Utility maximisation with time-lagged trading

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Abstract. In this paper, we study the effect of a delay in execution of trades on the solution to the classical Merton-Samuelson problem of optimal investment for an agent with CRRA utility. Such a delay is a ubiquitous feature of markets, more pronounced the less the liquidity of the market. We firstly consider a continuous-time problem where the single risky asset is a log-Lévy process, and show that if the investor is only allowed to change his portfolio at times which are multiples of some positive h, then the effect is at worst $O(h)$. To make this more precise, we take a discrete-time setting, where the effect of the delay is to constrain the agent to choose his portfolio one period h in advance. We then develop an expansion in powers of h for the delay effect, which we finally confirm by numerical calculations; the asymptotics derived prove to be very good.

Key words: Asymptotic, binomial tree, optimisation, portfolio choice time-lag

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1 Introduction

Our goal in this paper is to study the effects of delay in execution of trades on an agent trying to maximise the expected utility of terminal wealth in a market in which there is a single risky asset and a riskless asset. We can interpret a delay in execution as arising from a lack of liquidity in the risky asset, or perhaps a delay arising between the time an order is submitted to a broker until the time it is filled. Such delays are commonplace, and render the problem incomplete, which makes a huge qualitative difference. To understand the magnitude of this effect, we begin in Section 2 with a continuous-time problem where the log of the risky asset is a Lévy process (which includes the familiar log-Brownian model of the Black-Scholes world). The agent has a fixed time horizon T and aims to maximise the expected utility of his wealth at time T, where the utility is constant relative risk aversion (CRRA). If the agent is only allowed to alter his portfolio at times which are multiples of $h = T/N$, then he will do less well than an agent who is allowed to alter his portfolio at any time, and we shall prove that the order of this effect is at most $O(h)$. This is an interesting and surprising result; if we took the standard log-Brownian share, the risky asset moves of the order of $h^{1/2}$ over a time interval of length h, so why are we not getting a loss of the order of $h^{1/2}$ per time period? Indeed, the result shows that in some sense the loss per time period is $O(h^2)$. Moreover, this order of the loss per period remains correct even for share price processes with jumps, which at first sight is not obvious.

This result does not of course rule out the possibility that the effect could be of smaller order, but in Section 3 we go on to investigate a discrete-time analogue of the log-Brownian model, where we can obtain exact asymptotics, which turn out to be exactly $O(h)$, with expansions for the efficiency and optimal policy as power series in h. Although this gives an asymptotic, it does not tell us how good this asymptotic is; to investigate this, we carried out exact numerical calculations in Section 5 and compared the values obtained with those predicted by the asymptotic. We found that the agreement was virtually perfect. More interestingly, we found that the magnitude of the loss was extremely small, even for quite big values of h; as a consequence, there seems to be little motivation for the constant rebalancing required of the pure Merton solution.

Section 4 contains the asymptotic expansion results for a slightly different discrete-time lag problem from that considered in Section 3, and the final section, Section 6, concludes.

The basic problem of an agent trying to maximise the expected utility of terminal wealth was treated in discrete-time by Samuelson (1969) and in continuous time by Merton (1969). One of the most memorable conclusions of the analysis is that if the agent's utility is CRRA (constant relative risk aversion), then the agent will at all time keep a fixed proportion (the *Merton proportion*) of wealth in the risky asset. Though we speak of the Merton *proportion*, there is nothing preventing this being outside $[0, 1]$ in general. However, we shall throughout the paper assume that *the Merton proportion is in* $[0, 1]$. Cases where this does not hold have little economic reality.

The literature on delay problems of this type is not extensive. Ehrlich and Hamlen (1995) take an asset which follows exponential Brownian motion, and consider the optimal deterministic precommitment to investment and consumption strategies for a fixed time horizon. Rogers (1998) considers the situation of a risk-free asset and an asset which follows exponential Brownian motion, where the agent must choose at equally spaced times how much wealth to set aside for consumption in the next time period, and how to divide the remainder between the two assets. Rogers and Zane (1998) deals with a similar problem where the revision times are the times of a Poisson process, rather than being equally spaced. The methods are quite different, but there are clear similarities in the conclusions, in that (as in this study) the effects of the time lag (or precommitment) are actually quite small, and can be accurately approximated. A one-period time delay features in a very different type of study by Benninga and Proptopapadakis (1988), where a two-nation model is considered, with a shipping delay between the ordering of the other nation's good and its arrival for consumption.

2 The continuous-time problem

In this Section, we shall consider an economy with just two assets, a riskless asset with constant rate of return r, and a risky asset whose price at time t

is given by

$$S_t = S_0 \exp(Z_t),$$

where Z is a Lévy process of the form

$$Z_t = \sigma W_t + at + J_t, \tag{1}$$

where σ and a are constants, W is a standard Brownian motion, and J is a compound Poisson process, consisting of independent jumps with common distribution function F coming at the points of a Poisson process of rate λ. This is not the most general Lévy process possible, but all Lévy processes are limits of such processes, and these assumptions make the proof technically simpler; for the general Lévy process, we would have to consider separately the big jumps and the little jumps, where big and little are defined in terms of the step size h. Note that the standard log-Brownian share model is covered by our assumptions.

Within this economy, we shall consider the situation of an investor who acts as a price-taker, investing in the two assets without transaction costs. His wealth at time t, w_t, satisfies the equation

$$dw_t = rw_t dt + \theta_t \left[\frac{dS_t}{S_{t-}} - r dt \right], \tag{2}$$

where θ is the (previsible) portfolio process. The investor has a fixed time horizon $T > 0$, and his goal is to maximise his expected utility $EU(w_T)$ of wealth at the time horizon, where his utility is CRRA:

$$U(w) = \frac{w^{1-R}}{1-R},$$

where $R > 0$ is different from 1. The case $R = 1$ corresponds to log utility, and could be treated similarly, but we leave the details to the interested reader. We shall assume throughout that the asset price process has finite second moment, which is equivalent to

$$\int e^{2x} F(dx) < \infty. \tag{3}$$

We shall also insist that the agent is only allowed to choose *admissible* portfolio processes, which are those for which w remains non-negative for all time; this is to prevent 'doubling' strategies.

If we were to work with the discounted wealth process $\tilde{w}_t = e^{-rt} w_t$, then (2) becomes

$$d\tilde{w}_t = \theta_t \, d\tilde{S}_t / \tilde{S}_{t-},$$

where $\tilde{S}_t = e^{-rt} S_t$, and the agent's problem is equivalent to maximising $EU(\tilde{w}_T)$. Thus we may (and for this Section shall) assume that $r = 0$, so that the wealth equation is simply

$$dw_t = \theta_t \, \frac{dS_t}{S_{t-}}. \tag{4}$$

Our first result is the generalisation of the familiar Merton result.

Proposition 1. *The value function $V(t,w)$ for the agent, defined by*

$$V(t,w) \equiv \sup E[U(w_T)|w_t = w] \qquad\qquad (0 \le t \le T)$$

takes the form

$$V(t,w) = e^{\alpha(T-t)} U(w), \tag{5}$$

where α satisfies

$$\frac{\alpha}{1-R} = \sup_z \left[(a+\sigma^2/2)z - \frac{\sigma^2 R z^2}{2} + \frac{1}{1-R} \int ((1+z(e^x-1))^{1-R} - 1)\nu(dx) \right], \tag{6}$$

where $\nu \equiv \lambda F$ is the Lévy measure of the jumps. The optimal policy for the agent is to take always

$$\theta_t = \pi_* w_{t-}, \tag{7}$$

where π_* is the value of z achieving the sup in (6).

PROOF. See Appendix.

Notice that when there are no jumps, we recover the familiar Merton solution, with $\pi_* = (a + \sigma^2/2)/(\sigma^2 R)$. The function of z on the right-hand side of (6) is strictly concave in z, so the sup is unique and is attained. Notice also that if the support of F were not bounded above or below, then it must be that π_* is in [0,1], otherwise the integral expression in (6) would be $-\infty$ (by convention, we assume that U is defined on $(-\infty, 0)$ so as to be concave on the whole real line, thus $U(x) = -\infty$ if $x < 0$). This accords with our expectations; if the stock could fall arbitrarily low in a single unpredictable jump, we would never borrow cash to buy the stock, for example.

We are now going to consider what happens to an investor who is *not* able to change his portfolio continuously; we shall suppose that he is able to choose his portfolio at N equally-spaced time points $0, h, 2h, \ldots, (N-1)h$, where $h = T/N$. We call this investor the *h-investor*. Clearly, this investor cannot do as well as the unconstrained (Merton) investor whose optimal policy was determined in Proposition 1, and what we shall do is to show that his loss relative to the Merton investor is $O(h)$. To do this, we define the relative efficiency of two investment strategies.

Definition 1. *Suppose that investor j ($j = 0, 1$) starting with unit wealth and following investment strategy Π_j achieves expected utility of terminal wealth C_j. Then the* efficiency *of the strategy Π_0 relative to the strategy Π_1 is defined to be*

$$\Theta \equiv \Big(C_0/C_1\Big)^{1/(1-R)}. \tag{8}$$

The interpretation of the efficiency of strategy Π_0 relative to strategy Π_1 is that it is *the level of initial wealth that agent 1 would need in order to achieve the same payoff (using strategy Π_1) as agent 0 achieves (using strategy Π_0) starting from wealth 1.*

Our next result compares the performance of Agent 0, who uses a proportional policy (that is, chooses $\theta_t = p w_{t-}$ for some $p \in (0, 1)$), with the performance of Agent 1, who is an h-investor following a proportional policy with the same value of p (that is, at each of the decision times kh chooses to put a proportion p of his current wealth into the risky asset). We do *not* claim that this policy is optimal for Agent 1 (in general it is not), but what *is* clear is that Agent 1's payoff using this policy is a lower bound for his optimal payoff. The result says that the efficiency of Agent 1 relative to Agent 0 is $1 - O(h)$.

Proposition 2. *Agent 0 starting with initial wealth 1 who uses the policy*

$$\theta_t = p w_{t-} \tag{9}$$

for some $p \in (0, 1)$ will achieve payoff

$$C_0 \equiv EU(w_T) = \exp(\beta T) U(w_0), \tag{10}$$

where

$$\beta = (1-R)\left[(a+\sigma^2/2)p - \frac{\sigma^2 Rp}{2} + \frac{1}{1-R}\int((1+p(e^x-1))^{1-R}-1)\nu(dx)\right]. \tag{11}$$

Agent 1, who is an h-investor starting with initial wealth 1 and using the proportional policy with the same p, achieves a payoff C_1 which satisfies

$$\Theta = \Big(C_0/C_1\Big)^{1/(1-R)} = 1 + O(h)$$

as $h \downarrow 0$.

Proof. See Appendix.

We see from Proposition 2 that if $\pi_* \in (0,1)$ then the h-investor, Agent 1, loses efficiency of at most $O(h)$ relative to the optimal investor, Agent 0. The assumption is crucial however, because the h-investor will *never* borrow to buy shares, or sell shares short to invest in the riskless asset; in any time period of length h, the value of the risky asset could with positive probability climb unboundedly, or fall arbitrarily far, and such moves would push the investor into negative wealth (and therefore utility $-\infty$) if the proportion of wealth invested in the risky asset were not in $(0,1)$.

3 Asymptotics for the discrete-time model

We have now established that the effect of a time delay h on changes of portfolio is at worst $O(h)$, but is it perhaps of smaller order? What can we say of the small-h asymptotics of the efficiency? In this Section, we shall formulate these questions in a discrete-time context and solve them.

As in the previous Section, we have in mind an agent who is investing to maximise his expected utility of wealth at the fixed time horizon $T > 0$, but who now is investing in two *discrete-time* asset processes, one risky and the other riskless. The risky asset is the discrete-time analogue of the continous-time share price process S_t satisfying

$$dS_t = S_t(\sigma dW_t + \mu dt) \tag{12}$$

for some constants σ and μ, and the riskless asset is the discrete-time analogue of a savings account yielding interest at a continously-compounded rate r. We approximate the asset price dynamics (12) by a binomial asset price process, which moves in discrete time steps of size $h = T/N$, stepping either up from s to $sa > s$, or down to s/a with respective probabilities p and $1-p$. This way, we cut up the continuous-time parameter interval $[0, T]$ into N equal pieces. In order to match the first two moments of S, we choose

$$a = \frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4}}{2}$$

and

$$p = \frac{a\mathrm{e}^{\mu h} - 1}{a^2 - 1}.$$

Here, $\beta = \mathrm{e}^{-\mu h} + \mathrm{e}^{(\mu+\sigma^2)h}$. The riskless return over one period is

$$\rho = \mathrm{e}^{rh};$$

obviously we need $a > \rho > a^{-1}$ to preclude arbitrage.

We are going to compare the performance of the Merton investor (who chooses his holding of the share in period n knowing the share price at the end of period $n-1$) with an h-investor whose holding of the share in period n is decided at the end of period $n-2$. Let us begin by recording the optimal behaviour of the Merton investor in the following result before turning to the more complicated study of the h-investor.

Proposition 3. *The Merton investor chooses at each stage to invest a proportion*

$$\pi(h) \;\equiv\; \frac{\rho(\lambda - 1)}{a - \rho + \lambda(\rho - a^{-1})}, \tag{13}$$

of his wealth in the share, where

$$\lambda \;=\; \left(\frac{(a-\rho)p}{(\rho - a^{-1})(1-p)} \right)^{1/R}.$$

His maximised expected utility of wealth after N steps starting from initial wealth w is

$$\alpha^N U(w), \tag{14}$$

where

$$\alpha \equiv (p\lambda^{1-R} + 1 - p)\left(\frac{\rho(a-a^{-1})}{a-\rho+\lambda(\rho-a^{-1})}\right)^{1-R}. \tag{15}$$

Proof. If $v_n(w) \equiv \sup E[U(w_N)|w_n = w]$ is the value function for the Merton agent, then v_n satisfies the Bellman equation

$$v_n(w) = \sup_x \left[\, pv_{n+1}(\rho w + x(a-\rho)) + (1-p)v_{n+1}(\rho w + x(a^{-1}-\rho))\,\right]$$

with the initial condition $v_N(w) = U(w)$. The variable x is interpreted as the amount of current wealth invested in the risky asset. By induction, we prove that $v_n(w) = c_n U(w)$ for some constants c_n:

$$\begin{aligned} v_n(w) &= c_{n+1}\sup_x \left[\, pU(\rho w + x(a-\rho)) + (1-p)U(\rho w + x(a^{-1}-\rho))\,\right] \\ &= c_{n+1}w^{1-R}\sup_t \left[\, pU(\rho + t(a-\rho)) + (1-p)U(\rho + t(a^{-1}-\rho))\,\right] \\ &= c_{n+1}U(w)\alpha \end{aligned}$$

by routine calculus. ∎

Let us now set up some notation, and specify the problem of the h-investor precisely. The investor enters the nth time period $(nh, nh+h]$ with total wealth w_n, committed to investing x_n in the risky asset that period, and knowing the current price s_n of the risky asset. He next chooses the *number* θ_{n+1} of units of the risky asset which he is going to hold during the $(n+1)$th period; then the price $s_{n+1} = s_n Z$ of the risky asset for the $(n+1)$th period is revealed, where the random variable Z takes the value a with probability p, and the value $1/a$ with probability $(1-p)$. Thus at the end of the nth period, the investor's wealth w_{n+1} and the value x_{n+1} to be assigned to the risky asset in the $(n+1)$th period can be calculated:

$$w_{n+1} = \rho w_n + (Z-\rho)x_n, \qquad x_{n+1} = \theta_{n+1}s_n Z \equiv \theta_{n+1}s_{n+1}. \tag{16}$$

Thus the evolution of time period n can be summarised as follows:

- At time $nh+$: w_n, s_n, x_n are known; choose θ_{n+1}:

- At time $nh+h$: $s_{n+1} = s_n Z$ revealed, w_{n+1} and x_{n+1} calculated.

At first sight, the effect of this one-step delay is to alter the distribution of returns; an investment choice leads to 4 possible outcomes after 2 time steps, instead of 2 after one time step, and so we should be able read off the change in efficiency simply by making the appropriate perturbation of the volatility. However, this interpretation is *not* correct, because the 2-period returns are not independent, and a computation of the efficiency using this idea does indeed lead to the wrong answer.

The value function

$$V_n(w,x) \equiv \max E\Big[U(w_N)|w_n = w, x_n = x\Big]$$

of this problem solves the Bellman equations

$$\begin{aligned} V_n(w,x) &= \max_{\theta_{n+1}} \Big[pV_{n+1}(\rho w + x(a-\rho), \theta_{n+1} s_n a) \\ &\qquad\qquad + (1-p)V_{n+1}(\rho w + x(1/a-\rho), \theta_{n+1} s_n/a) \Big] \\ &= \max_{\xi} \Big[pV_{n+1}(\rho w + x(a-\rho), \xi a) \\ &\qquad\qquad + (1-p)V_{n+1}(\rho w + x(1/a-\rho), \xi/a) \Big] \qquad (17) \end{aligned}$$

together with the boundary condition

$$V_{N-1}(w,x) = pU(\rho w + x(a-\rho)) + (1-p)U(\rho w + x(1/a-\rho)).$$

It is easy to see that for each n the function V_n is concave as a function of its two arguments. Indeed, this is obvious for $n = N-1$, and then by induction we deduce the concavity of V_n from the concavity of V_{n+1} using the Bellman equations.

It is also easy to see that the value function must have the scaling property

$$V_n(\lambda w, \lambda x) = \lambda^{1-R} V_n(w,x) \qquad (18)$$

for any $\lambda > 0$. Let us therefore define

$$g_n(.) \equiv V_n(1,.),$$

which is a concave function by the concavity of V_n. Now substituting $t = x/w$, $\eta = \xi/w$ reduces the problem to a single variable:

$$g_n(t) = \max_\eta \Bigg[p(\rho + t(a-\rho))^{1-R} g_{n+1}\left(\frac{\eta a}{\rho + t(a-\rho)}\right) + (1-p)(\rho + t(1/a-\rho))^{1-R} g_{n+1}\left(\frac{\eta/a}{\rho + t(1/a-\rho)}\right)\Bigg] \quad (19)$$

with boundary condition

$$g_{N-1}(t) = \frac{p(\rho + t(a-\rho))^{1-R} + (1-p)(\rho + t(1/a-\rho))^{1-R}}{1-R}. \quad (20)$$

Finally, the maximised expected utility is given by

$$\begin{aligned} V_0(w_0, x^*) &= w_0^{1-R} g_0\left(\frac{x^*}{w_0}\right) \\ &= w_0^{1-R} g_0(t^*) \end{aligned}$$

where t^* is simply the value of t which maximises $g_0(t)$.

Explicit solution of the Bellman equations (19) and (20) is impossible, but we can make progress by studying the asymptotics of the problem as $h \downarrow 0$.

When we examine the expression on the right-hand side of (19), the variable η over which we maximise appears in the argument of g_{n+1} two times. In the first occurrence, we see argument

$$x_1 = \frac{\eta a}{\rho + t(a-\rho)} = \eta \left[1 + \sigma(1-t)\sqrt{h} + O(h)\right]$$

and in the second occurrence we see argument

$$x_2 = \frac{\eta/a}{\rho + t(1/a-\rho)} = \eta \left[1 - \sigma(1-t)\sqrt{h} + O(h)\right].$$

Since g_{n+1} is concave, it is unimodal, with maximum at α_{n+1}, say; and therefore it is clear that the maximising choice of η must have the property that the two arguments x_1 and x_2 lie on either side of α_{n+1}. Thus the maximising value of η will be within about $\alpha_{n+1}\sigma(1-t)\sqrt{h}$ of α_{n+1}; it will be close to α_{n+1} for a wide range of t. The optimal value α_{n+1} we expect to be close to the Merton proportion $\pi = (\mu - r)/\sigma^2 R$, and so if we are approximating the function g_{n+1} well enough in a neighbourhood of π, we should be able to identify the asymptotic effect of the delay h. How well do we need to approximate g_{n+1} near π? We know that we are looking for an overall effect of magnitude at most $O(h)$, which will be made up of an effect for each of the $N = T/h$ time-steps in the problem. Therefore if we have got each of these one-step effects correct to order $O(h^2)$, we should have the correct $O(h)$ effect overall. Since the range of η values we are interested in is $O(\sqrt{h})$, this tells us that we need to carry round the Taylor expansion of g_{n+1} up to order $M = 4$. In fact, we performed the calculations up to order $M = 6$ so as to obtain the term in h^2 in the expansion.

To study this directly, we define (recalling the definition of efficiency)

$$\tilde{g}_n(t) \equiv \alpha^{n-N} g_n(t),$$

which modifies the Bellman equations (19) to

$$\begin{aligned}\tilde{g}_n(t) \quad = \quad \max_{\eta} \Bigg[& p(\rho + t(a-\rho))^{1-R} \alpha^{-1} \tilde{g}_{n+1} \left(\frac{\eta a}{\rho + t(a-\rho)} \right) \\ & + (1-p)(\rho + t(1/a - \rho))^{1-R} \alpha^{-1} \tilde{g}_{n+1} \left(\frac{\eta/a}{\rho + t(1/a-\rho)} \right) \Bigg] \quad (21)\end{aligned}$$

The route followed now is to express

$$\log((1-R)\tilde{g}_{N-n}(\pi + u)) \doteqdot \sum_{i=0}^{M} s^i \sum_{j=0}^{M} b_{ij}(n) u^j,$$

where $s \equiv \sqrt{h}$. [‡] Next we assume that the optimal η in (21) can be expressed as a power series, which we truncate to $\pi + \sum_{k=1}^{M} c_k(n+1)s^k$. The coefficients $c_k(n+1)$ are computed from the optimality condition on the right-hand side of (21), and then by substituting back into the right-hand side of (21) we have an expression for $\tilde{g}_n(t)$, which we then expand to obtain the coefficients $b_{ij}(n)$. The expansion was done using Maple; the results are recorded in the following Proposition.

Proposition 4. *Defining $\tilde{b}_{ij}(n) \equiv b_{ij}(n)/(1-R)R$, the matrices $\tilde{b}(n)$ have the following structure.*

[‡]We choose to take logs because the effect per period is multiplicative.

(i) $\tilde{b}_{ij}(n) = 0$ *for all* $j = 0, \ldots, 6$ *if* $i = 0, 1, 3, 5$*;*

(ii) $\tilde{b}_{2j}(n) = -\sigma^2 \delta_{j2}/2$*;*

(iii) for $i = 4$ *we have for* $n \geq 2$

$$
\begin{aligned}
\tilde{b}_{4,0}(n) &= -(n-1)\sigma^4\pi^2(1-\pi)^2/2 \\
\tilde{b}_{4,1}(n) &= -\sigma^2\pi(3\sigma^2 + 6rR\pi + 8\sigma^2\pi^2 + 6\sigma^2R\pi + 6r\pi + 4\sigma^2R^2\pi^2 - 9\sigma^2\pi)/6 \\
\tilde{b}_{4,2}(n) &= -\sigma^2(4rR\pi + 4\sigma^2\pi^2 + 4\pi r + 2\sigma^2\pi^2R^2 + 2\sigma^2R\pi - 2\sigma^2\pi + \sigma^2)/4 \\
\tilde{b}_{4,3}(n) &= -(1+R)\sigma^2(2r - \sigma^2 + 2\sigma^2\pi)/6 \\
\tilde{b}_{4,4}(n) &= \sigma^4(R^2 - 3R - 1)/12 \\
\tilde{b}_{4,5}(n) &= \tilde{b}_{4,6}(n) = 0
\end{aligned}
$$

(iv) for $i = 6$ *we have for* $n \geq 2$

$$
\begin{aligned}
\tilde{b}_{6,0}(n) = & -(n-2)\sigma^4R\pi^2(1-\pi)(2\sigma^2\pi^3R^2 + 2\sigma^2R^2\pi^2 + 6\sigma^2\pi^2R + 6rR\pi + 7\sigma^2\pi^3 \\
& -17\sigma^2\pi^2 + 15\sigma^2\pi + 6\pi r - 3\sigma^2)/6 \\
& -\frac{1}{72}(48\,r\sigma^2R^3\pi^3 + 96\,\sigma^2\pi^3rR - 108\,\sigma^2\pi^2rR + 36\,\sigma^4R\pi + 90\,\sigma^4\pi \\
& -27\,\sigma^4 - 75\,\sigma^4\pi^2 + 84\,\sigma^4R^2\pi^2 + 16\,\sigma^4R^4\pi^4 - 72\,\sigma^4\pi^3R^2 \\
& +108\,\sigma^2\pi\,r + 48\,\sigma^4R^3\pi^3 + 108\,r\sigma^2R\pi - 180\,\sigma^2\pi^2r + 96\,\sigma^2\pi^3r \\
& +72\,r\sigma^2R^2\pi^2 + 48\,\sigma^2\pi^3rR^2 + 36\,r^2R^2\pi^2 + 16\,\pi^4\sigma^4 + 36\,\pi^2r^2 \\
& +24\,\sigma^4R\pi^3 - 36\,\sigma^4R\pi^2 + 40\,\pi^4\sigma^4R^2 + 72\,\pi^2r^2R)\sigma^2\pi^2. \qquad (22)
\end{aligned}
$$

Moreover, for each $j = 1, \ldots, 6$, $\tilde{b}_{6,j}(n) = \tilde{b}_{6,j}(3)$ *for all* $n \geq 3$.

(v) For $n \geq 4$*, the coefficients of the series expansion of the optimal choice of* η *satisfy* $c_1(n) = 0$ *and*

$$
\begin{aligned}
c_2(n) = & -\frac{\pi}{6}(4\sigma^2R^2\pi^2 + 6rR\pi + 6\sigma^2R\pi - 21\sigma^2\pi + 9\sigma^2 + 14\sigma^2\pi^2 + 6\pi r) \\
& +\sigma^2\pi(1-\pi)(R+2)u - \sigma^2(1+R)\pi u^2 + O(u^3)
\end{aligned}
$$

To obtain the asymptotics of the efficiency, we consider maximising over u the expression

$$
\exp\left(\frac{\sum_{i=0}^{M} s^i \sum_{j=0}^{M} b_{ij}(N)u^j}{1-R}\right)
$$

which gives the approximation to Θ up to order h^2 (in this instance - by including further terms in the expansion we could of course obtain higher-order terms). What we obtain in the end is the following result.

Theorem 1. *The efficiency* $\Theta(h)$ *has the expansion*

$$\begin{aligned}\Theta(h) \; &= 1 - \frac{1}{2}\sigma^4\pi^2(1-\pi)^2 RTh - Rh^2\pi^2\sigma^4(1-\pi)\Big[\frac{1}{2}(\pi-1) \\ &+\frac{1}{8}\sigma^4 R\pi^2(\pi-1)^3 T^2 + (\frac{7}{6}\pi^3\sigma^2 + rR\pi - \frac{1}{2}\sigma^2 + \frac{1}{3}\sigma^2R^2\pi^3 \\ &+\frac{1}{3}\sigma^2R^2\pi^2 + \frac{5}{2}\sigma^2\pi + \sigma^2\pi^2 R - \frac{17}{6}\sigma^2\pi^2 + \pi r)T\Big] + O(h^3) \end{aligned} \tag{23}$$

It is also interesting to compare this result with the result of Rogers (2000), Theorem 2. The situation there is considering the difference between the (continuous-time) Merton investor, who adjusts his portfolio continuously during the time interval $[0, h]$, and an investor who divides his wealth optimally at time 0 between the share and the riskless asset, and makes no adjustments to the portfolio thereafter. The efficiency of the latter investor is shown to be

$$1 - \frac{1}{4}\sigma^4\pi^2(1-\pi)^2 Rh^2 + O(h^3). \tag{24}$$

It is shown that the efficiency of an investor who invests throughout the interval $[0, T]$ making portfolio changes only at times which are multiples of h will be

$$1 - \frac{1}{4}\sigma^4\pi^2(1-\pi)^2 RhT + O(h^2).$$

Notice that this is *higher* than the efficiency obtained in Theorem 1, but this is not contradictory, as the problems are different, even though in some sense they become the same as $h \downarrow 0$. In the situation of Rogers (2000), Theorem 2, the investor achieves a continuous return distribution, as opposed to the two-point return distribution obtained in the problem considered here. It is intuitively natural that the continuous return distribution should do a better job of approximating the return for the Merton investor, and this is reflected in the difference in the results.

4 The asymptotics of the delay effect, II

In this Section, we record the analogous results if the h-investor must pre-commit the *cash value* of his holding in the share, rather than the number of shares, as was considered in the previous Section.

The analysis is very similar. The Bellman equations are modified to

$$V_n(w,x) = \max_{\theta_{n+1}} \Big[pV_{n+1}(\rho w + x(a-\rho), \theta_{n+1}) + (1-p)V_{n+1}(\rho w + x(1/a-\rho), \theta_{n+1}) \Big], \tag{25}$$

the scaling property (18) again holds, so the reduced form of the Bellman equations becomes

$$g_n(t) = \max_{\eta} \Big[p(\rho + t(a-\rho))^{1-R} g_{n+1}\left(\frac{\eta}{\rho + t(a-\rho)}\right) + (1-p)(\rho + t(1/a-\rho))^{1-R} g_{n+1}\left(\frac{\eta}{\rho + t(1/a-\rho)}\right) \Big] \tag{26}$$

The boundary condition (20) is as before.

Proposition 5. *Defining* $\tilde{b}_{ij}(n) \equiv b_{ij}(n)/(1-R)R$, *the matrices* $\tilde{b}(n)$ *have the following structure.*

(i) $\tilde{b}_{ij}(n) = 0$ *for all* $j = 0, \ldots, 6$ *if* $i = 0, 1, 3, 5$;

(ii) $\tilde{b}_{2j}(n) = -\sigma^2 \delta_{j2}/2$;

(iii) for $i = 4$ *we have for* $n \geq 2$

$$\begin{aligned}
\tilde{b}_{4,0}(n) &= -(n-1)\sigma^4\pi^4/2 \\
\tilde{b}_{4,1}(n) &= -\sigma^2\pi(3\sigma^2 + 6rR\pi + 8\sigma^2\pi^2 + 6\sigma^2R\pi + 6r\pi + 4\sigma^2R^2\pi^2 - 3\sigma^2\pi)/6 \\
\tilde{b}_{4,2}(n) &= -\sigma^2(4rR\pi + 4\sigma^2\pi^2 + 4\pi r + 2\sigma^2\pi^2R^2 + 2\sigma^2R\pi - 2\sigma^2\pi + \sigma^2)/4 \\
\tilde{b}_{4,3}(n) &= -\sigma^2(1+R)(2r - \sigma^2 + 2\sigma^2\pi)/6 \\
\tilde{b}_{4,4}(n) &= \sigma^4(R^2 - 3R - 1)/12 \\
\tilde{b}_{4,5}(n) &= \tilde{b}_{4,6}(n) = 0
\end{aligned}$$

(iv) for $i = 6$ we have for $n \geq 2$

$$\tilde{b}_{6,0}(n) = \sigma^6(2\pi^2R^2 + 7\pi^2 + 6R\pi + 3)\pi^4 n - \frac{1}{2}(2\sigma^2R^2\pi^2 + 4\sigma^2R\pi + 5\sigma^2\pi^2 - \sigma^2\pi + 2\pi r + 2\sigma^2 + 2rR\pi)$$

Moreover, for each $j = 1, \ldots, 6$, $\tilde{b}_{6,j}(n) = \tilde{b}_{6,j}(3)$ for all $n \geq 3$.

(v) For $n \geq 4$, the coefficients of the series expansion of the optimal choice of η satisfy $c_1(n) = 0$ and

$$c_2(n) = -\frac{\pi}{6}(4\sigma^2R^2\pi^2 + 6\sigma^2R\pi - 3\sigma^2\pi + 3\sigma^2 + 6rR\pi + 14\sigma^2\pi^2 + 6\pi r - 6r) - \sigma^2\pi^2(R+2)u - \sigma^2(1+R)\pi u^2 + O(u^3)$$

Finally, we have the asymptotic expansion of the efficiency, again obtained using Maple.

Theorem 2. *The efficiency $\Theta(h)$ has the expansion*

$$\Theta(h) = 1 - \frac{1}{2}\sigma^4\pi^4RTh + Rh^2\pi^4\sigma^4\left[\frac{1}{2} + \frac{1}{6}\sigma^2(2R^2\pi^2 + 3 + 6R\pi + 7\pi^2)T + \frac{1}{8}\pi^4\sigma^4RT^2\right] + O(h^3) \qquad (27)$$

5 Comparing asymptotics and exact calculation

As a check of the asymptotic solution derived above, we solved the Bellman equations (19) numerically for four different examples, with parameter values given by Table I. The method was to compute the values of g_n at a grid of points, and then interpolate between points using a cubic fit in order to perform the (numerical) maximisation required for (19). Notice that the investor must always keep his wealth positive with probability 1, so this forces him to choose his proportion of wealth in the interval $(-\rho/(a-\rho), a\rho/(a\rho - 1))$. Since the effect under study is very small, we needed to work to high accuracy, and found that it was necessary to put 1500 equally-spaced points into this interval. This slowed the calculations down substantially, so we reduced the interval to an interval of one tenth the size, centred at the Merton proportion.

From Table (II), we see that the asymptotic and numerical analyses are in virtually perfect agreement, and that the loss of efficiency (that is, $1 -$ efficiency), quoted in basis points, for realistic examples is very small, even with as few as 4 rebalancings during the year!

6 Conclusions

We have taken a simple model for the effect of delay in execution of trades, which could be interpreted in several ways: it could represent the effect of delay in the execution of an order placed through a broker; or it could be taken as a proxy for the effect of lack of liquidity, where an agent wishing to change his holding of the asset must wait until there is a counterparty to trade with. This latter interpretation should be viewed as a first step rather than a proper explanation, since lack of liquidity is frequently associated with large price moves, and should more properly be viewed as driven by a changing of information and perceptions.

We have firstly proved that the effect of a delay h is a loss of at most $O(h)$ in efficiency, when the agent has a CRRA utility, and the log asset process is a Lévy process with finite Lévy measure. The fact that inclusion of jumps does not alter the asymptotic from the Brownian case seems at first sight surprising, but it arises because neither the h-investor nor the unrestricted investor is able to anticipate the jumps; both are exposed to the jump risk.

Next, by working in a discrete-time binomial asset model, we are able to solve the problem by dynamic programming, though no closed-form solutions exist. Nonetheless, we have developed an asymptotic for the effect, and have compared this with exact values obtained by numerical solution of the Bellman equations. It turns out that

- the effect of a delay is small;
- the effect is well approximated by the asymptotic expression.

It should perhaps come as little surprise that the effect is quite small. In the Bellman equation, we are maximising a concave indirect utility function. If we make an error ε in the maximising value, the change in the maximised value will be $O(\varepsilon^2)$. If we simply used the optimal rule from the no-lag version of the problem, we should be investing very nearly the correct proportion of wealth in the risky asset, so we expect that the true optimal proportion will be performing well.

The rather surprising message is that the frantic rebalancing required of the Merton problem is not worth the effort, and a more relaxed approach will do nearly as well. This is reflected in the work on transactions costs of Constantinides (1986), Davis & Norman (1990), and Shreve (1995), where a small proportional transaction cost results in a wide 'no-transaction' region. Indeed, Shreve proves that (for the case $0 < R < 1$) if the proportional transaction cost is δ then the width of the no-transaction interval will be $O(\delta^{1/3})$, and the cost will be $O(\delta^{2/3})$; see also Rogers (1999). Part of the reason for this relaxed approach is the same as the reason for the small effect of delay here - being ε away from the optimal proportion only costs you $O(\varepsilon^2)$.

It is a natural question to ask what happens to the Merton consumption problem under a similar delay in execution; this is tackled in different forms in Rogers (2000) and Rogers & Zane (1998).

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A Appendix

Here we collect the various proofs from the main text.

PROOF OF PROPOSITION 1. Although we have a candidate (5) for the value function, it appears that there is no proof based on the martingale principle of optimal control (see, for example, Rogers & Williams (2000), V.15); an Itô expansion of the candidate value function (5) along the path gives only a *local* martingale term, and it appears impossible in general to determine when this is a supermartingale. In the case $0 < R < 1$, the Fatou inequality goes the right way, and we *can* obtain the result we want; but for $R > 1$, the Fatou inequality goes the wrong way, and we have to seek another approach. Such an approach is the following.

We have a candidate for the optimal wealth process, given by taking $\theta_t = \pi_* w_{t-}$ (see (7)). If we follow this, we have

$$\begin{aligned} dw_t &= \pi_* w_{t-} \frac{dS_t}{S_{t-}} \qquad (28) \\ &= \pi_* w_{t-} \left[dZ_t + \frac{1}{2}\sigma^2 dt + e^{\Delta J_t} - 1 - \Delta J_t \right] \\ &= \pi_* w_{t-} \left[\sigma dW_t + a dt + \frac{1}{2}\sigma^2 dt + e^{\Delta J_t} - 1 \right], \end{aligned}$$

which is solved by

$$w_t = w_0 \exp \left[\pi_* \sigma W_t + \pi_*(a + \frac{1}{2}\sigma^2)t - \frac{1}{2}\pi_*^2\sigma^2 t \right] \Pi_{s \le t}(1 + \pi_*(e^{\Delta J_s} - 1)).$$

If we follow this wealth process, then the marginal utility of terminal wealth, $U'(w_T)$, is proportional to

$$Y \equiv \exp \left[-R\pi_*\sigma W_t - R\pi_*(a + \frac{1}{2}\sigma^2)t + \frac{R}{2}\pi_*^2\sigma^2 t \right] \Pi_{s \le t}(1 + \pi_*(e^{\Delta J_s} - 1))^{-R}.$$

If we now introduce a measure $\tilde{P}$ equivalent to P defined by

$$\frac{d\tilde{P}}{dP} \equiv cY,$$

where c is the appropriate normalising constant, then under the measure $\tilde{P}$, $W_t = \tilde{W}_t - R\pi_*\sigma t$, where $\tilde{W}$ is a $\tilde{P}$-Brownian motion. The law of the jumping part of the process Z is also modified in a simple way:

$$\begin{aligned} \tilde{E}\big[\exp(i\theta J_T)\big] &= E\big[\exp\{i\theta J_T - R\sum_{s \le t} \log(1 + \pi_*(e^{\Delta J_s} - 1))\}\big] \\ &= \exp(\ \lambda T \int (\exp\{i\theta x - R\log(1 + \pi_*(e^x - 1))\} - 1)F(dx)) \\ &\propto \exp(\ \lambda T \int (e^{i\theta x} - 1)(1 + \pi_*(e^x - 1))^{-R}\, F(dx)\ ). \end{aligned}$$

Thus under $\tilde{P}$ the Lévy measure of the compound Poisson part of Z is

$$\tilde{\nu}(dx) = (1 + \pi_*(e^x - 1))^{-R} \nu(dx).$$

Accordingly, we can conclude that under $\tilde{P}$, S *is a martingale*: indeed,

$$\begin{aligned} \tilde{E}[S_t/S_0] &= \tilde{E}[\, \exp\{\sigma W_t + at + J_t\}\,] \\ &= \exp\left[\, \frac{1}{2}\sigma^2 t + at - \sigma^2 R \pi_* t + t \int (e^x - 1)(1 + \pi_*(e^x - 1))^{-R} \nu(dx) \,\right] \\ &= 1, \end{aligned}$$

using the fact that π_* is the maximising value of z in (6).

To complete the proof, notice that if x is any admissible wealth process with the same initial wealth w_0, then

$$U(x_T) \leq U(w_T) + U'(w_T)(x_T - w_T),$$

where w is the candidate optimal wealth process defined by (28); this follows from the concavity of U. Taking expectations, therefore, we learn that

$$\begin{aligned} EU(x_T) &\leq EU(w_T) + c\tilde{E}(x_T - w_T) \\ &\leq EU(w_T) + c(w_0 - \tilde{E}w_T), \end{aligned}$$

the last inequality being justified by the fact that x is a non-negative $\tilde{P}$-local martingale, and therefore a $\tilde{P}$-supermartingale. The final step is to check that $\tilde{E}w_T = w_0$, that is, that w is a $\tilde{P}$-martingale; but this results from a calculation similar to that used to prove that S is a $\tilde{P}$-martingale. In conclusion, we have proved that among all admissible wealth processes, the one with the largest payoff is the conjectured optimum, as required.

Proof of Proposition 2. Throughout the proof, V will denote a random variable with distribution F, independent of the Brownian motion W. We shall investigate the relative payoffs of the h-investor over one time period $[0, h]$ and the investor who continuously adjusts his portfolio according to $\theta_t = p w_{t-}$. We aim to show that the expansions of both of these up to and including the term linear in h agree, so that the difference is $O(h^2)$. There is no difficulty in obtaining the expansion up to order h^2 for the second of these, as it is given by (10); the main effort therefore is devoted to the first.

The investor who starts at time 0 with 1 and then puts proportion p of his wealth in the risky asset has at time h a wealth

$$1 + X \equiv 1 + p(e^{Z_h} - 1).$$

Thus his payoff at time h is $E(1+X)^{1-R}$, and it is this that we need to expand in powers of h, up to the linear term. In outline, we do this by decomposing the expectation according to the number of jumps made by time h; with probability $O(h^2)$ there will be more than 1 jump, and with probability $O(h)$ there will exactly one jump, which will have distribution F. If we write N_t for the number of jumps by time t, we have

$$\begin{aligned} E(1+X)^{1-R} &= E\big[\,(1+X)^{1-R}; N_h = 0\,\big] + E\big[\,(1+X)^{1-R}; N_h = 1\,\big] \\ &\qquad + E\big[\,(1+X)^{1-R}; N_h > 1\,\big] \\ &\equiv I + II + III, \end{aligned}$$

say. Taking these in turn, we have

$$\begin{aligned} I &= e^{-\lambda h} E\Bigg[(1 + p(e^{\sigma W_h + ah} - 1))^{1-R} \Bigg] \\ &= e^{-\lambda h} E\Bigg[\sum_{k=0}^{N-1} \frac{X^k}{k!} \frac{\Gamma(2-R)}{\Gamma(2-R-k)} + \frac{X^N}{N!} \frac{\Gamma(2-R)}{\Gamma(2-R-N)} (1+\theta X)^{1-R-N} \Bigg], \end{aligned}$$

where $\theta \in (0,1)$ is some random variable, and $X = p(e^{\sigma W_h + ah} - 1)$. We have proved elsewhere (see Rogers (2000)) that $E|X|^k \le C_k h^{k/2}$, so to get all terms out to order h^2 we have to take $N = 4$ in the above expansion. The remainder term is dealt with because

$$(1+\theta X)^{1-R-4} = (1+\theta X)^{-R-3} \le (1-p)^{-R-3}.$$

Expanding out (using Maple) gives

$$I = 1 + \{(1-R)p(a + \frac{1}{2}\sigma^2 - Rp\sigma^2/2) - \lambda\}h + O(h^2).$$

Turning now to the second term, we have

$$\begin{aligned} II &= \lambda h e^{-\lambda h} E\Big(1 + p(e^{\sigma W_h + ah + V} - 1)\Big)^{1-R} \\ &= \lambda h e^{-\lambda h} E\Big(1 + p(e^V - 1) + p(e^{\sigma W_h + ah} - 1)e^V\Big)^{1-R} \\ &= \lambda h e^{-\lambda h} E\Bigg[\big(1 + p(e^V - 1)\big)^{1-R} \left\{1 \; + \; \frac{pe^V(e^{\sigma W_h + ah} - 1)}{1 + p(e^V - 1)}\right\}^{1-R}\Bigg] \\ &\equiv \lambda h e^{-\lambda h} E\Bigg[\big(1 + p(e^V - 1)\big)^{1-R} \left\{1 \; + \; Y\right\}^{1-R}\Bigg], \end{aligned}$$

say. Thus

$$\begin{aligned} II = \; & \lambda h e^{-\lambda h} E\Bigg[\big(1 + p(e^V - 1)\big)^{1-R} \\ & \left\{ \sum_{k=0}^{N-1} \frac{Y^k}{k!} \frac{\Gamma(2-R)}{\Gamma(2-R-k)} + \frac{Y^N}{N!} \frac{\Gamma(2-R)}{\Gamma(2-R-N)} (1+\theta Y)^{1-R-N} \right\}\Bigg]. \end{aligned}$$

Taking $N = 2$, the remainder term in the sum will be bounded by a constant times

$$\begin{aligned} E(q + pe^V)^{1-R} Y^2 (1+\theta Y)^{-1-R} &\leq E(q + pe^V)^{1-R} Y^2 (1 \; - \; \frac{pe^V}{q + pe^V})^{-1-R} \\ &= E(q + pe^V)^2 Y^2 q^{-1-R} \\ &= p^2 q^{-1-R} E e^{2V} (e^{\sigma W_h + ah} - 1)^2 \\ &\leq ch \end{aligned}$$

for all h small enough. Here, $q \equiv 1 - p$, and c is some positive finite constant. This uses the integrability condition (3). The term $k = 1$ in the expansion is easily seen to contribute something at most $O(h^2)$ to II, leading us to the conclusion that

$$II = \lambda h E\Big[\; \big(1 + p(e^V - 1)\big)^{1-R}\Big] + O(h^2).$$

Finally, turning to III, we have that

$$\begin{aligned} III &= E\big[(1+X)^{1-R}|N_h > 1\big].P\big[\ N_h > 1\ \big] \\ &= (1-e^{-\lambda h}(1+\lambda h))E\big[(1+X)^{1-R}|N_h > 1\big] \\ &= O(h^2). \end{aligned}$$

The final expectation $E[(1+X)^{1-R}|N_h > 1]$ is clearly finite when $R > 1$, and for $0 < R < 1$ we use the inequality $(x+y)^{1-R} \leq x^{1-R} + y^{1-R}$ valid for non-negative x and y, together with the integrability assumption (3). Putting all this together,

$$E\big[(1+X)^{1-R}\big] = 1+h\Big\{(1-R)p(a+\frac{1}{2}\sigma^2-Rp\sigma^2/2)-\lambda+\lambda E\big(1+p(e^V-1)\big)^{1-R}\Big\}+O(h^2),$$

Comparing this with the expression (10) for the payoff of the agent who is allowed to rebalance continuously, we see that the two agree to order h. Thus the efficiency of the investor who follows the proportional investment rule (9) with horizon h relative to the investor who puts proportion p of his wealth in the share at time 0 and leaves it until time h will be $1 + O(h^2)$. Hence the efficiency of the proportional investor relative to the h-investor over the time horizon $T = Nh$ will be $1 + O(h)$, as claimed.

Table I: Parameter Values

Example	1	2	3	4
T	1	1	1	1
w_0	1	1	1	1
μ	ln(1.15)	ln(1.15)	ln(1.15)	ln(1.15)
σ	0.3	0.3	0.3	0.3
r	ln(1.1)	ln(1.05)	ln(1.1)	ln(1.05)
R	2	2	4	4
π	0.246954	0.505399	0.123477	0.252699

Table II: A comparison of the asymptotic solution with the numerical

1/h	Efficiency loss (bp): numerical values	Efficiency loss (bp): asymptotic values
Example 1		
4	0.551532307	0.565819157
8	0.314695502	0.316536482
16	0.166441208	0.166674967
32	0.085409202	0.085439165
64	0.043240257	0.043245002
128	0.021752099	0.021753856
256	0.010908295	0.010909767
512	0.005461606	0.005463093
1024	0.002732081	0.002733599
Example 2		
4	1.115708975	1.183294932
8	0.603760931	0.612156210
16	0.310160182	0.311205291
32	0.156753644	0.156884442
64	0.078745207	0.078762670
128	0.039457784	0.039461447
256	0.019748655	0.019750751
512	0.009878395	0.009880382
1024	0.004939423	0.004941443
Example 3		
4	0.364155216	0.369773302
8	0.210300724	0.211046009
16	0.111965973	0.112062844
32	0.057651546	0.057666382
64	0.029236875	0.029241931
128	0.014719002	0.014723150
256	0.007382902	0.007387121
512	0.003695604	0.003699947
1024	0.001847140	0.001851570
Example 4		
4	1.159947339	1.199362587
8	0.655838070	0.660912869
16	0.345119700	0.345764328
32	0.176625421	0.176709137
64	0.089295863	0.0893113121
128	0.044887035	0.0448948419
256	0.022499904	0.0225072174
512	0.011261046	0.0112685578
1024	0.005630322	0.0056380162