To appear in *Stochastics and Stochastics Reports*, 1994

EQUIVALENT MARTINGALE MEASURES AND NO-ARBITRAGE

by

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Abstract. We give here an elementary proof of the fundamental theorem of discrete-time asset pricing, due originally to Dalang, Morton and Willinger. The essence is a simple utility-maximisation argument, and no deep results from functional analysis are required.

Keywords: equivalent martingale measure, risk-neutral measure, arbitrage, utility.

1. Introduction. Let $(S_t)_{t=0}^N$ be an $\mathbb{R}^d$-valued discrete-time stochastic process, which we think of as modelling the evolution of the (discounted) price of d assets. There is a zeroth asset, cash, whose price at time t is always 1, and we include this in the notation by defining $\bar{S}_t^T \equiv (1, S_t^T)$. The process S is adapted to a filtration $(\mathcal{F}_t)_{t=0}^N$ defined on some probability space $(\Omega, \mathcal{F}, P)$. During day n, the investor holds a portfolio $\theta_n \equiv (\theta_n^1, \ldots, \theta_n^d)^T$ of the assets, $\bar{\theta}_n \equiv (\theta_n^0, \ldots, \theta_n^d)^T$, and at the end of the day the prices S_n for that day are revealed, yielding the investor a gain of $\theta_n \cdot (S_n - S_{n-1}) = \bar{\theta}_n \cdot (\bar{S}_n - \bar{S}_{n-1})$ for the day. In the light of what was known at the end of day n, the investor chooses the next day's portfolio $\bar{\theta}_{n+1}$ subject to the budget constraint

$$(\bar{\theta}_{n+1} - \bar{\theta}_n) \cdot \bar{S}_n = 0 \tag{1.1.}$$

Since there are no short-selling restrictions, what this means in effect is that the investor chooses θ_{n+1}, and then θ_n^0 is altered to pay for the new portfolio. Thus the gain over the whole time period up to N is simply

$$\sum_{n=1}^{N} \theta_n \cdot (S_n - S_{n-1}) \equiv (\bar{\theta} \cdot \bar{S})_N - (\bar{\theta} \cdot \bar{S})_0. \tag{1.2}$$

It is clear from the interpretation of the portfolio process that $\bar{\theta}_n$ must be $\mathcal{F}_{n-1}$–measurable for each n. When we speak of a portfolio process $\bar{\theta}$, we shall always assume that

$\bar\theta_n$ is $\mathcal{F}_{n-1}-$measurable for all n, and that $\bar\theta$ satisfies the self-financing budget constraint (1.1).

We call the portfolio process $(\bar\theta_n)_{n=1}^N$ an *arbitrage opportunity* if

$$(1.3)\qquad (\bar\theta\cdot\bar S)_N-(\bar\theta\cdot\bar S)_0\ge 0\quad \text{a.s., and } P[(\bar\theta\cdot\bar S)_N-(\bar\theta\cdot\bar S)_0>0]>0.$$

The result of Dalang, Morton & Willinger [2] is the following.

THEOREM 1 (Dalang, Morton & Willinger). *The following are equivalent:*

(i) *There exists a probability $\tilde P$ equivalent to P such that under $\tilde P$, $(S_n,\mathcal{F}_n)_{n=0}^N$ is a martingale;*

(ii) *There does not exist any arbitrage opportunity.*

Moreover, if either of these equivalent conditions holds, then it is possible to choose $\tilde P$ in such a way that $(d\tilde P/dP)$ is bounded.

The purpose of this paper is to give a simple proof of this result. The method used is essentially to maximise the expected utility of gains from trade over all possible trading strategies.

This problem in various forms has been considered by a number of authors over the years: Harrison & Kreps [6], Harrison & Pliska [7], Kreps [9], Duffie & Huang [4], Stricker [12], Back & Pliska [1], Föllmer & Schweizer [5], Delbaen [3], Schachermayer [10], Kabanov & Kramkov [8].

The basic idea of the present approach is easily sketched in the case of one time period. Take $U:\mathbb{R}\to(-\infty,0)$ to be a strictly concave, strictly increasing function with continuous derivative, and take X to be an $\mathbb{R}^d$-valued random variable (whose support is not contained in any proper subspace, for simplicity.) We define $\tilde U:\mathbb{R}^d\to(-\infty,0)$ by

$$\tilde U(a)\equiv EU(a\cdot X).$$

If $\tilde U$ is maximised at the point $a^*\in\mathbb{R}^d$, then the first-order condition implies that

$$0=EXU'(a^*\cdot X),$$

and so (an appropriate multiple of) $U'(a^*\cdot X)$ will serve as a change of measure which makes X into a zero-mean random variable. On the other hand, if $\sup_a\tilde U(a)$ is not attained, then it can be shown that there exists some $\theta \in \mathbb{R}^d\backslash\{0\}$such that $\tilde{U}(t\theta)$ remains bounded as $t \to \infty$, and this implies that $P[\theta \cdot X < 0] = 0$. Since the support of X is assumed not to be contained in a proper subspace, we have $P[\theta \cdot X > 0] > 0$, and so θ is an arbitrage opportunity.

Making this precise, and extending to the multi-period case, requires some care, but the essential idea is that just given. It may be possible to extend the method to continuous time.

In Section 2, we give the structure of the proof, using a number of technical propositions on the way. The proof of these is deferred until Section 3.

Acknowledgement. It is a pleasure to thank two referees, whose careful reading of this paper has greatly improved the presentation.

2. Proof of the result of Dalang, Morton & Willinger. The proof of the result is built on several simple propositions. The first says that there is an arbitrage opportunity during the time period up to N if and only if there is an arbitrage opportunity on one of those days.

PROPOSITION 2.1 *The following are equivalent.*

(2.1i) *There exists an arbitrage opportunity;*

(2.1ii) *For some* $n = 1, \ldots, N$*, there exists an* $\mathcal{F}_{n-1}-$*measurable* $\mathbb{R}^d$*-valued random variable* θ_n *such that*

$$\theta_n \cdot (S_n - S_{n-1}) \geq 0 \quad \text{a.s, and } P[\theta_n \cdot (S_n - S_{n-1}) > 0] > 0;$$

(2.1iii) *For some* $n = 1, \ldots, N$*, there exists an* $\mathcal{F}_{n-1}-$*measurable* $\mathbb{R}^{d+1}$*-valued random variable* $\bar{\theta}_n$ *such that*

$$\bar{\theta}_n \cdot (\bar{S}_n - \bar{S}_{n-1}) \geq 0 \quad \text{a.s, and } P[\bar{\theta}_n \cdot (\bar{S}_n - \bar{S}_{n-1}) > 0] > 0.$$

It is not too hard to show (and it is a special case of Proposition 2.2) that if $\varphi : \mathbb{R}^d \to \mathbb{R}^+$ is strictly convex, then *either* there exists a unique a^* minimising φ, *or* it is possible to find some $a \in \mathbb{R}^d\backslash\{0\}$ such that

$$\limsup_{t\to\infty} \varphi(ta) < \infty.$$

The purpose of the next proposition is to show that if we deal with a random strictly convex function, these choices can be made measurably.

PROPOSITION 2.2. *Suppose that* $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}^+$ *has the properties that*

(i) $\varphi(\omega, \cdot)$ *is strictly convex for each* ω*;*

(ii) $\varphi(\cdot, a)$ *is* $\mathcal{G}$*-measurable for each* a*, where* $\mathcal{G}$ *is a sub-*σ*-field of* $\mathcal{F}$*. Then the events*

$$A_0 \equiv \{\omega : \text{there exists } a^* \text{ such that } \varphi(\omega, a^*) \le \varphi(\omega, a) \text{ for all } a\}$$

$$A_1 \equiv \{\omega : \text{for each } a \in \mathbb{R}^d \backslash \{0\}, \lim_{t \to \infty} \varphi(\omega, ta) = +\infty\}$$

are the same, and are $\mathcal{G}$*-measurable. Moreover,* $a^* \equiv a^*(\omega)$ *is* $\mathcal{G}$*-measurable.*

If we further assume that

$$F(\omega) \equiv \{a \in S^{d-1} : \lim \varphi(\omega, ta) < \infty\} \tag{2.2}$$

is closed for all ω*, then it is possible to make a* $\mathcal{G}$*-measurable choice* $\alpha(\omega) \in F(\omega)$ *whenever* $F(\omega) \ne \emptyset$*.*

Proof of Theorem 1. (i) $\Rightarrow$ (ii). If there existed an equivalent martingale measure $\tilde{P}$ and an arbitrage opportunity, then by (2.1ii) there is some $\mathcal{F}_{n-1}$-measurable θ_n such that

$$\theta_n \cdot (S_n - S_{n-1}) \ge 0 \quad \text{a.s.}, \ \tilde{P}[\theta_n \cdot (S_n - S_{n-1}) > 0] > 0. \tag{2.3}$$

Replacing θ_n by $\theta_n |\theta_n|^{-1} I_{\{|\theta_n|>0\}}$, we may assume that θ_n is bounded, and then $\theta_n \cdot (S_n - S_{n-1}) \in L^1(\tilde{P})$, and has $\tilde{P}$-mean 0, since S is a $\tilde{P}$-martingale. This contradicts (2.3).

(ii) $\Rightarrow$ (i). We now suppose that there is no arbitrage, so, specifically, that (2.1ii) does not hold; we must construct an equivalent martingale measure.

Let us now fix our strictly concave, strictly increasing function $U : \mathbb{R} \to (-\infty, 0)$. We shall insist also that U satisfies the condition that for some $\gamma > 0$, for all $x \in \mathbb{R}$

$$U'(x) \le \gamma(1 + |U(x)|). \tag{2.4}$$

We aim to maximise the expected utility $E\Pi_{j=1}^N U(\theta_j \cdot \Delta S_j)$, where $\Delta S_j \equiv S_j - S_{j-1}$, but the first problem is that such an expectation may not be finite. This is easily overcome, however.

PROPOSITION 2.3. *There exists a bounded decreasing strictly positive function* $g : \mathbb{R}^+ \to (0, \infty)$ *such that for each* $a \in \mathbb{R}^d$

$$\sup_{x \in \mathbb{R}^d} |U(a \cdot x)| g(|x|) < \infty.$$

So if we immediately replace P by P', where $dP' \propto \Pi_{j=1}^N g(|\Delta S_j|)dP$, then the integrability of $\Pi_{j=1}^N U(\theta_j \cdot \Delta S_j)$ is assured for all $\theta_j \in \mathbb{R}^d$.

We shall construct inductively measures $P_N = P', P_{N-1}, \ldots, P_0 \equiv \tilde{P}$ equivalent to P of the form

$$dP_n = c_n \Pi_{k=n+1}^N U'(\xi_k \cdot \Delta S_k)dP', \tag{2.6}$$

where the ξ_k will be $\mathcal{F}_{k-1}$-measurable, and the c_n will be appropriate normalising constants. The measures P_n will satisfy the inductive hypothesis
(2.7i) for all $k \leq n$, for all $a \in \mathbb{R}^d$, $U(a \cdot \Delta S_k) \in L^1(P_n)$;
(2.7ii) $(S_k)_{k=n}^N$ is a $((\mathcal{F}_k)_{k=n}^N, P_n)$-martingale.

Clearly, therefore, once the inductive procedure is complete, the measure P_0 is our equivalent martingale measure, and the density will be bounded if the U we chose has bounded derivative.

Verification of the inductive hypothesis for $n = N$ is immediate; (2.7i) is assured by the construction of P', and (2.7ii) is vacuous. Suppose therefore that we have built $P_N, \ldots P_n$, and we now wish to construct P_{n-1}. Abbreviate $S_n - S_{n-1}$ to X, and let $\kappa(\cdot, \cdot)$ be a regular conditional P_n-distribution for X given $\mathcal{F}_{n-1}$.

PROPOSITION 2.4. *Let Π denote the compact metric space of all $d \times d$ orthogonal projection matrices. Then there exists an $\mathcal{F}_{n-1}$-measurable mapping $R : \Omega \to \Pi$ such that for almost all ω*

$$\ker R(\omega) = \text{lin}(\text{supp}(\kappa(\omega, \cdot))).$$

We now apply Proposition 2.2 to the $\mathcal{F}_{n-1}$-measurable random convex function

$$\begin{aligned} \varphi(\omega, a) &\equiv \int -U(a \cdot x)\kappa(\omega, dx) + |R(\omega)a|^2 \\ &= -E_n[U(a \cdot \Delta S_n)|\mathcal{F}_{n-1}] + |R(\omega)a|^2, \end{aligned}$$

which it is easy to check satisfies the hypotheses of Proposition 2.2. For each rational a, $\varphi(\omega, a) < \infty$ almost surely, since by the inductive hypothesis (2.7i), $U(a \cdot \Delta S_n)$ is in $L^1(P_n)$. Hence, except on a null set of ω, φ is finite-valued at every rational, and so is finite-valued everywhere.

Consider now what happens for $a \in F(\omega)$ (defined at (2.2)); we have

$$\lim \varphi(\omega, ta) < \infty.$$

Thus we conclude that $R(\omega)a = 0$, so $a \in \text{lin}\,(\,\text{supp}\,(\kappa(\omega,\cdot)))$, and we must have

$$\kappa(\omega, \{x : a \cdot x < 0\}) = 0, \tag{2.8}$$

otherwise the term $\int -U(ta \cdot x)\kappa(\omega, dx)$ would explode as t tends to infinity. For fixed ω, the set of $a \in S^{d-1}$ for which (2.8) holds is clearly closed, so we may use Proposition 2.2 to give an $\mathcal{F}_{n-1}$-measurable choice $\alpha(\omega)$ whenever $F(\omega) \neq \emptyset$. But then we must have

$$\kappa(\omega, \{x : \alpha(\omega) \cdot x > 0\}) > 0, \tag{2.9}$$

for otherwise $\kappa(\omega, \{x : \alpha(\omega) \cdot x = 0\}) = 1$, contradicting the definition of R. *This $\mathcal{F}_{n-1}$-measurable choice of $\alpha(\omega)$ would therefore be an arbitrage opportunity; the only conclusion is that $F(\omega)$ must be empty almost surely, and there exists an $\mathcal{F}_{n-1}$-measurable $a^*(\omega)$ which minimises $\varphi(\omega,\cdot)$.* Evidently, $R(\omega)a^*(\omega) = 0$ for this minimising choice.

All that remains is to carry out the first-order analysis. We have for any $v \in \mathbb{R}^d$ and $h > 0$

$$\begin{aligned} 0 &\leq \frac{1}{h}\int [U(a^*(\omega) \cdot x) - U(a^*(\omega) \cdot x + hv \cdot x)]\kappa(\omega, dx) \\ &= \frac{1}{h}\int_{\{v \cdot x < 0\}} [U(a^*(\omega) \cdot x) - U(a^*(\omega) \cdot x + hv \cdot x)]\kappa(\omega, dx) \\ &\quad + \frac{1}{h}\int_{\{v \cdot x > 0\}} [U(a^*(\omega) \cdot x) - U(a^*(\omega) \cdot x + hv \cdot x)]\kappa(\omega, dx) \end{aligned} \tag{2.10}$$

and letting $h \downarrow 0$, we use monotone convergence in each integral, the finiteness of the first being assured by hypothesis (2.7i). The conclusion is that

$$0 \leq \int v \cdot x \quad U'(a^*(\omega) \cdot x)\ \kappa(\omega, dx)$$

and since v is arbitrary, we have

$$0 = \int xU'(a^*(\omega) \cdot x)\ \kappa(\omega, dx), \tag{2.11}$$

and indeed $xU'(a^*(\omega) \cdot x) \in L^1(\kappa(\omega,\cdot))$ by modifying slightly the argument at (2.10). Before we allow ourselves to define P_{n-1} by the recipe (2.6), we must check that (writing now ξ_n for a^*, ΔS_n for X) $U'(\xi_n \cdot \Delta S_n) \in L^1(P_n)$. However, by the condition (2.4)

$$\begin{aligned} \int U'(\xi_n \cdot \Delta S_n) dP_n &\leq \gamma + \gamma \int -U(\xi_n \cdot \Delta S_n) dP_n \\ &= \gamma + \gamma \int E_n[-U(\xi_n \cdot \Delta S_n)|\mathcal{F}_{n-1}] dP_n \\ &\leq \gamma - \gamma U(0), \end{aligned}$$

since $\xi_n(\omega)$ is the choice of portfolio which maximises $E_n(U(\xi \cdot \Delta S_n)|\mathcal{F}_{n-1})$, so this must be at least as good as using 0!

Thus we can indeed define P_{n-1} using (2.6) and we have only to check that P_{n-1} satisfies the inductive hypothesis (2.7). But we have for $k \leq n-1$, $a \in \mathbb{R}^d$

$$
\begin{aligned}
\int |U(a \cdot \Delta S_k)| \, dP_{n-1} &\propto \int |U(a \cdot \Delta S_k)| \, U'(\xi_n \cdot \Delta S_n) \, dP_n \\
&\leq \gamma \int |U(a \cdot \Delta S_k)| \, \{1 - U(\xi_n \cdot \Delta S_n)\} \, dP_n \\
&\leq \gamma (1 - U(0)) \int |U(a \cdot \Delta S_k)| \, dP_n
\end{aligned}
$$

by firstly conditioning on $\mathcal{F}_{n-1}$ and using the above argument again. Thus (2.7i) holds, and for (2.7ii) it is clear that changing from P_n to P_{n-1} will not affect the martingale property of $(S_k)_{k=n}^N$, and the martingale property for ΔS_n is exactly what we worked so hard to establish at (2.11)! The proof is complete.

3. Proof of auxiliary propositions.

Proof of Proposition 2.1. Equivalence of (2.1ii) and (2.1iii) is trivial. The implication (2.1iii) $\Rightarrow$ (2.1i) is immediate, so let us now suppose conversely that (2.1i) holds, and that the portfolio process $(\bar{\theta}_n)_{n=1}^N$ is an arbitrage opportunity. Assume without loss of generality that $(\bar{\theta} \cdot \bar{S})_0 = 0$. We define

$$
m \equiv \inf\{n : (\bar{\theta} \cdot \bar{S})_n \geq 0 \quad \text{a.s., and } P[(\bar{\theta} \cdot \bar{S})_n > 0] > 0\},
$$

so that $m \geq 1$. Now *either* (a) $(\bar{\theta} \cdot \bar{S})_{m-1} = 0$ a.s., *or* (b) the probability of $A \equiv \{(\bar{\theta} \cdot \bar{S})_{m-1} < 0\}$ is positive. If (a) happens, then, using (1.1),

$$
\begin{aligned}
(\bar{\theta} \cdot \bar{S})_m &= (\bar{\theta} \cdot \bar{S})_m - (\bar{\theta} \cdot \bar{S})_{m-1} \\
&= \bar{\theta}_m \cdot (\bar{S}_m - \bar{S}_{m-1}) \\
&\geq 0 \quad \text{a.s.}
\end{aligned}
$$

and $P[\bar{\theta}_m \cdot (\bar{S}_m - \bar{S}_{m-1}) > 0] = P[(\bar{\theta} \cdot \bar{S})_m > 0] > 0$, so (2.1iii) holds.

If (b) happens, then on the event A

$$
\bar{\theta}_m \cdot (\bar{S}_m - \bar{S}_{m-1}) = (\bar{\theta} \cdot \bar{S})_m - (\bar{\theta} \cdot \bar{S})_{m-1} \geq -(\bar{\theta} \cdot \bar{S})_{m-1} > 0.
$$

Thus if $\tilde{\theta}_m \equiv I_A \bar{\theta}_m$, we have

$$
\tilde{\theta}_m \cdot (\bar{S}_m - \bar{S}_{m-1}) \geq 0, \qquad P[\tilde{\theta}_m \cdot (\bar{S}_m - \bar{S}_{m-1}) > 0] = P(A) > 0.
$$

So once again, (2.1iii) holds.

Proof of Proposition 2.2. Suppose that there exists a^* such that $\varphi(a^*) \leq \varphi(a)$ for all $a \in \mathbb{R}^d$, and that there exists $b \neq 0$ such that $\liminf_{t\to\infty} \varphi(tb) < \infty$. Thus there exist $t_j \to \infty$ such that $\varphi(t_j b) \leq c < \infty$. Now consider the convex combinations $a_j \equiv \theta_j a^* + (1-\theta_j)t_j b$, where θ_j is so chosen that a_j lies on the unit sphere centred at a^*. Convexity implies that

$$\varphi(a_j) \leq \theta_j \varphi(a^*) + (1-\theta_j)\varphi(t_j b) \leq \theta_j \varphi(a^*) + (1-\theta_j)c,$$

and $\theta_j \to 1$ as $j \to \infty$. Thus $\varphi(a_\infty) \leq \varphi(a^*)$, where a_∞ is the limit of the a_j. But also $\varphi(a_\infty) \geq \varphi(a^*)$, and strict convexity of φ yields a contradiction. Thus $A_0 \subseteq A_1$.

Conversely, suppose that for each $b \in \mathbb{R}^d \backslash \{0\}$, $\lim_{t\to\infty} \varphi(tb) = +\infty$. Now consider the closed subsets

$$F_n = \{x \in S^{d-1} : \varphi(nx) \leq \varphi(0) + 1\}$$

of the compact set S^{d-1}. Convexity of φ implies that the F_n decrease. Since $\cap_n F_n = \emptyset$ by hypothesis, it follows from the finite intersection property that for some N, $F_N = \emptyset$. This implies that $\varphi(\lambda x) \geq \varphi(0) + 1$ for all $x \in S^{d-1}$, for all $\lambda \geq N$, and hence that $\{x \in \mathbb{R}^d : \varphi \leq \varphi(0) + 1/2\}$ is compact. Thus the infimum of φ is attained, and we conclude that $A_1 \subseteq A_0$.

Next,

$$A_1 = \bigcap_n \bigcup_m \bigcap_{q \in \mathbb{Q}^d, |q| \geq m} \{\varphi(q) > n\}$$

is $\mathcal{G}$-measurable, and for any open ball $B \subseteq \mathbb{R}^d$, in view of the strict convexity of φ,

$$\{a^* \in B\} = \bigcup_{q \in \mathbb{Q}^d \cap B} \bigcap_{q' \in \mathbb{Q}^d \backslash B} \{\varphi(q) < \varphi(q')\}$$

is again $\mathcal{G}$-measurable.

We turn now to the final assertion of Proposition 2.2. Fix some dense sequence D in S^{d-1}, and observe that if K is a closed ball of positive radius, then

$$F \cap K \neq \emptyset \Leftrightarrow \exists M \quad \text{such that} \quad \forall k \in \mathbb{N}, \exists x \in D \cap K \quad \text{such that} \quad \varphi(kx) \leq M.$$

Only the implication from right to left is not obvious. But if there exist $x_k \in D \cap K$ such that $\varphi(kx_k) \leq M$, then by passing to a subsequence if necesary we can assume that $x_k \to x^*$, and that $x^* \in F$. Indeed, if for some m we have $\varphi(mx^*) > \varphi(0) \vee M$, we see that for $k > m$ we shall have

$$\varphi(mx_k) \leq \frac{m}{k}\varphi(kx_k) + \frac{k-m}{k}\varphi(0) \leq \varphi(0) \vee M$$

and $\varphi(mx_k) \to \varphi(mx^*)$ as $k \to \infty$, a contradiction.

Thus

$$\{F \cap K \neq \emptyset\} = \bigcup_M \bigcap_{k \geq 1} \bigcup_{x \in D \cap K} \{\varphi(\omega, kx) \leq M\}$$

is a $\mathcal{G}$-measurable event, and hence for any open V, $\{F \cap V \neq \emptyset\}$ is also $\mathcal{G}$-measurable.

Fix now some orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$. The events

$$B_{j\pm} = \{\omega : \pm e_j \cdot x > 0 \quad \text{for some} \quad x \in F(\omega)\}$$

are all in $\mathcal{G}$, and $\bigcup_{j=1}^d (B_{j+} \cup B_{j-}) = \{\omega : F(\omega) \neq \emptyset\}$. On the event B_{1+}, there is an α in $F(\omega)$ maximising $e_1 \cdot a$, and this α is unique, since $C = \{a \in \mathbb{R}^d : \limsup_{t \to \infty} \varphi(\omega, ta) < \infty\}$ is a convex cone. Now for any compact ball K,

$$\begin{aligned}
\{\alpha \notin K\} &= \{\sup_{x \in F} e_1 \cdot x > \sup_{x \in F \cap K} e_1 \cdot x\} \\
&= \bigcup_{q \in \mathbb{Q}} \{q \leq \sup_{x \in F} e_1 \cdot x\} \cap \{\sup_{x \in F \cap K} e_1 \cdot x < q\} \\
&\in \mathcal{G}.
\end{aligned}$$

Thus on B_{1+} we can make a measurable choice of α; and then on $B_{1-} \backslash B_{1+}$; and then on $B_{2+} \backslash (B_{1+} \cup B_{1-})$, and so on.

Proof of Proposition 2.3. Define for $t \geq 0$

$$\psi(t) = 1 \vee |U(-t)|$$

so that $|U(a \cdot x)| \leq \psi(|a|.|x|)$. Now we simply take $g(t) \equiv \psi(t^2)^{-1}$, the inequality for $a, b \geq 0$

$$\psi(ab) \leq \psi(a^2)\psi(b^2)$$

being trivial.

Proof of Proposition 2.4. If Π is the collection of all $d \times d$ orthogonal projection matrices, then Π is a compact metric space, $\Pi = \cup_{r=0}^d \Pi_r$, where Π_r is the subset of all rank-r projections, a closed subset of Π. Then Π_r contains a dense sequence, therefore

$$\gamma_r \equiv \inf\{E[|R\Delta S_n|^2 | \mathcal{F}_{n-1}] : R \in \Pi_r\}$$

is $\mathcal{F}_{n-1}$-measurable, as is

$$\nu \equiv \sup\{r \geq 0 : \gamma_r = 0\}.$$

The support of $P(\cdot|\mathcal{F}_{n-1})$ is $(d-\nu)$-dimensional and $\gamma_\nu = 0$ is attained at a unique R^*, which is $\mathcal{F}_{n-1}$-measurable.

Remark. An example of Schachermayer [11], Section 2.8, shows that if one allows an infinite sequence of price processes, instead of the finite sequence we took here, then Theorem 1 is no longer true.

References

[1] BACK, K. & PLISKA, S. R.: On the fundamental theorem of asset pricing with an infinite state space. *J. Math. Econom.* **20**, 1–18, 1990.

[2] DALANG, R. C., MORTON, A., & WILLINGER, W.: Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics and Stochastics Reports* **29**, 185–201, 1990.

[3] DELBAEN, F.: Representing martingale measures when asset prices are continuous and bounded. To appear, *Math. Finance*.

[4] DUFFIE, D. & HUANG, C.-F.: Multiperiod security markets with differential information. *J. Math. Econom.* **15**, 283–303, 1986.

[5] FÖLLMER, H. & SCHWEIZER, M.: Hedging of contingent claims under incomplete information. Preprint, 1991.

[6] HARRISON, J. M. & KREPS, D.: Martingales and arbitrage in multiperiod securities markets. *J. Econ. Theory* **20**, 381–408, 1979.

[7] HARRISON, J. M. & PLISKA, S. R.: Martingales and stochastic integrals in the theory of continuous trading. *Stoch. Proc. Appl.* **11**. 215–260, 1981.

[8] KABANOV, Y. M. & KRAMKOV, D. O.: No-arbitrage and equivalent martingale measures: an elementary proof of the Harrison-Pliska theorem. Preprint, 1992.

[9] KREPS, D.: Arbitrage and equilibrium in economics with infinitely many commodities. *J. Math. Econ.* **8**, 15–35, 1981.

[10] SCHACHERMAYER, W.: Martingale measures for discrete time processes with infinite horizon. Preprint, 1992.

[11] SCHACHERMAYER, W.: A Hilbert space proof of the fundamental theorem of asset pricing in finite discrete time. Preprint, 1992.

[12] STRICKER, C. Arbitrage et lois de martingale. *Ann. Inst. Henri Poincaré* **26**, 451–460, 1990.