previous Section it is enough to look only at terminal wealths of the form $X = \psi(\zeta_T)$ for decreasing functions ψ. However, in this case the quantiles of X are simply related to the quantiles of ζ_T;

$$F_X^{-1}(a) = \psi(F_{\zeta}^{-1}(1-a)). \tag{14}$$

where we write $F_\zeta \equiv F_{\zeta_T}$ as an obvious abbreviation. Hence the shortfall risk measure

$$\begin{aligned}
\rho_a &= -a^{-1} \int_0^a F_X^{-1}(x)\ dx \\
&= -a^{-1} \int_{1-a}^1 \psi(F_\zeta^{-1}(y))\ dy \\
&= -a^{-1} \int_{F_\zeta^{-1}(1-a)}^\infty \psi(z)\ F_\zeta(dz) \qquad (15)
\end{aligned}$$

is expressed in terms of the quantiles of ζ_T, and then the risk measure ρ^μ can be expressed in terms of the distribution of ζ_T also;

$$\begin{aligned}
\rho^\mu(X) &= -\int_0^1 \left\{ a^{-1} \int_{F_\zeta^{-1}(1-a)}^\infty \psi(z)\ F_\zeta(dz) \right\} \mu(da) \\
&= -\int \psi(z) \left\{ \int_{1-F_\zeta(z)}^1 a^{-1}\ \mu(da) \right\} F_\zeta(dz) \\
&= -E[\psi(\zeta_T) g_\mu(\zeta_T)], \qquad (16)
\end{aligned}$$

where we see that g_μ is non-negative increasing. The optimisation problem therefore becomes

$$\max_\psi EU(\psi(\zeta_T)) \quad \text{subject to} \quad w_0 = E[\zeta_T \psi(\zeta_T)], \quad E[\psi(\zeta_T) g_\mu(\zeta_T)] \geq b \quad \forall \mu \in \mathcal{M} \qquad (17)$$

where the function ψ is understood to be decreasing, and bounded below by $\underline{x}$.

We shall explore this problem under the simplifying assumption that

$$\mathcal{M} = \{\mu_1, \ldots, \mu_n\} \qquad (18)$$

is a finite set. Writing $g_i \equiv g_{\mu_i}$, we could (and shall) allow a slightly more general form of the problem (17), by taking $E[\psi(\zeta_T) g_i(\zeta_T)] \geq b_i$ for each i, where the b_i may be different. We shall also assume the (Inada) condition $\lim_{x\to\infty} U'(x) = 0$, and that

$$g_i(-\infty) = 0 \quad \forall i. \qquad (19)$$

This is a natural restriction. Indeed, the only way we could have $g_i(-\infty) > 0$ would be if μ_i charges 1, so the risk measure ρ^{μ_i} is a convex combination of $\rho_1(X)$ and some other coherent law-invariant risk measure. But the risk measure ρ_1 is special: $\rho_1(X) = -EX$, so the condition $E[\psi(\zeta_T) g_i(\zeta_T)] \geq b_i$ will be satisfied if the *mean* of terminal wealth $\psi(\zeta_T)$ is large enough. This can easily be achieved, by taking $\psi(\zeta_T)$ very large on the set where ζ_T is very small; the objective is affected very little, and the budget constraint is also affected very little. So any constraint for which $g_\mu(-\infty) > 0$ will in fact not be a constraint at all.

The Lagrangian form of the optimisation problem is to maximise over non-increasing ψ and non-negative slack variables z_i the Lagrangian[7]

$$\begin{aligned} L(\psi, z) &\equiv E\Big[\, U(\psi(\zeta)) + \lambda(w_0 - \zeta\psi(\zeta)) + \sum_{i=1}^{n} \alpha_i \{\psi(\zeta) g_i(\zeta) - b_i - z_i\} \,\Big] \\ &= E\Big[\, U(\psi(\zeta)) - \psi(\zeta)\big\{\lambda\zeta - \sum_{i=1}^{n} \alpha_i\ g_i(\zeta)\big\} - \alpha \cdot (z+b) \,\Big] + \lambda w_0. \end{aligned} \tag{20}$$

Dual-feasibility requires that $\alpha \geq 0$, and complementary slackness gives $\alpha \cdot z = 0$ at optimality. Moreover, dual-feasibility also requires that

$$\lambda \geq \sup_{x>0} \frac{\sum_{i=1}^{n} \alpha_i\ g_i(x)}{x}, \tag{21}$$

otherwise for some $x > 0$ we could make the objective unbounded by taking $\psi(x)$ very large.

The optimisation of L over non-increasing ψ is straightforward if the function

$$h(z) \equiv \lambda z - \sum_{i=1}^{n} \alpha_i\ g_i(z)$$

is monotone increasing, for then we simply use the pointwise maximisation $U'(\psi(z)) = h(z)$, which defines the value $\psi(z)$ uniquely. However, if h is not monotone increasing, the story is more subtle. To explain what happens, define

$$\begin{aligned} \tilde{h}(x) &\equiv h(F_\zeta^{-1}(x)) \\ &= \lambda F_\zeta^{-1}(x) - \sum_{i=1}^{n} \alpha_i \int_{1-x}^{1} a^{-1}\ \mu_i(da), \\ \tilde{\psi}(x) &\equiv \psi(F_\zeta^{-1}(x)), \end{aligned}$$

mapping $[0, 1]$ to $\mathbb{R}$, where we require that $\tilde{\psi}$ is non-increasing. The interesting part of the Lagrangian can be expressed as

$$E\big[\, U(\psi(\zeta)) - \psi(\zeta)h(\zeta) \,\big] = \int_0^1 \{U(\tilde{\psi}(x)) - \tilde{\psi}(x)\tilde{h}(x)\}\ dx, \tag{22}$$

to be optimised over non-increasing $\tilde{\psi}$ which decrease to $\underline{x}$. Now set

$$H(x) \equiv \int_0^x \tilde{h}(y)\ dy,$$

and let $\underline{H}$ be the *greatest convex minorant* of H, which we may express as

$$\underline{H}(x) = H(x) + \eta(x)$$

[7] We abbreviate ζ_T to ζ for this part of the discussion.

for a non-positive function η which is differentiable almost everywhere, and is equal to the integral of its derivative. In addition, $\eta(0) = \eta(1) = 0$. Since $\underline{H}$ is convex, its derivative $\tilde{h}(x) + \eta'(x)$ is non-decreasing, and so we may estimate (22)

$$\begin{aligned}
\int_0^1 \{U(\tilde{\psi}(x)) - \tilde{\psi}(x)\tilde{h}(x)\}\, dx &= \int_0^1 \{U(\tilde{\psi}(x)) - \tilde{\psi}(x)(\tilde{h}(x) + \eta'(x))\}\, dx + \int_0^1 \tilde{\psi}(x)\eta'(x)\, dx \\
&\leq \int_0^1 \tilde{U}(\tilde{h}(x) + \eta'(x))\, dx + [\tilde{\psi}(x)\eta(x)]_0^1 - \int_0^1 \eta(x)\, d\tilde{\psi}(x) \quad (23) \\
&\leq \int_0^1 \tilde{U}(\tilde{h}(x) + \eta'(x))\, dx, \quad (24)
\end{aligned}$$

where in (23) we have integrated by parts, and used the notation $\tilde{U}$ for the convex dual of U, and to reach (24) we have used the fact that η is non-positive and $\tilde{\psi}$ is decreasing. Moreover, the bound (24) is achieved when we use[8]

$$\tilde{\psi}(x) = I(\tilde{h}(x) + \eta'(x)), \quad (25)$$

because at any x where $\eta(x) < 0$, the greatest convex minorant $\underline{H}$ is strictly less than H, and so its slope is not changing; thus $d\tilde{\psi}$ does not charge the set $\{x : \eta(x) < 0\}$, and the second integral in (23) vanishes.

In our application, we shall require that the terminal wealth is always at least some minimum tolerated value $\underline{x}$, the level at which the agent is stopped and kicked out. While we could view this constraint as another risk measure constraint $\rho_0(w_T) \leq -\underline{x}$ and incorporate this into the Lagrangian, the simplest thing is just to work through the argument at steps (23) and (24) again, replacing $\sup_\psi\, [U(\psi) - \psi(\tilde{h}(x) + \eta'(x))]$ with

$$\sup_{\psi \geq \underline{x}}\, [U(\psi) - \psi(\tilde{h}(x) + \eta'(x))] = \tilde{U}((\tilde{h}(x) + \eta'(x)) \wedge \underline{y}),$$

where $\underline{y} = U'(\underline{x})$. For $\tilde{\psi}$ which satisfy the lower bound, this provides an upper bound for the value of (this part of) the Lagrangian, the bound being achieved when we take

$$\tilde{\psi}(x) = I(\tilde{h}(x) + \eta'(x)) \vee \underline{x} = I((\tilde{h}(x) + \eta'(x)) \wedge \underline{y}). \quad (26)$$

Finally we apply the results of Section 3. We assume that the principal's objective is expressed in terms of the gross wealth achieved at time T, rather than in terms of the wealth $w_T - \varphi(w_T)$ achieved net of fees to the agent. To do otherwise would be far more complicated, as then the optimum depends on the fees to be paid to the agent, which in turn depend on the optimum. In any case, we have seen in Section 2 that the two are quite close in practical situations; and we do not pretend that risk management is an exact science.

Given this, the risk-constrained principal will optimally choose a terminal wealth that is of the form $\psi(\zeta_T)$ for some decreasing function ψ; so the principal offers the agent a contract φ which satisfies (8), where u satisfies(7), and the principal's risk-constrained optimum becomes the agent's unconstrained optimum.

[8]The function I is the inverse marginal utility $(U')^{-1}$.

5 Some numerical examples.

We now consider two numerical examples, to give substance to the theoretical discussion of the previous section, and to show the kinds of contracts that a risk-constrained principal would offer to an agent. Throughout, we assume that the initial wealth is 1.

In the first example, there is just one constraint ($n = 1$ in (18)), which is of expected-shortfall type:

$$\rho(X) = \rho_a(X),$$

where we take $a = 0.05$ and $b = 0.5$. Thus the expected shortfall below the 5-percentile, given that the terminal wealth is below the 5-percentile, should not exceed 0.5. The lower bound $\underline{x}$ is taken to be 0.5. The results are plotted in Figures 2 and 3. In the first of these, we see the inverse marginal utility for the risk-constrained principal in the upper panel, with the utility $u \equiv U_a \circ \varphi$ for the agent shown in the lower panel, superimposed over the original utility for the agent. The top panel shows that the lower bound $\underline{x} = 0.5$ for the wealth appearing as a vertical wall in the inverse marginal utility, along with another vertical segment at $x_c \simeq 0.67$, where the utility has a discontinuity of gradient. The effect of this is that there will be a positive probability that the final wealth will be equal to this critical value, even though the optimally-controlled wealth process will not stop investing at any time $t < T$. The second figure, Figure 3, shows the actual contract offered, and the payment made as a fraction of the wealth generated. Again, the discontinuity at x_c is visible.

In the second example, we have $n = 3$. The first two constraints are simple expected-shortfall constraints,

$$\rho_{a_i} \leq -b_i \qquad (i = 1, 2),$$

with $a_1 = 0.65$, $b_1 = 1.04$, $a_2 = 0.05$, $b_2 = 0.7$. For the third risk measure, we take the expression (16) using $\mu(da) = I_{\{a<\beta\}}da$, and $\rho^\mu \leq -0.85$. Once again, the results of the calculation are displayed in Figures 4 and 5. This time, the inverse marginal utility shows *two* steps, so there are two values for the terminal wealth which will be achieved with positive probability. The derived utility and the contract show marked risk aversion.

6 Conclusions.

We have seen how the alignment of the objectives of principal and agent is possible by a very simple construction of a payment schedule based on the wealth generated by the agent. This leads on to a remarkably complete answer to the question of the construction of the optimal principal-agent contract in the situation where the principal aims to maximize his expected utility of terminal wealth subject to law-invariant coherent risk measure constraints.

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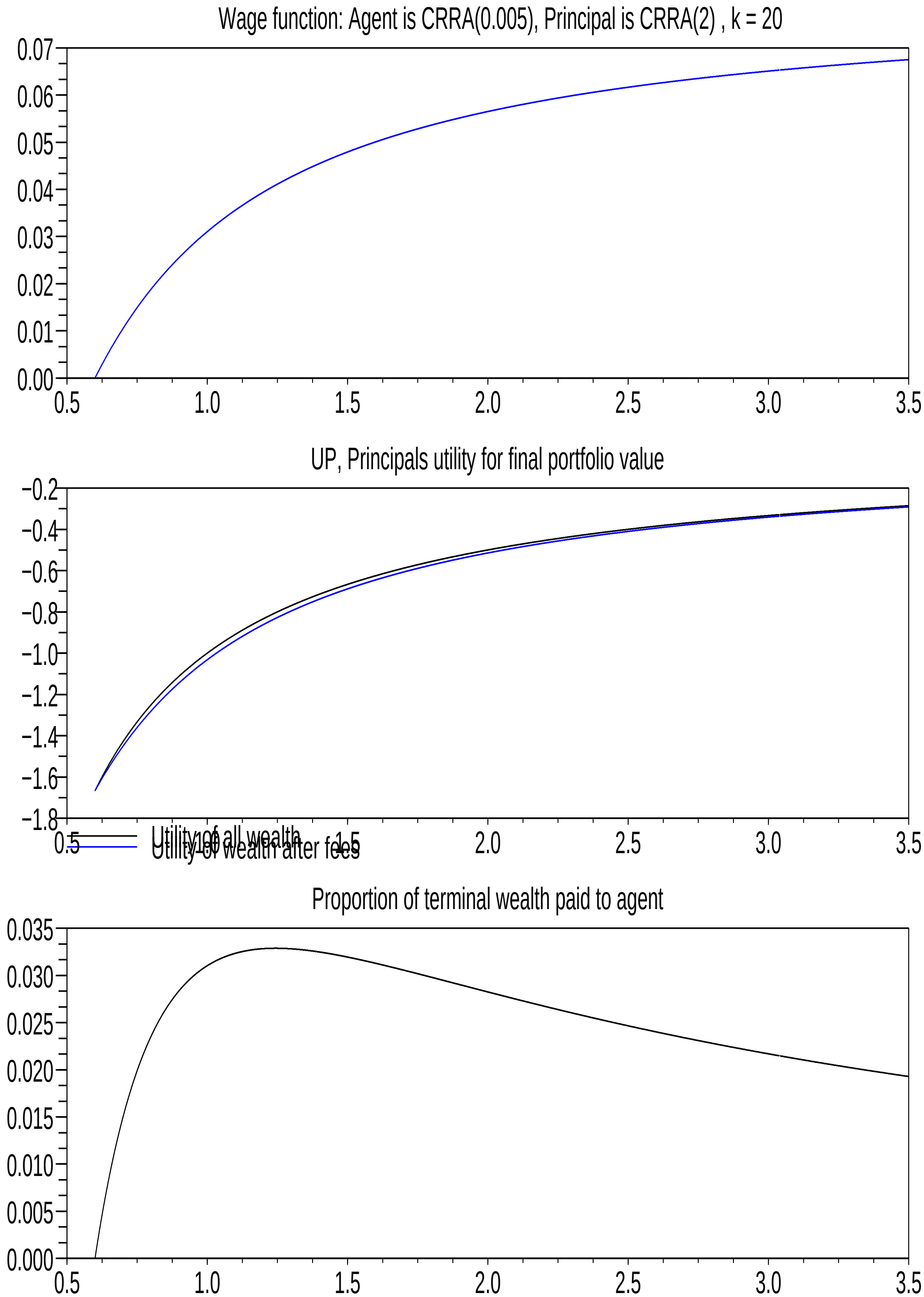


Figure 1: Contract alignment

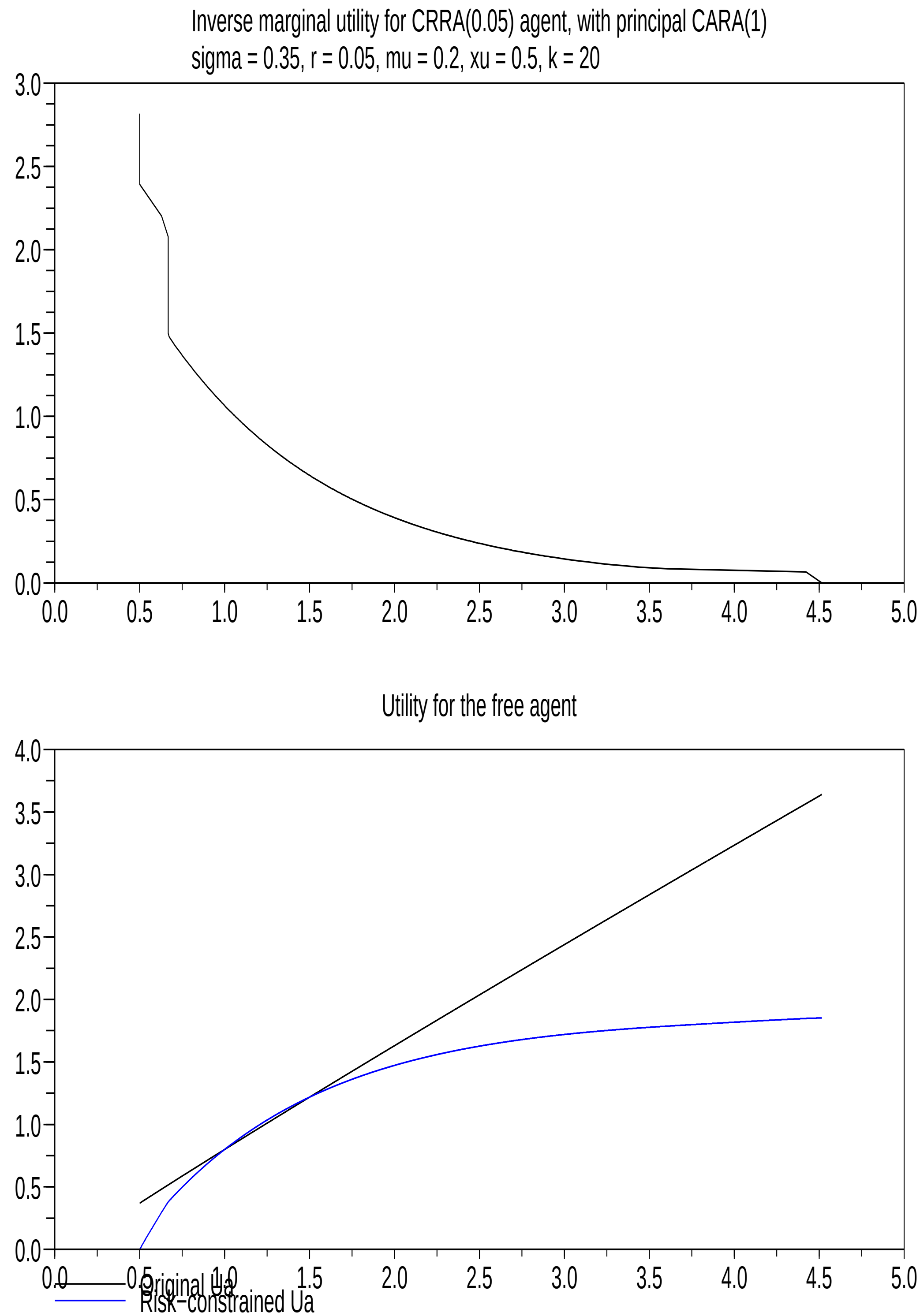


Figure 2: Single expected shortfall constraint

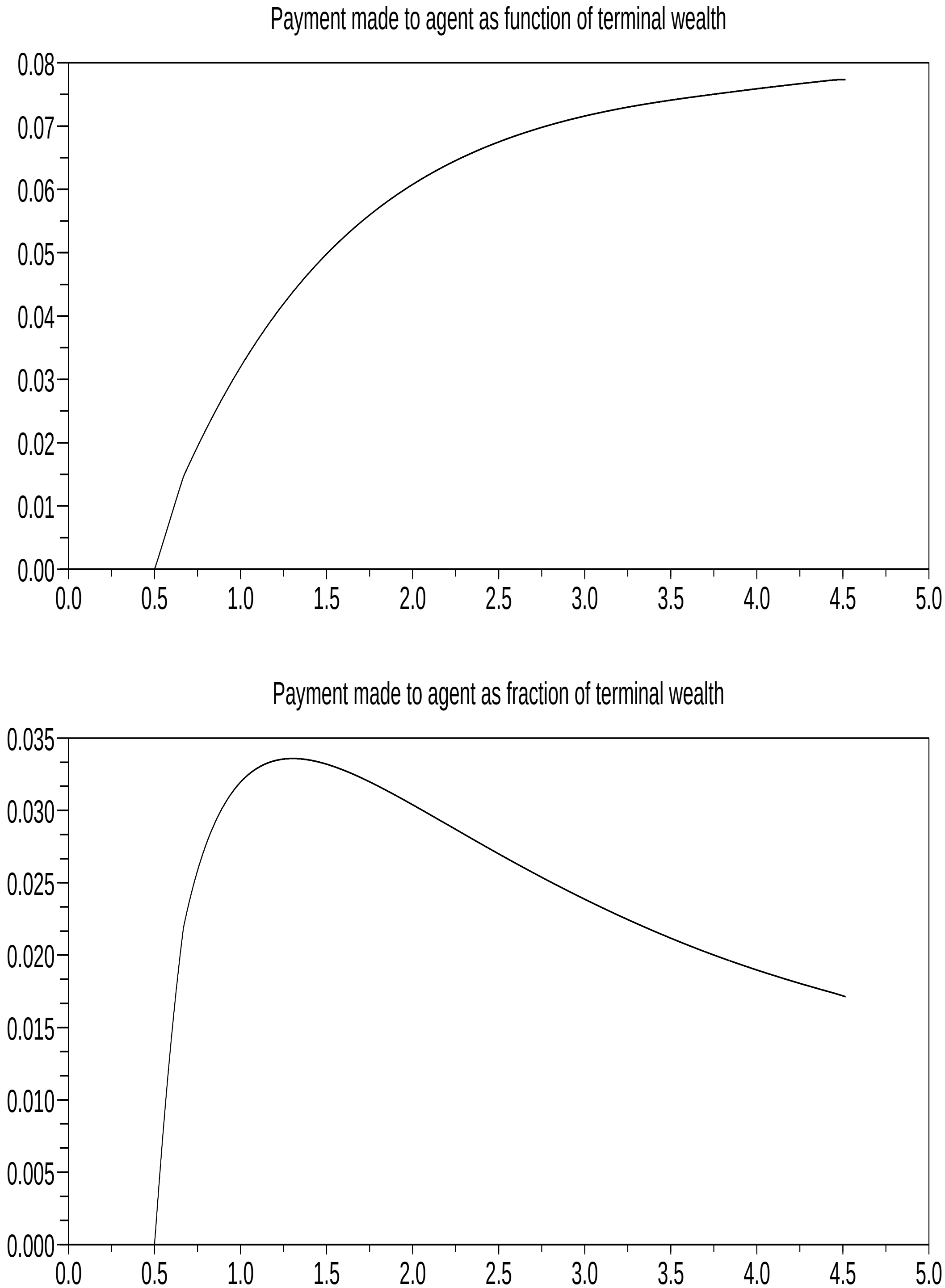


Figure 3: Single expected shortfall constraint

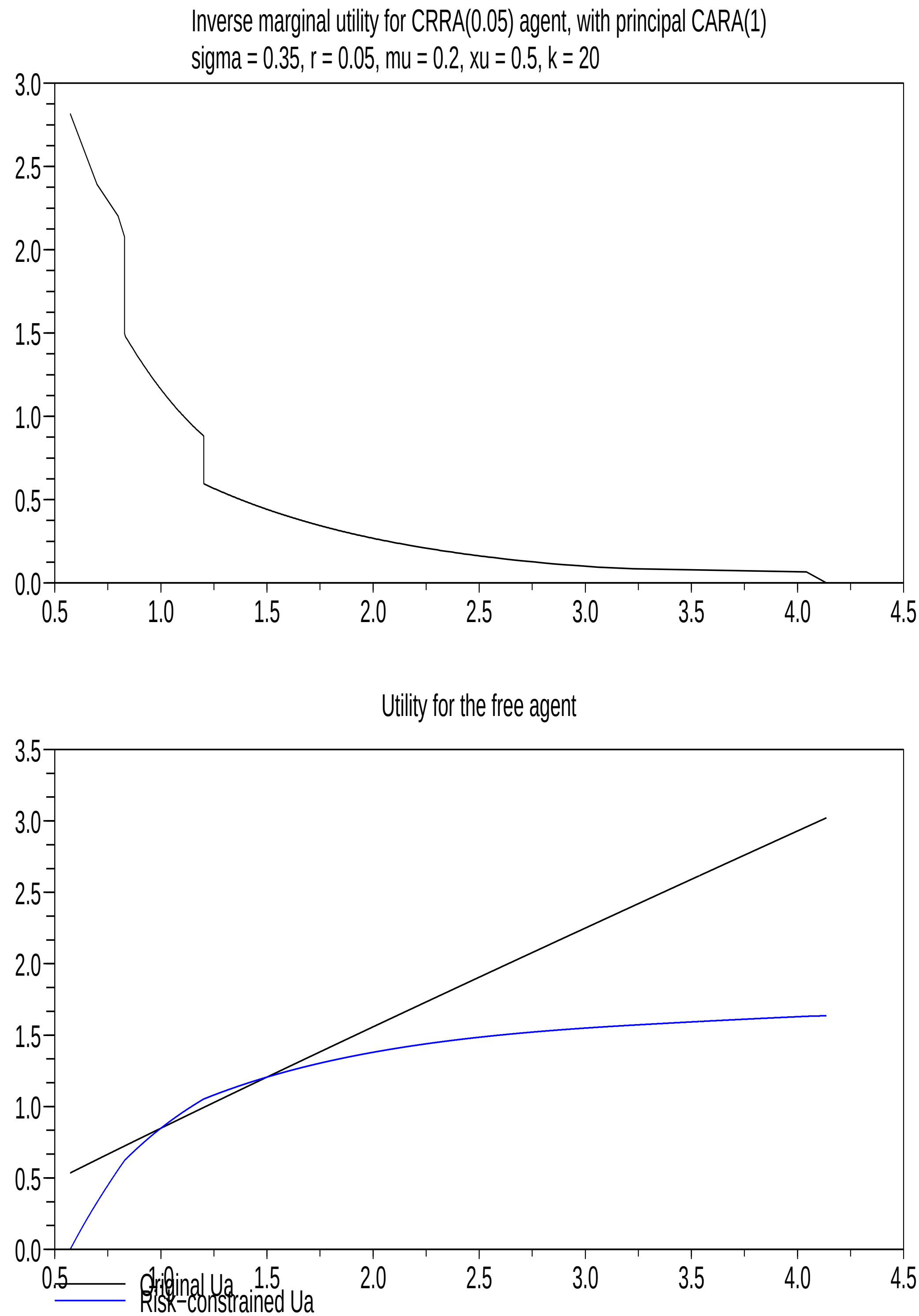
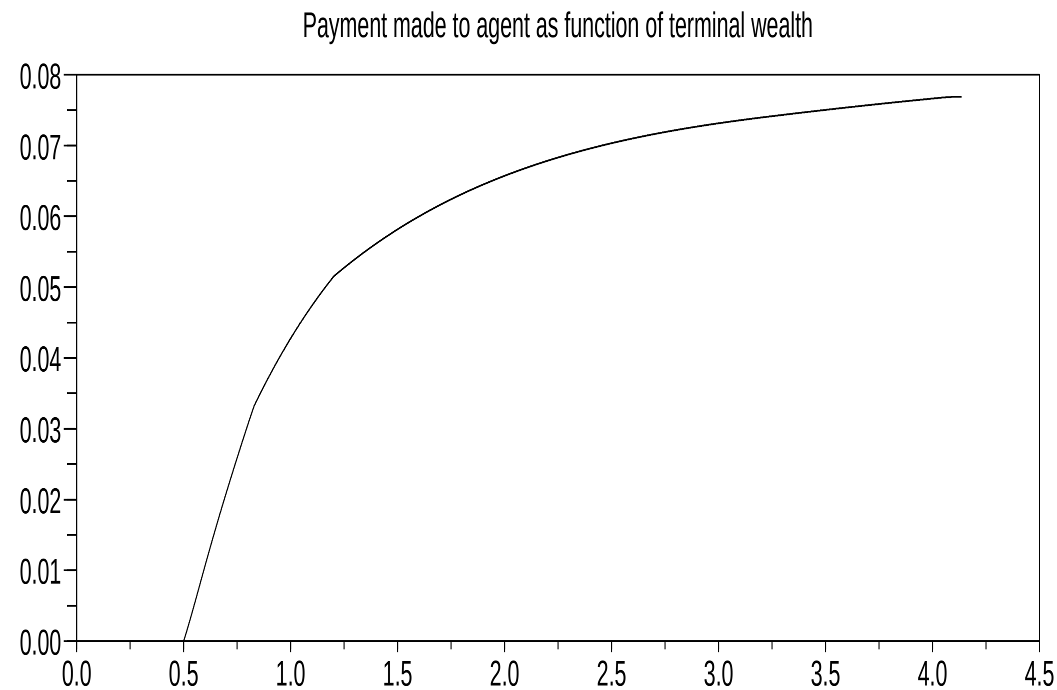


Figure 4: Three risk measure constraints



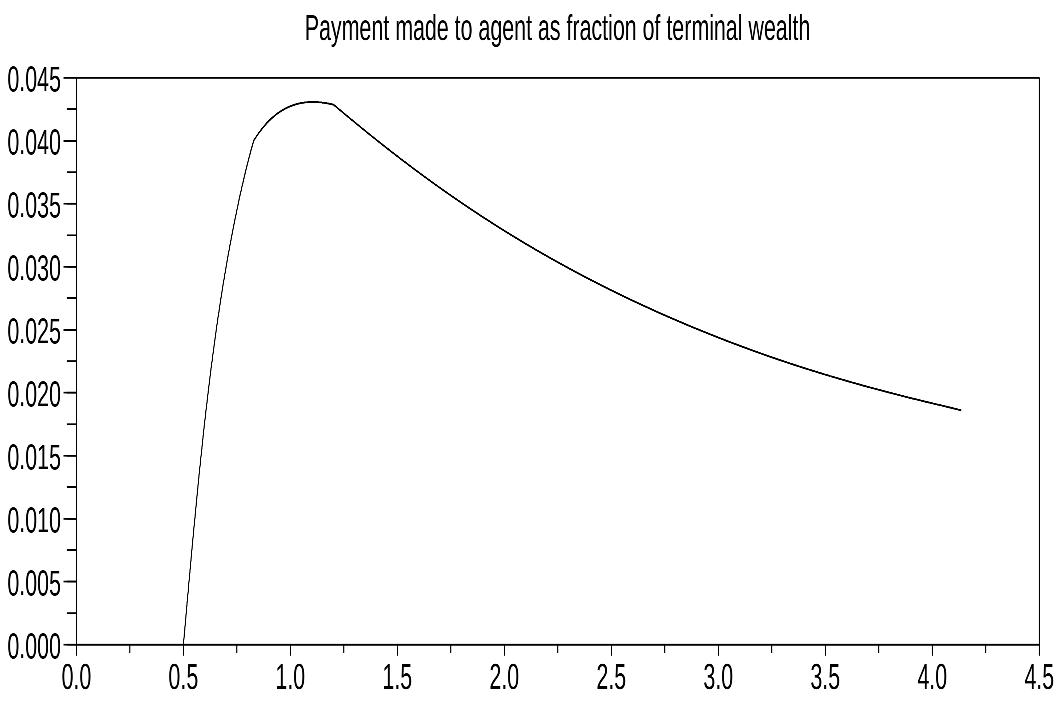


Figure 5: Three risk measure constraints