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A MARTINGALE APPROACH TO SOME WIENER-HOPF PROBLEMS, I,

by

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This is one of two companion papers. This paper, I, studies how certain of "Feller's Brownian motions on $[0,\infty)$" may be obtained from Brownian motion via time-substitutions based on fluctuating clocks. Paper II starts afresh with a look at time substitutions for symmetrizable Markov chains; and in that context it is possible to see rather more clearly what is going on. Much of the fascination of Wiener-Hopf theory lies in the difficulty of obtaining explicit answers in concrete cases. The second half of Paper II is a detailed analysis, partially motivated by our study of the chain case, of a concrete example of the problem discussed here in Paper I; and whether or not it makes good reading, it was fun to do.

1. Introduction and summary

1.1. Let $\{B_t : t \geq 0\}$ be a Brownian motion on $\mathbb{R}$ with $B_0 = 0$. Let $\{L_t(x) : t \geq 0, x \in \mathbb{R}\}$ denote the jointly continuous local-time process of B, normalised so that for each x,

$$|B_t - x| - L_t(x)$$

is a martingale. Hence L is twice the standard Brownian local time of Itô-McKean [4].

Let m be a measure on $(-\infty,0]$. [Note. 'Measure' always implies: 'taking values in $[0,\infty]$'.]

Define the additive functionals:

$$(1a) \qquad \phi_t^+ \equiv \int_0^t I_{(0,\infty)}(B_s)\,ds, \qquad \phi_t^- \equiv \int_{(-\infty,0]} L_t(x)\,m(dx),$$

We emphasize that throughout the whole paper, m is understood to satisfy convention (2).

The possibility that ϕ^- can jump to infinity requires us to specify the lifetimes $\zeta(X^-)$ of X^- and $\zeta(Y^-)$ of Y^- more precisely.
Let $\rho^- \equiv \inf\{t:\phi_t^- = \infty\}$. Then

(3) $$\zeta(X^-) \equiv \lim_{s\uparrow\rho^-} \phi_s^-, \qquad \zeta(Y^-) \equiv -\inf_{s<\rho^-} \phi_s.$$

1.2. We wish to study the law of the process Y^+. It is easy to show that

4(i) Y^+ is a strong Markov process with state-space $[0,\infty)$;

and it is clear that

4(ii) Y^+ behaves as a Brownian motion while inside the open interval $(0,\infty)$,

so that if G^+ is the infinitesimal generator of Y^+, then $G^+f = \frac{1}{2}f''$ within $(0,\infty)$.

The results 4(i) and 4(ii) exactly comprise the statement that Y^+ is a Feller Brownian motion in the sense of §5.7 of Itô-McKean [4]. Now the domain of the infinitesimal generator of an arbitrary Feller Brownian motion Z is specified by a side condition of the following type:

(5) $$p_1 f(0) - p_2 f'(0) + \tfrac{1}{2}p_3 f''(0) = \int_{(0,\infty)} [f(x) - f(0)]p_4(dx)$$

where p_1, p_2 and p_3 are nonnegative constants, p_4 is a measure on $(0,\infty)$ such that

(6) $$\int (1 - e^{-x})p_4(dx) < \infty,$$

and $f'(0) = f'(0+)$, $f''(0) = f''(0+)$.

$$\text{measure } \{t : Y_t^+ = 0\} = 0,$$

and this 'must' hold since ϕ^+ has 'no $L_t(0)$ component'. We shall give a proper (analytic) proof later.]

Of course, the 'abstract' statement of Theorem 7 needs to be complemented by the more interesting solution to the 'practical' problem: How does one make explicit the one-one correspondence between measures m and triples (p_1, p_2, J) (considered projectively)? The solution is described in §1.7 after we have introduced the necessary terminology.

1.4. Our basic method is the 'martingale-problem' approach to this type of problem employed in Barlow-Rogers-Williams [1] and Rogers-Williams [6].

For each $\theta > 0$, we find a bounded function f_θ on $\mathbb{R}$ such that

(10) $$M_t^\theta \equiv \exp(\tfrac{1}{2}\theta^2\phi_t) f_\theta(B_t) \text{ defines a martingale } M^\theta.$$

Since M^θ is bounded on each interval of the form $[0, \tau_t^+]$, we may apply the optional-sampling theorem to deduce that

$$\exp(\tfrac{1}{2}\theta^2 t) f_\theta(Y_t^+) \text{ is a martingale,}$$

whence, with G^+ again denoting the infinitesimal generator of Y^+, we have

(11) $$f_\theta \in \mathcal{D}(G^+) \quad (\text{and } G^+ f_\theta = -\tfrac{1}{2}\theta^2 f_\theta).$$

Our hope is that on feeding the information (11) into formula (5), we can determine the characteristics (p_1, p_2, p_3, p_4) of Y^+; and this proves to be justified.

Note. We need to be rather careful in checking the validity of the above application of the optional-sampling theorem because of the possibility that $\tau_t^+ = \infty$. Now, of course, $f_\theta(\partial) = 0$, by the usual convention. So the essential thing to prove is that (except on a null set of ω)

independence of the 'up' and 'down' excursion processes from 0, and some standard independent-increment properties, we have:

$$\text{(13)} \qquad E[M^{\theta}(\sigma_t^+)|\mathcal{G}_t] = \exp[\tfrac{1}{2}\theta^2 t - \tfrac{1}{2}\theta^2 c_\theta \Lambda(\sigma_t^+)] f_\theta(X_t^+),$$

where c_θ is determined via the equation:

$$\text{(14)} \qquad \exp(-c_\theta t) = E \exp[-\tfrac{1}{2}\theta^2 \phi^-(\Lambda^{-1}(t)],$$

where, of course, $\Lambda^{-1}(t) \equiv \inf\{u:\Lambda(u) > t\}$. The fact that the expression at (13) defines a martingale implies that

$$f_\theta \in \mathcal{D}(\mathcal{A}_\theta) \quad \text{and} \quad \mathcal{A}_\theta f = -\tfrac{1}{2}\theta^2 f ,$$

where $\mathcal{A}_\theta$ is the infinitesimal generator of elastic Brownian motion with killing constant c_θ. See §2.3 of Itô-McKean [4]. Hence, f_θ must satisfy the boundary condition:

$$f'_\theta(0) = c_\theta f_\theta(0),$$

and we have (using (14)):

$$\begin{aligned} f_\theta(0) = c_\theta^{-1} &= \int_{[0,\infty)} \exp(-c_\theta t)dt \\ &= E\int_{[0,\infty)} \exp[-\tfrac{1}{2}\theta^2\phi^-(\Lambda^{-1}(t))]dt \\ &= E\int_{[0,\zeta(X^-))} \exp(-\tfrac{1}{2}\theta^2 t)d\Lambda(\sigma_t^-) . \end{aligned}$$

By a further elementary application of the optional-sampling theorem to (10), the reader can easily show that

If we relax the assumption that m is finite and strictly positive on every compact subinterval of $(-\infty,0]$, then (17) still holds, but now we have

$$f_\theta(0) = \gamma + \int_{[0,\infty)} \frac{G(dr)}{r^2 + \theta^2}, \tag{18}$$

where $-\gamma = \inf\{u \le 0 : m[u,0] = 0\}$ and G is again a measure on $[0,\infty)$. [Note. A certain amount of poetic licence may be needed in the interpretation of (17) when $m[a,0] = \infty$ for some a. Then $f_\theta(a) = 0$, and we may need licence to interpret $0 \times \infty$].

We thought it instructive to derive the analytic form of f_θ from the assumption that M^θ is a martingale. We leave the reader to check the converse result, the one we really need: viz., that if f_θ has the analytic form we have described, then M^θ is indeed a martingale.

1.7. The deep and very remarkable inverse spectral theorem of Krein (see Dym-McKean [2]) tells us that (17) and (18) put measures m satisfying (2) into one-one correspondence with pairs (γ,G), where

$$\infty \ge \gamma \ge 0, \qquad \int (r^2+1)^{-1} G(dr) < \infty, \quad \text{and} \quad G = 0 \quad \text{if} \quad \gamma = \infty. \tag{19}$$

We shall prove that if the pair (γ,G) satisfies (19), and if $f_\theta(0)$ is defined by (18) and f_θ on $(0,\infty)$ via (12), then the quadruple (p_1,p_2,p_3,p_4) is determined uniquely (modulo multiplication by scalars) by the fact that f_θ satisfies (5) for all $\theta > 0$. If we temporarily assume (8), we are led via (18), (12), and (5) to the relation:

probabilistic insight.

However, it would be totally wrong to imagine that everything of interest in the present paper can be attributed in some way to the 'dominant' rôle of the process $\phi \circ \Lambda^{-1}$. Indeed, Paper II makes it clear that the way in which the spectral decomposition of the transition semigroup of Y^- governs the law of Y^+ reflects a general principle for Markov processes. Paper II also gives some explanation, rather than only verification, of why the p_4 measure for Y^+ is completely monotone.

1.9. In §3, we show that the martingales M^θ at (10) form a 'full' family in a stronger sense than is implicit in various uniqueness assertions made above. In particular, we show that for $x < 0$, the P^x law of Y_0^+ is UNIQUELY determined by the Wald identity (optional-sampling result):

$$E^x f_\theta(Y_0^+) = f_\theta(x) \qquad \forall \theta > 0.$$

This key uniqueness theorem is obtained as a consequence of the Wiener-Hopf factorization (20) of $f_\theta(0)$.

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Given $\varepsilon > 0$, we can first choose δ so that the first term on the right-hand side is less than $\frac{1}{2}\varepsilon$, and then choose θ_0 so large that the second term is less than $\frac{1}{2}\varepsilon$ when $\theta > \theta_0$. Hence,

$$\int (\theta^{-1}\sin\theta x)p_4(dx) = o(1) \qquad \text{as} \quad \theta \uparrow \infty.$$

Next,

$$\left|\int (1 - \cos\theta x)p_4(dx)\right| \le \int_0^1 |1 - \cos\theta x| p_4(dx) + 2\int_1^\infty p_4(dx)$$

$$\le \theta\int_0^1 xp_4(dx) + 2\int_1^\infty p_4(dx) = O(\theta).$$

Since we are ignoring the case when $f_\theta(0) = 0, \forall\theta$, we see from (22) that $\theta^2 f_\theta(0) \uparrow K \in (0,\infty]$ as $\theta \uparrow \infty$. On dividing (23) by $\theta^2 f_\theta(0)$, we see that

$$-\tfrac{1}{2}p_3 + O(\theta^{-1}) = \frac{p_2}{\theta^2 f_\theta(0)} + o(1).$$

If $K = \infty$, we see that $p_3 = 0$; and if $K < \infty$, we obtain $-\frac{1}{2}p_3 = K^{-1}p_2$, so that (since $p_2 \ge 0$ and $p_3 \ge 0$) we must have $p_3 = p_2 = 0$. □

[<u>Note</u>. The reader should perform the exercise of spelling out the more informative probabilistic proof described after the statement of Theorem 7.]

(25) THEOREM. <u>The quadruple</u> $(p_1, p_2, 0, p_4)$ <u>is uniquely determined (modulo scalar multiples) by the fact that equation (23) holds for every</u> $\theta > 0$.

<u>Proof</u>. This proof is a modification of the proof due to Kingman which was given in §5 of Rogers-Williams [6].

Let

$$\mathbb{H} \equiv \{z \in \mathbb{C} : \text{Im}(z) \ge 0\}, \qquad \mathbb{H}^+ \equiv \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

are determined up to a constant multiplier; and, by standard results, so too is the quadruple $(p_1, p_2, 0, p_4)$.

Note. In §3 below, we present a deeper uniqueness result which is more useful in practice.

2.2. We continue on the course mapped out in §1.7. If we assume (8) and substitute (8) and (22) into (23), then we obtain the following equation, previously labelled as (20):

$$\gamma + \int \frac{G(dr)}{r^2 + \theta^2} = \frac{p_2 + \int \frac{J(dr)}{r^2 + \theta^2}}{p_1 + \theta^2 \int \frac{J(dr)}{r(r^2 + \theta^2)}} \tag{30}$$

We shall prove the following theorem.

(31) THEOREM. Equation (30) sets up a one-one correspondence between pairs (γ, G) satisfying (19) and triples (p_1, p_2, J) (considered modulo scalar multiples) where $p_1 \geq 0$, $p_2 \geq 0$, and J satisfies (9).

Let us briefly recall the logic of the situation. A measure m determines a pair (γ, G). Part of Theorem 31 guarantees the existence of a triple (p_1, p_2, J) such that (30) holds. Theorem 25 guarantees that

$$p_4(dx) = dx \int e^{-rx} J(dr) \tag{32}$$

and also that (p_1, p_2, J) is unique. Conversely, if a triple (p_1, p_2, J) is given, then Theorem 31 guarantees the existence of a unique pair (γ, G) such that (30) holds, and Krein's inverse spectral theorem guarantees existence and uniqueness of the corresponding m.

of atoms of masses G_i at points $\sqrt{(\mu_i)}$ $(0 \le i \le n)$, then (30) holds where $p_1 = p_2 = 0$ and J consists of atoms of masses J_i at $\sqrt{(\nu_i)}$ $(0 \le i \le n)$.

(34) LEMMA. *Suppose that* $G_i > 0$ $(0 \le i \le n)$ *and that*

$$0 = \mu_0 < \mu_1 < \ldots < \mu_n.$$

Then there exist strictly positive constants J_i $(0 \le i \le n)$ *and* ν_i $(0 \le i \le n)$ *with*

$$\mu_0 < \nu_0 < \mu_1 < \nu_1 < \ldots < \mu_n < \nu_n \tag{35}$$

such that for all z *in* $\mathbb{C}$ *(with the obvious interpretation at various poles)*

$$\sum_i \frac{G_i}{z + \mu_i} = \frac{\sum_i \frac{J_i}{z + \nu_i}}{z \sum_k \frac{J_k}{(z + \nu_k)\sqrt{(\nu_k)}}} . \tag{36}$$

Proof of Lemma 34. First, assume that (36) holds. Let $z \to -\nu_j$ in (36) to obtain:

$$\sum_i \frac{G_i}{\mu_i - \nu_j} = -\frac{1}{\sqrt{(\nu_j)}} .$$

Hence, the values ν_j must be roots of the equation:

$$\sum_i \frac{G_i}{x - \mu_i} = \frac{1}{\sqrt{x}} . \tag{37}$$

But, on sketching the graphs of the two sides of (37), we see that (37) has exactly $(n+1)$ roots $\nu_0, \nu_1, \ldots, \nu_n$ within $(0,\infty)$, and that the order-relations (35) hold.

On putting $z = -\mu_i (i \neq 0)$ in (36), we obtain:

$$\gamma + \int_{[0,\infty)} \frac{G(dr)}{r^2 + \theta^2} = \int_{[0,\infty]} \frac{r^2 + 1}{r^2 + \theta^2} \hat{G}(dr)$$

where

(41) $$\hat{G}(dr) = (r^2 + 1)^{-1} G(dr) \quad \text{on} \quad (0,\infty), \quad \hat{G}\{\infty\} = \gamma.$$

Since we are ignoring the case when $\gamma = \infty$, the measure $\hat{G}$ is a bounded measure on $[0,\infty]$. In the sense of weak* convergence of bounded measures on $[0,\infty]$, we can approximate $\hat{G}$ by measures $\hat{G}^{(n)}$ each consisting of an atom at 0 together with a finite number of atoms within $(0,\infty)$. From Lemma 34, we know that

(42) $$\int_{[0,\infty]} \frac{r^2 + 1}{r^2 + \theta^2} \hat{G}^{(n)}(dr) = \frac{\int_{[0,\infty]} \frac{r(r+1)}{r^2 + \theta^2} \hat{J}^{(n)}(dr)}{\int_{[0,\infty]} \frac{\theta^2(r+1)}{r^2 + \theta^2} \hat{J}^{(n)}(dr)}$$

for some atomic measure $\hat{J}^{(n)}$ on $[0,\infty]$ which we can take to be a probability measure. If $\hat{J}$ is any weak* limit of $\hat{J}^{(n)}$ as $n \to \infty$, we have

$$\gamma + \int_{[0,\infty)} \frac{G(dr)}{r^2 + \theta^2} = \frac{\int_{[0,\infty]} \frac{r(r+1)}{r^2 + \theta^2} \hat{J}(dr)}{\int_{[0,\infty]} \frac{\theta^2(r+1)}{r^2 + \theta^2} \hat{J}(dr)}$$

But for $\mu > 0$,

$$\sum_{\nu_k > \mu} \frac{J_k}{\nu_k - \mu} > \sqrt{\mu} \sum_{\nu_k > \mu} \frac{J_k}{(\nu_k - \mu)\sqrt{(\nu_k)}},$$

$$-\sum_{\nu_k > \mu} \frac{J_k}{\nu_k - \mu} < -\sqrt{\mu} \sum_{\nu_k < \mu} \frac{J_k}{(\nu_k - \mu)\sqrt{(\nu_k)}}.$$

Hence for $i = 1,2,\ldots,n$,

$$\sum_k \frac{J_k}{\nu_k - \mu_i} > \sqrt{(\mu_i)} \sum \frac{J_k}{(\nu_k - \mu_i)\sqrt{(\nu_k)}} = 0,$$

and $G_i > 0$. Of course,

$$(44) \qquad G_0 = (\Sigma J_i/\nu_i)(\Sigma J_k/\nu_k^{3/2}) > 0.$$

To show that (36) must hold if the G_i $(0 \le i \le u)$ are defined via (43) and (44), we can apply the 'polynomial' argument at the end of §2.3, or else appeal to the Mittag-Leffler theorem.

The proof of Theorem 31 is now complete.

2.6. Notes on equation (21). The Greenwood-Pitman paper [3] explains very clearly the probabilistic significance of equation (20) viewed as a Wiener-Hopf factorization of $\phi \circ \Lambda^{-1}$, and equation (21) makes up one part of the Greenwood-Pitman path decomposition.

The partial result provided by equation (21) also admits a direct proof by our martingale method. If m consists only of a finite number of atoms within $(-\infty, 0]$, then we can find a bounded function g_θ on $\mathbb{R}$ such that

$$N_t^\theta \equiv \exp(-\tfrac{1}{2}\theta^2 t) g_\theta(B_t) \quad \text{defines a martingale } N^\theta ;$$

Notes.

(a) Observe that part (i) would be false if the interval $(0,\infty)$ were replaced by $[0,\infty)$. For if $p_1 = p_2 = 0$ and p_4 is a probability measure, then in the functional notation for measures, we have $p_4(f_\theta) = \delta_0(f_\theta)$, $\forall\theta > 0$, where δ_0 is the unit mass at 0.

(b) The case when $f(0) = \infty$ must of course be interpreted as the cosine-transform theorem. We continue to ignore that case.

Proof of (i). Suppose that μ_1 and μ_2 are probability measures on $(0,\infty)$ such that (47) holds. Because of the Wiener-Hopf factorization (30), we may rewrite (47) as follows:

$$\int F_\theta(y)\mu_1(dy) = \int F_\theta(y)\mu_2(dy), \qquad \forall\theta > 0, \tag{48}$$

where

$$F_\theta(y) = \left[\theta p_2 + \int\frac{\theta r K(dr)}{r^2+\theta^2}\right]\cos\theta y + \left[p_1 + \int\frac{\theta^2 K(dr)}{\theta^2 + r^2}\right]\sin\theta y,$$

$K(dr)$ being the $r^{-1}J(dr)$ of our previous notation. Recall that

$$\int (r+1)^{-1}J(dr) < \infty.$$

As before, define

$$\mathbb{H} \equiv \{z \in \mathbb{C}: \mathrm{Im}(z) \geq 0\}, \qquad \mathbb{H}^+ \equiv \{z \in \mathbb{C}: \mathrm{Im}(z) > 0\},$$

$$h(z) \equiv p_1 - ip_2 z - \int\frac{izK(dr)}{r - iz}.$$

Then h is analytic in $\mathbb{H}^+$ and continuous on $\mathbb{H}$. Moreover, if $z = a + ib$ (of course, a no longer has the significance it had at (2)), then

$$(55) \qquad \int (1 - e^{-\theta y})\mu_1(dy) = \int (1 - e^{-\theta y})\mu_2(dy) + c[p_1 + p_2\theta + \int (1 - e^{-\theta y})p_4(dy)$$

By examining what happens when $\theta \to \infty$, it is trivial to show that $c = 0$. [Note that it is here that we need the fact that the μ_j are probability measures on $(0,\infty)$ not $[0,\infty)$.] Hence $\mu_1 = \mu_2$.

We must now prove (54). To ensure that a function g, which is harmonic in the open first quadrant and continuous on the closed first quadrant except perhaps at 0, is determined by its values on the edges of the quadrant, it is enough to show that g is bounded near 0 and that $g(z) = O(|z|)$ as $|z| \to \infty$. We apply this principle not to the function $\Psi_j(z)$ but to the function $\Psi_j(1/z)$, that is, to the function $\Psi_j \circ z^{-1}$ defined on the fourth quadrant. Translating back to the first quadrant, we see that to prove (54), we need to establish:

(56) $\qquad \Psi_j(z)$ is bounded near ∞ (within the first quadrant),

(57) $\qquad z\Psi_j(z) \to 0$ as $z \to 0$ (within the first quadrant).

Note that $|\tilde{\mu}_j| \le 2$ on $\mathbb{H}$. From (49), $|h| \ge p_1$ on $\mathbb{H}$, so that if $p_1 > 0$ then (see (50)) $|\Psi_j| \le 2p_1^{-1}$ on $\mathbb{H}$, and (56) and (57) follow.

It remains to prove (56) and (57) when $p_1 = 0$. [As usual, we ignore the case when ($p_2 \ne 0$ and) the measure K is zero. The theorem is classical in that case.] From (49),

$$|h(z)| \ge \int \frac{b(r+b) + a^2}{(r+b)^2 + a^2} K(dr)$$

$$\ge \frac{1}{2}\int \frac{a^2 + b^2}{r^2 + a^2 + b^2} K(dr) \ge \frac{1}{2}\int \frac{1}{r^2 + 1} K(dr)$$

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