Gaussian Errors

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Among the models proposed for the spot rate of interest, Gaussian models are probably the most widely used; they have the great virtue that many of the prices of bonds and derivatives can be easily computed in closed form. One drawback is that the spot rate process r, being Gaussian, may occasionally take negative values, though it is often claimed that if the probability of negative values is small, then there is no need to worry. It turns out that in many cases this is true, but there are some derivatives whose prices are very sensitive to the possibility of negative rates. One example is a knockout bond, which is a bond which becomes worthless if ever the interest rate drops below zero. Since we do not really believe that interest rates can go negative, we can expect that Gaussian models will give prices at odds with intuition. But there are other, more subtle, examples, such as bonds of long maturity. The discrepancies which arise for these are happening because the bond price is of the form $E\exp(-X)$ for some Gaussian variable X and, although it may be very unlikely that X should be negative, when it *is*, we are exponentiating $-X$, and the contribution to the expectation can be overwhelming. For such derivatives, the prices which the Gaussian models predict can be absurd, yet we have no idea what the *true* price should be. This is because we have not clearly decided what should be the *true* interest rate model (which

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for convenience we approximate by a Gaussian). Until we address this, Gaussian models can continue to spring nasty surprises on us. In this article, we explore the problem, and suggest possible remedies.

To investigate this, we shall take the simplest model, the model of Vasicek [8], in which the spot rate process $(r_t)_{t\geq 0}$ solves a stochastic differential equation driven by a Brownian motion $(W_t)_{t\geq 0}$

$$dr_t = \sigma dW_t + \beta(\mu - r_t)dt, \tag{1}$$

where μ, β and σ are positive constants. [1]

Firstly, let's look at the prices of zero-strike caps and floors; if the zero-strike floor has a significantly positive price, this is a sign of trouble. In Table 1, you find the prices for zero-strike caps/floors for two different scenarios: in Scenario A, $\sigma = 0.01$, $\mu = 0.05$, $\beta = 0.125$ and $r_0 = 0.02$, while in Scenario B, $\sigma = 0.025$, $\mu = 0.10$, $\beta = 0.125$ and $r_0 = 0.05$. The prices are calculated assuming a sum borrowed of \$ 1000.

Table 1

	A		B	
	10 years	20 years	10 years	20 years
Cap	275.83	529.04	492.59	773.28
Floor	0.86	1.42	4.56	8.53

These are fairly typical values of parameters (see, for example, [7]). The prices of the floors are not very large, though they are a lot bigger in Scenario B, the high interest/high volatility scenario, accounting for about 1% of the cap price.

[1] As Hull & White [5] observed, a time-dependent version of the Vasicek model is analytically almost as nice, as are multi-dimensional generalisations.

We get a fuller picture of what is going on by plotting the diagnostic surface of the quantity floor/(cap+floor) - which is always between 0 and 1 - against the values of σ and $1/\beta$. The surface is shown in Figure 1. Regions where the height of the surface is near zero are regions where the price of the floor is negligible compared to the price of the cap. As the volatility σ rises or the strength of the mean reversion drops, the errors evidently rise, as one would expect. Contours of the diagnostic surface are plotted in Figure 2, to give quantitative information on the sizes of the effects.

Another comparison we could make is to price a knockout bond within the Vasicek framework; this is a bond which pays the agreed sum (\$1000 in this case) at maturity, unless the spot rate has fallen below zero at some time during the life of the bond. In reality, we would not expect to get this bond much more cheaply than the ordinary bond without the knockout feature. Table 2 shows what happens.

Table 2

	A		B	
	10 years	20 years	10 years	20 years
Vasicek price	725.03	472.37	511.97	235.25
Knockout price	576.79	334.47	325.46	100.75

The price reduction is certainly noticeable! As before, we display the diagnostic surface (Figure 3) of the ratio (Vasicek - knockout)/Vasicek plotted above the (σ, $1/\beta$) plane. The error surface has a similar shape to the surface in Figure 1, but rises far more steeply. The contour plot of the surface is displayed in Figure 4.

Now of course this derivative is going to highlight problems of negative interest rates starkly, and one reaction might be that one would simply charge the client the full (Vasicek) price for the knockout bond. But if a client came asking for a knockout bond with a knockout at 0.5%, he would expect *some* reduction on the Vasicek price, and the Gaussian framework has no answer. To resolve this, we should have in mind some 'true' model which can give sensible answers for such pricing questions. In order that the Gaussian calculations (which are very tractable) should still be a good approximation for 'true' bond and cap prices, we would want the spot rate process to behave very much like the Vasicek process when away from zero. Near zero, we could (for example) let the volatility drop to 0 to prevent the process going negative, or we could reflect the Vasicek process off zero. It would in practice be difficult to decide from data which model might be best among such alternatives, since there is no relevant data available; and slightly different behaviour near zero could make very big differences to knockout bond prices. We have to accept that although the Gaussian models behave badly on such questions, there are no clearly superior alternatives; maybe all one can do is refuse to deal in such derivatives!

But there is another area where Gaussian models can go astray, namely the pricing of long bonds. The essence of the problem is visible in a simple example, based on Brownian motion, W.

We know that $W_t - \frac{1}{4}t$ tends to $-\infty$ as $t \to \infty$, almost surely. It follows that $\exp(W_t - \frac{1}{4}t)$ tends to 0 as $t \to \infty$. We can even compute the probability that $W_t - \frac{1}{4}t$ is positive, [2] so for large t the probability of a positive value can be as small as we please. And yet $E \exp(W_t - \frac{1}{4}t) = e^{t/4}$ which *grows* exponentially!! Table 3 illustrates the point dramatically.

[2]This is $\bar{\Phi}(\sqrt{t}/4)$, which goes down faster than $e^{-t/32}$. Here, $\bar{\Phi}(\cdot)$ is the tail of the standard Normal $(0, 1)$ distribution

Table 3

t	$P(W_t - \frac{1}{4}t > 0)$	$E\exp(W_t - \frac{1}{4}t)$
25	0.106	518
50	0.039	268337
100	0.006	7.20×10^{10}
200	2×10^{-4}	5.18×10^{21}
400	6×10^{-7}	2.69×10^{43}

Why is this happening? As mentioned previously, the trouble is that, although it may be very unlikely that $W_t - t/4$ should be positive, when it *is*, we exponentiate it, and so we collect a huge contribution to the expectation.

Something very similar may happen with Gaussian interest rate processes. As t increases, the limiting distribution is $N(\mu, \sigma^2/2\beta)$, so by choosing the mean μ to be a reasonable number of standard deviations (say, 5) away from zero, we can ensure that the probability (in the limit, in the risk-neutral probability) of the spot rate r being negative is extremely small. Nevertheless, the long rate [3] in this example is $\mu - \sigma^2/2\beta^2$; so even if we were to take $\mu = 5\sigma/\sqrt{2\beta}$, five standard deviations away from zero, there is absolutely no guarantee even that the long rate is non-negative! Thus we could have the prices of pure discount bonds of large enough maturity actually exceeding 1!

Table 4 below shows the Vasicek prices of bonds of up to 50 years maturities, for a range of parameter values. For large volatilities, we see that the bond price does not have to decrease with maturity, and can even exceed 1! As in Scenario A, we have $\mu = 0.05$ and $\beta = 0.125$, but we have taken $r_0 = 0.05$ this time.

[3] The long rate is defined as $\lim_{t\to\infty} -t^{-1}\log P(0,t)$

Table 4

σ	Maturity (years)				
	10	20	30	40	50
0.01	0.7798	0.6109	0.4805	0.3790	0.2993
0.02	0.7830	0.6243	0.5059	0.4143	0.3414
0.03	0.7882	0.6472	0.5512	0.4807	0.4250
0.04	0.7957	0.6808	0.6215	0.5918	0.5776
0.05	0.8053	0.7265	0.7253	0.7733	0.8569
0.06	0.8172	0.7865	0.8758	1.0723	1.3878

Now volatilities of 4% for a situation where $\mu = 0.05$ are unrealistically high, but if something *obviously* stupid has happened in the last two rows of the Table 4, has something less obviously stupid happened in the earlier rows?

To understand this more deeply, we could calculate the bond prices assuming that the spot rate is given by the Vasicek process (1) reflected at zero. This will certainly be a non-negative process, and if the effects of possible negative values of r are not important, then the bond prices with a reflected Vasicek spot rate should not be much smaller than the Vasicek bond price (they always *will* be smaller). The reflected Vasicek process is a much more reasonable model for the spot rate. Another sensible alternative is to take the modulus of the Vasicek process; see Rogers [6]. Some specimen bond prices are shown in Table 5.

Table 5

	A		B	
	10 years	20 years	10 years	20 years
Vasicek price	725.03	472.37	511.97	235.25
Reflected Vasicek price	720.79	465.03	495.39	210.86

As with zero-strike floors, the discrepancy is more noticeable in Scenario B, where the difference in price on the 20-year bond is about 10 % .

You can see the effect as a function of $(\sigma,\ 1/\beta)$ in the picture Figure 5 of the diagnostic surface, and the corresponding contour plot Figure 6.

As Dybvig & Marshall [1] have recently pointed out, there can be opportunities for profit in the pricing and mispricing of long bonds. For example, if we consider Scenario B here, with $\mu = 0.10$, $\sigma = 0.025$ and taking $r_0 = 0.02$, the Vasicek price of a 50-year bond is 0.027307, and the reflecting-Vasicek price of a 25-year bond is 0.157282. Now the reflecting-Vasicek price of a 25-year bond assuming that $r_0 = 0.0$ is 0.164086, and this is the most expensive that a 25-year bond could be for the reflecting-Vasicek model. Put these facts together, and you find that if the reflecting Vasicek model is correct, you could buy today a 25-year bond which would deliver 0.164086 in 25 years' time, which would then certainly allow you to buy a 25-year bond which would deliver 1 after another 25 years, and the cost today of this strategy would be $0.157282 \times 0.164086 = 0.0258077$, *less than the price of the 50-year bond in the Vasicek model!*

The problems we have identified with the Vasicek model become more involved if one is dealing with time-dependent Gaussian models, or multifactor models. The simple diagnostic plots which we have used here to give a picture of the possible errors have no natural analogue, so we are forced to consider other methods. For a multifactor Gaussian model, the notion of a reflected Gaussian process also seems less natural, so at this stage probably taking the modulus of the Gaussian process and using that as the spot rate process is the best one can do simply. There are effective bounds for the error in bond prices when using the modulus of the Gaussian process (see Rogers [6]), and these are reasonably easy to compute. This then seems to be the best point of view to take, both theoretically and practically; (a) we have adopted a model with non-negative interest rates, so none of the lognormal explosions can occur; (b) we can compute derivative prices on the assumption that r is Gaussian; (c) we can frequently find effective bounds on the error committed by replacing the mod-Gaussian process (which we believe in) with the Gaussian process (which is easy to work with).

Conclusions. We have seen that in the Vasicek model of interest rates the possibility of negative rates can result in substantial mispricing, which gets worse as the volatility or maturity increases. Such problems are not unique to the Vasicek model; *any* Gaussian model (for example, the models of Hull & White [5], Ho & Lee [4], and Heath, Jarrow & Morton [3] when used with deterministic $\sigma(\cdot,\cdot)$) and more generally *any* model which allows negative spot rates (for example, Fong & Vasicek [2]) may suffer from these problems. Such models should ideally only be used in conjunction with checks on the magnitude of possible errors.

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