THE IMPLIED VOLATILITY SURFACE DOES NOT MOVE BY PARALLEL SHIFTS

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Abstract. This note explores the analogy between the dynamics of the interest rate term structure and the implied volatility surface of a stock. In particular, we prove an impossibility theorem conjectured by Steve Ross.

1. Introduction

Today the famous Black–Scholes formula [2] is rarely used to price vanilla call and put options, since for a wide range of strikes and expiries these options are so liquid that the market price cannot be disputed. Instead, the volatility implied by the Black–Scholes formula is used as common language for expressing the market prices of these liquid options.

In recent years there has been a growing interest in the modelling the stochastic dynamics of the Black–Scholes implied volatility surface of a stock. This approach is at some level erroneous, proposing to model a derived quantity rather than the fundamental from which it is derived; however, provided care is taken over the necessary consistency conditions, something may be done. The analogy is with the Heath-Jarrow-Morton approach to modelling of interest rates, though in the context of implied volatility surfaces the consistency conditions are more onerous; see the thesis [4] of Durrleman and the article [13] of Schönbucher for details.

In this note, we shall derive certain model-independent properties of the implied volatility surface, and use these properties to establish (under mild conditions) a conjecture of Steve Ross [12]. This conjecture says (informally) that the implied volatility surface cannot move by parallel shifts - the shape must also change. This is interesting and important because it shows that blindly imposing dynamics on the implied volatility surface (for example, postulating that it moves up and down by parallel shifts) may lead to inconsistency.

The article is structured as follows. Section 2 presents notation and Ross's conjecture. In Section 3 we prove that the implied volatility surface cannot make a uniform downward move, and in Section 4 we prove (under a mild regularity condition) that the implied volatility surface cannot make a uniform upward move, confirming Ross's conjecture. Finally, Section 5 presents some refined results on the flattening of the implied volatility surface which are of independent interest.

2. The implied volatility notation and assumptions

We consider a market with one stock, and European call options of all strikes and expiries. With no loss of generality[1] we assume that the interest rate is zero and that the stock pays no dividend. The probability P of our filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is taken to be the pricing probability, so the stock price process $(S_t)_{t\geq 0}$ is a non-negative P-martingale, which we suppose to start at $S_0 = 1$.

Define the Black–Scholes call price function $f : \mathbb{R} \times [0,\infty) \to [0,1)$ in terms of the tail of the standard Gaussian distribution[2] $\bar{\Phi}$ by

$$f(k,v) = \begin{cases} \bar{\Phi}\big(\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2}\big) - e^k \bar{\Phi}\big(\frac{k}{\sqrt{v}} + \frac{\sqrt{v}}{2}\big) & \text{if } v > 0 \\ (1-e^k)^+ & \text{if } v = 0 \end{cases} \tag{1}$$

The implied variance is the process $(V_t(k,\tau))_{t\geq 0, k\in\mathbb{R}, \tau\geq 0}$ defined implicitly by the formula

$$E\big[\big(\frac{S_{t+\tau}}{S_t} - e^k\big)^+ \big| \mathcal{F}_t\big] = f(k, V_t(k,\tau)), \tag{2}$$

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[1]... but considerable gain in transparency ...

[2]Explicitly, $\bar{\Phi}(x) = \int_x^\infty \exp(-y^2/2)\, dy/\sqrt{2\pi}$.

and in terms of this we define the implied volatility $\Sigma_t(k,\tau)$ as

$$\Sigma_t(k,\tau) = \sqrt{\frac{V_t(k,\tau)}{\tau}} \tag{3}$$

for $\tau > 0$. We have introduced notation for both the implied variance and the implied volatility since some of our results are more naturally stated in terms of one or the other. Throughout, we abbreviate $V_0(k,\tau)$ to $V(k,\tau)$, $\Sigma_0(k,\tau)$ to $\Sigma(k,\tau)$.

Our study is concerned with the following conjecture of Steve Ross:

> Suppose there exists a process $(\xi_t)_{t\geq 0}$ such that for all $t \geq 0, \tau > 0$ and $k \in \mathbb{R}$
>
> $$\Sigma_t(k,\tau) = \Sigma_0(k,\tau) + \xi_t; \tag{4}$$
>
> then $\xi_t = 0$ almost surely for all $t \geq 0$.

We will denote by $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ the standard normal density, and freely make use of the well-known bounds on the Mills's ratio

$$0 \leq 1 - \frac{x\bar{\Phi}(x)}{\phi(x)} \equiv \varepsilon(x) \leq \frac{1}{1+x^2} \tag{5}$$

for $x \geq 0$. The first partial derivatives of f will be used in what follows:

$$f_k(k,v) \;=\; -e^k\bar{\Phi}\left(\frac{k}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) \tag{6}$$

$$f_v(k,v) \;=\; \phi\left(\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right)/2\sqrt{v} \tag{7}$$

3. Long implied volatilities cannot fall

In this section we now study the dynamics of the implied volatility surface at long maturities. Notice that in order to define $V_t(k,\tau)$ via equation (2) we need $S_t > 0$ almost surely; we therefore make the

Assumption A1: $S_t > 0$ for all t almost surely.

The main result proved in this section is the following.

Theorem 3.1. *Under Assumption A1, for any $k_1, k_2 \in \mathbb{R}$, for $0 \leq s \leq t$, we have*

$$\limsup_{\tau\uparrow\infty} \left\{ \, \Sigma_t(k_1,\tau) - \Sigma_s(k_2,\tau) \, \right\} \geq 0$$

almost surely.

Given Theorem 3.1, it is immediate that if the representation (4) of the implied volatility surface holds, then ξ is non-decreasing.

The proof begins with several lemmas, the first being the result of Hubalek, Klein & Teichmann [7] which they use to prove the Dybvig-Ingersoll-Ross [5] result. We present the (short) proof for completeness.

Lemma 3.2. *Let $(X_p)_{p\geq 0}$ be a sequence of non-negative random variables with with finite mean for each $p \geq 0$. Then*

$$\liminf_{p\uparrow\infty} X_p^{1/p} \leq \liminf_{p\uparrow\infty} E(X_p)^{1/p}$$

almost surely.

Proof. Let

$$X = \liminf_{p\uparrow\infty} X_p^{1/p} \text{ and } x = \liminf_{p\uparrow\infty} E(X_p)^{1/p}.$$

By Fatou's lemma and Hölder's inequality, we have

$$\begin{aligned} E[X\mathbb{1}_{\{X>x\}}] &= E[\liminf_{p\uparrow\infty} X_p^{1/p}\mathbb{1}_{\{X>x\}}] \\ &\leq \liminf_{p\uparrow\infty} E[X_p^{1/p}\mathbb{1}_{\{X>x\}}] \\ &\leq \liminf_{p\uparrow\infty} E[X_p]^{1/p}E[\mathbb{1}_{\{X>x\}}]^{1-1/p}. \end{aligned}$$

The above computation implies $E[(X-x)\mathbb{1}_{\{X>x\}}] \leq 0$ and hence $X \leq x$ almost surely. ☐

The next lemma explains the condition (15), which will be satisfied for most models of interest.

Lemma 3.3. *The following are equivalent:*
(i) $S_t \to 0$ *as* $t \uparrow \infty$ *in distribution.*
(ii) $S_t \to 0$ *as* $t \uparrow \infty$ *almost surely.*
(iii) *For some* $K > 0$, $E(S_\tau - K)^+ \uparrow 1$ *as* $\tau \uparrow \infty$.
(iv) *For all* $K > 0$, $E(S_\tau - K)^+ \uparrow 1$ *as* $\tau \uparrow \infty$.
(v) *For some* $k > 0$, $V(k,\tau) \uparrow \infty$ *as* $\tau \uparrow \infty$.
(vi) *For all* $k > 0$, $V(k,\tau) \uparrow \infty$ *as* $\tau \uparrow \infty$.

Proof. The martingale convergence theorem establishes that $S_\tau \to S_\infty$ almost surely for some integrable limit S_∞; the equivalence of (i) and (ii) is immediate. In view of the identity

$$1 - E(S_\tau - K)^+ = E\left[\, S_\tau \wedge K \,\right],$$

(iii) implies (ii), and (ii) implies (iv). The equivalence of (iii) and (v), and of (iv) and (vi), are immediate. ☐

An important consequence of this result is the following.

Corollary 3.4. *If* $P(S_\infty > 0) > 0$, *then for all* k

$$\lim_{\tau\uparrow\infty} \Sigma(k,\tau) = 0.$$

Proof. According to Lemma 3.3, for each k the increasing limit $\lim_{\tau\uparrow\infty} V(k,\tau)$ is finite, and so

$$\Sigma(k,\tau) = \sqrt{V(k,\tau)/\tau} \to 0.$$

☐

Lemma 3.5. *If* $S_t \to 0$ *in distribution then for all* $M > 0$ *we have*

$$\inf_{k\in[-M,M]} V(k,\tau) \uparrow \infty.$$

Proof. By Lemma 3.3, $V(k,\tau) \uparrow \infty$ as $\tau \uparrow \infty$ for each $k \in \mathbb{R}$. Let $T^* > 0$ be so large that $\overline{k}(T^*) > M$ and $\underline{k}(T^*) < -M$. Then for $\tau \geq T^*$ the functions $k \mapsto 1/V(k,\tau)$ are positive and continuous on $[-M,M]$ and converge monotonically to 0 pointwise. The conclusion follows from Dini's theorem. ☐

The heart of the proof is in the following result, which expresses the limiting behaviour of the implied volatility surface as $\tau \uparrow \infty$.

Lemma 3.6. *For each* $t \geq 0$, *for each* $M > 0$, *we have*

$$\lim_{\tau\uparrow\infty} \sup_{k\in[-M,M]} \left| \Sigma_t(k,\tau) - \left(-\frac{8}{\tau} \log E[S_{t+\tau} \wedge 1 | \mathcal{F}_t] \right)^{1/2} \right| = 0. \tag{8}$$

Proof. On the event $\{P(S_t \to 0|\mathcal{F}_t) < 1\}$ the claim is true, since we have both $\lim_{\tau\uparrow\infty} \Sigma_t(0,\tau) = 0$ (by Corollary 3.4), and $\lim_{\tau\uparrow\infty} E[S_{t+\tau} \wedge 1|\mathcal{F}_t] > 0$.

So assume $S_t \to 0$ almost surely. Using (5), and writing $x_1 \equiv (v/2-k)/\sqrt{v}$, $x_2 \equiv (v/2+k)/\sqrt{v}$, we have whenever $v > 2k$ that

$$\begin{aligned}
1 - f(k,v) &= \bar{\Phi}\left(\frac{v/2-k}{\sqrt{v}}\right) + e^k \bar{\Phi}\left(\frac{v/2+k}{\sqrt{v}}\right) \\
&\equiv \bar{\Phi}(x_1) + e^k \bar{\Phi}(x_2) \\
&= \phi(x_1)\left\{\frac{1}{x_1}(1-\varepsilon(x_1)) + \frac{1}{x_2}(1-\varepsilon(x_2))\right\} \\
&= \phi(x_1)\left\{\frac{\sqrt{v}}{v/2-k} + \frac{\sqrt{v}}{v/2+k} - \frac{\varepsilon(x_1)}{x_1} - \frac{\varepsilon(x_2)}{x_2}\right\} \\
&= \phi(x_1)\left\{\frac{v^{3/2}}{v^2/4-k^2} - \frac{\varepsilon(x_1)}{x_1} - \frac{\varepsilon(x_2)}{x_2}\right\}
\end{aligned}$$

We apply this when $|k| \leq M$ and $v = V_t(k,\tau)$, for then

$$\begin{aligned}
E[\left(\frac{S_{t+\tau}}{S_t}\right) \wedge e^k \mid \mathcal{F}_t] &= 1 - f(k,v) \\
&= \phi(x_1)\left\{\frac{v^{3/2}}{v^2/4-k^2} - \frac{\varepsilon(x_1)}{x_1} - \frac{\varepsilon(x_2)}{x_2}\right\}
\end{aligned}$$

and if τ is large enough we have from Lemma 3.5 that v is much larger than M, so $\varepsilon(x_1)/x_1 \leq 2x_1^{-3} \leq 50v^{-3/2}$, and $\varepsilon(x_2)/x_2 \leq 50v^{-3/2}$. Thus

$$\begin{aligned}
-8\log(1-f(k,v)) &= \frac{(v-2k)^2}{v} + 4\log(v) + \delta(v), \\
&= v + \eta(v)
\end{aligned}$$

where $|\delta(v)| \to 0$ as $\tau \to \infty$, and there exist constants A and B such that $|\eta(v)| \leq A + B\log(v)$ for all large enough τ. We therefore have

$$\lim_{\tau\uparrow\infty} \sup_{|k|\leq M} \left| \sqrt{V_t(k,\tau)} - \left\{-8\log E[\left(\frac{S_{t+\tau}}{S_t}\right) \wedge e^k \mid \mathcal{F}_t]\right\}^{1/2} \right| = 0.$$

Dividing by $\sqrt{\tau}$, we deduce that

$$\lim_{\tau\uparrow\infty} \sup_{|k|\leq M} \left| \Sigma_t(k,\tau) - \left\{-\frac{8}{\tau}\log E[\left(\frac{S_{t+\tau}}{S_t}\right) \wedge e^k \mid \mathcal{F}_t]\right\}^{1/2} \right| = 0,$$

and the elementary inequality for positive a, b, x

$$1 \wedge \left(\frac{a}{b}\right) \leq \frac{x \wedge a}{x \wedge b} \leq 1 \vee \left(\frac{a}{b}\right)$$

leads to the result (8). □

Notice the following Corollary of Lemma 3.6, which expresses in a quite precise sense the flattening of the implied volatility surface.

Corollary 3.7.

(9) $$\lim_{\tau\uparrow\infty} \sup_{k_1,k_2\in[-M,M]} |\Sigma(k_2,\tau) - \Sigma(k_1,\tau)| = 0$$

Now we come to the proof of the main theorem of this section.

Proof of Theorem 3.1. We present here the case $s = 0$ as the general case $0 \le s \le t$ is essentially the same. Let $M_t(\tau) = E[S_\tau \wedge 1|\mathcal{F}_t]$ so that $(M_t(\tau)/M_0(\tau))_{t\in[0,\tau]}$ is a martingale for each $\tau > 0$. By Lemma 3.2 we have that

$$\limsup_{\tau\uparrow\infty}\left\{-\frac{8}{\tau}\log(M_t(\tau)) + \frac{8}{\tau}\log(M_0(\tau))\right\} \ge 0.$$

It is easy to see that if $\limsup_{\tau\uparrow\infty} a(\tau)^2 - b(\tau)^2 \ge 0$ for positive functions a and b, then $\limsup_{\tau\uparrow\infty} a(\tau) - b(\tau) \ge 0$. Now taking $a(\tau)^2 = -\frac{8}{\tau}\log(M_t(\tau))$ and $b(\tau)^2 = -\frac{8}{\tau}\log(M_0(\tau))$, an application of Lemma 3.6 yields

$$\limsup_{\tau\uparrow\infty} \Sigma_t(k_1, \tau - t) - \Sigma_0(k_2, \tau) \ge 0.$$

The proof is completed by noting that $\tau \mapsto V_t(k_1, \tau)$ is increasing so that $\Sigma_t(k_1, \tau) \ge \sqrt{(\tau - t)/\tau}\,\Sigma_t(k_1, \tau - t)$ for $\tau \ge t$. □

Remark 1. We now exhibit a model such that the long volatitity strictly increases. Flip a coin at time 0 and let

$$S_t = \begin{cases} 1 & \text{with probability } 1/2 \\ e^{-t/2+W_t} & \text{with probability } 1/2. \end{cases}$$

Since $P(S_t \to 0) = 1/2 < 1$ we have $\lim_{\tau\uparrow\infty} \Sigma_0(k, \tau) = 0$ for all $k \in \mathbb{R}$. But when $t > 0$ we have $\Sigma_t(k, \tau) = 1 > 0$ with probability $1/2$.

4. The implied volatility surface cannot move in parallel shifts

In this section we prove a version of a conjecture of Ross: If the implied volatility surface moves in parallel shifts, the surface must be constant. Again, Assumption A1 is in force for this section.

Theorem 4.1. *Suppose for all $t \ge 0, \tau > 0$ and $k \in \mathbb{R}$ that*

$$\Sigma_t(k, \tau) = \Sigma_0(k, \tau) + \xi_t$$

for some process $(\xi_t)_{t\ge0}$. Define the function g_p by

$$g_p(t) = \begin{cases} \frac{1}{p(p-1)}\log E(S_t^p) \text{ if } p \ne 0, p \ne 1 \\ E(S_t \log S_t) \text{ if } p = 1 \\ -E(\log S_t) \text{ if } p = 0. \end{cases}$$

If for all $t \ge 0$ there exists a $p \in \mathbb{R}$ and $\tau > 0$ such that

$$g_p(t + \tau) \le g_p(t) + g_p(\tau) < \infty, \tag{10}$$

then $\xi_t = 0$ almost surely for all $t \ge 0$.

Remark 2. By Jensen's inequality, the function g_p is positive and increasing for all $p \in \mathbb{R}$, and is finite at least for $0 < p < 1$. Note that if $S_t = e^{-\sigma^2 t/2+\sigma W_t}$ then $g_p(t) = \sigma^2 t/2$ for all $p \in \mathbb{R}$.

Proof. Note that by hypothesis

$$\xi_t = \Sigma_t(0, \tau) - \Sigma_t(0, \tau).$$

By considering the limit superior of the right-hand side as $\tau \uparrow \infty$ we see from Theorem 3.1 that $\xi_t \ge 0$ almost surely.

By the fact that $v \mapsto f(k, v)$ increases for all $k \in \mathbb{R}$, we have

$$\begin{aligned} E\big[\big(\frac{S_{t+\tau}}{S_t} - K\big)^+|\mathcal{F}_t\big] &= f(\log K, \tau\Sigma_t(\log K, \tau)^2) \\ &\ge f(\log K, \tau\Sigma_0(\log K, \tau)^2) \\ &= E[(S_\tau - K)^+] \end{aligned} \tag{11}$$

and

$$\begin{aligned} E\big[\big(K - \frac{S_{t+\tau}}{S_t}\big)^+|\mathcal{F}_t\big] &= Kf(-\log K, \tau\Sigma_t(\log K, \tau)^2) \\ &\ge E[(K - S_\tau)^+] \end{aligned}$$

for all $K > 0$. For twice-differentiable, convex $G : (0, \infty) \to \mathbb{R}$ we have the identity

$$G(s) = G(1) + (s-1)G'(1) + \int_1^\infty (s-K)^+ G''(K)dK + \int_0^1 (K-s)^+ G''(K)dK$$

so that we can conclude that the following inequality holds almost surely for convex G for which $G(1) = G'(1) = 0$:

$$E\big[G\big(\frac{S_{t+\tau}}{S_t}\big)|\mathcal{F}_t\big] \geq E[G(S_\tau)]. \tag{12}$$

Letting G be the convex function $G_p(S) = \frac{1}{p(p-1)}(S^p - pS - (1-p))$ for $p \neq 0, p \neq 1$, we have from inequality (12), after multiplying both sides by S_t^p and taking expectations and logarithms, that

$$g_p(t+\tau) \geq g_p(t) + g_p(\tau).$$

A similar argument shows that the above inequality holds also for $p = 0$ and $p = 1$.

But by assumption, there exists a $p \in \mathbb{R}$ and a $\tau > 0$ such that $g_p(t+\tau) \leq g_p(t) + g_p(\tau) < \infty$, and hence the inequality is, in fact, an equality. Inequality (12) implies that there exists an event $\Omega_0 \in \mathcal{F}_t$ with $P(\Omega_0) = 1$ such that

$$E\big[G_p\big(\frac{S_{t+\tau}}{S_t}\big)|\mathcal{F}_t\big] = E[G_p(S_\tau^p)]$$

for all $\omega \in \Omega_0$. Fixing an $\omega \in \Omega_0$ inequality (11) yields

$$E\big[\big(\frac{S_{t+\tau}}{S_t} - K\big)^+|\mathcal{F}_t\big] = E[(S_\tau - K)^+].$$

for almost all $K > 0$. Hence $\xi_t = 0$ on Ω_0 as claimed. □

Remark 3. Here is a cautionary example which shows that the conjecture is false for implied average variance $V_t(k,\tau)/\tau = \Sigma_t(k,\tau)^2$.

Take $S_t = e^{-t^4/2 + W_{t^2}}$. Then $V_t(k,\tau) = (t+\tau)^2 - t^2$ so that $\Sigma_t(k,\tau)^2 = \tau + 2t$. Hence this example has $\xi_t = 2t$ and

$$\Sigma_t(k,\tau)^2 = \Sigma_0(k,\tau)^2 + \xi_t$$

almost surely for all $k \in \mathbb{R}$, $\tau > 0$, and $t \geq 0$. Note that this example, although not a counterexample to Ross's conjecture, is not included in Theorem 4.1 as $\log E(S_t^p)$ grows quadratically here

The final result shows that if the implied volatility surface is constant then the stock price is the exponential of a Levy process.

Theorem 4.2. *Suppose for all $t \geq 0, \tau > 0$ and $k \in \mathbb{R}$ that*

$$\Sigma_t(k,\tau) = \Sigma_0(k,\tau).$$

If $S_t \to 1$ in distribution as $t \downarrow 0$ (or equivalently, if $V_0(k,\tau) \downarrow 0$ for each $k \in \mathbb{R}$), then $(S_t)_{t\geq 0}$ is an exponential Levy process.

Proof. By assumption we have

$$E\big[\big(\frac{S_{t+\tau}}{S_t} - K\big)^+|\mathcal{F}_t\big] = E(S_\tau - K)^+$$

for all $K \geq 0$. This shows that $\log(S_t)$ has independent and identically distributed increments. □

5. The implied volatility surface flattens at long maturities

The main result of this section proves that the implied volatility smile/skew $\Sigma(\cdot,\tau)$ becomes very flat at long maturities. This is a consequence of the following result, which estimates the derivative of the implied variance with respect to log strike.

Theorem 5.1. *Let $[\underline{k}(t), \overline{k}(t)]$ be the smallest interval containing the support of S_t. Then:*

(i) The right derivative $D_+V(k,\tau)$ of V with respect to k exists for $k \neq \underline{k}(\tau)$, and for all $k \geq 0$

$$D_+V(k,\tau) < 4; \tag{13}$$

(ii) The left derivative $D_-V(k,\tau)$ of V with respect to k exists for $k \neq \overline{k}(\tau)$, and for all $k \leq 0$

$$D_-V(k,\tau) > -4; \tag{14}$$

(iii) Whenever both one-sided derivatives exist, $D_-V(k,\tau) \leq D_+V(k,\tau)$.

(iv) Provided

$$S_t \to 0 \quad \textit{in distribution as } t \uparrow \infty \tag{15}$$

then for all $M > 0$ the following inequalities hold:

$$\limsup_{\tau\uparrow\infty} \sup_{k\in[-M,M]} \max\{|D_-V(k,\tau)|, |D_+V(k,\tau)|\} \leq 4. \tag{16}$$

(v) The bound (16) is sharp in the sense that there exists a martingale $(S_t)_{t\geq 0}$ such that $DV(k,\tau) \to -4$ as $\tau \uparrow \infty$ uniformly for $k \in [-M,M]$.

Remark 4. The flattening of implied volatility as $\tau \uparrow \infty$ has been noticed before in the context of specific models, and the phenomenon has been incorrectly attributed to the central limit theorem; for instance, see Section 7.3 of Rebonato's book [11]. A proof of the flattening of the surface under some additional smoothness and finiteness assumptions has been given by Carr and Wu [3]. The sharp constant appears to be new.

The next lemma asserts that the map $k \mapsto V(k,\tau)$ is rather smooth for each $\tau \geq 0$.

Lemma 5.2. *For each $\tau \geq 0$, the function $k \mapsto V(k,\tau)$ is continuous on $\mathbb{R}$. The left derivative $D_-V(k,\tau)$ exists for all $k \neq \overline{k}(\tau)$ and right derivative $D_+V(k,\tau)$ exists for all $k \neq \underline{k}(\tau)$.*

Proof. Define the function $I : \{(k,c) \in \mathbb{R} \times [0,\infty) : (1-e^k)^+ \leq c < 1\} \to [0,\infty)$ implicitly by the formula

$$f(k, I(k,c)) = c. \tag{17}$$

The function I is continuous on $\{(1-e^k)^+ \leq c < 1\}$ and differentiable (in fact, infinitely-differentiable) on $\{(1-e^k)^+ < c < 1\}$. Calculus gives $I_c = 1/f_v$, and $I_k = -f_k/f_v$. Since $V(k,\tau) = I(k, E[(S_\tau - e^k)^+])$, we have the explicit calculation (omitting appearance of the arguments (k,τ))

$$\begin{aligned} D_+V &= I_k + I_c\, D_+E[(S_\tau - e^k)^+] \\ &= -\frac{f_k}{f_v} - \frac{P(S_\tau > e^k)}{f_v} \qquad (18) \\ &= 2\sqrt{V}\,\frac{\bar{\Phi}\big(k/\sqrt{V} + \sqrt{V}/2\big) - P(S_\tau > e^k)}{\phi\big(k/\sqrt{V} + \sqrt{V}/2\big)}, \qquad (19) \end{aligned}$$

and

$$D_-V = 2\sqrt{V}\,\frac{\bar{\Phi}\big(k/\sqrt{V} + \sqrt{V}/2\big) - P(S_\tau \geq e^k)}{\phi\big(k/\sqrt{V} + \sqrt{V}/2\big)} \tag{20}$$

for $\underline{k}(\tau) < k < \overline{k}(\tau)$. The conclusion now follows since $V(k,\tau) = 0$ for all $k \leq \underline{k}(\tau)$ and for all $k \geq \overline{k}(\tau)$. $\square$

We now turn to the proof of the main result of this section.

Proof of Theorem 5.1. For $\underline{k}(\tau) < k < \overline{k}(\tau)$ it is clear that $D_-V(k,\tau) \leq D_+V(k,\tau)$ by equations (19) and (20), establishing claim (iii). To establish claim (i), note that for $0 \leq k < \overline{k}(\tau)$ we have the following

$$D_+V(k,\tau) < -\frac{f_k}{f_v} = 2\sqrt{V}\,\frac{\bar{\Phi}\big(k/\sqrt{V} + \sqrt{V}/2\big)}{\phi\big(k/\sqrt{V} + \sqrt{V}/2\big)} \leq \frac{4}{k/V + 1} < 4; \tag{21}$$

by inequality (5). Furthermore, for $k \geq \overline{k}(\tau)$ we have $D_+V(k,\tau) = 0$, and claim (i) is established.

As a first step to proving claim (iv), note that if $S_t \to 0$ in distribution there exists a $T^* > 0$ such that $k/V(k,\tau) > -1/4$ for all $k \geq -M$ and all $\tau \geq T^*$ by Lemma 3.5. Hence (21) holds for all $\tau \geq T^*$ and $k \geq -M$.

Now the identities

$$1 - e^{-k} + e^{-k} f(k,v) = f(-k,v)$$

and

$$1 - e^{-k} + e^{-k} E[(S - e^k)^+] = E[(1 - Se^{-k})^+]$$

imply the alternative representation of $V(k,\tau)$ as

$$V(k,\tau) = I(-k, E[(1 - S_\tau e^{-k})^+]).$$

Differentiating yields the explicit formula and the bound

$$\begin{aligned} D_- V(k,\tau) &= 2\sqrt{V}\, \frac{-\bar{\Phi}\big(\sqrt{V}/2 - k/\sqrt{V}\big) + E[S_t\ ;\ S_t < e^k]}{\phi\big(-k/\sqrt{V} + \sqrt{V}/2\big)} \\ &\geq -2\sqrt{V}\, \frac{\bar{\Phi}\big(\sqrt{V}/2 - k/\sqrt{V}\big)}{\phi\big(-k/\sqrt{V} + \sqrt{V}/2\big)} \\ &\to -4 \end{aligned} \tag{22}$$

uniformly on $(-\infty, M]$ as before. Claims (ii) and (iv) follow.

Finally to claim (v); the proof needs to use the following limit:

$$\sqrt{v}\, \frac{1 - f(k,v)}{\phi(-k/\sqrt{v} + \sqrt{v}/2)} \to 4 \tag{23}$$

uniformly for $k \in [-M, M]$ as $v \uparrow \infty$.

We construct a martingale S from two independent random variables, an exponentially-distributed random variable ξ of mean 1, and a random time with distribution

$$P(T \geq t) = \min\{1, t^{-1}\}.$$

The martingale is defined by

$$S_t = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ t\xi & \text{if } 1 \leq t < T \\ 0 & \text{if } T \leq t \end{cases}$$

It follows that

$$E[(S_t - K)^+] = e^{-K/t}$$

for $t \geq 1$. We claim that for all $M > 0$

$$DV(k,t) \to -4$$

uniformly for $k \in [-M, M]$ as $t \uparrow \infty$.

In light of (22), it is sufficient to show that

$$\sqrt{V(k,t)}\, \frac{E[S_t\ ;\ S_t < e^k]}{\phi\big(k/\sqrt{V(k,t)} - \sqrt{V(k,t)}/2\big)} \to 0$$

uniformly. However, from (23), we need only show that (with $K = e^k$)

$$\frac{E[S_t\ ;\ S_t < K]}{1 - E[(S_t - K)^+]} \to 0$$

uniformly. However, simple calculations give

$$\frac{E[S_t\ ;\ S_t < K]}{1 - E[(S_t - K)^+]} = \frac{1}{t}$$

and this is enough. Note that this example does not satisfy Assumption A1. ☐

Remark 5. It is interesting to compare Theorem 5.1 with the following result, which is a slightly stronger formulation of Lemma 3.1 of Lee [8]. We include a proof for completeness.

Theorem 5.3. *For each $\tau > 0$*

$$\lim_{k\uparrow\infty} \sqrt{2k} - \sqrt{V(k,\tau)} = \infty \tag{24}$$

and there exists a $k^* > 0$ such that

$$D_+V(k,\tau) < 2 \tag{25}$$

for all $k \geq k^*$.

Proof. As $k \uparrow \infty$, we have $f(k,v) \downarrow 0$. Also, the AM-GM inequality gives

$$k/\sqrt{v} + \sqrt{v}/2 \geq \sqrt{2k} \tag{26}$$

for all $k, v > 0$ whence

$$e^k \bar{\Phi}(k/\sqrt{V(k,\tau)} + \sqrt{V(k,\tau)}/2) \leq e^k \bar{\Phi}(\sqrt{2k}) < \frac{1}{2\sqrt{\pi k}} \downarrow 0.$$

Hence $\bar{\Phi}(k/\sqrt{V(k,\tau)} - \sqrt{V(k,\tau)}/2) \to 0$ as $k \uparrow \infty$, and the first statement follows quickly.

Using (21), the bound (26) and the bound (5) on the Mills' ratio, it follows also that

$$D_+V(k,\tau) < \sqrt{\frac{2V(k,\tau)}{k}}$$

for all $\tau \geq 0$ and $k > 0$. But by (24) there exists a $k^* > 0$ such that $V(k,\tau) < 2k$ for all $k \geq k^*$, proving the result. □

The above inequalities are sharp. See Benaim and Friz [1] to find exact asymptotics of the implied volatility surface for large absolute log-moneyness.

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