Utilities Bounded Below

Roman Muraviev*

L. C. G. Rogers†

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Abstract

It is common to work with utilities which are not bounded below, but it seems hard to reconcile this with common sense; is the plight of a man who receives only one crumb of bread a day to eat really very much worse than the plight of a man who receives two? In this paper we study utilities which are bounded below, which necessitates novel modelling elements to prevent the question becoming trivial. What we propose is that an agent is subjected to random reviews of his finances. If he is reviewed and found to be bankrupt, then he is thrown into jail, and receives some large but finite negative value. In such a framework, we find optimal investment and consumption behaviour very different from the standard story. As the agent's wealth goes negative, he gradually abandons hope of ever becoming honest again, and plunders as much as he can before being caught. Agents with very high wealth act like standard Merton investors.

1 Introduction.

Although non-negative wealth is a condition frequently imposed in dynamic optimal investment problems, in practice this may be violated if an agent secures a loan against some asset whose value subsequently declines. The lender may then call in the loan, but quite likely he will not, reasoning that the costs of recovering the loan are so great that it would be better to let matters continue in the hope that the collateral value rises and the borrower returns to positive wealth. Similarly, an agent may have various credit possibilities - different store cards, for example - and could run up some quite large debts without being stopped. Moreover, if the agent was found to be in negative wealth, the consequences for the agent would

*Department of Mathematics and RiskLab, ETH Zürich.

†Corresponding author: Statistical Laboratory, Wilberforce Road, Cambridge CB3 0WB, United Kingdom. lcgr1@cam.ac.uk.

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be bad but typically not catastrophic[1]. Importantly, the negative consequences would not get unboundedly bad as the size of the agent's shortfall got bigger and bigger; at worst the agent would be thrown in jail, and that would be the same outcome whether he owed 4.9bn EUR[2] or 490mn EUR.

So if we are going to consider the simple problem of an agent who is able to invest in some standard log-Brownian market, and who aims to maximize his objective $EU(w_T)$, where w_T is the wealth generated by terminal time T, we claim that it is completely realistic to suppose that U is *bounded below*[3]. Accepting this leads into uncharted territory; firstly, U cannot be concave. Secondly, in order that the problem should be well posed, we have to find some way to rule out the kind of doubling strategy which is the reason we usually assume wealth is bounded below. The mechanism we propose is *random reviews of the agent's financial status:* if the agent is reviewed and found to be in negative wealth, then he is declared bankrupt and suffers some substantial penalty. The way in which reviews happen has to be constructed suitably to make the problem well posed, but the modelling assumptions are reasonable; we present the model in a simple context in Section 2.

Developing this, we next address in Section 3 the problem where the agent's objective is $E[\int_0^\infty e^{-\rho t} U(c_t)\, dt]$, but wealth is again allowed to go negative. In this situation, we can allow U to be a conventional felicity function, defined only on $(0, \infty)$, and unbounded below. If things got really bad for the agent, he would not reduce his consumption lower and lower; he would prefer to go into negative wealth while maintaining a higher level of consumption. After all, the worst that could happen is that he gets found out and thrown into jail.

Literature: A special class of utilities bounded below consisting of the so-called S-shaped preferences, arises naturally as an essential ingredient in the prospect theory (cumulative prospect theory) of Kahnemann and Tversky (1979) (Kahnemann and Tversky (1992)). Motivated by the postulate that decision makers act non-rationally (where rationality can be regarded as a notion encoded within the axioms of Von Neumann and Morgenstern (1944)) and exhibit a risk-seeking behavior in some cases, many variations on prospect theory involving these utility functions have been attracting great attention over the last decades, in a number of economic settings and financial applications. Benartzi and Thaler (1995) employ prospect theory in an attempt to explain the equity premium puzzle. Shefrin and Statman (2000) develop behavioral portfolio theory for discrete-time models. Levy and Levy (2004) establish a correspondence between S-shaped preferences and mean-variance theory in simple single period models. Gomes (2005) analyzes equilibrium and studies the implications on volume trading patterns, in the setting of single period models and S-shaped utilities.

[1]... if the money was owed to legitimate lenders ...

[2]... as in the case of disgraced SocGen trader Jerome Kerviel..

[3]As Kenneth Arrow has argued, it is absurd to imagine that a utility should be unbounded above; for then you would prefer an infinitesimal chance of gaining more wealth than exists in the universe to the certainty of gaining all the wealth you could consume in ten lifetimes. We may accept unbounded utilities if they lead to tractable analyses, but should be wary if our conclusions are critically dependent on the exact nature of the infinite asymptotics.

A rather recent body of literature attempts to study non-concave utilities in rather complex frameworks such as continuous trading and incomplete markets. Jin and Zhou (2008) have studied cumulative prospect theory (involving both distorted probabilities and S-shaped preferences) in general complete markets driven by Ito diffusions, and established a class of well-posed problems and provided solutions. In Reichlin (2011), a complete market terminal wealth problem has been considered under general assumptions on the density of the equivalent martingale measure. Rasonyi and Rodrigues (2012) explore further the well-posedness conditions of the optimization problem in general semi-martingale markets. Carassus and Rasonyi (2012) study the corresponding problem in general incomplete discrete-time models.

The behaviour we find for the rational agent in the situation we study here exhibits surprising qualitative features, such as some claim can only be explained by abandoning rationality, and proposing, for example, that an agent may sometimes be risk seeking, or may misevaluate probabilities. However, we find apparent risk-seeking preferences when the agent is in negative wealth; he is in an undesirable situation, and is prepared to take risks to give him a chance of escaping from it. As he knows that the worst that can happen to him is bounded below, he is ready to take those risks, and it is completely rational for him to do so.

2 Terminal wealth.

We suppose that an agent is able to invest in a riskless bank account bearing constant interest at rate r, and in a risky stock, modelled as a log-Brownian motion. Thus the wealth of the agent evolves as

$$dw_t = rw_t\,dt + \theta_t(\sigma dW_t + (\mu - r)\,dt) \tag{2.1}$$

where σ and μ are constants, and W is a standard Brownian motion. The previsible process θ is the portfolio process, with θ_t denoting the cash value invested in the stock at time t. We shall suppose that the agent's finances are reviewed with intensity process

$$\lambda_t = G(w_t, \theta_t^2) \tag{2.2}$$

where G is non-negative, and decreasing in w. We may[4] (and shall) suppose that G is zero for $w \geq 0$. The reason to introduce the dependence on θ in the intensity is that if this were absent, then the agent could choose enormous values of θ, so that the evolution of w would be like a Brownian motion running very quickly; if this were allowed, then the agent could cause the wealth to reach any chosen high value in an arbitrarily short period of time, during which the chances he would be reviewed would be negligible. By including dependence on θ_t^2 in the intensity λ_t we rule out this kind of strategy and create an interesting question. The modelling hypothesis is not without intuitive content; an agent who is betting big in the stock

[4]If the agent was reviewed while $w \geq 0$, he would be allowed to continue, so this review would have no effect.

is likely to attract attention to himself, and is therefore more likely to get reviewed than one who is living quietly.

We shall let τ denote the time of first review; at this time, the agent is found out and thrown into jail, earning a utility $-K$; otherwise, he gets to time T and receives utility $U(w_T)$. Hence if $V(t,w)$ denotes the value function

$$V(t,w) = \sup E[\, U(w_T) I_{\{T<\tau\}} - K I_{\{T\geq\tau\}} \mid w_t = w, \tau > t\,] \tag{2.3}$$

we can write down the Hamilton-Jacobi-Bellman (HJB) equation for the value function:

$$\begin{aligned} 0 &= \sup_\theta [\, V_t + (rw + \theta(\mu - r))V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} - G(w,\theta^2)(K+V)\,], \qquad (2.4) \\ U(w) &= V(T,w), \end{aligned}$$

where the final term in (2.4) compensates for the downward jump of magnitude $V(w_t) + K$ coming at rate $G(w, \theta^2)$.

The value function for this problem will be time-dependent, which is a complicating feature. Since we treat this example more for illustration before embarking on the main problem in Section 3, let us simplify the question by assuming that $\mu = r = 0$, so that the agent's wealth process is a continuous local martingale. We shall also assume that

$$G(w,\theta^2) = g(w)\theta^2 \tag{2.5}$$

for some decreasing function g. In this case, the value function will not depend on time:

$$V(t,w) = v(w) \qquad \text{for any } t \in (0,1)$$

because any value which could be achieved starting with wealth w at some time $t \in [0,T)$ could equally be achieved starting with wealth w at any other time $s \in [0,T)$, by following the same portfolio process at different speed. Thus the problem is an optimal stopping problem, whose solution $v(w)$ must have no downward jumps of derivative[5] and must satisfy the variational inequality

$$\max\Big[\, \sup_\theta \Big\{\, \tfrac{1}{2}\sigma^2\theta^2 v_{ww} - g\,\theta^2(K+v)\,\Big\}, U - v\,\Big] = 0,$$

which is easily seen to be equivalent to the variational inequality

$$\max\Big[\, \tfrac{1}{2}\sigma^2 v_{ww} - g(w)\,(K+v),\ U - v\,\Big] = 0. \tag{2.6}$$

By writing $K + v \equiv f$ this becomes

$$\max\Big[\, \tfrac{1}{2}\sigma^2 f_{ww} - gf,\ U + K - f\,\Big] = 0. \tag{2.7}$$

[5]The process $v(w_{t\wedge\tau})$ has to be a supermartingale for any stopping time τ, and a martingale for the optimal τ. If there was an upward jump of v' at some point, then the supermartingale property would not hold because there would be an increasing local time contribution in the Itô expansion of $v(w_t)$.

This is a fairly conventional optimal stopping problem:

$$f(w) = \sup_\tau E\Big[\exp\big(-\int_0^\tau g(W_s)\, ds\big)\ (K + U(W_\tau)) \ \Big|\ W_0 = w \Big] \tag{2.8}$$

for a Brownian motion.

Example. Here is a simple example which can be dealt with fairly explicitly and illustrates the solution. We shall take for the utility

$$\begin{aligned} U(w) &= \max\{-K, -\exp(-\gamma w)\} && (2.9) \\ &= -\exp\{-\gamma(w \vee w_*)\} && (2.10) \end{aligned}$$

for some $K > 1$ and $\gamma > 0$, where $w_* = -\gamma^{-1}\log(K) < 0$. For the rate of reviewing, we choose

$$g(w) = \tfrac{1}{2}\varepsilon^2 I_{\{w \le 0\}}$$

for some $\varepsilon > 0$. So the agent here has a standard CARA utility but bounded below, and runs a constant risk of being discovered when his wealth is negative. Looking back to (2.7), we see that the differential equation to be solved in the region where the agent has not stopped is

$$\tfrac{1}{2}\sigma^2 \varphi_{ww} - g\varphi = 0, \tag{2.11}$$

which has the solution vanishing at $-\infty$ of the form

$$\varphi(w) = \begin{cases} Ae^{\varepsilon w} & (w \le 0) \\ A(1 + \varepsilon w) & (w \ge 0) \end{cases}$$

for some constant $A \ge 0$. We seek a solution of the form that the agent will only stop if his wealth is at least b for some $b > 0$, because a solution where the agent does nothing for all positive wealth values would not be very interesting. If the agent chooses to stop at $b > 0$, then matching the values of $\varphi(b)$ and $K + U(b)$ gives the condition

$$A(1 + \varepsilon b) = K - e^{-\gamma b},$$

and hence the constant A is

$$A = \frac{K - e^{-\gamma b}}{1 + \varepsilon b}.$$

The agent will want to choose b to make this as large as possible; routine calculus leads to the implicit equation

$$\varepsilon K e^{\gamma b} = \gamma + \varepsilon + \gamma\varepsilon b \tag{2.12}$$

for b, which has a strictly positive solution if and only if

$$\gamma + \varepsilon > \varepsilon K. \tag{2.13}$$

This condition has a natural interpretation. If we hold γ and ε fixed, we see that K must not be too large, which is what we would expect: if the penalty K for being in negative wealth is not too large, then we will be willing to take a chance and invest in the risky asset for larger values of w, but if the penalty becomes too big, it will not be worth taking the chance.

As can be readily verified, the equality (2.12) holds if and only if the derivative of φ matches the derivative of U at b. Now we confirm that the function f defined by

$$f(w) = \begin{cases} \varphi(w) & (w \leq b) \\ K + U(w) & (w \geq b) \end{cases}$$

satisfies (2.7). While $w > w_*$ the function $w \mapsto \varphi(w) - U(w) - K$ is convex; at b it vanishes along with its derivative; therefore it is non-negative throughout $[w_*, b]$. For $w < w_*$, the function is equal to $\varphi(w)$, which is still convex, so we conclude that $\varphi(w) - U(w) - K$ is non-negative throughout $(-\infty, b)$. Therefore $U(w) + K - f(w) \leq 0$ everywhere. Looking at (2.7), we see that $\frac{1}{2}\sigma^2 f_{ww} - gf = 0$ everywhere in $(-\infty, b)$ by construction, and the only thing that remains to be checked is that $\frac{1}{2}\sigma^2 f_{ww} - gf \leq 0$ to the right of b. But to the right of b the function f is concave, so this property holds also.

3 Running consumption.

In this section, we change the story so that the agent is consuming continuously from his wealth, which may be allowed to go negative[6]. The wealth dynamics are now

$$dw_t = rw_t\, dt + \theta_t(\sigma dW_t + (\mu - r)\, dt) - c_t\, dt \tag{3.1}$$

and the objective is

$$V(w) = \sup_{\theta,c} E\left[\int_0^\tau e^{-\rho t} U(c_t)\, dt - e^{-\rho\tau} K \,\middle|\, w_0 = w \right] \tag{3.2}$$

where τ is the review time, coming with intensity $\lambda_t = G(w_t, \theta_t^2)$, depending on current wealth, and current portfolio. We propose the form

$$G(w, \theta^2) = (b\,|w|^m + a\theta^2) I_{\{w<0\}} \tag{3.3}$$

which differs from (2.5) by having terms bw^m and $a\theta^2$ for some positive constants a and b, and $m > 0$. The first one is required to stop the problem becoming ill-posed: if it were not there, the agent could take a zero position in the risky asset, consume greedily, and never get found out. Of course, his wealth would be getting ever more negative, but that is not

[6] In this, our problem differs from the situation considered in [7], where bankruptcy of the agent is an allowed eventuality, but negative wealth is not. In the study of [7], bankruptcy occurs at the (previsible) first time that wealth hits zero.

ruled out by the modelling assumptions. This feature has some economic content as well as being required mathematically; a bank might have various triggers for action in place which would cause scrutiny of a heavily overdrawn customer, and the more heavily overdrawn the customer becomes the more emphatic the bank's response.

For this problem, the HJB equation for the value function V becomes

$$0 = \sup_{c,\theta} \big[-\rho V + U(c) + (rw + \theta(\mu - r) - c)V' + \tfrac{1}{2}\sigma^2\theta^2 V'' - G(w, \theta^2)(K + V) \big]. \tag{3.4}$$

Transforming the HJB equation. Certain transformations of (3.4) make it easier to work with. Firstly, we set $f(w) \equiv V(w) + K \geq 0$, so that the equation becomes

$$\begin{aligned} 0 &= \sup_{c,\theta} \big[-\rho f + \rho K + U(c) + (rw + \theta(\mu - r) - c) f' + \tfrac{1}{2}\sigma^2\theta^2 f'' - I_{\{w<0\}}(bw^m + a\theta^2) f \big] \\ &= \sup_{c,\theta} \big[-(\rho + bw^m I_{\{w<0\}}) f + \rho K + U(c) - f'c + (rw + \theta(\mu - r)) f' + \theta^2 (\tfrac{1}{2}\sigma^2 f'' - aI_{\{w<0\}} f) \big]. \end{aligned}$$

Now we introduce the increasing (respectively, decreasing) positive solution ψ_+ (respectively, ψ_-) to

$$\tfrac{1}{2}\sigma^2 \psi'' = aI_{\{w<0\}}\psi \tag{3.5}$$

and we write

$$y(w) = \psi_+(w)/\psi_-(w), \tag{3.6}$$

a positive increasing function. We notice that

$$y' = \frac{W}{\psi_-^2}, \tag{3.7}$$

where $W \equiv \psi_- D\psi_+ - \psi_+ D\psi_-$ is the Wronskian, a positive constant. Since $g = 0$ in $\mathbb{R}^+$, it must be that $\psi_\pm$ are both linear in that region, and we lose no generality in assuming that $\psi_-(w) = 1$ in $\mathbb{R}^+$, $D\psi_+(0) = 1$, so that $W = 1$.

We now write f in the form

$$f(w) = \psi_-(w) h(y(w)), \tag{3.8}$$

so that

$$f' = \psi_-' h(y) + \psi_- y' h'(y), \qquad f'' = \psi_-'' h(y) + (y')^2 \psi_- h''(y). \tag{3.9}$$

Substituting this back into the HJB equation gives us

$$\begin{aligned} 0 = \sup_{c,\theta} \big[& -(\rho + I_{\{w<0\}} bw^m)\psi_- h + \rho K + U(c) - (\psi_-' h + \psi_- y' h')c \\ & + (rw + \theta(\mu - r))(\psi_-' h + \psi_- y' h') + \tfrac{1}{2}\sigma^2\theta^2 (y')^2 \psi_- h'' \big]. \end{aligned} \tag{3.10}$$

This form of the HJB equation turns out to be easier to work with because the solution h *must be concave*, otherwise the optimization over θ yields an infinite value. The variable y takes positive values only, so h is a function of positive values only.

In the right half-line, (3.10) simplifies to

$$\begin{aligned} 0 &= \sup_{c,\theta}\big[\, -\rho h + \rho K + U(c) - h'c + (rw + \theta(\mu - r))h' + \tfrac{1}{2}\sigma^2\theta^2 h'' \,\big] \\ &= -\rho h + \rho K + \tilde{U}(h') + rwh' - \frac{(\kappa h')^2}{2h''} \end{aligned} \tag{3.11}$$

where $\kappa \equiv (\mu - r)/\sigma$, and $\tilde{U}$ is the convex dual function of U. In view of the assumption that $D\psi_+(0) = 1$, we see that

$$y(w) = \psi_+(0) + w \qquad \text{for } \; w \geq 0, \tag{3.12}$$

and so (3.11) becomes

$$0 = -\rho h + \rho K + \tilde{U}(h') + r(y - A)\, h' - \frac{(\kappa h')^2}{2h''}\,, \tag{3.13}$$

using the abbreviation $A \equiv \psi_+(0)$. In this form, the HJB equation is amenable to the standard change of variables $z \equiv h'$, $J(z) = h(y) - yz$, from which we see that $J'(z) = -y$, $J''(z) = -1/h''(y)$, and the HJB equation (3.13) take the simple *linear* form

$$0 = -\rho J + (\rho - r)zJ' + \tfrac{1}{2}\kappa^2 z^2 J'' + \rho K + \tilde{U}(z) - Arz. \tag{3.14}$$

This ODE can be solved explicitly almost completely if we assume that U is CRRA:

$$U(x) = \frac{x^{1-R}}{1-R}, \qquad \tilde{U}(x) = -\frac{R\, x^{(R-1)/R}}{R-1}. \tag{3.15}$$

Indeed, if we introduce the notation

$$Q(t) = \tfrac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho, \tag{3.16}$$

we see that the quadratic Q has roots $-\alpha < 0$, $\beta > 1$, and the solution to (3.14) is

$$\begin{aligned} J(z) &= A_0 z^{-\alpha} + B_0 z^{\beta} - \frac{\rho K}{Q(0)} - \frac{\tilde{U}(z)}{Q(1-1/R)} + \frac{Arz}{Q(1)} \\ &= A_0 z^{-\alpha} + B_0 z^{\beta} + K + \gamma_M^{-1}\, \tilde{U}(z) - Az \end{aligned} \tag{3.17}$$

for some constants A_0, B_0, where γ_M is the proportional rate of consumption in the standard Merton problem:

$$\gamma_M = R^{-1}\big[\rho + (R-1)(r + \kappa^2/2R)\big] = -Q(1 - 1/R).$$

Since we are supposing that the utility is bounded above, we have that $R > 1$, and J is bounded above by K. If $A_0 \neq 0$, then J given by (3.17) would either fail to be bounded above near zero, or would fail to be convex. What we conclude is that *for small enough positive z, the dual value function is of the form*

$$J(z) = B_0 z^\beta + K + \gamma_M^{-1}\, \tilde{U}(z) - Az \tag{3.18}$$

for some constant B_0. This is a powerful conclusion; up to some constant B_0 to be discovered, we know the solution to the HJB equation for positive wealth, and this allows us to state the boundary condition at $0-$ to be satisfied by the HJB solution for negative wealth.

4 Numerical examples.

In this section, we present outputs of some numerical examples. The numerical method used is policy improvement, with a zero boundary condition for h for some suitably large negative value, and with the correct derivative condition at a suitably chosen high value[7].

In addition to the assumption that U is CRRA, we take the form (3.3) for G and set $a = \sigma^2\nu^2/2$, so that the ODE (3.5) determining the solutions $\psi_\pm$ gives us the solutions

$$\psi_-(w) = \begin{cases} \cosh \nu w & (w \leq 0) \\ 1 & (w \geq 0) \end{cases}$$

and

$$\psi_+(w) = \begin{cases} e^{\nu w}/\nu & (w < 0) \\ (1+\nu w)/\nu & (w \geq 0) \end{cases}$$

This gives us $A \equiv \psi_+(0) = \nu^{-1}$. We have likewise that $y(w) \equiv \psi_+(w)/\psi_-(w)$ has the form

$$y(w) = \begin{cases} \frac{2}{\nu(1+e^{-2\nu w})} & (w < 0) \\ (1+\nu w)/\nu & (w \geq 0). \end{cases}$$

We note that $y(0) = A = \psi_+(0) = \nu^{-1}$; the region $w < 0$ corresponds to the region $0 < y < \nu^{-1}$. It is easy to verify that

$$y'(w) = \begin{cases} \operatorname{sech}^2 \nu w & (w < 0) \\ 1 & (w \geq 0). \end{cases}$$

Having noted this, the equation (3.10) becomes

$$\begin{aligned} 0 \;=\; \sup_{c,\theta} \Big[& -(\rho + bw^m) h \cosh(\nu w) + \rho K + U(c) - (\nu h \sinh(\nu w) + h' \operatorname{sech}(\nu w)) c \\ & + (rw + \theta(\mu - r))(\nu h \sinh(\nu w) + h' \operatorname{sech}(\nu w)) + \tfrac{1}{2}\sigma^2\theta^2\, \frac{h''}{\cosh^3 \nu w} \Big]. \end{aligned} \tag{4.1}$$

[7] In more detail, we know from our study of the dual value function J that for small arguments J is close to the dual value function of the standard Merton problem; accordingly, the value function for high wealth levels will be close to the Merton value function. We therefore impose the condition $(1-R)h(y^*) - y^* h'(y^*) = 0$ at the upper boundary point y^*.

The change of variables from w to y is reversed by the inverse transformation

$$w = \frac{1}{2\nu}\, \log\biggl(\frac{\nu y}{2-\nu y}\biggr), \qquad (0 < y \leq \nu^{-1}) \tag{4.2}$$

from which we learn that

$$\cosh \nu w = \frac{1}{\sqrt{\nu y (2-\nu y)}}\,, \qquad \sinh \nu w = \frac{\nu y - 1}{\sqrt{\nu y (2-\nu y)}} \qquad (0 < y \leq \nu^{-1}). \tag{4.3}$$

The HJB equation (4.1) can therefore be expressed in terms of the independent variable y as

$$\begin{aligned} 0 \;\; = \;\; \sup_{c,\theta}\Bigl[& -\frac{(\rho + b w^m) h}{\sqrt{\nu y(2-\nu y)}} + \rho K + U(c) - Qc + \biggl(\frac{r}{2\nu}\, \log\biggl(\frac{\nu y}{2-\nu y}\biggr) + \theta(\mu - r)\biggr) Q + \\ & + \tfrac{1}{2}\sigma^2\theta^2\, \{\nu y(2-\nu y)\}^{3/2} h'' \Bigr] \end{aligned} \tag{4.4}$$

where we abbreviate

$$Q \equiv \nu h \sinh(\nu w) + h' \mathrm{sech}(\nu w) \equiv \frac{\nu(\nu y - 1) h}{\sqrt{\nu y (2-\nu y)}} + \sqrt{\nu y (2-\nu y)}\; h'. \tag{4.5}$$

Optimizing over c and θ is achieved in the usual way, leading to the non-linear second-order ODE

$$0 = -\frac{(\rho + b w^m) h}{\sqrt{\nu y(2-\nu y)}} + \rho K + \tilde{U}(Q) + \frac{rQ}{2\nu}\, \log\biggl(\frac{\nu y}{2-\nu y}\biggr) - \frac{\kappa^2 Q^2}{2h''(\nu y(2-\nu y))^{3/2}}\,. \tag{4.6}$$

Throughout we kept fixed values for $R = 2$, $\rho = 0.02$, $\sigma = 0.35$, $\mu = 0.14$ and $r = 0.05$, and $m = 2$. The values of K, b and ν were varied from one run to the next, taking the base case $K = 60$, $b = 10$, $\nu = 2$, which is illustrated in Figures 1 and 2. We present two plots in each case, one for positive w and one for negative w. The reason to do this is that the aspect ratios are very different for positive w and for negative w in all instances. For positive w, we see consumption and portfolio holdings rise gradually over quite a large range of w values, but for negative w the values of consumption and portfolio holdings grow very rapidly as wealth goes negative. It is hard to appreciate these qualitative features in the overall pictures; when comparing figures, do note the vertical scale. For negative w, we have capped the plotted values of c and θ, otherwise very little of the behaviour of the solution could be discerned. In each of the plots with positive w, we include the consumption rate and the portfolio holdings which would arise from the standard Merton problem. As is to be expected, the graphs we see for the Merton agent always lie below the graphs for the agent whose utility is bounded below.

We can compare the base case where $K = 60$ with two other values of K to try to understand the effects of varying the severity of the penalty. In positive w, the plots Figures

1 ($K = 60$), 7 ($K = 6$) and 5 ($K = 600$) look at first glance to be very similar, until one appreciates that the scale of the axes is roughly proportional to the size of K. When the penalty for being found out is small, the investor chooses a more lavish lifestyle, as would be expected. In negative w, the plots Figures 2 ($K = 60$), 8 ($K = 6$) and 6 ($K = 600$) exhibit very similar behaviour for the investment θ in the risky asset, but the level of consumption is much smaller for larger values of K. Intuitively, the level of investment mainly influences the speed of evolution when it is high, and these plots indicate that the agent will try to get the bad times over one way or another as quickly as he can. However, high levels of consumption cause the wealth process to go negative faster, which increases the likelihood of detection, so when the penalty for detection is higher the agent will avoid large consumption rates to try to delay detection.

Increasing ν (Figures 3, 4) makes little difference to the solution for positive w but for negative w we find that the agent invests less heavily in the risky asset for the larger value of ν, as would be expected, since this would accelerate his detection. Interestingly, levels of consumption turn out to be higher for the larger value of ν.

Increasing b (Figures 9, 10) makes little difference to the solution for positive w but the differences can be seen for negative w in the portfolio holdings. Here, we see that as wealth falls the agent with higher b (and therefore greater rate of detection) ramps up his holdings of the risky asset much less rapidly.

In all the examples here, we see common features: consumption does not decrease to zero as $w \downarrow 0$, though it does appear to decrease, and then begin to rise again; holdings of the risky asset decrease with decreasing positive wealth, and then at some point start to rise again as w continues to fall. Once wealth goes negative, we see that consumption increases as wealth goes more negative. The profile of risky asset holdings is similar.

5 Conclusions.

In this paper we have taken a fairly conventional rational agent, but have modified his behaviour in two respects: instead of proposing that he must stay with strictly positive wealth at all times, we allow him to get into debt; and he is subject to random reviews of his financial status as an incentive not to live on borrowed/stolen money. If he is found to be in negative wealth, he is thrown into prison, incurring a fixed negative payoff.

Even though he is completely rational, calculating his standard von Neuman-Morgenstern objective, with standard expectations, he is found to act very differently from the conventional Merton investor. As his (positive) wealth falls, he reduces consumption, but not down to zero; he will still have a positive consumption rate even at zero wealth. As his (positive) wealth falls, his holdings of the risky asset fall, but then at some point turn around and start to rise again as his wealth continues to fall. The interpretation is that the risky asset has a superior rate of return, and the agent takes a chance that this will counteract the additional riskiness and help to bring his wealth level back up. As wealth goes negative, he gradually gets more desperate. He begins to consume more rapidly, which cannot help his financial position, but he is beginning to realize that liberty is starting to slip away from him, so he may as well enjoy what he can while he still can. Along with this, he starts to substantially raise his risky investments in a gamble that this may carry him back to solvency. In some cases, if ν is too large, he will for low enough negative wealth come out of the risky asset altogether, and hope to evade detection for a little longer by that means.

The kind of complex non-monotone dependence of consumption and portfolio on wealth which appears in this study requires no 'behavioural' modifications of the objectives or the agent's perceptions of probability. This conclusion supports the thesis of Ross (2005) that apparent paradoxes in finance which are used to justify the introduction of behavioural notions are usually resolved by an appropriate analysis using the traditional tools.

References

[1] Benartzi, S., Thaler, R.: Myopic Loss Aversion and the Equity Premium Puzzle. *Quart. J. Econ.* **10**, 73–92 (1995)

[2] Carassus, L., Rasonyi, M.: On optimal investment for a behavioural investor in multiperiod incomplete market models. *Preprint - arXiv:1107.1617* (2012)

[3] Gomes, F., J.: Portfolio choice and trading volume with loss-averse investors. *J. Business* **78**, 675–706 (2005)

[4] Jin, H., Zhou, X. Y.: Behavioural portfolio selection in continuous time. *Math. Finance* **18**, 385–426 (2008)

[5] Kahneman, D., Tversky, A.: Prospect theory: an analysis of decision under risk. *Econometrica* **47**, 263–291 (1979)

[6] Kahneman, D., Tversky, A.: Advances in prospect theory: cumulative representation of uncertainty. *J. Risk Uncertainty* **5**, 297–323 (1992)

[7] Karatzas, I., Lehoczky, J. P., Sethi, S. P., Shreve, S. E.: Explicit solution of a general consumption/investment problem. *Mathematics of Operations Research***11**, 261–294 (1986)

[8] Levy, H., Levy, M.: Prospect theory and mean-variance analysis. *Rev. Financial Stud.* **17**, 1015–1041 (2004)

[9] von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton, NJ. *Princeton University Press* (1944)

[10] Rasonyi, M., Rodrigues, A.: Optimal portfolio choice for a behavioural investor in continuous-time markets. *Preprint - arXiv:1202.0628* (2012)

[11] Reichlin, C.: Non-concave utility maximization with a given pricing measure. *NCCR FINRISK - Working Paper* **517** (2011)

[12] Ross, S. A.: *Neoclassical Finance.* Princeton University Press, Princeton, NJ (2005)

[13] Shefrin, H., Statman, M.: Behavioral portfolio theory. *J. Finance Quant. Anal.* **35**, 127-151 (2000)

Figure 1: Base case, positive w

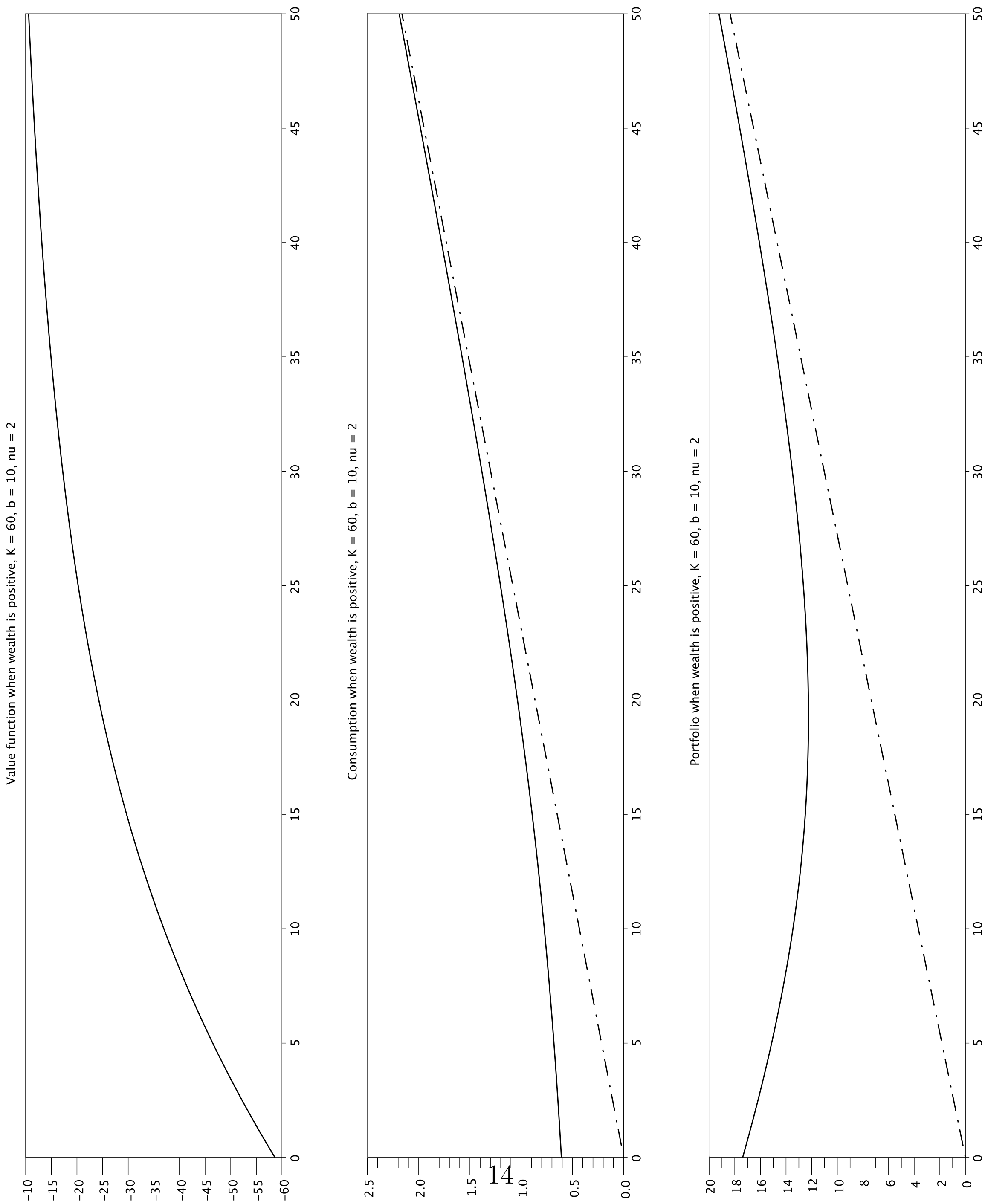


Figure 2: Base case, negative w

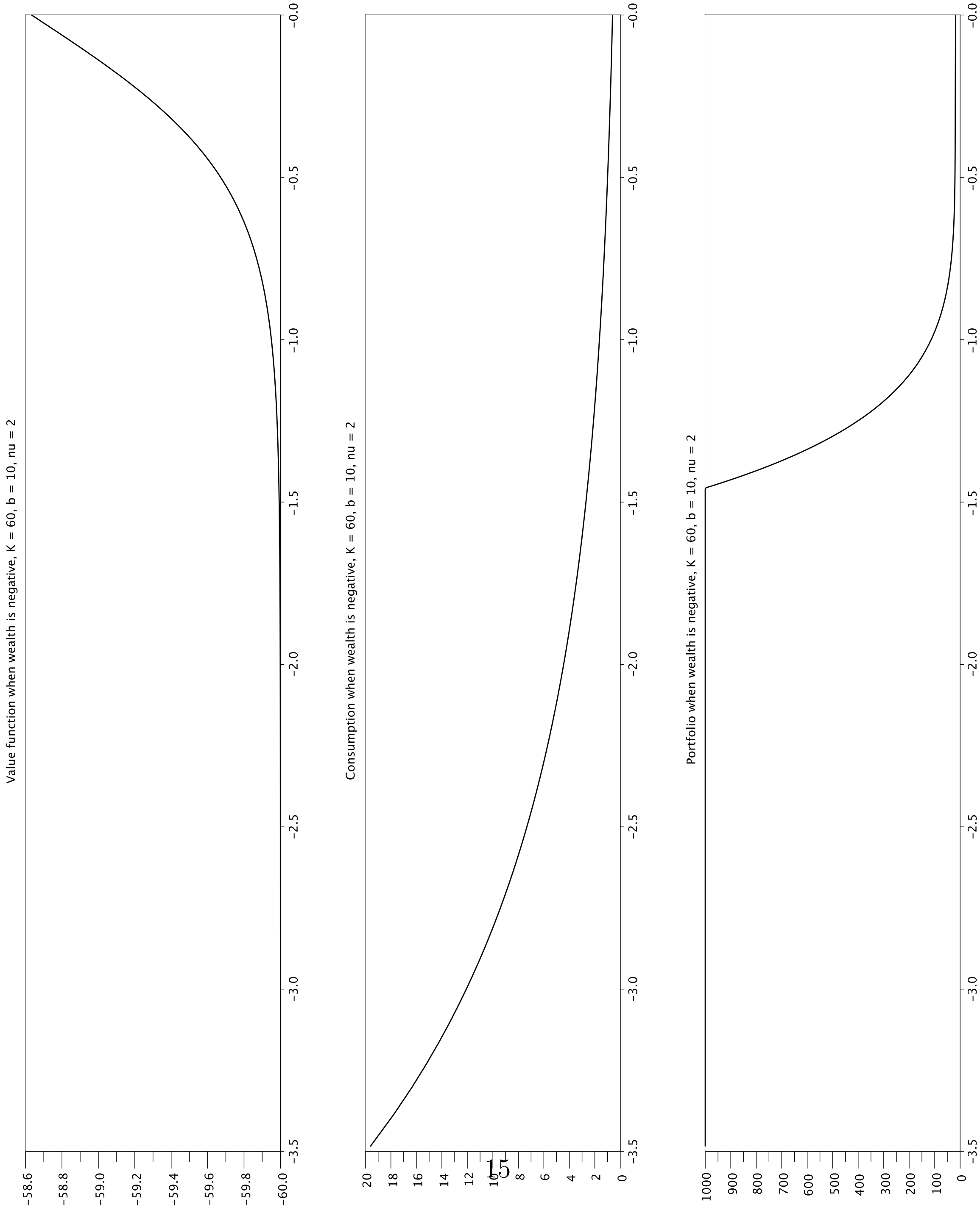
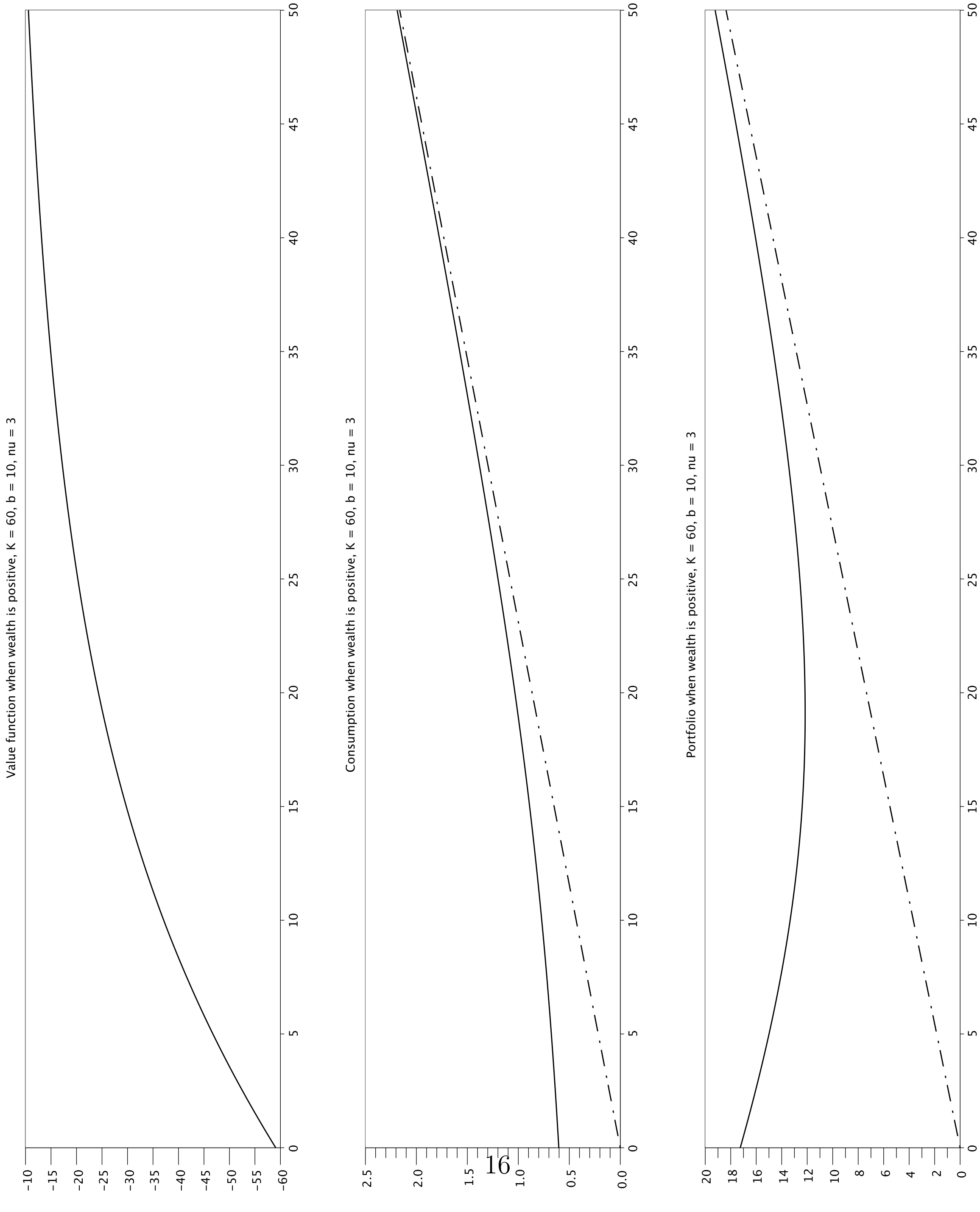


Figure 3: Higher ν, positive w



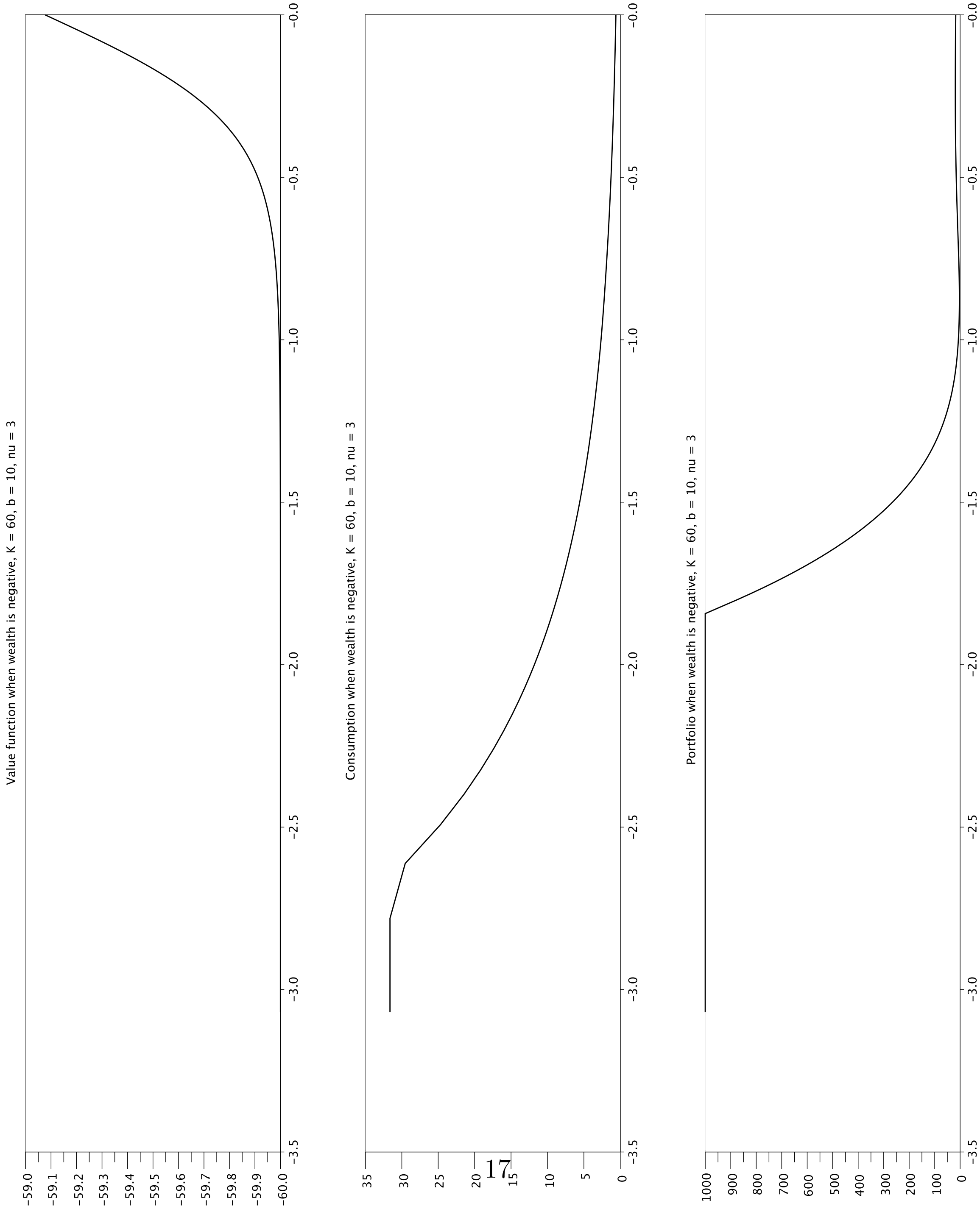


Figure 4: Higher ν, negative w

Figure 5: Higher K, positive w

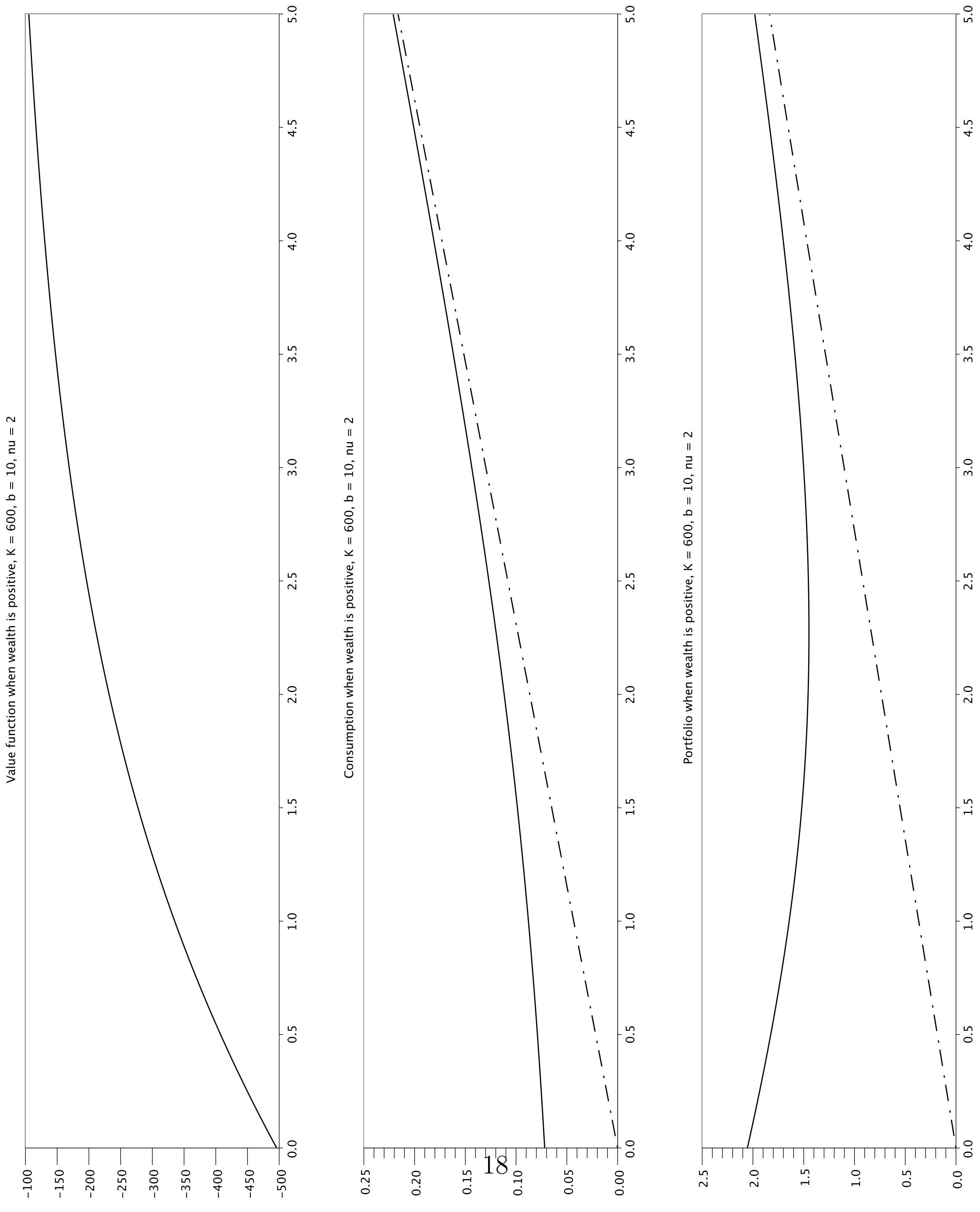


Figure 6: Higher K, negative w

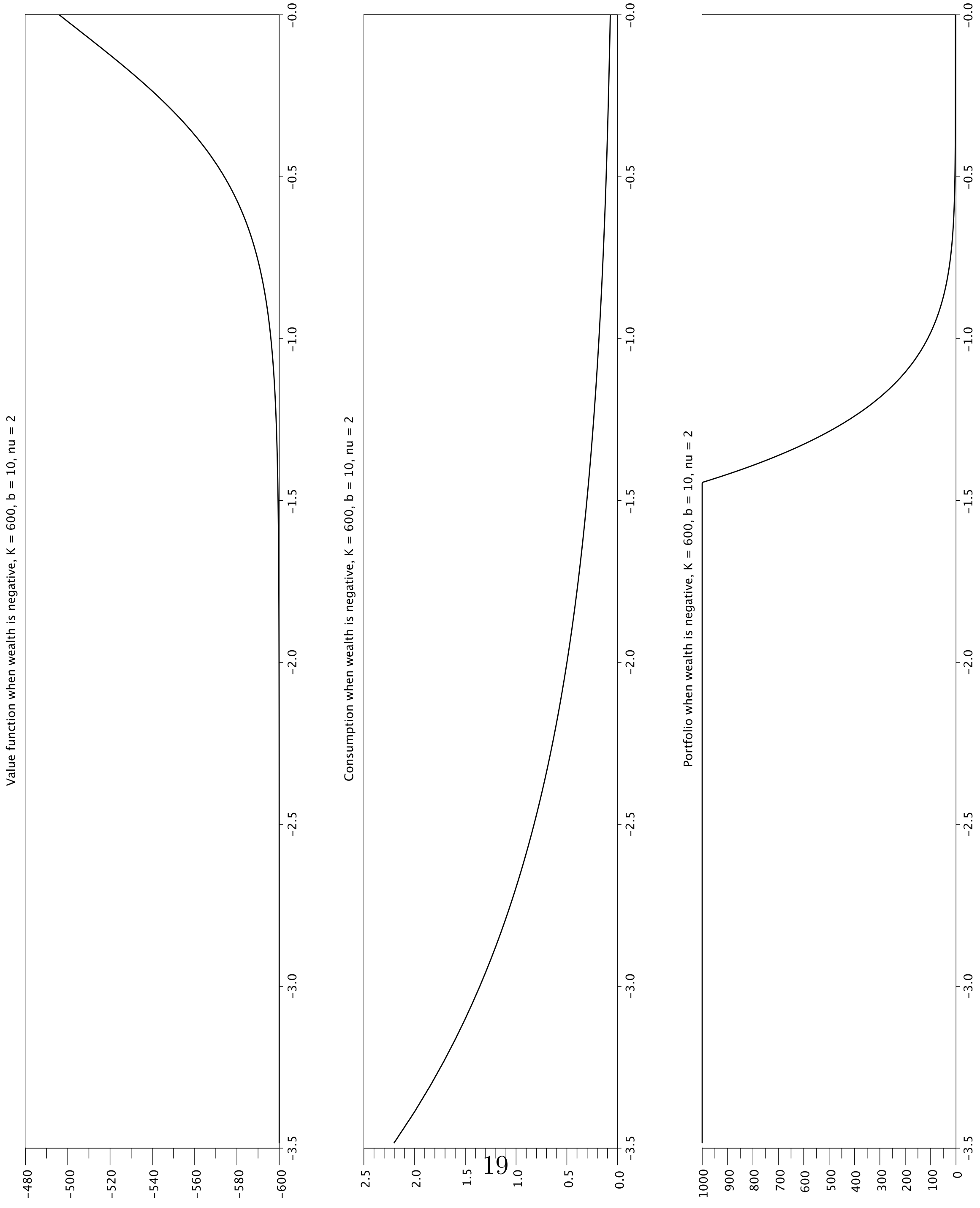


Figure 7: Lower K, positive w

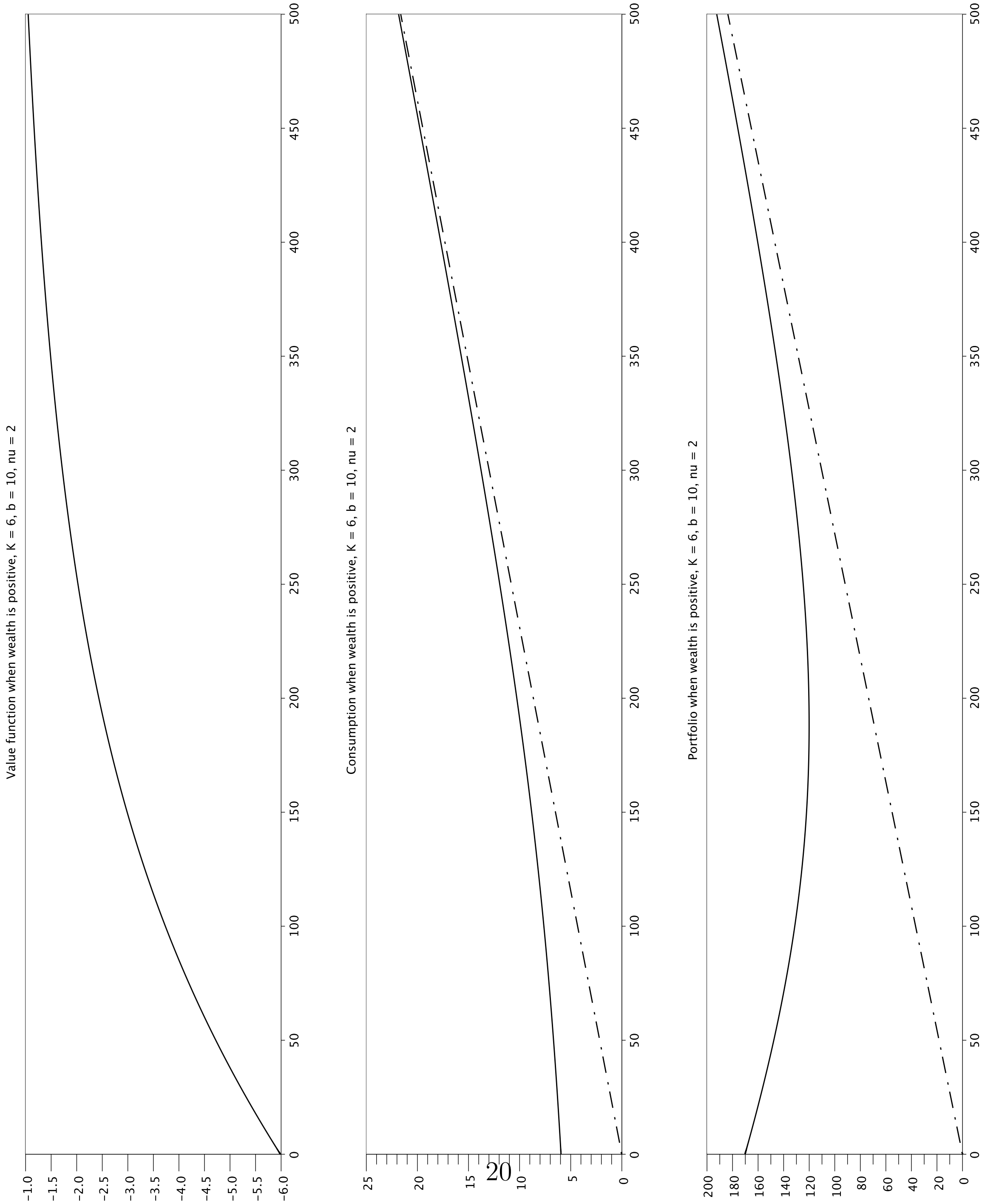


Figure 8: Lower K, negative w

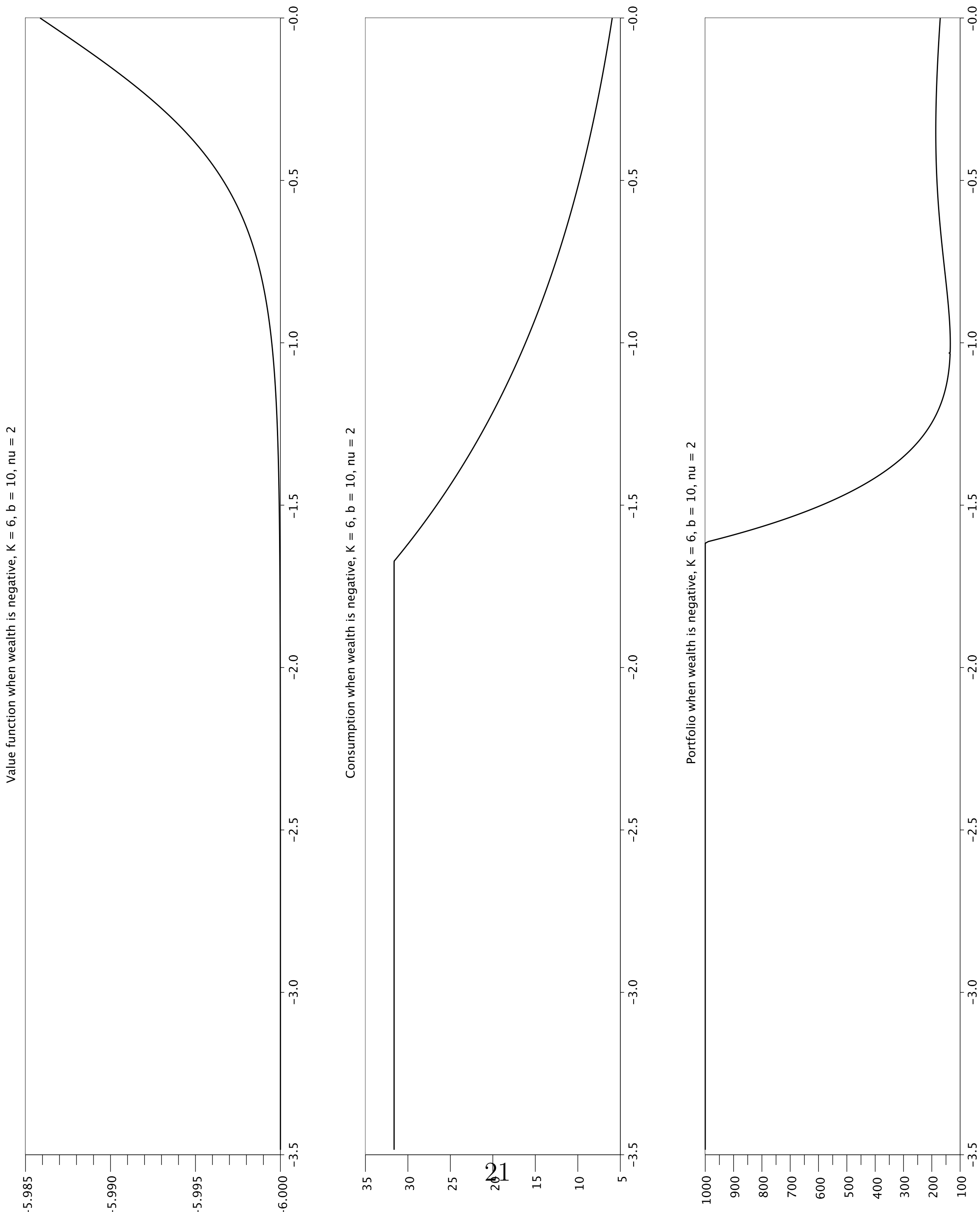


Figure 9: Higher b, positive w

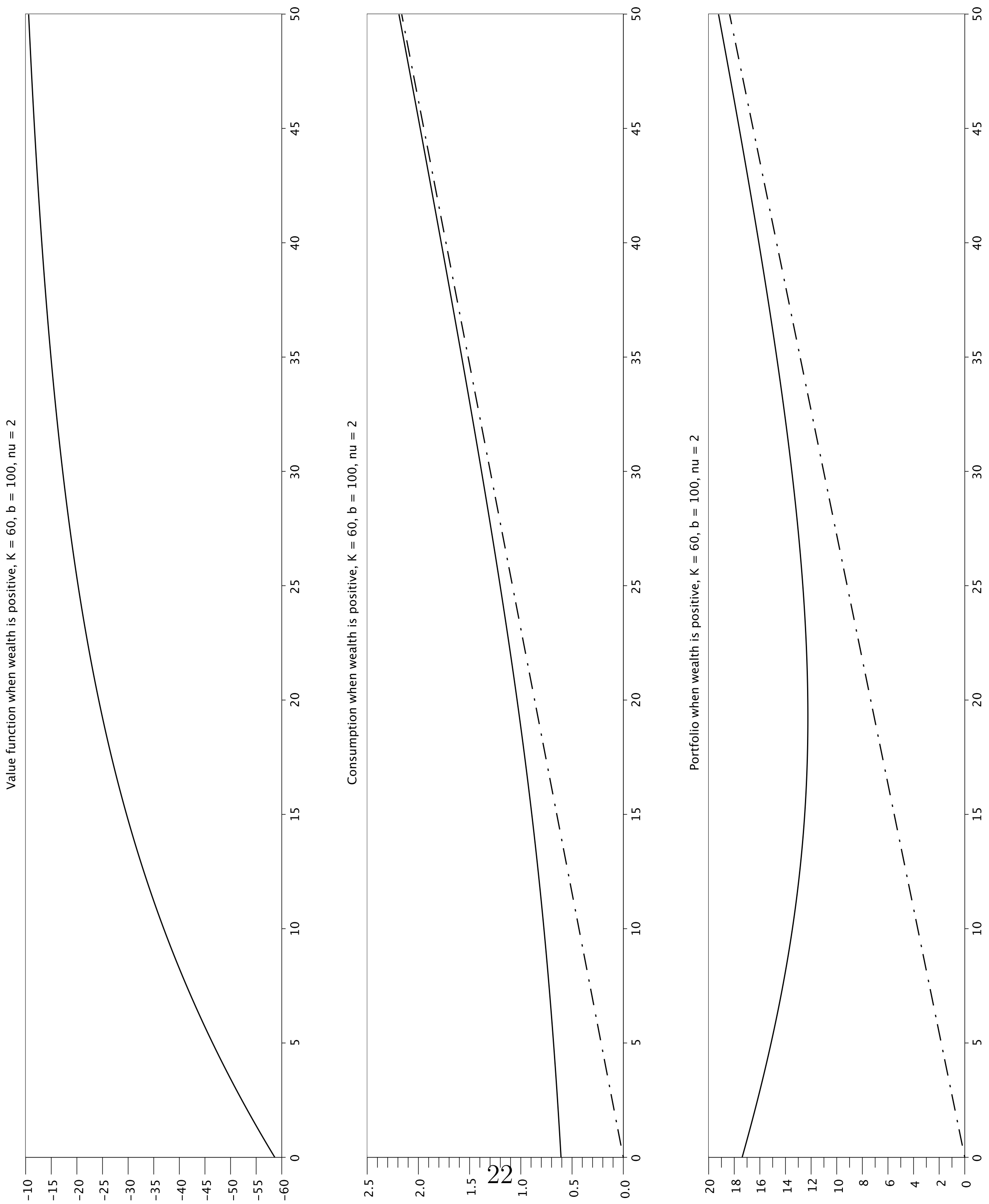


Figure 10: Higher b, negative w