A Stochastic Volatility Alternative to SABR

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Abstract

We present two new stochastic–volatility models in which option prices for European plain vanilla options have closed–form expressions. The models are motivated by the well-known SABR model but use modified dynamics of the underlying asset. The asset process is modelled as a product of functions of two independent stochastic processes: a Cox–Ingersoll–Ross process and a geometric Brownian motion. An application of the model to options written on foreign currencies is studied.

Keywords: SABR; European options; volatility smile;

1 Introduction

There is a growing interest in stochastic volatility models in all areas of financial mathematics: see for example Hull & White (1987), Hull & White (1988), Scott (1987), Wiggins (1987), Johnson & Shanno (1987), Stein & Stein (1991), Heston (1993), Hofmann et al. (1992), Dupire (1992). One stochastic volatility models which has gained great popularity with practitioners in particular for modelling the foreign exchange market is the so-called SABR model Hagan et al. (2002). As presented in Hagan et al. (2002) it has the advantage that it allows asset prices and market smiles to move in the same direction. Moreover, a closed–form (approximate) formula for the implied volatility is given. This implied volatility is not constant but a function of the strike price and some other model parameters. Hence the market prices and market risk, including Vanna and Volga risk, can be obtained very easily. Moreover, the SABR model is said to fit the implied volatility smile quite well. However, the SABR option pricing formula is not the option price corresponding to the underlying stochastic process, but is obtained by using an approximation, and as such must be treated with caution; the asymptotic is based on the assumption that the time-to-expiry is small, and recent work of Benaim (2007) shows that the extreme-strike behaviour of the formula is not consistent with arbitrage-free pricing.

The aim of this paper is to build an alternative model which retains many of the desirable features of the SABR model but also has *exact closed–form* expressions for the price of a European call option. The expressions involve a one-dimensional integral of elementary functions.

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We begin by applying certain natural transformations to the SABR model, which suggest varying the model in such a way that the discounted asset price process becomes the product of two independent processes, whose transition densities are known in closed form. The explicit formulae for the option prices follow easily from this representation.

We then generalise the model in Section 4. We define the discounted stock price as the product of a geometric Brownian motion and a *function* of the CIR process. The function used is essentially a confluent hypergeometric function. This choice makes the discounted asset price a martingale without restricting the choice of model parameters, creating a new model with seven parameters, in contrast to the four parameters of SABR, and the three of our original variant.

We therefore end up with a stochastic volatility model which is consistent with arbitrage-free pricing for all strikes and maturities. We do not rely on approximation techniques to derive the option prices for European plain-vanilla options but get closed–form formulae. This is rarely possible for other stochastic volatility models. Another not too common feature of our model is the fact that we constructed an asset price process which is a martingale and not only a local martingale and has finite higher moments, see e.g. Sin (1998) and Andersen & Piterbarg (2007) for further discussion on this matter.

The recent preprint Jäckel & Kahl (2007) presents a model similar to the ones we consider here.

2 Motivation

The SABR model is a stochastic volatility model in which the asset price and the volatility are correlated. The stock price S is assumed to solve the SDE

$$dS = \sigma S^{\beta} dW, \qquad d\sigma = \eta\sigma dB, \qquad dBdW = \rho dt,$$

for some constants $\beta \in (0,1)$, $\eta > 0$, $\rho \in (-1,1)$ and W, B are Brownian motions. In this model, singular perturbation techniques are used to obtain European option prices. Closed–form approximations to the option price and the implied volatility are stated in Hagan et al. (2002). Here, we transform the basic SABR model, making various changes along the way, to arrive at a new model for which option prices are available in closed form. The prices are represented as one–dimensional integrals. It should be emphasised that this section is purely for motivation; we take the basic SABR model and carry out various transformations, changing the dynamics in various ways when it suits us, and making whatever simplifying choices appear helpful at the time. The reader for whom such free–form mathematics is anathema should immediately pass to the next section, where an explicit model is proposed *ab initio*, inspired by, but completely independent of, the account of this section.

Recall the constant elasticity of variance model (CEV model, Cox (1996)) where the stock price solves $dS = \sigma S^{\beta} dW$ for a constant $\sigma > 0$. In this model it can be shown that the process $Y = S^{\gamma}$, $\gamma = 2(1-\beta)$, solves the SDE of a time–changed squared Bessel process. In particular, $Y_t = X(\gamma^2\sigma^2 t/4)$ where X is a Bessel process with dimension $2(1-\gamma^{-1})$, that is, $dX = 2(1-\gamma^{-1})dt + 2\sqrt{X}d\tilde{B}$, see for example Delbaen & Shirakawa (2002). Then,

$$dY = \gamma\sigma\sqrt{Y}dW + \frac{1}{2}\gamma(\gamma-1)\sigma^2 dt.$$

Let us apply this transformation to the SABR model. We have to account for the fact that in the SABR model the two Brownian motions B and W are correlated. We therefore define the process Y' as the sum of two independent squared Bessel processes, $Y' = y + \tilde{y}$, where

$$dy = \gamma\sigma\sqrt{y}dW' + a\sigma^2 dt,$$
$$d\tilde{y} = \gamma\sigma\sqrt{\tilde{y}}dB + b\sigma^2 dt,$$

for constants a, b which sum to $\gamma(\gamma-1)/2$. Then the correlation is $dY'd\sigma = \eta\sigma^2\gamma\sqrt{\tilde{y}}dt$ instead of $dYd\sigma = \eta\sigma^2\gamma\sqrt{Y}\rho dt$, so that now the constant ρ changes to the variable $\sqrt{\tilde{y}/Y}$. We can set some initial value for ρ by choice of $Y_0, \tilde{y_0}$, but notice that we cannot model *negative* correlation this way.
A particularly obliging choice of b is to take $b = \gamma^2/4$ since then

$$d\sqrt{\tilde{y}} = \frac{\sigma\gamma}{2}\, dB = \frac{\gamma}{2\eta}\, d\sigma$$

one solution of which is

$$\tilde{y} = \left(\frac{\gamma\sigma}{2\eta}\right)^2.$$

The corresponding choice for a will be $a = -\beta(1-\beta)$ and if $x = y/\sigma^2$ we find that

$$dx = \gamma\sqrt{x}dW' - 2\eta x dB + (a + 3\eta^2 x)dt.$$

For tractability, we propose instead to take $y = \sigma^2 x'$ where

$$dx' = \gamma\sqrt{x'}dW' + (a + 3\eta^2 x')dt,$$

which is of course a different model, having the virtue that x' and σ are independent. This leads to the model

$$Y_t = y_t + \tilde{y}_t = \sigma_t^2\left(\left(\frac{\gamma}{2\eta}\right)^2 + x_t'\right),$$

where σ and x' are independent. However we will not necessarily have $Y^{1/\gamma}$ a local martingale.

3 First Alternative to SABR

3.1 Model Description

Guided by the argument of the preceding section, we propose to represent the discounted asset price process by

$$S_t = Y_t^{\frac{1}{\gamma}} = \left(\sigma_t^2 z_t\right)^{\frac{1}{\gamma}}, \tag{1}$$

with z and σ the diffusions

$$\begin{aligned} dz &= (a_1 - a_2 z)dt + 2\sqrt{z}dW, \\ d\sigma &= \eta\sigma dB, \end{aligned} \tag{2}$$

where $0 < \eta$ and $0 < \gamma < 2$ are constants and W and B are two independent Brownian motions. The constants a_1 and a_2 are given by

$$a_1 = \frac{2(\gamma - 1)}{\gamma}, \quad a_2 = \frac{(2-\gamma)\eta^2}{\gamma}, \tag{3}$$

values which (as we shall shortly see) make S a martingale.

Remark 3.1. If $a_1 < 0$ the process z will hit 0 almost surely. Let $\tau := \inf\{0 \leq t : z_t = 0\}$. For $a_1 < 0$ we consider the stopped process $z_{t\wedge\tau}$ rather than z.

Definition 3.2. *We refer to the model for the asset price defined by (1), (2) and (3) as the* stochastic volatility model (SV1)*.*

We show in the following lemma that S is a martingale.

Lemma 3.3. *Suppose the diffusions z and σ satisfy (2) and the parameters are as in (3). Then the process $S_t = \sigma_t^{\frac{2}{\gamma}} z_t^{\frac{1}{\gamma}}$ is a martingale and solves the SDE*

$$dS = S\frac{2}{\gamma}\left(\frac{dW}{\sqrt{z}} + \eta dB\right).$$

A proof is given in the appendix.

Before we can compute the prices of European put and call prices we formulate the following lemma which specifies the transition density of the process z.

Lemma 3.4. *Suppose the diffusion z satisfies (2). We define for $t < T$*

$$c := \frac{2a_2}{4(1-\exp(-a_2(T-t)))}, \qquad u := cz_t\exp(-a_2(T-t)),$$
$$v := cz_T, \qquad q := \frac{a_1}{2} - 1.$$

Then

1. *Given z_t, z_T is distributed as $\frac{1}{2c}$ times a noncentral χ^2 random variable with a_1 degrees of freedom and noncentrality parameter $2u$:*

$$z_T = \frac{1}{2c}\chi^2_{a_1}(2u).$$

2. *For $a_1 > 0$ the transition density from z_t to z_T is given by*

$$p(z_t, z_T) = c\exp(-u-v)\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}), \tag{4}$$

where $I_q(\cdot)$ denotes the modified Bessel function of the first kind of order q.

3. *If $a_1 < 0$ the CIR process will hit 0 a.s.. Since we require that it is then absorbed at 0, the distribution of z has point mass $1 - \int_0^\infty p(z_t, z)dz > 0$ at zero. For $a_1 < 0$ the transition density is given by*

$$p(z_t, z_T) = c\exp(-u - v)\left(\frac{v}{u}\right)^{q/2} I_{|q|}(2\sqrt{uv}).$$

The proof is given in Göing-Jaeschke & Yor (2003). For additional information we refer also to Cox et al. (1985) and (Glasserman, 2004, Chapter 3.4).

In the following we exploit the independence of the two processes σ and z and compute prices for European put and call options by conditioning. This allows us to get analytic expressions for the option prices as the next theorem states.

Theorem 3.5 (SV1 Model). *Suppose $S_t = \sigma_t^{\frac{2}{\gamma}} z_t^{\frac{1}{\gamma}}$, where the diffusions z and σ satisfy (2) and the parameters are as in (3). Let r denote the interest rate and $\tilde{S}_t := e^{rt}S_t$ is the underlying asset price. Then the time–0–price of a European put option P^{SV1} and of a European call option C^{SV1} with expiry T and strike price K is given by*

$$P^{SV1}(S_0, T, K, r, \eta, z_0, \gamma) = \mathbb{E}\left[(e^{-rT}K - S_T)^+\right] = \int_0^\infty h_1(z)p_T(z)dz \tag{5}$$

and

$$C^{SV1}(S_0, T, K, r, \eta, z_0, \gamma) = \mathbb{E}\left[(S_T - e^{-rT}K)^+\right] = \int_0^\infty h_2(z)p_T(z)dz, \tag{6}$$

where

$$h_1(z) := e^{-rT}K\Phi(-d_2) - \sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}} \exp\left(\frac{\eta^2 T}{\gamma}\left(\frac{2}{\gamma} - 1\right)\right)\Phi\left(-d_1\right),$$
$$h_2(z) := \sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}} \exp\left(\frac{\eta^2 T}{\gamma}\left(\frac{2}{\gamma} - 1\right)\right)\Phi\left(d_1\right) - e^{-rT}K\Phi(d_2).$$

Here

$$d_1 := d_2 + \frac{2\eta}{\gamma}\sqrt{T}, \quad d_2 := \frac{\gamma}{2\eta\sqrt{T}}\left(\log\left(\frac{\sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}}}{e^{-rT}K}\right) - \frac{\eta^2}{\gamma}T\right),$$

$\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and $p_T(z) := p(z_0, z_T)$ is the probability density function of the non–central χ^2 distribution as specified in Lemma (3.4).

The integrals (5) and (6) can be rewritten as the definite integral

$$\int_0^1 h\left(\frac{1-x}{x}\right) p_T\left(\frac{1-x}{x}\right)\frac{dx}{x^2}$$

and can be evaluated by numerical integration. Alternatively they can be evaluated by Monte Carlo methods by sampling from a noncentral χ^2 distribution.

The theorem is proved in the appendix.

Remark 3.6. In this model, the correlation between the asset and its volatility will always be positive, in contrast to the SABR model. Hagan et al. (2002) state that with FX options, a key feature of the asset dynamics is that if the spot rises, then the place where the implied volatility is minimal should also rise, and this was a feature that they claim is not reflected by many stochastic volatility models. The model we propose here allows for this feature to some extent, since a shift of σ has the effect of multiplying S by some constant, but not altering the dynamics in any other way. Thus if the spot moves upward due to an increase in σ, then the implied volatility surface also shifts to the right. However, the effect of a change in z is ambivalent.

Remark 3.7. In the context of FX options, with Y denoting the price of one unit of foreign currency in domestic currency units we have that $S_t = \exp((r_f - r_d)t)Y_t$ is a martingale. Then the time–0–price of a European put option is given by

$$\mathbb{E}\left[e^{-r_d T}(K - Y_T)^+\right] = e^{-r_f T}\mathbb{E}\left[(Ke^{-(r_d - r_f)T} - S_T)^+\right]$$

We denote the corresponding put and call prices in our model by
$P^{SV1}(S_0, T, K, r_d, r_f, \eta, z_0, \gamma)$ and $C^{SV1}(S_0, T, K, r_d, r_f, \eta, z_0, \gamma)$.

3.2 Empirical Analysis

This section presents some empirical results. We consider data used by Bisesti et al. (2005). In the FX market option prices are not quoted directly. The quotes are in terms of the Black Scholes implied volatility. We consider EUR/USD volatility quotes as of 12 February 2004. On that day the spot exchange rate was 1.2832. The data contain observations for nine different maturities (1 and 2 weeks; 1, 2, 3, 6, 9 months; and 1 and 2 years) and 7 different strikes.

3.2.1 The Fitting Criterion

In the classical Black Scholes model the exchange rate process solves the SDE

$$dS^{BS} = S^{BS}((r_d - r_f)dt + \sigma^{BS}dW),$$

where r_d, r_f denote the constant domestic and foreign interest rate respectively. The volatility σ^{BS} is assumed to be constant. The price for a European call C^{BS} and put P^{BS} at time 0 with maturity T and strike K is then given by

$$\begin{aligned} C^{BS}(S_0, T, K, r_d, r_f, \sigma^{BS}) &= e^{-r_d T}\left[S_0 e^{(r_d - r_f)T}\Phi(d_1^{BS}) - K\Phi(d_2^{BS})\right], \\ P^{BS}(S_0, T, K, r_d, r_f, \sigma^{BS}) &= e^{-r_d T}\left[K\Phi(-d_2^{BS}) - S_0 e^{(r_d - r_f)T}\Phi(-d_1^{BS})\right], \end{aligned}$$

where $d_{1,2}^{BS} = \left\{ \log(S_0/K) + \left(r_d - r_f \pm \frac{1}{2}(\sigma^{BS})^2\right)T \right\} / \sigma^{BS}\sqrt{T}$,
from which implied volatilities are computed from prices.

A common feature in the FX market is that the implied volatilities are not constant but U–shaped (volatility smile). In the following we try to fit the SV1 model to the data such that we minimise the squared difference between the observed implied volatilities and the model implied volatilities. We denote by $\sigma^{(SV1)} := \sigma^{(SV1)}(\eta, z_0, \gamma)$ a model implied volatility meaning that it solves

$$P^{BS}(S_0, T, K, r_d, r_f, \sigma^{(SV1)}) - P^{SV1}(S_0, T, K, r_d, r_f, \eta, z_0, \gamma) = 0.$$

Suppose $(\sigma_1^{implied}, \ldots, \sigma_N^{implied})^T \in \mathbb{R}^N$ is the vector containing the observed implied volatilities for European options corresponding to the vector $(K_1, \ldots, K_N)^T \in \mathbb{R}^N$ of strike prices, $(T_1, \ldots, T_N)^T \in \mathbb{R}^N$ of maturities, and $(r_{d,1}, \ldots, r_{d,N})^T$, $(r_{f,1}, \ldots, r_{f,N})^T \in \mathbb{R}^N$ of domestic and foreign interest rates respectively. S_0 denotes the asset price at time zero. In the following we minimise the squared difference between the observed implied volatility and the implied volatility derived from the model price, i.e. we compute

$$\min_{\eta, z_0, \gamma} \sum_{i=1}^{N} \left(\sigma_i^{(SV1)}(\eta, z_0, \gamma) - \sigma_i^{implied} \right)^2 .$$

3.2.2 Implementation

The computation of the option price involves a numerical evaluation of an integral

$$\int_0^\infty h(z) p_T(z) dz,$$

see Theorem 3.5. This requires some care since in many examples the integrand has a very high and small peak. Simple integration routines might miss this point and might therefore compute too small prices. To overcome this problem we did the following.

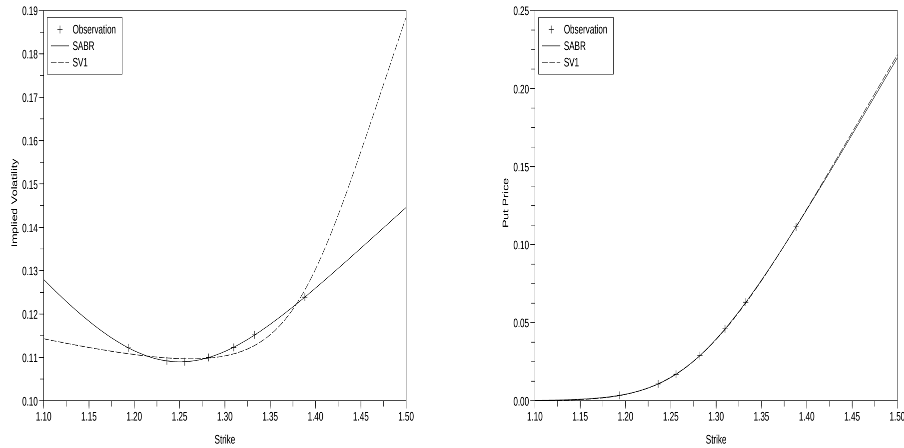
The integrand consists essentially of a product of a Black–Scholes type formula $h(\cdot)$ and the density of a non central χ^2 random variable $p_T(\cdot)$. So there are special functions involved: the cumulative distribution function of the standard normal distribution Φ and the modified Bessel function of first kind I_q. Both function can cause problems (regarding numerical precision) when considered with very small or large arguments. We therefore expressed Φ in terms of the logarithm of the complementary error function $\log\left(\frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2) dt\right)$. Moreover we did not compute I_q directly, but its scaled version $I_q(x) e^{-|x|}$.

Then we considered the logarithm of the integrand rather than the integrand itself. We used an optimisation routine to determine the maximum of the logarithm of the integrand $x^\star$. We then split the area of integration and computed

$$\int_0^{x^\star} h(z) p_T(z) dz + \int_{x^\star}^\infty h(z) p_T(z) dz.$$

Therefore we ensured that the numerical integration routine did not miss the main mass. The pricing routine was implemented in C using the *GNU Scientific Library*.

We used different optimisation routines to fit the data. We used gradient search methods, simulated annealing and the simplex algorithm by Nelder & Mead (1965).



(a) Implied Volatilities (b) European Put Prices

Model	Optimal parameters			
SABR	$\beta = 0.99$	$\eta = 1.0052314$	$\sigma_0 = 0.1078418$	$\rho = 0.147685$
SV1	$\eta = 0.01$	$z_0 = 91.576027$	$\gamma = 1.9980815$	

Figure 1: Comparing the fit of the SV1 model to the fit of the SABR model.

3.2.3 Empirical Results

In this section we present the empirical results from the analysis for one maturity (3 months) only. The model parameters are as follows: The exchange rate at time 0 is $S_0 = 1.2832$, the maturity is $T = 0.2493$ and the interest rates are $r_d = 0.0112995$ and $r_f = 0.0209007$. We consider 7 observations. Figure 1 shows the results. We see that our model (SV1) and the SABR model seem to fit the European put prices well. However, if we consider the implied volatilities we see that our model does not fit the implied volatilities as well as SABR does. SABR seems to fit the observed smile perfectly. However we have to bear in mind that our model contains only three parameters η, z_0, γ (since $\sigma_0 = \sqrt{\frac{S_0^\gamma}{z_0}}$) whereas the SABR model contains 4 parameters.

4 Second Alternative to SABR

4.1 Model Description

We now generalise the approach of the previous section. We still assume that the discounted stock price can be written as a product of two independent processes. However, we now assume that the discounted stock price S is a product of a geometric Brownian motion and a *general* function of a CIR process:

$$S_t = \sigma_t g(z_t), \tag{7}$$

where σ is a geometric Brownian motion and z is a CIR–process, i.e.

$$\begin{aligned} d\sigma &= \sigma(\mu dt + \eta dB), \\ dz &= (a_1 - a_2 z)dt + 2\sqrt{z}dW. \end{aligned} \tag{8}$$

The two Brownian motions B and W are assumed to be independent, which makes the analysis tractable, but restricts the correlation between asset and volatility to be non–negative. The function g solves the following second order ODE

$$2zg''(z) + (a_1 - a_2 z)g'(z) + \mu g(z) = 0. \tag{9}$$

[Observe that $g(z) = z^{1/\gamma}$ is a solution if $a_1 = 2(1-\gamma^{-1})$ and $\mu = (2-\gamma)\eta^2/\gamma^2$, so model SV1 is a special case of model SV2.] The ODE (9) is almost a Whittaker ODE. Its solution g can therefore be expressed in terms of the Whittaker's function W_M and W_W and is given by

$$\begin{aligned} g\left(z\right) = C_1\, e^{\frac{a_2 z}{4}} z^{-\frac{a_1}{4}} W_M \left(\frac{a_1 a_2 + 4\mu}{4a_2}, -\frac{1}{2} + \frac{a_1}{4}, \frac{a_2 z}{2} \right) \\ + C_2\, e^{\frac{a_2 z}{4}} z^{-\frac{a_1}{4}} W_W \left(\frac{a_1 a_2 + 4\mu}{4a_2}, -\frac{1}{2} + \frac{a_1}{4}, \frac{a_2 z}{2} \right), \end{aligned}$$

where C_1, C_2 are some constants. The Whittaker functions W_M and W_W are related to the Kummer functions $M(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ as follows, see Abramowitz & Stegun (1964):

$$\begin{aligned} W_M(\mu,\nu,z) &= \exp(-1/2z) z^{1/2+\nu} M(1/2+\nu-\mu, 1+2\nu, z), \\ W_W(\mu,\nu,z) &= \exp(-1/2z) z^{1/2+\nu} U(1/2+\nu-\mu, 1+2\nu, z). \end{aligned}$$

The Kummer functions are defined by

$$M(a,b,z) := \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!},$$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$, $(a)_0 = 1$ and

$$U(a,b,z) := \frac{\pi}{\sin(\pi b)} \left(\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right).$$

Therefore g is given by

$$g(z,a_1,a_2,\mu) = C_1 M\left(-\frac{\mu}{a_2}, \frac{a_1}{2}, \frac{a_2 z}{2}\right)\left(\frac{a_2}{2}\right)^{\left(\frac{a_1}{4}\right)} + C_2 U\left(-\frac{\mu}{a_2}, \frac{a_1}{2}, \frac{a_2 z}{2}\right)\left(\frac{a_2}{2}\right)^{\left(\frac{a_1}{4}\right)}. \tag{10}$$

Definition 4.1. *We refer to the model for the asset price defined by (7), (8) and (10) as the* stochastic volatility model (SV2). *Moreover we require that $\mu < 0$ and $a_1 > 2, a_2 > 0$ in the following.*

With this choice of g we have found a martingale:

Lemma 4.2. *Suppose the diffusions z and σ satisfy (8) and g is given by (10). Then the process $S_t = \sigma_t g(z_t)$ is a martingale and solves the SDE*

$$dS_t = S_t \left(\frac{g'(z_t)}{g(z_t)} 2\sqrt{z_t} dW_t + \eta dB_t \right).$$

A proof is given in the appendix.

Then the prices of European put and call option can again be derived in closed form by conditioning.

Theorem 4.3 (SV2 Model)**.** *Suppose $S_t = \sigma_t g(z_t)$, where the diffusions z and σ satisfy (8) and g is given by (10). Let r denote the interest rate and $\tilde{S}_t := e^{rt} S_t$ is the underlying asset price process. Then the time–0–price of a European put option P^{SV2} and of a European call option C^{SV2} with expiry T and strike price K is given by*

$$P^{SV2}(S_0, T, K, r, a_1, a_2, z_0, \mu, \eta) = \mathbb{E}\left[(e^{-rT}K - S_T)^+\right] = \int_0^\infty \tilde{h}_1(z) p_T(z) dz \tag{11}$$

and

$$C^{SV2}(S_0, T, K, r, a_1, a_2, z_0, \mu, \eta) = \mathbb{E}\left[(S_T - e^{-rT}K)^+\right] = \int_0^\infty \tilde{h}_2(z) p_T(z) dz, \tag{12}$$

where

$$\tilde{h}_1(z) := e^{-rT} K \Phi(-\tilde{d}_2) - \sigma_0 g(z) e^{\mu T} \Phi\left(-\tilde{d}_1\right),$$
$$\tilde{h}_2(z) := \sigma_0 g(z) e^{\mu T} \Phi\left(\tilde{d}_1\right) - e^{-rT} K \Phi(\tilde{d}_2).$$

Here

$$\tilde{d}_1 = \frac{1}{\eta\sqrt{T}} \left(\log\left(\frac{\sigma_0 g(z) e^{\mu T}}{K} \right) + \left(r + \frac{\eta^2}{2} \right) T \right)$$
$$\tilde{d}_2 = \tilde{d}_1 - \eta\sqrt{T}$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and $p_T(z) := p(z_0, z_T)$ is the probability density function of the non–central χ^2 distribution as specified in Lemma (3.4).

Again the proof is given in the appendix.

Remark 4.4. In this stochastic volatility model the option price is effectively an average of Black Scholes prices. Recall that in the classical Black Scholes model, where the stock price S is given by $S_t = S_0 \exp\left(\left(r - \frac{\eta^2}{2}\right) t + \eta W_t\right)$, the put price is given by

$$P^{BS}(S_0) := e^{-rT} K \Phi(-d_2^{BS}) - S_0 \Phi\left(-d_1^{BS}\right),$$

where

$$d_1^{BS} := \frac{1}{\eta\sqrt{T}}\left(\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\eta^2}{2}\right)T\right),$$
$$d_2^{BS} := d_1^{BS} - \eta\sqrt{T}.$$

Therefore if we substitute S_0 in this formula by the random variable $\sigma_0 e^{\mu T} g(z)$ where z is a non–central χ^2 distributed random variable we find that $\tilde{h}_1(z) = P^{BS}(\sigma_0 e^{\mu T} g(z))$. Moreover

$$P^{SV2}(S_0, T, K, r, a_1, a_2, z_0, \mu, \eta) = \mathbb{E}\left[P^{BS}(\sigma_0 e^{\mu T} g(z))\right].$$

Remark 4.5. The new stochastic volatility model SV2 contains 7 model parameters: $a_1, a_2, z_0, \mu, \eta, C_1, C_2$. Again σ_0 can be derived from $\sigma_0 = \frac{S_0}{g(z_0)}$.

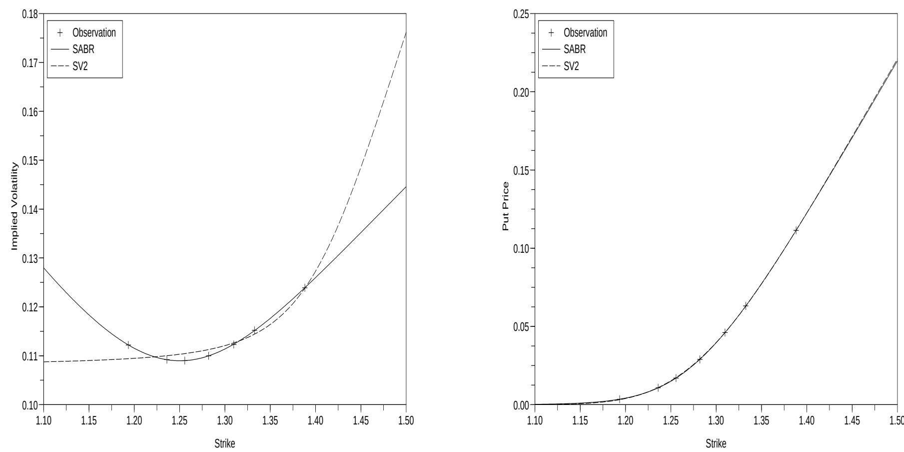
Remark 4.6. This modelling approach can be modified by replacing the CIR process by an Ornstein–Uhlenbeck process. The corresponding function $g(\cdot)$ can then still be expressed in terms of the Kummer functions. For this extension it is possible to allow for correlation between the Brownian motion driving the geometric Brownian motion and the Brownian motion driving the OU process. European option prices can still be obtained in closed form.

4.2 Empirical Results

We now fit the second stochastic volatility model SV2 to the same example considered already in the previous section. We consider a European put option with three months expiry. Figure 2 shows the implied volatilities and the fitted option prices compared to the observations and the SABR model. We find that the both the SV2 and the SABR model fit the put prices well. However, the SABR model still seems to fit the implied volatility smile better.

5 Summary

The aim of the paper was to construct a stochastic volatility which is close in spirit to the popular SABR model but does not rely on approximation techniques. Moreover we focused on the analytical and numerical tractability when choosing the dynamics and relationship of the stochastic processes involved. We obtained two stochastic volatility model which satisfy these criteria. In the first model the discounted asset price is modelled as a product of two independent processes: a geometric Brownian motion and a power of a CIR process. In the second, which generalises the first, we express the discounted asset price as a product of two independent processes: a geometric Brownian motion and a confluent hypergeometric function of a CIR process. For both models we derive analytic expressions for prices of European put and call options which is rarely possibly in other stochastic volatility models. The prices can be expressed as integrals of elementary functions and can therefore be computed very efficiently. The models fit well to FX option prices, and quite well to FX option implied volatilities.



(a) Implied Volatilities (b) European Put Prices

Model	Optimal parameters			
SABR	$\beta = 0.99$	$\eta = 1.0052314$	$\sigma_0 = 0.1078418$	$\rho = 0.147685$
SV2	$a_1 = 3.977$	$a_2 = 0.849$	$\mu = -0.0000439$	$\eta = 0.1079$
	$z_0 = 0.04305$	$C_1 = 3.68$	$C_2 = 3.81$	

Figure 2: Comparing the fit of the SV2 model to the fit of the SABR model.

A Proofs

Proof of Lemma 3.3. First, we show that S is a local martingale, using Itô calculus, and finally we argue a bound on S to show that S is a martingale.
Applying Itô 's formula to the functions $x \mapsto x^{\frac{2}{\gamma}}$ and $x \mapsto x^{\frac{1}{\gamma}}$ we get

$$\begin{aligned}
d\sigma^{\frac{2}{\gamma}} &= \frac{2}{\gamma}\sigma^{\frac{2}{\gamma}-1}d\sigma + \frac{1}{2}\frac{2}{\gamma}\left(\frac{2}{\gamma}-1\right)\sigma^{\frac{2}{\gamma}-2}\sigma^2\eta^2 dt \\
&= \sigma^{\frac{2}{\gamma}}\left(\frac{2-\gamma}{\gamma^2}\eta^2 dt + \frac{2\eta}{\gamma}dB\right), \\
dz^{\frac{1}{\gamma}} &= \frac{1}{\gamma}z^{\frac{1}{\gamma}-1}dz + \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma}-1\right)z^{\frac{1}{\gamma}-2}4z dt \\
&= \frac{z^{\frac{1}{\gamma}}}{\gamma}\left(\left(\frac{a_1}{z} - a_2 + \frac{2(1-\gamma)}{\gamma z}\right)dt + \frac{2}{\sqrt{z}}dW\right).
\end{aligned} \tag{13}$$

Using the product rule and the independence of the Brownian motions B and W gives

$$\begin{aligned}
dS = d\left(\sigma^{\frac{2}{\gamma}}z^{\frac{1}{\gamma}}\right) &= \sigma^{\frac{2}{\gamma}}dz^{\frac{1}{\gamma}} + z^{\frac{1}{\gamma}}d\sigma^{\frac{2}{\gamma}} \\
&= \sigma^{\frac{2}{\gamma}}z^{\frac{1}{\gamma}}\frac{1}{\gamma}\left(\left(\frac{a_1}{z} - a_2 + \frac{2(1-\gamma)}{\gamma z}\right)dt + \frac{2}{\sqrt{z}}dW + \frac{2-\gamma}{\gamma}\eta^2 dt + 2\eta dB\right) \\
&= S\frac{1}{\gamma}\left(\left(\frac{a_1}{z} - a_2 + \frac{2(1-\gamma)}{\gamma z} + \frac{2-\gamma}{\gamma}\eta^2\right)dt + \frac{2}{\sqrt{z}}dW + 2\eta dB\right).
\end{aligned}$$

Plugging in the definition of a_1 and a_2 we get

$$dS = S\frac{2}{\gamma}\left(\frac{1}{\sqrt{z}}dW + \eta dB\right) = S\frac{2}{\gamma}\sqrt{\frac{1}{z} + \eta^2}d\tilde{W},$$

where $\tilde{W}$ is a Brownian motion.

Since σ is a geometric Brownian motion, it is easy to see that $\sup_{0\le t\le T}\sigma_t \in L^p$ for any $p > 1$, and similarly $\sup_{0\le t\le T} z_t \in L^p$ for any $p > 1$, since $\sup_{0\le t\le T} z_t$ is bounded in law by the supremum of the squared Euclidean norm of an OU process in high enough dimension. Therefore S is a martingale. □

Proof of Theorem 3.5. We only show the expression for the put price (5) since (6) is similar. From Lemma 3.3 we know that S is a martingale. Hence the put price is the expectation in (5). Since the processes z and σ are independent we can compute the expectation by conditioning as follows.

$$\begin{aligned}
P^{SV1}(S_0, T, K, r, \eta, z_0, \gamma) &= \mathbb{E}\left[(e^{-rT}K - S_T)^+\right] = \mathbb{E}\left[(e^{-rT}K - \sigma_T^{\frac{2}{\gamma}} z_T^{\frac{1}{\gamma}})^+\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[(e^{-rT}K - \sigma_T^{\frac{2}{\gamma}} z_T^{\frac{1}{\gamma}})^+ \middle| z_T = z\right]\right] = \mathbb{E}\left[h_1(z)\right] = \int_0^\infty h_1(z) p_T(z) dz,
\end{aligned}$$

where

$$\begin{aligned}
h_1(z) &= \mathbb{E}\left[(e^{-rT}K - \sigma_T^{\frac{2}{\gamma}} z_T^{\frac{1}{\gamma}})^+ \middle| z_T = z\right] \\
&= \int_{-\infty}^{\infty}\left(e^{-rT}K - \sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}} \exp\left(\left(\frac{2-\gamma}{\gamma^2}\eta^2 - \frac{2\eta^2}{\gamma^2}\right)T + \frac{2\eta}{\gamma}\sqrt{T}x\right)\right)^+ \frac{\exp(-\frac{x^2}{2})}{\sqrt{2\pi}}dx \\
&= \int_{-\infty}^{a} e^{-rT}K\frac{\exp(-\frac{x^2}{2})}{\sqrt{2\pi}}dx - \sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}} e^{-\frac{\eta^2 T}{\gamma}} \int_{-\infty}^{a} \exp\left(-\frac{1}{2}\left(x^2 - \frac{4\eta\sqrt{T}}{\gamma}x\right)\right)\frac{dx}{\sqrt{2\pi}} \\
&= Ke^{-rT}\Phi(a) - \sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}} e^{-\frac{\eta^2 T}{\gamma}} e^{\frac{2\eta^2 T}{\gamma^2}} \Phi\left(a - \frac{2\eta\sqrt{T}}{\gamma}\right) \\
&= Ke^{-rT}\Phi(a) - \sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}} e^{\frac{\eta^2 T}{\gamma}(\frac{2}{\gamma}-1)} \Phi\left(a - \frac{2\eta\sqrt{T}}{\gamma}\right),
\end{aligned}$$

where $a := \frac{\gamma}{2\eta\sqrt{T}}\left(\log\left(\frac{e^{-rT}K}{\sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}}}\right) + \frac{\eta^2 T}{\gamma}\right)$. Then with $d_2 = -a$ and $d_1 = -a + \frac{2\eta\sqrt{T}}{\gamma}$ the result follows. □

Proof of Lemma 4.2. We shall have need of the following result, which can be seen as an application of Theorem 1.3.5 in Stroock & Varadhan (1979), though we present a direct proof here.

Proposition A.1. *Suppose that I is a non-empty open interval, and that $\sigma, b, \tilde{b} : I \to \mathbb{R}$ are locally Lipschitz in I, $\sigma > 0$ throughout I. Let P (respectively, $\tilde{P}$) be the law on path space $C(\mathbb{R}^+, I)$ under which the canonical process X solves the SDE*

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x_0 \tag{14}$$

respectively,

$$dX_t = \sigma(X_t)dW_t + \tilde{b}(X_t)dt, \quad X_0 = x_0 \tag{15}$$

for some fixed $x_0 \in I$. If $\tau \equiv \inf\{t : X_t \notin I\}$, and Z is the 'change-of-measure' local martingale

$$dZ_t = Z_t f(X_t) dW_t \tag{16}$$

where $f(x) \equiv \sigma(x)^{-1}(\tilde{b}(x) - b(x))$, then Z is a true martingale if and only if

$$\tilde{P}(\tau = \infty) = 1. \tag{17}$$

Proof. First suppose that (17) holds. Take compact intervals $K_n \subset I$, increasing to I, and let $\tau_n = \inf\{t : X_t \notin K_n\}$. Then $Z^n_t \equiv Z_{t\wedge\tau_n}$ is a martingale for each n, because

$$dZ^n_t = Z^n_t f(X_t) I_{\{t \le \tau_n\}} dW_t$$

and the drift in the change-of-measure is bounded. Under the probability P^n given by

$$\left.\frac{dP^n}{dP}\right|_{\mathcal{F}_t} = Z^n_t$$

the process X solves the SDE

$$dX_t = \sigma(X_t)dW_t + \tilde{b}(X_t)I_{\{t \le \tau_n\}}dt + b(X_t)I_{\{t > \tau_n\}}dt.$$

Notice that for any $T \in \mathbb{R}^+$

$$\begin{aligned} 1 = E[Z^n_T] &= E[Z^n_{\tau_n} : \tau_n \le T] + E[Z^n_T : \tau_n > T] \\ &= E[Z^n_{\tau_n} : \tau_n \le T] + \tilde{P}(\tau_n > T) \\ &= E[Z_{\tau_n} : \tau_n \le T] + \tilde{P}(\tau_n > T). \end{aligned}$$

By hypothesis, $\tilde{P}(\tau_n > T) \to 1$ as $n \to \infty$, and therefore

$$\begin{aligned} E[Z_T] &= E[Z_T : \tau_n \le T] + E[Z_T : \tau_n > T] \\ &= E[Z_T : \tau_n \le T] + E[Z^n_T : \tau_n > T] \\ &= E[Z_T : \tau_n \le T] + \tilde{P}(\tau_n > T) \\ &\to 1 \end{aligned}$$

Conversely, if Z is a martingale, then the laws $\tilde{P}$ and P are equivalent on each $\mathcal{F}_t$, so the event $\{\sup_n \tau_n \le t\}$ has probability zero under both P and $\tilde{P}$.

□

For a diffusion (14) on $I = (0, \infty)$, it is well known (see, for example Rogers & Williams (2000), Chapter V) that 0 is inaccessible if and only if $s(0+) = -\infty$ and $+\infty$ is inaccessible if and only if $s(\infty) = \infty$, where s is the scale function defined up to irrelevant affine transformations by

$$s'(x) = \exp\{-\int^x \frac{2b(z)}{\sigma(z)^2}\, dz\}.$$

Routine calculations prove that 0 and ∞ are inaccessible for the diffusion z satisfying (8) provided $a_1 \geq 2$ and $a_2 > 0$; for this diffusion,

$$\sigma(x) = 2\sqrt{x}, \quad b(x) = a_1 - a_2 x, \quad s'(x) = x^{-a_1/2} e^{\frac{a_2}{2}x}. \tag{18}$$

It remains only to analyse the scale function of the drift-transformed version of the diffusion, for which

$$\sigma(x) = 2\sqrt{x}, \quad \tilde{b}(x) = a_1 - a_2 x + \frac{4xg'(x)}{g(x)}, \quad \tilde{s}'(x) = s'(x)/\left(g(x)\right)^2. \tag{19}$$

This will require the asymptotics of the Kummer functions at 0 and ∞.

We use the first order approximation of the Kummer functions. According to (Abramowitz & Stegun, 1964, Chapter 13) for $z > 0$ and $z \to \infty$

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}\left(1 + O(|z|^{-1})\right), \qquad U(a,b,z) = z^{-a}\left(1 + O(|z|^{-1})\right).$$

Hence, for large z we can write

$$\begin{aligned} g(z, a_1, a_2, \mu) &= \left(\frac{a_2}{2}\right)^{\frac{a_1}{4}} \left(C_1 M\left(-\frac{\mu}{a_2}, \frac{a_1}{2}, \frac{a_2 z}{2}\right) + C_2 U\left(-\frac{\mu}{a_2}, \frac{a_1}{2}, \frac{a_2 z}{2}\right)\right) \\ &= O(e^{a_2 z/2} z^{-\frac{\mu}{a_2} - \frac{a_1}{2}}) \end{aligned} \tag{20}$$

From the equations (18), (19) and (20), we see that

$$\int^\infty \tilde{s}'(x)\, dx = +\infty.$$

All that remains is to show that $\int_{0+} \tilde{s}'(x)\, dx = +\infty$, and for this we need the asymptotics near zero of g.

For small z, the Kummer functions can be approximated, see (Abramowitz & Stegun, 1964, Chapter 13). For $|z| \to 0$, $M(a,b,0) = 1$. For $U(\cdot,\cdot,\cdot)$ there are several approximation dependent on the value of the second parameter, see (Abramowitz & Stegun, 1964, Chapter 13, formulae 13.5.6 - 13.5.11). Also the order of the approximation varies. Since we require $a_1 > 2$, we get for small z

$$U\left(-\frac{\mu}{a_2}, \frac{a_1}{2}, \frac{a_2}{2}z\right) = \frac{\Gamma\left(\frac{a_1}{2} - 1\right)}{\Gamma\left(-\frac{\mu}{a_2}\right)} \left(\frac{a_2}{2}z\right)^{1-\frac{a_1}{2}} + \begin{cases} O\left(|z|^{\frac{a_1}{2}-2}\right) & : \quad a_1 > 4, \\ O\left(\log\left|\frac{a_2}{2}z\right|\right) & : \quad a_1 = 4, \\ O(1) & : \quad 2 < a_1 < 4. \end{cases} \tag{21}$$

Whichever of these obtains, we see immediately that

$$g(z) = O(z^{1-\frac{a_1}{2}}). \tag{22}$$

as $z \to 0$. Combining (18), (19) and (22), we see that

$$\int_{0+} \tilde{s}'(x)\, dx = +\infty,$$

and the proof is complete by applying Proposition A.1.

□

Proof of Theorem 4.3. From Lemma 4.2 we know that S is a martingale. Hence the option prices are indeed the expectations in (11) and (12). These expectations can again be computed by conditioning under z.

$$\begin{aligned} P^{SV2}(S_0, T, K, r, a_1, a_2, z_0, \mu, \eta) &= \mathbb{E}\left[(e^{-rT}K - S_T)^+\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[(e^{-rT}K - \sigma_T g(z_T))^+ \middle| z_T = z\right]\right] = \mathbb{E}\left[\tilde{h}_1(z)\right] = \int_0^\infty \tilde{h}_1(z) p(z) dz \end{aligned}$$

and

$$\begin{aligned} \tilde{h}_1(z) &:= \mathbb{E}\left[(e^{-rT}K - S_T)^+ \middle| z_T = z\right] = \mathbb{E}\left[(e^{-rT}K - \sigma_T g(z_T))^+ \middle| z_T = z\right] \\ &= \int_{-\infty}^{\infty} \left(e^{-rT}K - \sigma_0 g(z) e^{\mu T} \exp\left(\frac{-\eta^2 T}{2} + \eta\sqrt{T}x\right)\right)^+ \frac{e^{-\frac{x^2}{3}}}{\sqrt{2\pi}} dx. \end{aligned}$$

We compute the integration boundary

$$\begin{aligned} & 0 \le e^{-rT}K - \sigma_0 g(z) e^{\mu T} \exp\left(\frac{-\eta^2 T}{2} + \eta\sqrt{T}x\right) \\ \iff & x \le \frac{1}{\eta\sqrt{T}}\left(\log\left(\frac{K}{\sigma_0 g(z) e^{\mu T}}\right) - \left(r - \frac{\eta^2}{2}\right) T\right) =: \tilde{a}. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{h}_1(z) &= \int_{-\infty}^{\tilde{a}} \left(e^{-rT}K - \sigma_0 g(z) e^{\mu T} \exp\left(\frac{-\eta^2 T}{2} + \eta\sqrt{T}x\right)\right) \frac{\exp(-0.5x^2)}{\sqrt{2\pi}} dx \\ &= e^{-rT}K\Phi(\tilde{a}) - \sigma_0 g(z) e^{\mu T}\Phi(\tilde{a} - \eta\sqrt{T}). \end{aligned}$$

Then setting $\tilde{d}_2 := -\tilde{a}$ and $\tilde{d}_1 := -a + \eta\sqrt{T}$ yields the result. □

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