Market Selection: Hungry Misers and Happy Bankrupts*

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Abstract. The Market Selection Hypothesis is a principle which (informally) proposes that 'less knowledgeable' agents are eventually eliminated from the market. This elimination may take the form of starvation (the proportion of output consumed drops to zero), or may take the form of going broke (the proportion of asset held drops to zero), and these are not the same thing. Starvation may result from several causes, diverse beliefs being only one. We firstly identify and exclude these other possible causes, and then prove that starvation is equivalent to inferior belief, under suitable technical conditions. On the other hand, going broke cannot be characterized solely in terms of beliefs, as we show. We next present a remarkable example with two agents with different beliefs, in which one agent starves yet amasses all the capital, and the other goes broke yet consumes all the output – the hungry miser and the happy bankrupt. This example also serves to show that although an agent may starve, he may have long-term impact on the prices. This relates to the notion of price impact introduced by Kogan et al. (2009), which we correct in the final section, and then use to characterize situations where asymptotically equivalent pricing holds.

Keywords: Market selection, asset pricing, heterogeneous beliefs.

JEL classification: D51, D01, G12,

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*The financial support by the Ministry of Education, Culture, Sports, Science and Technology - Japan, "Special Coordination Funds for Promoting Science and Technology" is greatly appreciated by the first author.

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1 Introduction

Standard microeconomic theory is usually based on the assumption that agents are all rational and thus focuses only on efficient markets and rational expectations equilibria. This assumption is supported by the hypothesis that if there are some agents who are irrational or inferior at forecasting the future, they will eventually be driven out of the market. The hypothesis is the so-called *Market Selection Hypothesis*, originally proposed by Alchian (1950) and Friedman (1953). If this evolutionary mechanism is true, any asset price in actual financial markets is eventually determined by rational investors with accurate forecasting and converges to its fundamentals based only on their trading behavior.

Rigorous theoretical analysis with mathematical modeling was first started by De Long et al. (1991) and Blume and Easley (1992). Rather surprisingly, they obtain a negative answer to the hypothesis, finding that agents with inferior forecasting ability can survive and become dominant in the market. For instance, De Long et al. (1991) show that agents with optimistic beliefs can survive in the market because they choose portfolio with a higher growth rate and dominate as a result. However, these papers employ a partial equilibrium analysis and take only one particular risky asset market into account, not examining how other markets evolve over time. For example, it is assumed in Blume and Easley (1992) that agents can borrow without restriction in the money market with no impact on the interest rate.

In a general equilibrium framework, several papers investigate the issue in the case of complete markets. When the preferences of agents are convex, an analysis on complete markets is rather tractable since the equilibrium solution is characterized by the central planner's problem, maximizing a linear combination of agents' individual utility functions. This method can be applied to the case of diverse subjective beliefs (see, for example, Brown and Rogers, 2010). Sandroni (2000) and Blume and Easley (2006) study the case where aggregated endowment is bounded away from zero and infinity and show that only agents with beliefs closest to the true probability measure can survive in a consumption market in the long run. The closeness of beliefs is measured by the relative entropy with respect to the true probability measure. This result gives a positive answer to the hypothesis. Yan (2008) considers a complete market with a log Brownian endowment process, and derives a similar survival index to Sandroni (2000). The paper also indicates that if heterogeneity lies only in agents' beliefs, the market selection holds and agents with inaccurate beliefs are finally driven out of a consumption market.

Kogan et al. (2006) study this issue from another point of view. They consider a setting where two agents with different beliefs consume only at a terminal time T, and study the possible equilibria of this situation. One finding is that for log agents, the presence of an incorrectly-informed agent can have (at intermediate times) a big impact on the equilibrium price of the asset, even though at that intermediate time the wealth of the incorrectly-informed agent is very small.[1] However, their results rely heavily on the assumption that an agent's utility comes only from consumption at the horizon time, and hence their observations cannot simply be compared to the ones in Sandroni (2000) and Blume and Easley (2006).

More recently, Kogan et al. (2009) consider a setting in which agents consume continuously in time, and they derive necessary and sufficient conditions for agents to survive. One of their results is that an agent with inaccurate beliefs becomes extinct in the long-run and has no price impact if his relative risk aversion is bounded or the aggregate

[1] On the other hand, Fedyck and Walden (2009) argue that market selection is rather efficient in theory and it takes about fifty years for an irrational investors to be wiped out of the U.S. stock market.

endowment is bounded. They also consider the price system in the long run. They define a notion of *no price impact* of an irrational agent and show (under certain conditions) that there is no price impact if the likelihood ratio of irrational agents' subjective probability measures go to zero.

In this paper, we consider an infinite-horizon complete market model where agents consume continuously as in Kogan et al. (2009). To focus on the effect of different beliefs, it is assumed in the main part that heterogeneity lies only in their subjective beliefs. More precisely, the agents have identical von Neumann-Morgenstern preferences, except for their subjective probability measures.

We first identify two types of elimination from the market: *starvation* and *going broke*. We say that an agent*starves* if his consumption, expressed as a fraction of the total output, tends to zero in the long run. We say that an agent *goes broke* if his wealth as a proportion of the total wealth of the market tends to zero in the long run. As we will see later, and as Kogan et al. (2009) emphasize , these two notions are completely different. The classical Market Selection Hypothesis concerns itself with starvation of ill-informed agents; our results show that the Market Selection Hypothesis does not hold without additional assumptions, but that if relative risk aversion is bounded and bounded away from zero, then the Market Selection Hypothesis is equivalent to convergence to zero of the ill-informed agent's likelihood-ratio martingale. In contrast, we give an example[2] to show that whether an agent goes broke *cannot* be characterized solely by conditions on beliefs; the nature of the output process matters here.

One of our main results is an example with two agents, *one of whom starves, and the other of whom goes broke!* This example also shows that the price of the market portfolio can differ for ever from the price in a reference economy where all agents know and agree on the true probability measure. This is despite the fact that the incorrectly-informed agent has no price impact according to the definition of Kogan et al. (2009), which we therefore conclude is not capturing the effect of interest.

To examine properly the impact of beliefs on prices, we introduce the notion of *asymptotically equivalent pricing*. This notion corrects the definition of no price impact proposed by Kogan et al. (2009) and characterizes the effect of different beliefs on the price system in the long run. Our main result on asymptotically equivalent pricing is that this notion is equivalent to certain natural conditions on the state-price densities, similar to those of Kogan et al. (2009), but required to hold *uniformly* over future time.

The rest of the paper is organized as follows. Section 2 sets up our market model and presents some basic results for equilibrium consumption allocation and asset pricing. In Section 3, we consider what determines starvation and then characterize the impact of diverse beliefs on the consumption allocation. Section 4 presents a two-agent example where changing the output process changes which agent goes broke. Section 5 constructs the two-agent example where one agent starves and the other goes broke; this also proves that it can happen that an agent may starve and yet have lasting price impact. Section 6 presents the notion of asymptotically equivalent pricing, and develops its main properties. Some concluding remarks are given in Section 7.

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[2]This example has constant relative risk aversion preferences, in contrast to the example of Section 3.2 of Kogan et al. (2009).

2 The Modelling Situation

We study a pure-exchange continuous-time economy with one productive asset which produces a continuous stream of one perishable consumption good. Future uncertainty is described by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Here $\mathbb{P}$ denotes a reference probability measure, which might be the 'real-world' measure, but does not have to be. We present the basic situation here for J agents, though once we move on to consider examples we will just take $J = 2$ for simplicity.

We suppose that agent i ($i = 1, \ldots, J$) has von Neumann-Morgenstern preferences over consumption streams c represented by

$$\mathcal{U}^i(c) \equiv \mathbb{E}^i\left[\int_0^\infty \mathrm{e}^{-\rho_i t} u_i(c_t)\mathrm{d}t\right] \tag{2.1}$$

$$= \mathbb{E}\left[\int_0^\infty \Lambda_t^i \mathrm{e}^{-\rho_i t} u_i(c_t)\mathrm{d}t\right] \tag{2.2}$$

where $\mathbb{E}^i$ is an expectation operator with respect to $\mathbb{P}^i$, which describes the subjective belief of agent i. We write Λ^i for the Radon-Nikodym derivative process

$$\Lambda_t^i \equiv \left.\frac{\mathrm{d}\mathbb{P}^i}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}, \tag{2.3}$$

which allows us to re-express the expectation (2.1) in terms of an expectation (2.2) with respect to the reference measure $\mathbb{P}$. The agents' felicity functions u_i will be assumed to be strictly concave, C^1, and to satisfy the Inada conditions. The familiar argument presented in Breeden (1979), suitably modified (see, for example, Brown and Rogers, 2010) to handle the change of measure, shows that agent i will at time t value a contingent claim Y_s to be received at time $s > t$ at marginal price

$$Y_t = \frac{1}{\zeta_t^i}\mathbb{E}[\zeta_s^i Y_s | \mathcal{F}_t] \tag{2.4}$$

where the state-price density process ζ^i is given by

$$\zeta_t^i = \Lambda_t^i \mathrm{e}^{-\rho_i t} u_i'(c_t^i). \tag{2.5}$$

This allows the consumption c^i of agent i to be expressed in terms of the state-price density c^i as

$$c_t^i = I_i(\mathrm{e}^{\rho_i t}\zeta_t^i / \Lambda_t^i), \tag{2.6}$$

where I_j is the inverse of marginal utility u_j. We shall assume either a complete market setting, or else a central planner equilibrium, so that all agent have the same state-price density, up to positive multiples:

$$c_t^i = I_i(\mathrm{e}^{\rho_i t}\alpha_i \zeta_t / \Lambda_t^i), \tag{2.7}$$

for some positive constants α_i and common process ζ.

The single productive asset in this economy outputs a continuous stream of consumption good at rate $\delta_t \mathrm{d}t$, where the adapted process δ is assumed to be strictly positive.

The state-price density process ζ is determined by *market clearing*: at all times, the total consumption rate of all agents equals the rate at which consumption good is being produced, so

$$\delta_t = \sum_j c_t^j = \sum_j I_j(e^{\rho_j t}\alpha_j \zeta_t / \Lambda_t^j). \tag{2.8}$$

Once the constants α_j are fixed, this uniquely determines the state-price density process ζ.

One of the concepts of elimination from the market which interests us is *elimination from the consumption market*.

Definition 1. *We say that agent j* starves *if*

$$\lim_{t\to\infty} \frac{c_t^j}{\delta_t} = 0. \tag{2.9}$$

The constants α_j appearing in (2.8) are related to the initial wealth of the individual agents. Again by a familiar argument, the time-t wealth of agent j is given by the expression

$$w_t^j \equiv \frac{1}{\zeta_t} \mathbb{E}_t \left[\int_t^\infty \zeta_s c_s^j \mathrm{d}s \right] \tag{2.10}$$

and in principle (though usually only by numerical means in practice) the dependence between the α_j and the initial wealth distribution $(w_0^j)_{j=1,\dots,J}$ can be inverted.

The wealth of all agents is denoted by $\bar{w}_t \equiv \sum_{j=1}^J w_t^j$, and is related to the output process δ by the analogue of (2.10):

$$\bar{w}_t = \frac{1}{\zeta_t} \mathbb{E}_t \left[\int_t^\infty \zeta_s \delta_s \mathrm{d}s \right]. \tag{2.11}$$

Another concept of elimination is *elimination from the asset market.*

Definition 2. *We say that agent j* goes broke *if*

$$\lim_{t\to\infty} \frac{w_t^j}{\bar{w}_t} \equiv \lim_{t\to\infty} \frac{\mathbb{E}_t \left[\int_t^\infty \zeta_s c_s^j \mathrm{d}s \right]}{\mathbb{E}_t \left[\int_t^\infty \zeta_s \delta_s \mathrm{d}s \right]} = 0.$$

Various questions naturally arise:

1. If agent j starves, does he necessarily go broke?
2. If agent j goes broke, does he necessarily starve?
3. Can we characterize situations in which an agent will starve?
4. Can we characterize situations in which an agent will go broke?

The answers to the first two are both 'No', as we shall see. For the third, once we have ruled out starvation arising from differences in preferences or initial wealth, we shall in the next section explain how starvation relates to market selection and inferior beliefs. The answer to the fourth question is less clear-cut; preferences and beliefs on their own are not sufficient to decide when an agent will go broke – the nature of the output process δ must also be taken into account.

3 Starvation

From now on, we focus on the situation where there are just two agents: $J = 2$. This does not restrict generality, but is sufficient to illustrate the principles at work. To begin with, we see the possible reasons why an agent may starve, and narrow the possible causes down to differences in belief. We then see when the Market Selection Hypothesis holds.

Example: starvation arising from different ρ. Suppose that $u_1 = u_2$, $\Lambda^1 = \Lambda^2 \equiv 1$, and $\rho_1 > \rho_2$. If the felicity function is CRRA, agent i's consumption at t is given by $c_t^i = \mathrm{e}^{-\rho_i t/R} I(\nu_j \zeta_t)$. Then[3] $c_t^1/c_t^2 = \kappa \exp(-(\rho_1 - \rho_2)t/R) \to 0$ as $t \to \infty$ and the more impatient agent starves because of his impatience. Henceforth we set $\rho_1 = \rho_2 = \rho$ to eliminate this possibility. □

Example: starvation arising from different u. Suppose that $\Lambda^1 = \Lambda^2 \equiv 1$, and $\rho_1 = \rho_2 = \rho$, but that the two agents have CRRA preferences with coefficients $R_1 > R_2$ of relative risk aversion. Then

$$\frac{c_t^1}{c_t^2} = \frac{(\nu_1 \mathrm{e}^{-\rho t}\zeta_t)^{-1/R_1}}{(\nu_2 \mathrm{e}^{-\rho t}\zeta_t)^{-1/R_2}} \propto (\mathrm{e}^{-\rho t}\zeta_t)^{\frac{1}{R_2} - \frac{1}{R_1}}.$$

Notice from the market clearing condition that $\delta_t = \sum_j (\nu_j \mathrm{e}^{-\rho t}\zeta_t)^{-1/R_j}$. If $\delta_t \to \infty$ as $t \to \infty$, then $\mathrm{e}^{-\rho t}\zeta_t$ must converge to zero and so agent 1 starves in this case. On the other hand, if $\delta_t \to 0$, then similarly agent 2 starves. To eliminate this as a possible cause of starvation, we henceforth assume that $u_1 = u_2$. □

Example: starvation arising from different initial wealth. Even allowing that both agents have identical preferences and identical beliefs, it may still happen that one of them starves. Informally, this is because the coefficient of relative risk aversion may approach zero, and an agent who starts ahead gets ever further ahead.

To explain in more detail, the ratio of the two consumption processes is

$$\frac{c_t^1}{c_t^2} = \frac{I(\mathrm{e}^{\rho t}\alpha_1 \zeta_t)}{I(\mathrm{e}^{\rho t}\alpha_2 \zeta_t)}.$$

Suppose that there exists $x_n \downarrow 0$ such that for some $p < 1$

$$\lim_n \frac{I(x_n)}{I(px_n)} = 0. \tag{3.1}$$

Then by taking $\alpha_2 = p$, $\alpha_1 = 1$, and giving ourselves a state-price density ζ such that $\mathrm{e}^{\rho t}\zeta_t \downarrow 0$ through the values (x_n) we see that agent 1 starves, even though his beliefs, utility, and rate of time preference are exactly the same as agent 2. What happens is that agent 2 starts ahead, and agent 1 just falls ever further behind. □

What this example shows is that in order to eliminate starvation due to different initial allocations, we have to insist that for each $p \in (0, 1)$

$$\sup_{x>0} \frac{I(px)}{I(x)} < \infty. \tag{3.2}$$

An obviously equivalent condition is that

$$\sup_{x>0} \frac{I(x)}{I(2x)} < \infty. \tag{3.3}$$

[3]Here κ denotes an unimportant constant.

Writing $f(x) \equiv \log I(e^x)$, this condition is equivalent to the existence of some $K < 1$ such that

$$0 \leq f(x-1) - f(x) \leq K \quad \forall x \in \mathbb{R}. \tag{3.4}$$

Thus f cannot fall to $-\infty$ faster than linearly as $x \uparrow \infty$, nor can it rise to ∞ faster than linearly as $x \downarrow -\infty$.

A related condition is that there exists some $\epsilon > 0$ such that

$$\epsilon \geq f(x-1) - f(x) \quad \forall x \in \mathbb{R}. \tag{3.5}$$

This condition says that as $x \uparrow \infty$, f must decrease at least linearly, and that as $x \downarrow -\infty$ the growth of f must be *at least* linear. Thus conditions (3.4) and (3.5) are complements of each other. Notice that for u which is constant relative risk aversion, the function f is exactly linear, so both conditions (3.4) and (3.5) hold. If we suppose that u is C^2, these conditions are related to conditions on the relative risk aversion of u. Defining the relative risk aversion of u in the usual way,

$$R(x) \equiv -\frac{xu''(x)}{u'(x)} \tag{3.6}$$

it is easy to see that $f'(x) = -1/R(I(e^x))$. Hence condition (3.4) is implied by the statement that R is bounded away from 0, and condition (3.5) is implied by the statement that R is bounded.

The conditions (3.4) and (3.5) are the conditions we need on the utility in order that the Market Selection Hypothesis should hold. The following result confirms the intuition of the Market Selection Hypothesis once the suitable conditions[4] have been imposed.

Theorem 1 (Market Selection Hypothesis). *Consider the case where the two agents are identical except for their beliefs.*

(i) Suppose that agent 2 has inferior beliefs:

$$\lim_{t\to\infty} \frac{\Lambda_t^2}{\Lambda_t^1} = 0 \quad a.s., \tag{3.7}$$

and suppose that condition (3.5) *holds. Then agent 2 starves.*

(ii) Suppose that condition (3.4) *holds and that agent 2 starves. Then agent 2 has inferior beliefs* (3.7)*.*

Proof. Write $x_j \equiv \log(e^{\rho t}\alpha_j \zeta_t/\Lambda_t^j)$, so that

$$c_t^j = I(\exp(x_t^j)) = \exp(f(x_t^j)).$$

The hypothesis (3.7) of the first statement is that $x_t^2 - x_t^1 \to \infty$. Since condition (3.5) implies that $f(x-n) - f(x) \geq n\epsilon$ for all positive integer n, the result follows.

The argument for the second statement is similar. Condition (3.4) implies that for all positive integer n the bound

$$f(x) - f(x-n) > -nK \tag{3.8}$$

[4]For CRRA utility u, the required conditions hold, and we have that starvation is equivalent to inferior beliefs.

holds uniformly in x. If agent 2 starves, then

$$f(x_t^2) - f(x_t^2 - \Delta_t) \equiv f(x_t^2) - f(x_t^1) \to -\infty$$

where $\Delta_t \equiv x_t^2 - x_t^1$. This is only consistent with (3.8) if $\Delta_t \to \infty$, which is equivalent to the inferior beliefs statement (3.7). $\square$

Corollary 1. *If conditions* (3.4) *and* (3.5) *both hold, then*

$$\textit{Agent 2 starves} \Leftrightarrow \lim_{t\to\infty} \frac{\Lambda_t^2}{\Lambda_t^1} = 0 \ \textit{a.s.}$$

4 Going broke.

While Theorem 1 tells us quite precisely in terms of agents' beliefs when one of the agents will starve, the story for going broke is not so clear cut. The reason is that going broke is not decided by beliefs alone, but depends in an essential way on what the output process δ is, as we shall demonstrate by constructing an example.

For the example, we will suppose that the two agents are identical except for their beliefs, each having CRRA felicity with coefficient $R \neq 1$ of relative risk aversion. The results (2.7), (2.8) of Section 2 apply here to give us

$$\begin{aligned} \delta_t &= \zeta_t^{-1/R} e^{-\rho t/R} \sum_j (\Lambda_t^j/\alpha_j)^{1/R}, \\ c_t^j &= \pi_t^j \delta_t, \\ \pi_t^k &= \frac{(\Lambda_t^k/\alpha_k)^{1/R}}{\sum_j (\Lambda_t^j/\alpha_j)^{1/R}}. \end{aligned}$$

The pricing identity (2.10) gives us that

$$\begin{aligned} w_t^i &= \zeta_t^{-1} E_t\Big[\int_t^\infty \zeta_s \delta_s \pi_s^i \, ds\Big] \\ &= \zeta_t^{-1} E_t\Big[\int_t^\infty e^{-\rho s} \delta_s^{1-R} \big\{\sum_j (\Lambda_t^j/\alpha_j)^{1/R}\big\}^R \pi_s^i \, ds\Big] \\ &= \zeta_t^{-1} E_t\Big[\int_t^\infty e^{-\rho s} \delta_s^{1-R} \big\{\sum_j (\Lambda_t^j/\alpha_j)^{1/R}\big\}^{R-1} (\Lambda_s^i/\alpha_i)^{1/R} \, ds\Big] \\ &\equiv \zeta_t^{-1} E_t\Big[\int_t^\infty e^{-\rho s} \lambda_s (\Lambda_s^i/\alpha_i)^{1/R} \, ds\Big] \qquad (4.1) \end{aligned}$$

where the process λ is defined to be

$$\lambda_s = \delta_s^{1-R} \big\{\sum_j (\Lambda_t^j/\alpha_j)^{1/R}\big\}^{R-1}.$$

We therefore have that the ratio of the two agents' wealths is given by

$$\frac{w_t^1}{w_t^2} = \frac{\lambda_t^{-1} E_t\Big[\int_t^\infty e^{-\rho s} \lambda_s (\Lambda_s^1/\alpha_1)^{1/R} \, ds\Big]}{\lambda_t^{-1} E_t\Big[\int_t^\infty e^{-\rho s} \lambda_s (\Lambda_s^2/\alpha_2)^{1/R} \, ds\Big]}. \qquad (4.2)$$

If we ensure that the process λ is a positive martingale, then we may interpret it as a change-of-measure martingale, and the numerator and denominator in (4.2) are (up to irrelevant positive constants) expectations of the future discounted value of $\Lambda_s^{1/R}$. We will specify the choice of output process δ equivalently by choice of the positive martingale λ, in terms of which the properties claimed will be more evident.

Example 1. In the example, we shall construct two agents with beliefs given by the likelihood-ratio martingales Λ^1, Λ^2 and two candidates δ^1 and δ^2 for the output process such that if the output process is δ^1, then agent 2 goes broke, and if the output process is δ^2 then agent 1 goes broke. What this proves is that whether an agent goes broke is not determined by the beliefs of the agents alone, but depends also on the properties of the output process. The example is constructed using a continuous-time Markov chain with countable statespace $I = \mathbb{Z} \cup i\mathbb{N} \subset \mathbb{C}$. We describe the statespace as consisting of three *ladders*, the positive ladder $\mathbb{Z}^+$, the negative ladder $-\mathbb{Z}^+$ and the imaginary ladder $i\mathbb{Z}^+$. The possible transitions of the chain are from points in $\mathbb{Z}$ to their nearest neighbours, from 0 to i, and from a point $ik \in i\mathbb{N}$ to $i(k+1)$, or to $\pm k$. If the chain is in state 0, then the only possible transition is to i, at rate 1. We shall assume that $X_0 =$; the process starts at the origin.

In the reference measure, the jump rates are given for $k > 0$ by

$$\begin{aligned} q_{ik,k} &= q_{ik,-ik} = 2^{-k}, \\ q_{ik,i(k+1)} &= 1, \\ q_{k,k-1} &= q_{-k,-k+1} = 2, \\ q_{k,k+1} &= q_{-k,-k-1} = 1. \end{aligned}$$

It is not hard to see that in the reference measure, the Markov chain may make a number of entries to the positive ladder and the negative ladder, but will eventually enter the imaginary ladder and climb it to infinity.

According to agent 1's beliefs, however, the chain behaves quite differently. Agent 1 thinks that the transition rates are for $k > 0$

$$\begin{aligned} q_{ik,k} &= 2^k \\ q_{ik,i(k+1)} &= q_{ik,-k} = 1 \\ q_{k,k+1} &= q_{-k,-k+1} = 2 \\ q_{k,k-1} &= q_{-k,-k-1} = 1 \end{aligned}$$

with $q_{0i} = 1$ as before. According to agent 1's beliefs, the process may make a number of entries to the imaginary and negative ladders, but will in the end enter the positive ladder and climb it to infinity without ever again visiting the other ladders.

Agent 2's beliefs are defined by taking the mirror image in the imaginary axis of agent 1's transition rates - so for agent 2, ultimate escape to infinity along the negative ladder is certain.

The likelihood-ratio martingales Λ_t^i have a particular form, being representable as

$$\exp(\int_0^t \varphi(X_s)\, ds) \prod_{s \le t} \alpha(X_{s-}, X_s)$$

where the function φ arises because of the possible differences in the mean residence times in states, and the α terms are contributed by the jumps ($\alpha(k,k) = 1 \quad \forall k$); see, for example, Rogers and Williams (2000), IV.22. However, for both agents, the mean times in all states coincide, so the ratio Λ^1_t/Λ^2_t is made up entirely of the jump contributions:

$$\frac{\Lambda^1_t}{\Lambda^2_t} = \prod_{s\leq t} \gamma(X_{s-}, X_s) \tag{4.3}$$

where $\gamma = \alpha^1/\alpha^2$.

From the expression (4.3) it would appear that in order to evaluate the likelihood ratio $L_t \equiv \Lambda^1_t/\Lambda^2_t$ at any time, it will be necessary to keep track of the entire history of jumps of the chain. However, the construction has been designed to simplify the expression: L_t *is a function of* X_t *only!* To understand how this can be, notice that the likelihood-ratio stays equal to 1 until the first time that X jumps off the imaginary ladder. Suppose that this first jump is to position k on the positive ladder. At that moment, the numerator in L gains a factor of 2^k, while the denominator gets a factor of 1, so L changes to 2^k. Thereafter, the process performs asymmetric random walk on $\mathbb{Z}^+$. If at some later time it reaches 0 again, having performed a total of m steps right and j steps left, then $m+k = j$, and the likelihood ratio is now

$$L = \frac{2^k \times 2^m \times 1^j}{1 \times 1^m \times 2^j} = 1. \tag{4.4}$$

Thus every time that the process is on the imaginary ladder, the value of L will be 1. If the process has jumped off the imaginary ladder to the positive ladder at position k, and then made a total of m steps right and j steps left, to arrive at position $n = k + m - j$, then exactly the same calculation (4.4) shows that the likelihood-ratio L is now 2^n. The conclusion is that the likelihood ratio is simply

$$L_t = 2^{\mathrm{Re}(X_t)}. \tag{4.5}$$

To finish the construction, we propose to take λ^j to be the change-of-measure martingale which converts the reference probability into agent j's probability. To fix ideas, suppose we first consider using $\lambda = \lambda^1$ in (4.2). We aim to show that as $t \to \infty$ this ratio tends to ∞. We have already remarked that in the reference probability the Markov chain will eventually ascend the imaginary ladder for ever, so we only need to understand what the wealth ratio looks like when $X_t = iK$ for very large K. However, the wealth ratio is the ratio of two expectations, of $\int_t^\infty e^{-\rho s}(\Lambda^j_s)^{1/R}\, ds$, taken in agent 1's probability. According to this probability, it is overwhelmingly likely that the process will jump almost immediately to K, since the jump rates out of iK are 1 to $-K$, 1 to $i(K+1)$ and 2^K to K. When the process does jump to K, the value of the integrand in the numerator is $(\Lambda^1)^{1/R} = 2^K/R$ whereas the value of the integrand in the denominator is just 1. Thus the expectation in the numerator is overwhelmingly bigger than that in the denominator. Of course, further analysis is needed to prove the required result conclusively, but this is a technical matter we leave to the reader; it is now entirely obvious that the wealth ratio w^1/w^2 tends to infinity. Switching to $\lambda = \lambda^2$, the conclusion is reversed by symmetry: $w^1_t/w^2_t \to 0$ as $t \to \infty$. □

5 A Remarkable Example

In this Section, we construct an example with some quite surprising properties, and then we develop it to provide another example with even more astonishing properties. We begin with the basic example.

Example 2. To start with, we construct a two-agent example with the following surprising properties:

(i) Both agents have same constant relative risk aversion felicities;

(ii) Agent 1 knows the correct probability law: $\Lambda_t^1 \equiv 1$;

(iii) Agent 1 consumes at constant rate 1;

(iv) Agent 2 starves;

(v) With positive probability, agent 2 does not go broke.

Given the properties we seek, the modelling effort is entirely to do with the construction of agent 2's likelihood-ratio martingale. Since agent 1's beliefs are the correct beliefs, we shall simplify notation through this section by writing Λ_t in place of Λ_t^2. We shall assume the agents are both CRRA with relative risk-aversion coefficient R, and suppose that a consumption allocation is given by $c_t^1 \equiv 1$ and $c_t^2 = (\Lambda_t)^{1/R}$. Then the allocation constitutes an equilibrium since

$$\frac{\mathrm{e}^{-\rho t} u'(c_t^1)}{u'(c_0^1)} = \frac{\mathrm{e}^{-\rho t} u'(c_t^2)\Lambda_t}{u'(c_0^2)}.$$

In this case, the state-price density with respect to $\mathbb{P}$ is given by

$$\zeta_t = \mathrm{e}^{-\rho t} \tag{5.1}$$

and thus we have

$$w_t^1 = 1/\rho$$

for any t. On the other hand, agent 2's wealth at t is given by

$$w_t^2 = \mathbb{E}_t\left[\int_t^\infty \mathrm{e}^{-\rho(s-t)}(\Lambda_s)^{1/R}\mathrm{d}s\right] \tag{5.2}$$

As a first example, we construct a positive martingale Λ such that $c_t^2 \to 0$ and yet $w_t^2 \not\to 0$ almost surely.

The positive martingale Λ will drift downwards until a random time τ, at which time a single upward jump occurs. As we see later, τ may be infinite with positive probability. If $\tau < \infty$, then Λ does not change after τ until $\inf\{k \in \mathbb{Z}^+ : k > \tau\}$, after which it starts to evolve as $\mathrm{d}\Lambda = \Lambda \mathrm{d}W$, where W is a Wiener process.

For the construction of our example, we take $a_n = 2^{-n}$, $n = 0, 1, \ldots$ and suppose that while $a_{n+1} < \Lambda_t < a_n$ and before the jump happened, Λ evolves deterministically as

$$\dot{\Lambda}_t = -2^{-n-1} \equiv b_n,$$

so that if there is no jump, it takes unit time to cross a_n to a_{n+1}.

The size of the upward jump in Λ varies depending on which interval it is in; denote the size of the upward jump while $a_{n+1} < \Lambda_t < a_n$ by ξ_n and assume that the jump occurs with intensity ν_n. Then, from the martingale condition, the following equation must hold:

$$\nu_n \xi_n = b_n.$$

Assuming that $\tau > n$, from (5.2) we see that agent 2's wealth at time n satisfies the following inequality:

$$\begin{aligned}
w_n^2 \geq & \mathbb{E}_n \left[\int_n^{n+1} \mathrm{e}^{-\rho(s-n)} \Lambda_s^{1/R} \mathrm{d}s \right] \geq \mathbb{E}_n \left[1_{\{\tau \leq n+1\}} \int_n^{n+1} \mathrm{e}^{-\rho(s-n)} \Lambda_s^{1/R} \mathrm{d}s \right] \\
\geq & \mathbb{E}_n \left[1_{\{\tau \leq n+1\}} \int_\tau^{n+1} \mathrm{e}^{-\rho(s-n)} (a_{n+1} + \xi_n)^{1/R} \mathrm{d}s \right] \\
= & (a_{n+1} + \xi_n)^{1/R} \int_0^1 \nu_n \mathrm{e}^{-\nu_n t} \left(\int_t^1 \mathrm{e}^{-\rho s} \mathrm{d}s \right) \mathrm{d}t \\
= & (a_{n+1} + \xi_n)^{1/R} \int_0^1 \nu_n \mathrm{e}^{-\nu_n t} \left(\mathrm{e}^{-\rho t} - \mathrm{e}^{-\rho} \right) \mathrm{d}t \\
= & \frac{(a_{n+1} + \xi_n)^{1/R}}{\rho} \left[\frac{\nu_n}{\nu_n + \rho} (1 - \mathrm{e}^{-(\rho + \nu_n)}) - e^{-\rho} + \mathrm{e}^{-\rho - \nu_n} \right] \\
= & \frac{(a_{n+1} + \xi_n)^{1/R}}{\rho(\rho + \nu_n)} \mathrm{e}^{-\rho} \left[\nu_n (\mathrm{e}^\rho - 1) - \rho(1 - \mathrm{e}^{-\nu_n}) \right].
\end{aligned}$$

If $\nu_n \longrightarrow 0$, we have $\mathrm{e}^{-\nu_n} \sim 1 - \nu_n$ and thus

$$w_n^2 \geq \frac{(a_{n+1} + \xi_n)^{1/R}}{\rho(\rho + \nu_n)} \mathrm{e}^{-\rho} (\mathrm{e}^\rho - 1 - \rho) \nu_n. \tag{5.3}$$

Now take $\xi_n = 2^n$, $R = 1/2$, $\nu_n = b_n / \xi_n = 2^{-2n-1}$. Then we have that the right-hand side of (5.3) converges to

$$\frac{\mathrm{e}^{-\rho}(\mathrm{e}^\rho - 1 - \rho)}{2\rho^2} > 0$$

as $n \longrightarrow \infty$. Note from the construction of our example that $c_t^2 \rightarrow 0$ almost surely but $w_t^i \not\rightarrow 0$ on the event $\{\tau = \infty\} \equiv \cap_n \{\tau > n\}$. It is also easily verified that $\mathbb{P}\{\tau = \infty\} > 0$. This implies that agent 2 starves almost surely while he does not go broke with a strictly positive probability. □

Example 3. We now develop Example 2 to produce an example with quite amazing properties:

(i) Both agents have same constant relative risk aversion felicities;

(ii) Agent 1 knows the correct probability law: $\Lambda_t^1 \equiv 1$;

(iii) Agent 1 consumes at constant rate 1;

(iv) Agent 2 starves;

(v) Agent 1 goes broke.

As before, we consider a setting where $c_t^1 \equiv 1$ and $c_t^2 = (\Lambda_t)^{1/R}$, so that the consumption allocation constitutes an equilibrium and agent 2's wealth is given by (5.2). In addition, we choose $\rho > 1$, which is simply a relatively unimportant condition on the scaling of time. Let $R = 1/2$ and the constant A be given by

$$A \equiv \frac{2R^2}{2\rho R^2 + R - 1} \geq \frac{1}{\rho}.$$

Recall that we write $a_n = 2^{-n}$, $n \in \mathbb{Z}^+$. The martingale Λ will take values in $\{a_n : n \geq 0\} \cup (1, \infty)$. While in $(1, \infty)$ it evolves as

$$d\Lambda_t = \Lambda_t dW_t. \tag{5.4}$$

Once it enters $S \equiv \{a_n : n \geq 0\}$ it evolves by discrete jumps until such time as it re-enters $(1, \infty)$, when the evolution (5.4) resumes. To explain the motion while in S, we specify that when Λ reaches some point $a_k \in S$, it remains there for unit time, and then jumps, either to ξ_n with probability p_n or to a_{n+1} with probability $1 - p_n$. Here the (p_n) and (ξ_n) are to be determined later (see (5.6), (5.9)).

If Λ starts at a_n, we define the wealth value

$$z_n = \mathbb{E}^{a_n}\left[\int_0^\infty e^{-\rho s}\Lambda_s^2 \, ds\right] \equiv \mathbb{E}\left[\int_0^\infty e^{-\rho s}\Lambda_s^2 \, ds \,\middle|\, \Lambda_0 = a_n\right]$$

We intend to choose the (p_n) and (ξ_n) so that

$$z_n = A(n+1). \tag{5.5}$$

One condition we will have to ensure is the martingale condition:

$$a_n = p_n \xi_n + (1 - p_n) a_{n+1}. \tag{5.6}$$

Next we develop an expression for z_n by splitting at time 1, when the process makes its first jump:

$$\begin{aligned} z_n =& \mathbb{E}^{a_n}\left[\int_0^1 e^{-\rho s}\Lambda_s^2 ds\right] + e^{-\rho}\mathbb{E}^{a_n}\left[\int_1^\infty e^{-\rho s}\Lambda_s^2 ds\right] \\ =& \frac{1 - e^{-\rho}}{\rho} a_n^2 + e^{-\rho}\{p_n h(\xi_n) + (1 - p_n) z_{n+1}\}. \end{aligned} \tag{5.7}$$

Here, the function h in (5.7) is defined by

$$h(\xi) = \mathbb{E}^\xi\left[\int_0^\infty e^{-\rho s}\Lambda_s^2 ds\right] = \mathbb{E}^\xi\left[\int_0^{H_1} e^{-\rho s}\Lambda_s^2 ds + e^{-\rho H_1} z_0\right]$$

with stopping time $H_1 \equiv \inf\{t \geq 0 : \ \Lambda_t = 1\}$. Some routine calculations show that with the special choice $z_0 \equiv A$, we have

$$h(\xi) = A\xi^{1/R} = A\xi^2 \tag{5.8}$$

for $\xi \geq 1$. Suppose we abbreviate

$$B \equiv \frac{1 - e^{-\rho}}{\rho A} < 1 - e^{-\rho} < 1.$$

Then the two conditions which p_n, ξ_n must satisfy are the martingale condition (5.6) and (from (5.5))

$$n + 1 = Ba_n^2 + e^{-\rho}\{p_n \xi_n^2 + (1 - p_n)(n + 2)\}. \tag{5.9}$$

Rearrangement gives

$$e^\rho(n + 1 - Ba_n^2) = \frac{(a_n - (1 - p_n)a_{n+1})^2}{p_n} + (1 - p_n)(n + 2).$$

Note that the left-hand side of the above equation is positive; the right-hand side diverges to positive infinity if $p_n \to 0$; and the right-hand side is a_n^2 when $p_n = 1$. Thus for there to exist a root p_n, we need

$$e^{\rho}(n + 1 - Ba_n^2) > a_n^2, \tag{5.10}$$

equivalently, $n+1 > (B + \mathrm{e}^{-\rho})a_n^2$, which is ensured by the assumption that $1 > B + \mathrm{e}^{-\rho}$.

Now clearly the p_n must converge to zero as $n \to \infty$. Indeed we have

$$p_n \sim \{(\mathrm{e}^{\rho} - 1)n\}^{-1}(a_n - a_{n+1})^2 \sim \frac{2^{-2(n+1)}}{n} \tag{5.11}$$

so the p_n get small geometrically fast and the ξ_n get big of order of $n2^n$.

Since $\sum_n p_n < \infty$, there is, starting from a_0, a positive probability that there will never an upward jump, implying that $\Lambda_t \to 0$ on this event. But if there is an upward jump, Λ will diffuse back down to a_0 with probability one and the story starts again. Eventually, Λ_t will head off to 0, and so $c_t^2 \to 0$ almost surely.

On the other hand, in $(0, 1]$, eventually the wealth values pass sequentially through the values z_0, z_1 and so on. From (5.8), z_n diverges to infinity as $n \to \infty$, we have $w_t^2 \to \infty$, even though $w_t^1 \equiv 1/\rho$ for ever.

Remarks. Notice that we could approximate this discontinuous likelihood ratio martingale with a continuous one and get the same qualitative conclusions; the discontinuous paths are not essential.

Example 4. This example is identical to Example 3, but is used to produce a strong answer to a quite different question, relating to the difference between prices in this equilibrium, and the equilibrium in a benchmark economy where all agents know the true probability measure.

In such a benchmark economy, suppose that all agents have the same subjective belief $\mathbb{P}$. Then the equilibrium solution is characterized by coefficients α_j such that

$$\alpha_1 \mathrm{e}^{-\rho_1 t} u_1'(c_t^1) = \cdots = \alpha_J \mathrm{e}^{-\rho_J t} u_J'(c_t^J), \tag{5.12}$$

where the α_j must be set so as to clear the market and match the initial wealths of the individual agents. Denote the state-price density in this economy by $\zeta^\star(\alpha)$, which in general depends on the coefficients α_j. Kogan et al. (2009) define the notion of no price impact in the following manner.

Definition 3 (Kogan et al., 2009)**.** *Suppose that $\Lambda_t^j \neq 1$. Agent j is said to have* no price impact *if there exist positive coefficients $\alpha_i^\star$ such that for $T \geq 0$,*

$$\lim_{t\to\infty} \frac{\zeta_{t+T}/\zeta_t}{\zeta_{t+T}^\star(\alpha^\star)/\zeta_t^\star(\alpha^\star)} = 1 \ \ a.s. \tag{5.13}$$

We shall show that Example 3 provides an example where

(i) Agent 2 starves and has no price impact;

(ii) If p_t is the time-t price of the future output, and p_t^* is the same quantity for the benchmark economy, then almost surely

$$\frac{p_t}{p_t^*} \to \infty \qquad (t \to \infty) \tag{5.14}$$

We confirm that agent 2 in Example 3 has no price impact according to Definition 3 as follows. In the example, the state-price density in the reference economy is independent of (α_j) and is given by

$$\frac{\zeta^\star_{t+T}}{\zeta^\star_t} = \mathrm{e}^{-\rho T}\left(\frac{\delta_{t+T}}{\delta_t}\right)^{-R}. \tag{5.15}$$

Since $\delta_t = 1 + (\Lambda^2_t)^{1/R}$ and $\Lambda^2_t \to 0$ almost surely, then (5.13) holds for any T.

The price of the market portfolio at time t in the original economy, denoted by p_t, is given by

$$p_t = \mathbb{E}_t\left[\int_t^\infty \mathrm{e}^{-\rho(t-s)}\delta_s \mathrm{d}s\right] = \mathbb{E}_t\left[\int_t^\infty \mathrm{e}^{-\rho(t-s)}(c^1_s + c^2_s)\mathrm{d}s\right] = w^1_t + w^2_t.$$

On the other hand, in the benchmark economy where both agents have the same beliefs, $\Lambda_t \equiv 1$, the state-price density $\zeta^\star_s$ is given by (5.15) and thus we have

$$\begin{aligned}
p^*_t &= \frac{1}{\zeta^\star_t}\,\mathbb{E}_t\left[\int_t^\infty \zeta^\star_s\delta_s \mathrm{d}s\right] \\
&= \frac{1}{\delta_t^{-R}}\,\mathbb{E}_t\left[\int_t^\infty \mathrm{e}^{-\rho(t-s)}\delta_s^{1-R} \mathrm{d}s\right] \\
&\le \frac{\rho}{\delta_t^{-R}}\left(\mathbb{E}_t\left[\int_t^\infty \rho\mathrm{e}^{-\rho(t-s)}\delta_s \mathrm{d}s\right]\right)^{1-R} \\
&= \frac{\rho^{-R}(w^1_t + w^2_t)^{1-R}}{\delta_t^{-R}},
\end{aligned}$$

where the inequality follows from Jensen's inequality and the fact that $\zeta_s/\zeta_t = \mathrm{e}^{-\rho(s-t)}$.

Hence, the ratio of the two prices is

$$\frac{p_t}{p^*_t} \ge \frac{w^1_t + w^2_t}{(w^1_t + w^2_t)^{1-R}/[1 + I(1/\Lambda_t)]^{-R}} \sim (w^1_t + w^2_t)^R \to \infty$$

almost surely as t goes to infinity. This observation shows that the prices of the market portfolio in the two cases differs for ever in our example, even though agent 2 with an inaccurate belief becomes extinct in the consumption market. □

From this result, two important findings are obtained. First, Definition 3 is not sufficient to capture the effect of beliefs on pricing and it can be that the price differs from the one in the homogeneous case even though the state-price densities are asymptotically equivalent. Second, starvation does not imply no long-run impact on prices: agents with inaccurate beliefs can have an influence on prices for ever.

Remarks. In Sandroni (2000) and Blume and Easley (2006), aggregate output is assumed to be bounded away from zero and infinity. In their setting, the price of any asset converges to the one in a reference economy since the dominated convergence theorem can be applied. However, in a general economy with diffusion processes, or even in a log-normal economy, this strong condition of boundedness on the output will not hold.

6 Price Impact and Asymptotically Equivalent Pricing

The example of Section 4 shows that Definition 3 does not appropriately capture the effect of different beliefs on prices. We will now introduce a notion of *asymptotically equivalent pricing (AEP)* which correctly embodies the features we seek to describe.

To set the scene, consider a situation where at any time $t > 0$ a future cashflow $(c_s)_{s\geq t}$ is priced according to the recipe (2.10):

$$\pi_t(c) = \frac{1}{\zeta_t}\mathbb{E}_t\left[\int_t^\infty \zeta_s c_s \mathrm{d}s\right] \tag{6.1}$$

where ζ is a strictly positive adapted process. We are thus thinking of a family $(\pi_t)_{t\geq 0}$ of pricing operators,

$$\pi_t : \mathcal{C}_t \equiv \{(c_t)_{s\geq t} : c \text{ non-negative, adapted}\} \to L_0^+(\mathcal{F}_t)$$

mapping non-negative adapted future cashflows to $\mathcal{F}_t$-measurable random variables. We restrict to non-negative cashflows to allow for the possibility that a pricing operator might assign an infinite value to a cashflow. If we change the process ζ to another strictly positive adapted process $\tilde{\zeta}$, we will arrive at a different family $(\tilde{\pi}_t)_{t\geq 0}$ of pricing operators; when do we consider these two families to be asymptotically the same?

Definition 4. *Families $(\pi_t)_{t\geq 0}$ and $(\tilde{\pi}_t)_{t\geq 0}$ are said to be asymptotically equivalent if there exists some stopping time t_0 such that*

(i) for all $t \geq t_0$,

$$\mathcal{A}_t \equiv \{c \geq 0; \pi_t(c) < \infty\} = \tilde{\mathcal{A}}_t \equiv \{c \geq 0; \tilde{\pi}_t(c) < \infty\}, \tag{6.2}$$

(ii)

$$\sup_{c\in\mathcal{A}_t, c\leq 1} \frac{\pi_t(c)}{\tilde{\pi}_t(c)} \to 1, \qquad \sup_{c\in\mathcal{A}_t, c\leq 1} \frac{\tilde{\pi}_t(c)}{\pi_t(c)} \to 1 \quad (t\to 1). \tag{6.3}$$

Remarks. The first condition of the definition says that the two families of pricing operators should eventually agree on what future cashflows should be assigned a finite value – surely a minimal requirement! The second says that for all such cashflows which are also bounded, the ratio of the two prices should converge to one, uniformly in the bounded cashflow process.

Theorem 2. *Families $(\pi_t)_{t\geq 0}$ and $(\tilde{\pi}_t)_{t\geq 0}$ of pricing operators generated by state-price density processes ζ and $\tilde{\zeta}$ respectively are asymptotically equivalent if and only if there exists positive adapted processes α and β and a stopping time t_0 such that*

(i) for all $t \geq t_0$,

$$\alpha_t \leq \frac{\zeta_{t,s}}{\tilde{\zeta}_{t,s}} \equiv \frac{\zeta_s/\zeta_t}{\tilde{\zeta}_s/\tilde{\zeta}_t} \leq \beta_t \quad \forall s \geq t; \tag{6.4}$$

(ii)

$$\frac{\alpha_t}{\beta_t} \to 1 \quad (t \to \infty). \tag{6.5}$$

The proof makes use of the following little result.

Lemma 1. *Suppose that $\mathbb{Q} \ll \mathbb{P}$ are two probability measures, and $Z = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is the density of $\mathbb{Q}$ with respect to $\mathbb{P}$. If the support of Z is unbounded, that is, $\mathbb{P}(Z > t) > 0$ for all t, then there is a random variable X whose $\mathbb{P}$-expectation is finite, but whose $\mathbb{Q}$-expectation is infinite.*

PROOF. Suppose that $F(t) \equiv \mathbb{P}(Z \le t)$ is the distribution function of Z. We shall construct a random variable $X = \varphi(Z)$ with the desired properties, where φ is increasing and differentiable, $\varphi(0) = 0$. To do this, we define

$$G(z) \equiv \int_0^z t \; F(dt),$$

which is another distribution function in view of the fact that $\mathbb{E}(Z) = 1$. Now define $\bar{G}(t) \equiv 1 - G(t)$, and

$$\varphi'(t) = \frac{1}{(1+t)\bar{G}(t)},$$

and notice that

$$\int_0^\infty z\varphi(z) \; F(dz) = \int_0^\infty \varphi(z) \; G(dz) = \int_0^\infty \varphi'(t)\bar{G}(t) \; dt = \int_0^\infty \frac{dt}{1+t} = \infty.$$

However, $\bar{G}(z) = \int_z^\infty x \; F(dx) \ge z\bar{F}(z)$, and so

$$\int_1^\infty \varphi(z) \; F(dz) = \int_1^\infty \varphi'(t)\bar{F}(t) \; dt = \int_1^\infty \frac{\bar{F}(t)}{(1+t)\bar{G}(t)} \; dt \le \int_1^\infty \frac{dt}{(1+t)t} < \infty. \tag{6.6}$$

$\square$

PROOF OF THEOREM 2. If conditions (i) and (ii) of the Theorem hold, then it is clear that for all $t \ge t_0$ the sets $\mathcal{A}_t$ and $\tilde{\mathcal{A}}_t$ of finite-price consumption streams coincide; hence condition (i) of the definition of asymptotic price equivalence holds. Next, since clearly $\alpha_t \le 1 \le \beta_t$, we know that $\alpha_t \to 1$ and $\beta_t \to 1$ as $t \to \infty$. Thus for any $\epsilon > 0$, for all large enough t we have $1 - \epsilon \le \alpha_t \le \beta_t \le 1 + \epsilon$, and so the ratio of the two prices $\pi_t(c) = \pi_t(c)$ for any $c \in \mathcal{A}_t$ must lie in the interval $(\alpha_t, \beta_t) \subset (1-\epsilon, 1+\epsilon)$, and we deduce that condition (ii) of the definition of asymptotic price equivalence holds.

Now we deal with the converse assertion, that asymptotic price equivalence implies (6.4) and (6.5). Suppose that $t \ge t_0$. Then we can define two conditional probabilities m and $\tilde{m}$ on $\Omega \times [t, \infty)$ with the optional σ-field by setting

$$m(Y) \propto \mathbb{E}_t \left[\int_t^\infty \frac{\zeta_{t,s} \mathrm{e}^{-(s-t)}}{1 + \zeta_{t,s} + \tilde{\zeta}_{t,s}} \; Y_s \; \mathrm{d}s \right]$$

for any bounded non-negative optional process Y , where $\tilde{m}$ is defined analogously by interchanging the rôles of $\tilde{\zeta}$ and ζ. Now according to Lemma 1, unless the Radon-Nikodym derivative of $\tilde{m}$ with respect to m is essentially bounded then there exists an optional process Y whose m-expectation is infinite and $\tilde{m}$-expectation is finite (or vice versa). However, according to the assumption (6.2) of asymptotic equivalence, this cannot happen for $t \ge t_0$, and so the first property (6.4) is established.

To prove that the second property (6.5) must also hold, we argue by contradiction, constructing a particular cashflow process which gets valued significantly differently by the two families of pricing operators. Firstly, by considering

$$\bar{\beta}_t \equiv \operatorname{essinf}\left\{ b : \mathbb{E}_t \left[\int_t^\infty 1_{\{\zeta_{t,s}/\tilde{\zeta}_{t,s} > b\}} \frac{\zeta_{t,s} \mathrm{e}^{-(s-t)}}{1 + \zeta_{t,s} + \tilde{\zeta}_{t,s}} \, \mathrm{d}s \right] = 0 \right\}$$

and the analogously-defined quantity $\bar{\alpha}_t$ by the essential supremum, we see that it will be sufficient to prove that $\bar{\beta}_t/\bar{\alpha}_t \to 1$. Now suppose that $\bar{\beta}_t = 1 + 2\lambda > 0$ for a fixed t, and set $b = 1 + \lambda$. Then we consider the cashflow process

$$c_s = 1_{\{\zeta_{t,s}/\tilde{\zeta}_{t,s} > b\}} \frac{\mathrm{e}^{-(s-t)}}{1 + \zeta_{t,s} + \tilde{\zeta}_{t,s}}$$

which is bounded and in $\mathcal{A}_t$. Moreover, $\pi_t(c) > 0$ since $b < \bar{\beta}_t$. Hence

$$\pi_t(c) = \mathbb{E}_t \left[\int_t^\infty \zeta_{t,s} c_s \, \mathrm{d}s \right] \geq b \mathbb{E}_t \left[\int_t^\infty \tilde{\zeta}_{t,s} c_s \, \mathrm{d}s \right] = b \tilde{\pi}_t(c)$$

Thus

$$\frac{\tilde{\pi}_t(c)}{\pi_t(c)} \leq \frac{1}{b} = \frac{1}{1+\lambda} \tag{6.7}$$

However, according to (6.3), the ratio $\tilde{\pi}_t(c) = \pi_t(c)$ tends to 1, and so the hypothesis that $\bar{\beta}_t = 1 + 2\lambda > 1$ must eventually fail. Thus $\bar{\beta}_t \to 1$ almost surely as $t \to \infty$. A symmetrical argument gives that $\bar{\alpha}_t \to 1$ almost surely as $t \to \infty$. □

Remarks. The condition (6.4) is similar to Kogan et al. (2009)'s definition of no price impact, but the key difference is our requirement that the convergence of $\zeta_{t,s}/\tilde{\zeta}_{t,s}$ to 1 should be uniform in s.

7 Conclusions

This paper has made several contributions to the Market Selection Hypothesis. We have begun by isolating two different notions that correspond to different ways in which an agent might be thought to be eliminated from the market, either by starving, or by going broke. Starvation may arise from a number of different causes, only one of which is differences in beliefs. Once we have eliminated all other possible causes of starvation, we are able to show that (under boundedness conditions on relative risk aversion) starvation is equivalent to inferior beliefs. On the other hand, we show through an example with two CRRA agents that going broke depends in general on the output process of the economy as well as on the beliefs of the agents.

Our next contribution is to build an example with two CRRA agents where the second agent (with the inferior beliefs) starves - in accordance with out first result - and yet the first agent (with the correct beliefs) goes broke. This example also shows that an agent may starve, may have no price impact in the sense of Kogan et al. (2009), and yet his presence in the economy profoundly alters prices for all time. Our final contribution is to unravel this conundrum; the notion of no-price-impact of Kogan et al. (2009) is not correct, and we present a similar concept which we prove does indeed capture the desired effect. Our concept of asymptotically equivalent pricing can be characterized in terms of the state-price density quite simply.

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