Robust hedging of lookbacks with a few call options (31/10/98)

(i) Suppose we wish to hedge the contingent claim $\varphi(\bar{S}_1)$ robustly, but that all that we know is the price of N call options of strikes $K_1 < K_2 < \dots < K_N$. It turns out that we can specify the cheapest super-replication very easily.

If we know $c_1(x) \equiv E(S_1 - x)^+$ for every x, then the extremal law is obtained by Azéma-Yor embedding, and it can be shown that

$$P(\bar{S}_1 > y) = \inf_{x<y} \frac{c_1(x)}{y-x}.$$

Now we make this as big as possible by making $c_1(\cdot)$ the piecewise-linear interpolation of $C(K_i)$, $i=1,\dots,N$. There is a little indeterminacy in the call prices for strikes above K_N, but by putting an atom at $K^* \gg K_N$ and letting K^* go to infinity, we get the supremum.

(ii) What about the superreplicating strategy? If we look at the Lagrangian form

$$\Lambda \equiv \varphi(y) + \int_0^y (x-a)\,\theta(da) - \sum_{i=1}^N \gamma_i (x-K_i)^+$$

then at each K_i the extremal law is concentrated on $\{(K_i, y) ; \eta_{i-1} \le y \le \eta_i\}$ for appropriate constants η_i, whence

for $\eta_{i-1} < y < \eta_i$, $\quad \dfrac{\theta(dy)}{dy} = \dfrac{\varphi'(y)}{y-K_i} \qquad (i=1,\dots,N)$

If $\sup_y \Lambda = 0$ for each K_i, we must have for each j

$$\sum_{i=1}^N \gamma_i (K_j - K_i)^+ = \varphi(\eta_{j-1}) + \sum_{i=1}^{j-1} \int_{\eta_{i-1}}^{\eta_i} (K_j - a) \frac{\varphi'(a)\,da}{a-K_i}$$

$$= \sum_{i=1}^{j-1} \int_{\eta_{i-1}}^{\eta_i} \frac{K_j - K_i}{a - K_i} \varphi'(a)\,da$$

so that

$$\boxed{\gamma_i = \int_{\eta_{i-1}}^{\eta_i} \frac{\varphi'(a)\,da}{a-K_i}}$$

(iii) Note in particular if $\varphi(x) = x$, as we let $\eta_N \equiv K^* \uparrow \infty$, $\gamma_N \uparrow \infty$, so we have to hold more and more of the call of highest strike... so it's an ill-posed problem! The sup of $E\bar{S}_1$ subject to $E(S_1 - K_j)^+ = c_1(K_j)$ is infinite.

Occasional portfolio review (3/11/98)

(i) Let's consider an agent who can invest in the classical bond of constant rate-of-return r, or in the share, $dS_t = (\sigma dW_t + \alpha dt) S_t$. The agent faces proportional transactions costs $\varepsilon \geq 0$, and is for the moment going to consider a policy of the form "We choose some π_0, and constants $a < 1 < b$, then divide our wealth so that a proportion π_0 is in the risky asset. We don't touch the shares until the first time that the proportion of wealth in the share exits $(a\pi_0, b\pi_0)$, at which time we rebalance so as to return the proportions to π_0 in share." The agent aims to obtain

$$\bar{V}(w) \equiv \max E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \,\middle|\, w_0 = w \right]$$

where U is CRRA, $U'(x) = x^{-R}$, and we'll suppose that the consumption rate process c_t is of the form $c_t = w_t f(\pi_t)$, where π_t is the proportion of wealth in the share at time t.

(ii) Let ν_t be the number of shares held at time t, so that (away from rebalances) the wealth process satisfies

$$dw_t = r w_t\,dt - c_t\,dt + \nu_t (dS_t - r S_t\,dt),$$

and in terms of $\pi_t \equiv \nu_t S_t / w_t$ the equation is

$$\boxed{dw_t = w_t \left[r\,dt - f(\pi_t)dt + \pi_t \{ \sigma dW_t + (\alpha - r)dt \} \right],}$$

so that

$$\boxed{d\pi_t = \pi_t (1-\pi_t) \left[\sigma\, dW_t + (\alpha - r)dt + \frac{f(\pi_t)}{1-\pi_t} dt - \sigma^2 \pi_t\, dt \right].}$$

From this, we find that

$$\boxed{d \log\left(\frac{\pi}{1-\pi}\right) = \sigma dW + \left\{ \alpha - r + \frac{f(\pi_t)}{1-\pi_t} - \tfrac{1}{2}\sigma^2 \right\} dt}$$

Noting $\boxed{f(\pi) \equiv (1-\pi) F(\pi)}$ results in various simplifications, among them

$$\boxed{w_t = w_0 \left(\frac{1-\pi_0}{1-\pi_t} \right) \exp\left\{ rt - \int_0^t F(\pi_s)\,ds \right\}.}$$

(iii) What happens at the rebalancings? If we rebalance from wealth w, proportion $b\pi_0$ to wealth w', proportion π_0, we have

$$w' = \frac{1-\varepsilon\pi_0 b}{1-\varepsilon\pi_0}\, w,$$

and if the rebalancing is from $a\pi_0$ to π_0 we find that the wealth afterwards is

$$w' = \frac{1+a\varepsilon\pi_0}{1+\varepsilon\pi_0}\, w.$$

We shall therefore have that the value V satisfies

$$V(w) = E\Big[\int_0^\tau \frac{e^{-\rho t}}{1-R} \Big(w(1-\pi_0)\, F(\pi_t) \exp\{rt - \int_0^t F(\pi_s)\,ds\}\Big)^{1-R} dt$$

$$+ I_{\{\pi_{\tau-} = b\pi_0\}} V\Big(\frac{1-\varepsilon\pi_0 b}{1-\varepsilon\pi_0} \cdot w \frac{1-\pi_0}{1-b\pi_0} e^{r\tau - \int_0^\tau F(\pi_s)\,ds} \Big)$$

$$+ I_{\{\pi_{\tau-} = a\pi_0\}} V\Big(\frac{1+a\varepsilon\pi_0}{1+\varepsilon\pi_0}\, w \frac{1-\pi_0}{1-a\pi_0} e^{r\tau - \int_0^\tau F(\pi_s)\,ds} \Big)\Big].$$

Clearly from scaling we know that $V(w) = \theta w^{1-R}/(1-R)$, so we deduce that θ must satisfy

$$\theta = E\Big[\int_0^\tau e^{-\rho t} \Big(F(\pi_t)\, e^{rt - \int_0^t F(\pi_s)\,ds}\Big)^{1-R} (1-\pi_0)^{1-R}\, dt \Big]$$

$$+ \theta \Big(\frac{1-\varepsilon\pi_0 b}{1-\varepsilon\pi_0} \cdot \frac{1-\pi_0}{1-b\pi_0} \Big)^{1-R} E\Big[e^{(r\tau - \int_0^\tau F(\pi_s)\,ds)(1-R)} ; \ \pi_{\tau-} = b\pi_0 \Big]$$

$$+ \theta \Big(\frac{1+a\varepsilon\pi_0}{1+\varepsilon\pi_0} \cdot \frac{1-\pi_0}{1-a\pi_0} \Big)^{1-R} E\Big[e^{(r\tau - \int_0^\tau F(\pi_s)\,ds)(1-R)} ; \ \pi_{\tau-} = a\pi_0 \Big].$$

<u>Markov chain approximations to an OU process</u> (20/12/98)

Suppose we want to build a Markov-chain approximation ξ to

$$dX_t = \sigma\, dW_t - \beta X_t\, dt ;$$

how do we go about it? Here are some suggestions.

1) If Z is a standard Brownian motion, the process $\tilde{X}_t \equiv a e^{-\beta t} Z(e^{2\beta t})$ is the same in law as X, where $a = \sigma/\sqrt{2\beta}$. Since we have for each θ that

$$\exp\left[\theta Z_t - \tfrac{1}{2}\theta^2 t\right] \equiv \sum_{n\ge 0} \frac{\theta^n}{n!} H_n(t, Z_t)$$

is a martingale, we shall also have that

$$\exp\left[\theta a Z(e^{2\beta t}) - \tfrac{1}{2}\theta^2 a^2 e^{2\beta t}\right] = \exp\left[\theta e^{\beta t}\tilde{X}_t - \tfrac{1}{2}\theta^2 a^2 e^{2\beta t}\right]$$

$$= \sum_{n\ge 0} \frac{(\theta e^{\beta t})^n}{n!} H_n(a^2, \tilde{X}_t)$$

is a martingale, which implies that for each n,

$$\boxed{e^{n\beta t} H_n\left(\frac{\sigma^2}{2\beta}, X_t\right) \text{ is a martingale}}$$

If we try to build a b+d chain on $I = \{x_{-N}, x_{-N+1}, \dots, x_N\}$ by using the martingale property of $e^{\beta t}X_t$ and $e^{2\beta t}(X_t^2 - \sigma^2/2\beta)$ at $\tau = \inf\{t : \xi_t = x_{j-1} \text{ or } x_{j+1}\}$ we shall have

$$(*) \quad \begin{cases} (\lambda_j + \mu_j - \beta)x_j = \lambda_j x_{j+1} + \mu_j x_{j-1} \\ (\lambda_j + \mu_j - 2\beta)x_j^2 + \sigma^2 = \lambda_j x_{j+1}^2 + \mu_j x_{j-1}^2 \end{cases}$$

These are solved by

$$\lambda_j = \frac{\sigma^2 + \beta b(a-b)}{(c-a)(c-b)}, \qquad \mu_j = \frac{\sigma^2 + \beta b(c-b)}{(c-a)(b-a)} \qquad \begin{pmatrix} a \equiv x_{j-1} \\ b \equiv x_j \\ c \equiv x_{j+1} \end{pmatrix}$$

There are some possible snags with this:

(i) for large $x_j > 0$, it may be that $\lambda_j < 0$, and likewise for $x_j < 0$, could get $\mu_j < 0$

(ii) If we wanted to let $x_N \to \infty$, we'd have $\lambda_{N-1} \to 0$, but $\lambda_{N-1} x_N^2 \to \sigma^2 + \beta x_{N-1}(x_{N-2} - x_{N-1})$ so the equations solved by μ_{N-1} wouldn't just be got by setting $\lambda_{N-1} = 0$ in $(*)$; the martingale $e^{2\beta t}(X_t^2 - \sigma^2/2\beta)$ stopped at the first hit on x_{N-2} isn't UI.

2) We could try making $A_t = \sum_{j=-N}^{N} L(t, x_j)$ as an additive functional, and then time change using it. This will make a Markov chain on $I = \{x_{-N}, \dots, x_N\}$. How do we find the jump intensities? By considering Itô's formula on the functions

$$f(x) \equiv (s(x) - s(b))^+, \quad g(x) \equiv (s(b) - s(x))^+$$

We have $f(X_t) - f(X_0) = M_t + \frac{1}{2} s'(b) L(t,b)$, $g(X_t) - g(X_0) = \tilde{M}_t + \frac{1}{2} s'(b) L(t,b)$ so using OST at $\tau \equiv \inf\{t : X_t = a \text{ or } X_t = c\}$ $\quad (x_{j-1} \equiv a < x_j \equiv b < x_{j+1} \equiv c)$ we get

$$\begin{cases} (s(c) - s(b))\, P(X_\tau = c) = \frac{1}{2} s'(b)\, E L(\tau, b) \\ (s(b) - s(a))\, P(X_\tau = a) = \frac{1}{2} s'(b)\, E L(\tau, b) \end{cases}$$

so

$$P(X_\tau = c) = \frac{s(b) - s(a)}{s(c) - s(a)} = \frac{\lambda_j}{\lambda_j + \mu_j}$$

$$E L(\tau, b) = \frac{2\,(s(b) - s(a))\,(s(c) - s(b))}{s'(b)\,(s(c) - s(a))} = \frac{1}{\lambda_j + \mu_j}$$

from which we conclude that

$$\boxed{\begin{aligned} \lambda_j &= \frac{s'(b)}{2} \cdot \frac{1}{(s(c) - s(b))} \\ \mu_j &= \frac{s'(b)}{2} \cdot \frac{1}{(s(b) - s(a))} \end{aligned}}$$

For the OU process, we have $s'(x) = \exp\left[\beta x^2/\sigma^2\right]$, and from this we may compute λ_j, μ_j.

3) If we didn't insist on nearest-neighbour jumps only, could we do better? Suppose we have $h_n(x) = H_n(\sigma^2/2\beta, x)$ and wanted $e^{n\beta t} h_n(\xi_t)$ to be a martingale, $n = 0, \dots, 2N$, when ξ takes values in $I = \{-x_N, \dots, x_N\}$. Then we'd need

$$Q h_n + n\beta h_n = 0$$

which would mean that if $H_{ij} = h_j(x_i)$, $\Delta = \mathrm{diag}(n\beta, n = 0, \dots, 2N)$ we'd get $QH + H\Delta = 0$, so $Q = -H \Delta H^{-1}$. We can make a Q-matrix for $N = 1$, but not $N = 2$, it appears from some Maple investigations.

<u>Random fields in interest rate models (13/1/99)</u>

(i) Doug Kennedy models the forward rates as a Gaussian process $\{F_{st} : s,t \in \mathbb{R}\}$, where by convention $F_{st} = r_t \equiv F_{tt}$ for $s \geq t$. He shows that the field has the independent increments property

$$F_{s_2 t} - F_{s_1 t} \perp \sigma(\{F_{s u} : s_1 \geq s,\ u \in \mathbb{R}\}) \equiv \mathcal{F}_{s_1}$$

for all $s_2 \geq s_1$, and that the means of F are determined by initial conditions and the covariance of F. Thus by a slight abuse of notation we'll write F for the centred random field $F - E[F]$. If we also suppose the process is <u>stationary</u>, that is, $\forall a \in \mathbb{R}$

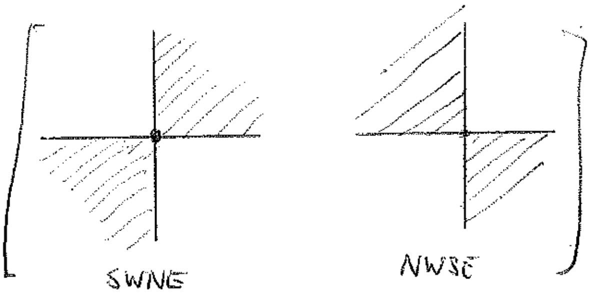
$$\{F_{s+a,t+a} : s,t \in \mathbb{R}\} \overset{\mathcal{D}}{=} \{F_{st} ; s,t \in \mathbb{R}\}$$

Then the covariance is determined by $f(x,y) \equiv E(F_{0x} F_{0y})$, and this contains essentially all the interest.

(ii) Giles Thompson in his thesis considers two 'Markov properties' of the random field. If $\mathcal{F}^{\pm}_{s,t} \equiv \sigma(\{F_{uv} : u \leq s,\ \pm v \geq \pm t\})$, $\mathcal{G}^{\pm}_{s,t} \equiv \sigma(\{F_{u,v} \mid u \geq s,\ \pm v \geq \pm t\})$ then the field is

SWNE Markov if $\mathcal{F}^{-}_{st} \perp \mathcal{G}^{+}_{st} \mid F_{st}$

NWSE Markov if $\mathcal{F}^{+}_{st} \perp \mathcal{G}^{-}_{st} \mid F_{st}$



One of his results is that

(a) <u>The stationary r.f. F is SWNE Markov iff</u>

$$f(x,y) = \exp\{g(x \wedge y) - \mu|x-y|\}$$ <u>for some decreasing f^n</u> g

<u>where</u> $-2\mu \leq \dfrac{g(y)-g(x)}{y-x} \leq 0$

(b) <u>The stationary r.f. F is NWSE Markov iff for some constant μ and decreasing f^n</u> g

$$f(x,y) = \exp\{g(x \vee y) - \mu|x-y|\}$$ <u>where</u> $\dfrac{g(y)-g(x)}{y-x} \leq 2\mu \wedge 0$.

In each case, it's the necessity that's the interesting question. Thompson uses an auxiliary notion of the covariance SWNE-factorising, but in fact this is not necessary; the result can be proved directly.

The key stepping stone is the following Lemma.

<u>Lemma</u> If $\{\xi_t : t \in I\}$ is a Gaussian random field indexed by some subinterval I of $\mathbb{R}$, with continuous covariance, and if ξ is also Markovian, then there exist functions $\varphi, \psi : I \to \mathbb{R}$ such that

$$\mathrm{cov}(\xi_s, \xi_t) = \varphi(s)\psi(t) \qquad (s \le t)$$

<u>Proof</u> The Markov statement is that for $s < t < u$ in I,

$$\mathrm{cov}(\xi_s, \xi_u) \cdot \mathrm{var}(\xi_t) = \mathrm{cov}(\xi_s, \xi_t)\,\mathrm{cov}(\xi_t, \xi_u)$$

Fixing t, we see that $\mathrm{cov}(\xi_s, \xi_u)$ has the required form in $s < t < u$. Now vary t and paste together. $\square$

The proof of (a) and (b) makes use of two observations. Firstly, if either the SWNE or NWSE Markov property holds then for any s the process $\{F_{s,s+u} : u \ge 0\}$ is Markov; and, secondly, the independent increments property plus stationarity ensures that $f(t,t) = \mathrm{var}(F_{0t})$ is decreasing in t.

<u>Proof of (a)</u> In this case, for any $a \ge 0$ the process $(F_{s,s+a})_{s \in \mathbb{R}}$ is a stationary Markov Gaussian process, therefore an OU process, and so

$$\mathrm{cov}\left(F_{s,s+a}, F_{t,t+a}\right) = \theta_a \exp\left(-\gamma_a |s-t|\right)$$

for constants θ_a, γ_a. By the above observation, θ_a is decreasing in a. Independent increments implies that for $s \le t$

$$\mathrm{cov}\left(F_{s,s+a}, F_{t,t+a}\right) = \mathrm{cov}\left(F_{s,s+a}, F_{s,t+a}\right) = f(s+a, t+a)$$

$$= \theta_a \exp\{-\gamma_a(t-s)\}.$$

The Markovian factorisation of f implies that $\gamma_a = \mu$, same for all a, and the inequality $-2\mu \le (g(y)-g(x))/(y-x) \le 0$ follows from Cauchy-Schwarz.

<u>Proof of (b)</u> The NWSE property implies that $\tilde{X}_t \equiv F_{-t,t}$ is Markovian, and we exploit this together with the Markov property of $F_{0,t}$.

We have $\mathrm{cov}(\tilde{X}_s, \tilde{X}_t) = \mathrm{cov}\left(F_{-s,s}, F_{-t,t}\right) = \mathrm{cov}\left(F_{t-s,s+t}, F_{0,2t}\right) = f(s+t, 2t)$

for $0 \le s \le t$, and the Markov property implies that for functions φ, ψ,

$$f(s+t, 2t) = \mathrm{cov}(\tilde{X}_s, \tilde{X}_t) = \varphi(s)\psi(t) \qquad (0 \le s \le t)$$

But also there exist functions φ_* and ψ_* such that for $0 \le x \le y$

$$f(x,y) = \varphi_*(x)\psi_*(y).$$

Putting these together gives us for $x \equiv y$

$$\varphi_*(x)\ \psi_*(y) = \varphi(x - \tfrac{y}{2})\ \psi(y/2).$$

Writing $x - y/2 \equiv \theta$, $y/2 \equiv \eta$ gives $\varphi_*(\theta+\eta) = \varphi(\theta) . \psi(\eta)/\psi_*(2\eta)$, which says that at least locally $\varphi(\theta) = \exp(\alpha\theta)$, $\psi(\eta)/\psi_*(2\eta) = A\exp(\alpha\eta)$, and $\varphi_*(x) = A^{-1}\exp(\alpha x)$. The result follows.

(iii) In the case of the SWNE-Markov interest rates, we can build a process $(X_t)_{t\geq 0}$ having the covariance structure $\mathrm{cov}(X_s, X_t) = f(s,t)$ by simply expressing

$$\boxed{X_t = e^{-\mu t}\left(X_0 + \int_0^t \sigma_s\, dW_s\right)}$$

where X_0 is a zero-mean Gaussian, indept of W, with variance $e^{g(0)}$, and σ is determined by the condition from $\mathrm{var}(X_t) = e^{g(t)}$:

$$\boxed{e^{g(0)} + \int_0^t \sigma_s^2\, ds = \exp\left(2\mu t + g(t)\right).}$$

For NWSE-Markov interest rates, there is a similar expression writing

$$\boxed{X_t = e^{\mu t}\int_t^\infty \sigma_s\, dW_s}$$

where

$$\boxed{\int_t^\infty \sigma_s^2 = e^{g(t) - 2\mu t}}$$

(If $\mu < 0$, we may have to allow $X_t = e^{\mu t}\int_t^\infty \sigma_s\, dW_s + X_\infty$).

(iv) There's a further result concerning finding random field models with $F_{st} = K_{st}\, W_{\varphi(s,t)}$ for continuous $\varphi : (\mathbb{R}^+)^2 \to (\mathbb{R}^+)^2$, K, which I reckon can also be done more neatly.

Stylized models of differential information (25/1/99)

(i) In the paper 'Market liquidity, hedging and crashes', Gennotte + Leland consider a situation where different investors have different information about the value of some underlying variable, and about demand. All variables are Gaussian, and all agents have CARA utilities. An agent with utility $-\exp(-x/a)$ who may buy (for p_0 each) shares in an enterprise which will deliver $Z \sim N(\mu, \sigma^2)$ will choose the number θ of those shares so as to maximise

$$-E\ \exp(-\theta(Z-p_0)/a) = -\exp\left[\frac{1}{2}\frac{\theta^2\sigma^2}{a^2} - \theta(\mu-p_0)/a\right]$$

and therefore takes

$$\boxed{\theta = a(\mu-p_0)/\sigma^2}$$

(ii) We may abstract the situation considered by G+L by supposing that there is some MVN random vector X (without loss of generality, with zero mean, and I covariance) such that the firm will deliver $\mu + \gamma^T X$ per share, μ, γ known and constant. The assumption is that the equilibrium price is a function of $\lambda^T X$ for some λ ($|\lambda|=1$, wlog) which is to be determined. Agent j sees the MVN vector $A_j X$, where $A_j A_j^T$ is the identity, wlog (though of smaller dimension than X), together with the Gaussian variable $\lambda^T X$. Let's write $P_j \equiv A_j^T A_j$ for the orthogonal projection onto the subspace which agent j sees, and write $Q_j \equiv I - P_j$.

Given what agent j knows, $Y_j \equiv \begin{pmatrix}\lambda^T \\ A_j\end{pmatrix} X$, X is MVN with mean $\Gamma_j Y_j$ and variance V_j, where

$$\boxed{\begin{aligned} \Gamma_j Y_j &\equiv \Gamma_j \begin{pmatrix}\lambda^T \\ A_j\end{pmatrix} X = \left(P_j + \frac{Q_j\lambda\,(Q_j\lambda)^T}{|Q_j\lambda|^2}\right) X \\ V_j &= Q_j - \frac{(Q_j\lambda)(Q_j\lambda)^T}{|Q_j\lambda|^2} \end{aligned}}$$

The interpretation is clear: $\Gamma_j \begin{pmatrix}\lambda^T \\ A_j\end{pmatrix}$ is orthogonal projection onto the subspace spanned by the rows of A_j, together with λ; the covariance is the variance of what's left after this projection.

(iii) The aggregate demand of all agents is

$$\sum_j \frac{a_j}{\gamma^T Q_j \gamma - (\gamma^T Q_j \lambda)^2/|Q_j\lambda|^2} \left\{\gamma^T\left(P_j + \frac{(Q_j\lambda)(Q_j\lambda)^T}{|Q_j\lambda|}\right) X + \mu - p_0\right\}$$

which is required to match the total supply

$$\bar{m} + \pi(p_0) + \beta^T X$$

where $\bar{m}$ is constant, β is a known constant and $\pi(\cdot)$ is a known function (hedging demand in the G+L story). By making specific assumptions, G+L are able to take this further: at an abstract level, if $\tilde{P}_j = I - \tilde{Q}_j = P_j + (Q_j\lambda)(Q_j\lambda)^T / |Q_j\lambda|^2$, we have to find λ of unit length such that

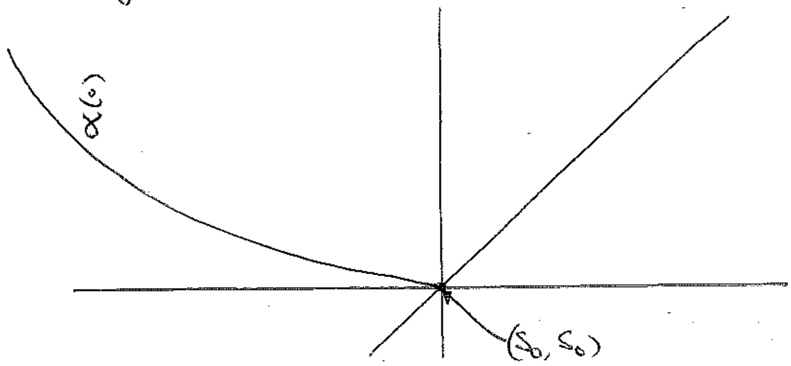
$$\boxed{-\beta + \sum_j \frac{q_j}{\gamma^T \tilde{Q}_j \gamma} \tilde{P}_j \gamma = \text{const.}\, \lambda}$$

The min-max embedding made easy (2/2/99)

(i) Here's a nice way to do the embedding of a martingale with given terminal law and continuous paths which has the stochastically smallest maximum. For simplicity, we'll assume that the law of the martingale $(S_t)_{0 \le t \le 1}$ at time 1 has a positive density f. Since the proof is inspired by ideas from finance, we'll introduce call and put prices

$$C(K) \equiv E(S_1 - K)^+ \quad , \quad P(K) \equiv E(K - S_1)^+ = C(K) + K - S_0 .$$

(ii) The idea of the embedding is to do something like Azéma-Yor:



We seek some function

$$\alpha : [S_0, \infty) \to (-\infty, S_0]$$

which is decreasing, and some non-negative $\theta : [S_0, \infty) \to \mathbb{R}^+$ in order to make the construction.

We'll start a Brownian motion W at S_0, and let $\bar{W}_t = \sup_{u \le t} W_u$

Then we shall stop at the maximum using intensity θ, or we stop when first $W_t < \alpha(\bar{W}_t)$, whichever is sooner.

The problem is that we have <u>two</u> unknown functions θ and α. However, the results of 'Robust hedging of barrier options' Lemma 2.6 tells us that

$$P\left[\sup_{u \le 1} S_u \ge B\right] \ge P\left[S_1 \ge B\right] + \sup_{y < B} \frac{C(B) - P(y)}{B - y}$$

and this gives us the key: <u>pick</u> $\alpha(B)$ <u>to be the</u> y <u>which attains the supremum</u>.

(iii) Differentiation gives us that $\alpha = \alpha(B)$ is the unique solution to

$$\boxed{\frac{P'(\alpha)}{C(B) - P(\alpha)} = \frac{1}{B - \alpha}}$$

Rearranging,

$$(B - \alpha) . F(\alpha) = C(B) - P(\alpha)$$

and differentiating with respect to B gives us

$$(1 - \alpha')F(\alpha) + \alpha' f(\alpha)(B - \alpha) = -\bar{F}(B) - \alpha' F(\alpha)$$

so that

$$\boxed{\alpha' = -\frac{\bar{F}(B) + F(\alpha)}{(B - \alpha) f(\alpha)}}$$

which we need presently.

From the construction used, we shall have

$$P[S_1 < \alpha(B) \text{ or } S_1 > B] = F(\alpha(B)) + \bar{F}(B)$$
$$= \exp\left[-\int_{S_0}^{B} \left\{ \theta_u + \frac{1}{u - \alpha(u)} \right\} du \right]$$

which gives us

$$\theta_B + \frac{1}{B-\alpha} = \frac{f(B) - \alpha'(B) f(\alpha)}{\bar{F}(B) + F(\alpha)}$$
$$= \frac{f(B)}{\bar{F}(B) + F(\alpha)} + \frac{1}{B-\alpha}$$

Therefore

$$\boxed{\theta_B = \frac{f(B)}{\bar{F}(B) + F(\alpha(B))}}$$

which is clearly non-negative. Verifying that we have embedded the desired law is immediate. Have we attained the bound?

$$P\left[\sup_{u \le 1} S_u > B\right] = \exp\left[-\int_{S_0}^{B} \left\{ \theta_u + \frac{1}{u - \alpha(u)} \right\} du \right]$$
$$= \bar{F}(B) + F(\alpha(B))$$
$$= \bar{F}(B) + \frac{C(B) - P(\alpha(B))}{B - \alpha(B)}$$
$$= \bar{F}(B) + \sup_{y<B} \frac{C(B) - P(y)}{B - y}$$

So we have indeed got down to the lower bound. That's it!

More on the big investor story (9/2/99)

(i) Recall the setup: if the big investor holds θ shares, the market price of a share is assumed to be $f(\theta, \xi)$, where ξ is the level of some economic fundamental. Writing

$$F(\theta,\xi) \equiv \int_0^\theta f(z,\xi)\,dz$$

then the liquidation value at time t, v_t, of the agent's portfolio will obey

$$\boxed{dv_t = r\,(v_t - F(\theta_t,\xi_t))\,dt + F_\xi(\theta_t,\xi_t)\,d\xi_t + \tfrac{1}{2}F_{\xi\xi}(\theta_t,\xi_t)\,d\langle\xi\rangle_t - c_t\,dt}$$

For simplicity, let's assume

$$\boxed{d\xi_t = dW_t + \alpha\,dt}$$

so

$$dv_t = F_\xi\,dW_t - c_t\,dt + r v_t\,dt + \left[\tfrac{1}{2}F_{\xi\xi} + \alpha F_\xi - rF\right]dt$$

$$\equiv F_\xi\left[dW_t + h_t\,dt\right] - c_t\,dt + r v_t\,dt$$

where we have

$$\boxed{h_t \equiv \frac{\tfrac{1}{2}F_{\xi\xi} + \alpha F_\xi - rF}{F_\xi}(\theta_t,\xi_t).}$$

(ii) Let Z be the change-of-measure mg $Z_t = \left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t}$, where $dW_t^* \equiv dW_t + h_t\,dt$ is a P^*-Brownian motion. If $\tilde{v}_t = e^{-rt}v_t$ is the discounted wealth process, it follows that

$$\boxed{d\tilde{v}_t = e^{-rt}F_\xi\,dW_t^* - e^{-rt}c_t\,dt}$$

so if the agent aims to max $E\int_0^\infty e^{-\rho t}U(c_t)\,dt$, by the usual story we get for optimal c^*

$$U'(c_t^*) = \lambda e^{\rho t}\zeta_t \equiv \lambda e^{\rho t}e^{-rt}Z_t$$

for some constant $\lambda > 0$ to be determined by budget constraint.

(iii) Simplifying assumption: $U(x) = \log x$

This gives us $c_t^* = 1/\lambda e^{\rho t}\zeta_t$ and the budget constraint

$$v_0 = E\int_0^\infty \zeta_t c_t^*\,dt = \frac{1}{\lambda\rho} \quad\Rightarrow\quad \boxed{\lambda = \frac{1}{\rho v_0},\quad c_t^* = \rho v_0 e^{-\rho t}/\zeta_t}$$

Now

$$E_t^*\left[\int_0^\infty e^{-rs}c_s^*\,ds\right] = \int_0^t e^{-rs}c_s^*\,ds + \tilde{v}_t = \int_0^t e^{-rs}c_s^*\,ds + \frac{1}{Z_t}\int_t^\infty e^{-ru}E_t(Z_u c_u^*)\,du$$

$$= \int_0^t e^{-rs} c_s^* \, ds + \frac{1}{Z_t} v_0 e^{-\rho t}$$

Thus we have the equality of the two martingales (P^*)

$$\int_0^t e^{-rs} c_s^* \, ds + \tilde{v}_t = \int_0^t e^{-rs} c_s^* + \frac{v_0 e^{-\rho t}}{Z_t}$$

so matching the two Ito expansions

$$d\tilde{v}_t \doteq e^{-rt} F_\xi \, dW^*$$

$$= v_0 e^{-\rho t} \frac{1}{Z_t} h_t \, dW_t^*$$

$$\therefore \quad \boxed{\frac{1}{Z_t} = \frac{e^{(\rho - r)t} F_\xi(\theta_t, \xi_t)}{v_0 \, h_t}}$$

Now this exhibits $Z_t^{-1} = e^{(\rho-r)t} \psi(\theta_t, \xi_t)$ for a function ψ which is known <u>explicitly</u> in terms of the given F. Hence if we suppose that $d\theta_t = \sigma_t \, dW_t^* + \mu_t \, dt$, we can explore the implications for σ, μ. We have

$$d(Z_t^{-1}) = Z_t^{-1} h_t \, dW_t^*$$

$$= Z_t^{-1} (\rho - r) \, dt + e^{(\rho-r)t} \left\{ \psi_\theta \, d\theta + \psi_\xi \, d\xi + \left(\tfrac{1}{2} \sigma_t^2 \psi_{\theta\theta} + \sigma_t \psi_{\theta\xi} + \tfrac{1}{2} \psi_{\xi\xi}\right) dt \right\}$$

$$= Z_t^{-1} \Big[(\rho - r) \, dt + \frac{1}{\psi} \Big\{ \psi_\theta (\sigma dW^* + \mu dt) + \psi_\xi (dW^* + \alpha dt - h dt)$$
$$+ \left(\tfrac{1}{2} \sigma^2 \psi_{\theta\theta} + \sigma \psi_{\theta\xi} + \tfrac{1}{2} \psi_{\xi\xi}\right) dt \Big\} \Big]$$

Hence we have

$$\begin{cases} h = \dfrac{\sigma \psi_\theta + \psi_\xi}{\psi} \\ 0 = (\rho - r)\psi + \mu \psi_\theta + (\alpha - h)\psi_\xi + \tfrac{1}{2} \sigma^2 \psi_{\theta\theta} + \sigma \psi_{\theta\xi} + \tfrac{1}{2} \psi_{\xi\xi} \end{cases}$$

so that

$$\boxed{\begin{aligned} \sigma &= (\psi h - \psi_\xi) / \psi_\theta \\ \mu &= -\psi_\theta^{-1} \left\{ (\rho - r)\psi + (\alpha - h)\psi_\xi + \tfrac{1}{2}\sigma^2 \psi_{\theta\theta} + \sigma \psi_{\theta\xi} + \tfrac{1}{2} \psi_{\xi\xi} \right\} \end{aligned}}$$

(iv) (1/3/99) We may proceed without the simplifying assumption $U(x) = \log x$ and see what comes up. It seems a reasonable assumption that optimal θ and c will depend only on ξ_t, v_t; Assuming the dependence of θ on v is 1-1, we may write c and v as functions of θ and ξ:

$$c_t = c^*(\theta_t, \xi_t) \quad , \quad v_t = v^*(\theta_t, \xi_t)$$

and develop this. If we have $d\theta_t = \sigma_t\, dW_t + \mu_t\, dt$ under optimal behaviour, we've got two sources of info, the first being the equation for dv, the second the equation for dZ. So we have from the first

$$\begin{aligned} dv^* &= F_\xi \{dW + h\,dt\} - c^*\,dt + r v^*\,dt \\ &= (v^*_\theta \sigma_t + v^*_\xi)\,dW + \{\mu_t v^*_\theta + \alpha v^*_\xi + \tfrac{1}{2}\sigma_t^2 v^*_{\theta\theta} + \sigma_t v^*_{\theta\xi} + \tfrac{1}{2} v^*_{\xi\xi}\}\,dt \end{aligned}$$

and from the second

$$\begin{aligned} dZ_t &= -h Z_t\, dW_t \\ &= d\ \lambda^{-1} e^{(r-\rho)t}\, U'(c^*(\theta_t, \xi_t)) \qquad\qquad U'\circ c^* \equiv \tilde{g}, \text{ say} \\ &= (r-\rho) Z_t\, dt + \frac{Z_t}{\tilde{g}(\theta_t,\xi_t)} \Big[\, \tilde{g}_\theta \sigma_t\, dW_t + \tilde{g}_\xi\, dW_t \\ &\qquad + \{\mu_t \tilde{g}_\theta + \alpha \tilde{g}_\xi + \tfrac{1}{2}\sigma_t^2 \tilde{g}_{\theta\theta} + \sigma_t \tilde{g}_{\theta\xi} + \tfrac{1}{2}\tilde{g}_{\xi\xi}\}\,dt \Big] \end{aligned}$$

Collecting this,

$$\boxed{\begin{aligned} &(1) \quad F_\xi = v^*_\theta \cdot \sigma_t + v^*_\xi \\ &(2) \quad r v^* - c^* + h F_\xi = \mu_t v^*_\theta + \alpha v^*_\xi + \tfrac{1}{2}\sigma_t^2 v^*_{\theta\theta} + \sigma_t v^*_{\theta\xi} + \tfrac{1}{2} v^*_{\xi\xi} \\ &(3) \quad -h = (\tilde{g}_\theta \sigma_t + \tilde{g}_\xi)/\tilde{g} \\ &(4) \quad 0 = (r-\rho)\tilde{g} + \mu_t \tilde{g}_\theta + \alpha \tilde{g}_\xi + \tfrac{1}{2}\sigma_t^2 \tilde{g}_{\theta\theta} + \sigma_t \tilde{g}_{\theta\xi} + \tfrac{1}{2}\tilde{g}_{\xi\xi} \end{aligned}}$$

We expect that $\sigma_t = \sigma(\theta_t, \xi_t)$, $\mu_t = \mu(\theta_t, \xi_t)$, so we can reexpress the quadratic bits appearing here as

$$\begin{cases} \sigma F_{\theta\xi} + F_{\xi\xi} = \sigma^2 v_{\theta\theta} + 2\sigma v_{\xi\theta} + v_{\xi\xi} + \sigma \sigma_\theta v_\theta + \sigma_\xi v_\theta \\ h^2 \tilde{g} - \tilde{g}(\sigma h_\theta + h_\xi) = \sigma^2 \tilde{g}_{\theta\theta} + 2\sigma \tilde{g}_{\theta\xi} + \tilde{g}_{\xi\xi} + \tilde{g}_\theta(\sigma\sigma_\theta + \sigma_\xi) \end{cases}$$

Multi-agent equilibria (8/3/99)

(i) Suppose we have N agents with initial wealths $w_j(0)$, $j = 1, \dots, N$, each trying to maximise $E\int_0^\infty U_j(t, c_j(t))\,dt$ subject to budget constraint. Suppose that there are a shares in total, each share producing a flow $\delta_t \equiv \delta(X_t)$ of good, where (X_t) is some Markov process.

The usual story tells us that

$$c_j(t) = I_j(t, \lambda_j \zeta_t), \qquad j = 1, \dots, N, \qquad \zeta_0 = 1,$$

$$a\,\delta(X_t) = \sum_{j=1}^N c_j(t) = \sum_{j=1}^N I_j(t, \lambda_j \zeta_t),$$

$$w_j(0) = E\left[\int_0^\infty \zeta_t\, c_j(t)\,dt\right] = E\left[\int_0^\infty \zeta_t\, I_j(t, \lambda_j \zeta_t)\,dt\right].$$

In order to obtain time-homogeneous solutions, let's take $U_j(t,x) = e^{-\beta_j t} U_j(x)$. Assuming uniqueness of solutions is justified, we find a solution

$$\begin{cases} \lambda = \lambda^*(\underline{w}(0), X_0; a), \\ \zeta_t = \varphi(a\delta(X_t), \lambda_j^*(w(0), X_0; a)e^{\beta_j t}, j = 1, \dots, N) \end{cases}$$

If we write $\zeta_t \equiv \zeta_{0t}$ instead, to emphasize the initial condition $\zeta_0 = 1$ in use, then we must have

$$\boxed{\zeta_{tT} = \zeta_{0T} / \zeta_{0t}}$$

so this implies

$$\zeta_{t,T} = \varphi(a\delta(X_T), \lambda_j^*(w(t), X_t; a)e^{\beta_j(T-t)})$$

$$= \varphi(a\delta(X_T), \lambda_j^*(w(0), X_0; a)e^{\beta_j T}) / \varphi(a\delta(X_t), \lambda_j^*(w(0), X_0; a)e^{\beta_j t})$$

By considering the consumption process c_j at time t in two different ways, we also have

$$\lambda_j^*(w(t), X(t); a) = e^{\beta_j t} \lambda_j^*(w(0), X(0); a) \cdot \zeta_{0t}.$$

Can we extract more from these relations?

(ii) Exponential utility. Let's have $U_j(t,x) = -\gamma_j^{-1} \exp(-\rho_j t - \gamma_j x)$, so that

$$c_j(t) = \gamma_j^{-1}\{-\rho_j t - \log \lambda_j - \log \zeta_t\}$$

and so the market-clearing condition gives

$$-a\delta(X_t) = t\sum_{j=1}^N \rho_j/\gamma_j + \sum_{j=1}^N \gamma_j^{-1} \log \lambda_j + (\log \zeta_t) \cdot \sum_{j=1}^N \gamma_j^{-1}$$

Note: if we sum over j,

$$\sum_j w_j(0) = E \int_0^\infty \zeta_t \, a \, \delta(X_t) \, dt ,$$

So if we set $\Gamma^{-1} \equiv \sum_1^N \gamma_j^{-1}$, we shall have

$$\zeta_t = \exp\Big\{ -\Gamma\{a\delta(X_t) + t\sum \rho_j\gamma_j^{-1} + \sum \gamma_j^{-1}\log\lambda_j\}\Big\}$$

$$\equiv \exp\Big[-\Gamma\{a\delta(X_t) + \kappa t + \mu\}\Big]$$

for short.

The budget constraints are

$$w_j(0) = E\int_0^\infty e^{-\Gamma(a\delta(X_t)+\kappa t+\mu)}\,\gamma_j^{-1}\{-\rho_j t - \log\lambda_j + \Gamma(a\delta(X_t)+\kappa t+\mu)\}\,dt$$

$$= E\int_0^\infty e^{-\Gamma(a\delta(X_t)+\kappa t+\mu)}\Big[(\Gamma\kappa-\rho_j)t + a\Gamma\delta(X_t) + (\mu\Gamma - \log\lambda_j)\Big]\,dt/\gamma_j$$

and so we need to compute $E\Big[\int_0^\infty \exp(-\alpha\delta(X_t)-\beta t)\,dt \,\Big|\, X_0 = x\Big] \equiv g_{\alpha,\beta}(x)$, say. This solves

$$\boxed{(\beta - \mathcal{G})\,g_{\alpha\beta}(x) = e^{-\alpha\delta(x)}}$$

If we have $\mathcal{G} \equiv \frac{1}{2}\sigma(x)^2 D^2 + \mu(x)D$, and we write $g(x) = e^{-\alpha x}\psi(x)$, with $\delta(x) = x$, then the de for ψ is

$$\boxed{\psi\{\beta - \tfrac{1}{2}\alpha^2\sigma^2 + \alpha\mu\} + \psi'(\alpha\sigma^2 - \mu) - \tfrac{1}{2}\sigma^2\psi'' = 1}$$

(iii) Suppose we look initially at the special case where $\sigma(x) = \sigma x$, $\mu(x) = \mu x$: then we are trying to solve

$$\beta g(x) - \mu x g'(x) - \frac{x^2}{2}\sigma^2 g''(x) = e^{-\alpha x}.$$

The homogeneous equation has a general solution of the form

$$\varphi(x) = a_1 x^{\theta_1} + a_2 x^{\theta_2}$$

where θ_i are the roots of $\frac{1}{2}\sigma^2\theta(\theta-1) + \mu\theta - \beta = 0$. Write the roots as

$$\theta_1, \theta_2 = \tfrac{1}{2} - \mu\sigma^{-2} \pm \lambda\sigma^{-2}, \quad \lambda \equiv \big((\mu - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2\beta\big)^{1/2}.$$

For this problem, Maple comes up with a particular integral (much to its credit!):

$$\boxed{g(x) = \frac{1}{\lambda}\Big[x^{\theta_2}\int_x^\infty t^{-1-\theta_2}e^{-\alpha t}\,dt - x^{\theta_1}\int_x^\infty t^{-1-\theta_1}e^{-\alpha t}\,dt\Big] + a_1 x^{\theta_1} + a_2 x^{\theta_2}}$$

$\theta_2 = \frac{1}{2} - \mu\sigma^{-2} + \lambda\sigma^{-2}$

$\theta_1 = \frac{1}{2} - \mu\sigma^{-2} - \lambda\sigma^{-2}$

We know that $\theta_2 > 0 > \theta_1$, and the solution must be bounded between 0 and 1, $g(0) = 1/\beta$. Boundedness at $\infty \Rightarrow a_2 = 0$. Unravelling the boundary condition at 0 gives us the explicit solution

$$\boxed{g(x) = \frac{1}{\lambda}\left[x^{\theta_2} \int_x^\infty t^{-1-\theta_2} e^{-\alpha t}\, dt + x^{\theta_1} \int_0^x t^{-1-\theta_1} e^{-\alpha t}\, dt \right]}$$

This is just the resolvent, of course... Writing the solution as $g(x, \alpha, \beta)$, we get from the budget equation for agent j that

$$\boxed{\log \lambda_j = \mu\Gamma - (\Gamma\kappa - \rho_j)\frac{g_3}{g} - a\Gamma \frac{g_2}{g} - \gamma_j e^{\mu\Gamma} w_j(0)/g}$$

and

$$\boxed{\mu = -aX_0}$$ from the condition $\zeta_0 = 1$,

where the g's are evaluated at $(x, a\Gamma, \kappa\Gamma) \equiv (X_0, a\Gamma, \kappa\Gamma)$. The way it works out, therefore, is that

$$\boxed{g_j(t) = \frac{\Gamma\kappa - \rho_j}{\gamma_j} t + \frac{a\Gamma}{\gamma_j}\delta(X_t) + \frac{\mu - \log\lambda_j}{\gamma_j}}$$

which holds in generality (don't need $\delta(X_t) = X_t$, nor the form of the diffusion for X); it exhibits the consumption as a term linear in t, a contribution from the dividend stream, plus a constant, which is the sole influence of the initial distribution of wealth

<u>BUT</u> for some j, $\Gamma\kappa - \rho_j < 0$, so eventually that agent is consuming at arbitrarily large <u>negative</u> rate, which is absurd.

(iv) Let's go back, and take $U_j(t,x) = e^{-\rho_j t} x^{1-R}/(1-R)$ (same coefficient of relative risk aversion for all agents). Then in the usual way

$$g_j(t) = \left(\lambda_j e^{\rho_j t} \zeta_t\right)^{-1/R},$$

so market clearing forces

$$a\delta(X_t) = \zeta_t^{-1/R} \sum_j \left(\lambda_j e^{\rho_j t}\right)^{-1/R} \Rightarrow \boxed{\zeta_t = (a\delta(X_t))^{-R}\left(\sum \lambda_i^{-1/R} e^{-\rho_i t/R}\right)^R}$$

and from $\zeta_0 = 1$ we get

$$a\delta(X_0) = \sum_j \lambda_j^{-1/R}.$$

The budget constraint is

$$w_j(0) = a^{1-R} X_0^{1-R} \int_0^\infty e^{(1-R)(b - \frac{1}{2}R\sigma^2)t} \left\{\sum_i \lambda_i^{-1/R} e^{-\rho_i t/R}\right\}^{R-1} \lambda_j^{-1/R} e^{-\rho_j t/R}\, dt$$

if $dX = X(\sigma dW + b\,dt)$ as before. Nothing closed-form available here without further assumptions, for example, all agents identical, <u>or that</u> R <u>is an integer</u> ≥ 1.

(v) If we pursue the possibility that R is an integer, we have to have

$$\sum_j w_j(0) = E\int_0^\infty \zeta_t \; a\delta_t \; dt$$

$$= E\int_0^\infty (a\delta_t)^{1-R} \Big\{ \sum_i \lambda_i^{-1/R} e^{-\rho_i t/R} \Big\}^R dt$$

$$= a^{1-R} X_0^{1-R} \int_0^\infty e^{(1-R)(b-R\sigma^2/2)t} \Big\{ \sum_i \lambda_i^{-1/R} e^{-\rho_i t/R} \Big\}^R dt$$

Case A: $R=1$. Then

$$w_j(0) = 1/\rho_j\lambda_j \quad , \quad \sum \lambda_j^{-1} = aX_0 \equiv a\delta_0$$

since $\zeta_0 = 1$. Thus the consumption stream at time zero gets divided up as

$$g_j(0) \equiv \gamma_j = \lambda_j^{-1}$$

and at any later time

$$\begin{cases} g_j(t) = \gamma_j e^{-\rho_j t} \zeta_t^{-1} = a\delta_t \dfrac{\gamma_j e^{-\rho_j t}}{\sum_i \gamma_i e^{-\rho_i t}} , \\ \\ w_j(t) = \rho_j^{-1} g_j(t). \end{cases}$$

This gives us

$$\zeta_t = \frac{\sum \gamma_i e^{-\rho_i t}}{\sum \gamma_i} e^{-\sigma W_t - (b-\frac{1}{2}\sigma^2)t} = \sum \gamma_i e^{-\rho_i t} / aX_t$$

so that the discount factor and change-of-measure martingale are

$$\beta_t \equiv e^{-R_t} = \frac{\sum \gamma_i e^{-\rho_i t}}{\sum \gamma_i} e^{-bt+\sigma^2 t} , \qquad Z_t = e^{-\sigma W_t - \frac{1}{2}\sigma^2 t} ,$$

and

$$r_t = b - \sigma^2 + \frac{\sum \rho_i \gamma_i e^{-\rho_i t}}{\sum \gamma_i e^{-\rho_i t}} ,$$

$$S_t = X_t \cdot \frac{\sum \rho_j^{-1} \gamma_j e^{-\rho_j t}}{\sum \gamma_j e^{-\rho_j t}} = \sum \rho_j^{-1} \gamma_j e^{-\rho_j t} / a\zeta_t .$$

From this we deduce that

$$\boxed{\theta_j(t) = \rho_j^{-1} \gamma_j e^{-\rho_j t} \zeta_t^{-1} / S_t = a\, w_j(t) / \sum w_i(t)}$$

which tells us in particular in this case that each agent holds all his wealth in the productive asset.

Case B: R=2. The budget equation for agent j now reduces to

$$w_j(0) = \frac{\lambda_j^{-1/2}}{aX_0} \sum_i \frac{2\,\lambda_i^{-1/2}}{\rho_i + \rho_j + 2(b - \sigma^2 R)}$$

$$= \frac{\gamma_j}{aX_0} \sum_i \frac{2\gamma_i}{\rho_i + \rho_j + 2(b - \sigma^2 R)}$$

where $\gamma_j \equiv \lambda_j^{-1/2}$ is the consumption rate at time 0 of agent j. We have further

$$\zeta_t = \left(\frac{\sum_j \gamma_j e^{-\rho_j t/2}}{aX_t} \right)^2$$

$$c_j(t) = \gamma_j e^{-\rho_j t/2} \zeta_t^{-1/2}$$

$$w_j(t) = \frac{aX_t}{\left(\sum_i \gamma_i e^{-\rho_i t/2}\right)^2} \cdot \sum_i \frac{2\gamma_i \gamma_j e^{-(\rho_i + \rho_j)t/2}}{\rho_i + \rho_j + 2(b - \sigma^2)}$$

$$aS_t = \sum_i w_i(t)$$

and exactly as in case A we deduce that

$$\boxed{\theta_j(t) = w_j(t)/S_t = aw_j(t) / \sum w_i(t)}$$

and once again all agents keep all their wealth in the risky asset.

Case C: $R > 0$ general. The computations don't go through as cleanly, but we can easily develop the same structural story:

$$w_j(t) = aX_t\,\varphi_j(t), \qquad S_t = X_t \sum \varphi_i(t)$$

so again, each agent keeps all wealth in the risky asset.

(vi) Can this be a general observation, or does it depend on the special form of X? We can see whether it is general if we were to suppose instead that $Y_t \equiv X_t^{1-R}$ is a CIR process,

$$dY_t = \sigma\sqrt{Y_t}\,dW_t + \beta(\alpha - Y_t)\,dt.$$

In that case

$$w_j(t) = \frac{1}{\zeta_t} E_t\left[\int_t^\infty \zeta_u^{1-1/R} \gamma_j e^{-\rho_j u/R} du\right] = \frac{a^{1-R}}{\zeta_t} E_t\left[\int_t^\infty Y_u \left(\sum \gamma_i e^{-\rho_i u/R}\right)^{R-1} \gamma_j e^{-\rho_j u/R} du\right]$$

$$= X_t^R \{ \psi_j(t) + \varphi_j(t) X_t^{1-R} \}$$

for some functions ψ_j, φ_j, and

$$a S_t = X_t^R \{ \bar\psi(t) + \bar\varphi(t) X_t^{1-R} \} \qquad \left[\bar\psi(t) \equiv \sum_j \psi_j(t), \ \bar\varphi(t) = \sum_j \varphi_j(t) \right]$$

After a few calculations, we conclude that

$$\theta_j(t) = \alpha \frac{R X_t^{R-1} \psi_j(t) + \varphi_j(t)}{R X_t^{R-1} \bar\psi(t) + \bar\varphi(t)} \neq w_j(t)/S_t$$

if $R \neq 1$. So this won't hold in general.

Using Maple, we can obtain explicit expressions for $\varphi_j(t)$ and $\psi_j(t)$:

$$\varphi_j(t) = 2A_j c_j a \Big\{ \rho_j^3 (A_1c_1 + A_2c_2) + \rho_j^2 \big(A_1c_1(\rho_1 + 2\beta + 2\rho_2) + A_2c_2(\rho_2 + 2\beta + 2\rho_1)\big)$$
$$+ \rho_j \big(A_1c_1(\rho_2^2 + 2\rho_1\beta + 2\rho_2\beta + 2\rho_1\rho_2) + A_2c_2(\rho_1^2 + 2\beta(\rho_1+\rho_2) + 2\rho_1\rho_2)\big)$$
$$+ \rho_1\rho_2 \big(A_1c_1(\rho_2 + 2\beta) + A_2c_2(\rho_1 + 2\beta)\big) \Big\} \Big/ (\rho_j+\rho_1)(\rho_j+\rho_2)(\rho_1+\rho_j+2\beta)(\rho_2+\rho_j+2\beta)(c_1A_1+c_2A_2)^2$$

$$A_i \equiv e^{-\rho_i t/2}$$

$$\psi_j(t) = 2ac_jA_j \Big\{ 2\alpha\beta\rho_j^2 (c_1A_1 + c_2A_2) + 4\rho_j\alpha\beta\big(\beta(c_1A_1 + c_2A_2) + \rho_2c_1A_1 + \rho_1c_2A_2\big)$$
$$+ 2\alpha\beta\big(2\beta(A_1c_1\rho_2 + A_2c_2\rho_1) + \rho_1^2A_2c_2 + \rho_2^2A_1c_1\big) \Big\} \Big/ (\rho_1+\rho_j)(\rho_2+\rho_j)(\rho_1+\rho_j+2\beta)(\rho_2+\rho_j+2\beta)(c_1A_1+c_2A_2)^2$$

CHAPTER 5

The Knaster-Kuratowski-Mazurkiewicz lemma

5.0 Remark
The K-K-M lemma (Corollary 5.4) is quite basic and in some ways more useful than Brouwer's fixed point theorem, although the two are equivalent.

5.1 Theorem (Knaster-Kuratowski-Mazurkiewicz [1929])
Let $\Delta = co\ \{e^0, \ldots, e^m\} \subset \mathbf{R}^{m+1}$ and let $\{F_0, \ldots, F_m\}$ be a family of closed subsets of Δ such that for every $A \subset \{0,\ldots,m\}$ we have

$$co\ \{e^i : i \in A\} \subset \bigcup_{i \in A} F_i. \tag{5.2}$$

Then $\bigcap_{i=0}^{m} F_i$ is compact and nonempty.

5.3 Proof (Knaster-Kuratowski-Mazurkiewicz [1929])
The intersection is clearly compact, being a closed subset of a compact set. Let $\varepsilon > 0$ be given and subdivide Δ into subsimplexes of diameter $\leqslant \varepsilon$. (See 3.10 for example.) For a vertex v of the subdivision belonging to the face $e^{i_0} \cdots e^{i_k}$, by 5.2 there is some index i in $\{i_0, \ldots, i_k\}$ with $v \in F_i$. If we label all the vertexes this way, then the labeling satisfies the hypotheses of Sperner's lemma so there is a completely labeled subsimplex ${}^{\varepsilon}p^0 \cdots {}^{\varepsilon}p^m$, with ${}^{\varepsilon}p^i \in F_i$ for each i. As $\varepsilon \downarrow 0$, choose a convergent subsequence ${}^{\varepsilon}p^i \to z$. Since F_i is closed and ${}^{\varepsilon}p^i \in F_i$ for each i, we have $z \in \bigcap_{i=0}^{m} F_i$.

5.4 Corollary
Let $K = co\ \{a^0, \ldots, a^m\} \subset \mathbf{R}^k$ and let $\{F_0, \ldots, F_m\}$ be a family of closed sets such that for every $A \subset \{0,\ldots,m\}$ we have

$$co\ \{a^i : i \in A\} \subset \bigcup_{i \in A} F_i. \tag{5.5}$$

Then $\bigcap_{i=0}^{m} F_i$ is nonempty.

5.6 Proof
Again compactness is immediate. Define the mapping $\sigma : \Delta \to K$ by $\sigma(z) = \sum_{i=0}^{m} z_i a^i$. If $\{a^0, \ldots, a^m\}$ is not an affinely independent set, then σ is not injective, but it is nevertheless continuous. Put $E_i = \sigma^{-1}[F_i \cap K]$ for each i. Since σ is continuous, each E_i is a closed subset of Δ. It is straightforward to verify that 5.2 is satisfied by $\{E_0, \ldots, E_m\}$ and so let $z \in \bigcap_{i=0}^{m} E_i \neq \varnothing$. Then $\sigma(z) \in \bigcap_{i=0}^{m} F_i \neq \varnothing$.

5.7 Corollary (Fan [1961])
Let $X \subset \mathbf{R}^m$, and for each $x \in X$ let $F(x) \subset \mathbf{R}^m$ be closed. Suppose:

(i) For any finite subset $\{x^1, \ldots, x^k\} \subset X$,

$$co\ \{x^1, \ldots, x^k\} \subset \bigcup_{i=1}^{k} F(x^i).$$

(ii) $F(x)$ is compact for some $x \in X$.

Then $\bigcap_{x \in X} F(x)$ is compact and nonempty.

5.8 Proof
The conclusion follows from Corollary 5.4 and the fact that in a compact set, a family of closed sets with the finite intersection property has a nonempty intersection. (Rudin [1976, 2.36].)

If $\mathcal{F}_n = \sigma(\{Z_j : j \leq n\})$ then all processes are adapted in this account, except if indicated to contrary

Need to check F_j closed

Discrete-time version of multi-agent equilibria (19/5/99)

1). Let's suppose that there is a discrete-time dividend stream $(\delta(m))_{m\geq 0}$ of non-negative dividends from a productive asset, and that $\delta(m) = \delta(m-1) \cdot Z_m$, where the Z_i are IID, $P(Z=a) = \alpha$, $P(Z=a^{-1}) = 1-\alpha$. This way we have market completeness and the story is easier.

Agent j aims to maximise $E_n\left[\sum_{m\geq n} U_j(m, c_j(m))\right]$, where $c_j(m)$ is his consumption on day m, for each $j = 1, \dots, J$. Agent j enters day n with $\theta_j(n-1)$ shares, and $\xi_j(n-1)$ of food, the dividend $\delta(n)$ is revealed, a market opens up in food, shares, and promises of food tomorrow, an equilibrium is attained, and then each agent consumes what he wants, and passes to day $(n+1)$ holding $\theta_j(n)$ shares and $\xi_j(n)$ in food.

Let's suppose $U_j(m,x) = -\infty$ for $x<0$, which forces agents to undertake non-negative consumption.

2). By the usual story, we consider for any $p \in \Delta \equiv \{x \in \mathbb{R}^J : x_i \geq 0, \sum x_i = 1\}$ the representative agent with utility $U(m,c;p) = \max\{\sum U_j(m,c_j)p_j : \sum c_j = c, c_j \geq 0\}$, and find that the state price density must satisfy

$$(1) \qquad \begin{cases} p_j \, U_j'(m, c_j(m)) = \zeta_m & \forall m\geq n, \quad \forall j \\ \delta(m) = \sum_j c_j(m) = \sum_j I_j(m, \zeta_m/p_j) = I(m, \zeta_m; p) \end{cases}$$

$$(2) \qquad \text{so} \quad \boxed{\zeta_m = U'(m, \delta(m); p).}$$

What we want is to find some p for which the budget constraints hold, that is

$$(3) \qquad \theta_j(n-1)\left(S_n + \delta(n)\right) + \xi_j(n-1) = E_n\left[\sum_{m\geq n} \zeta_m \, c_j(m)\right] \Big/ \zeta_n,$$

where

$$(4) \qquad S_n = S(n;p) = E_n\left[\sum_{m>n} \zeta_m \, \delta(m)\right] \Big/ \zeta_n$$

We shall of course have that $\sum_j \xi_j(n-1) = 0$, $\sum_j \theta_j(n) = 1$, and we shall also have that whatever $\theta_j(n-1)$, $\xi_j(n-1)$, we must always have $\theta_j(n-1)\{S_n + \delta(n)\} + \xi_j(n-1) \geq 0$ (else the portfolio $\theta_j(n-1)$, $\xi_j(n-1)$ would be a disaster for agent j, and he wouldn't choose it.)

3). For the existence of such a p, we use the KKM theorem again, as in Karatzas et al. Let

$$(5) \qquad F_j \equiv \left\{p\in\Delta : \theta_j(n-1)\left(S(n;p) + \delta(n)\right) + \xi_j(n-1) \leq E_n\left[\sum_{m\geq n} \zeta_m \, c_j(m)\right]\Big/\zeta_n \right\}$$

where $\zeta_m \equiv U'(m, \delta(m); p)$, $c_j(m) = I_j(m, \zeta_m/p_j)$. For any nonempty $B \subseteq \{1, \dots, J\}$, We have the further condition $\sum g(n) = \delta(n)$. Equilibrium would be the statement that for all feasible perturbations

$$\Delta c_j \, U_j' + \sum_i \Delta\theta_i \frac{\partial V_j}{\partial \theta_i} + \sum_i \Delta\xi_i \frac{\partial V_j}{\partial \theta_i} < 0$$

for at least one j.

we have $\Delta_B \equiv \{x \in \Delta : x_j = 0 \ \forall j \notin B\} \subseteq \bigcup_{j \in B} F_j$, because if not, the LHS of (3) is more than the RHS for $j \in B$, and the LHS of (3) is at least zero = RHS of (3) for $j \notin B$. But summing both sides of (3) over j should result in equality. Hence by KKM, $\bigcap_j F_j \neq \emptyset$, and that's enough.

As for uniqueness, we shall have to return to this issue. Observe that in general the same p will do for many $(\underline{\theta}(n-1), \underline{\xi}(n-1))$.

4). Assuming the initial $\underline{\theta}(n-1)$, $\underline{\xi}(n-1)$ uniquely determines p, we have at the end of day n an ex-dividend price for the share

(6) $$S_n = E_n\left[\sum_{m>n} \zeta_m \delta_m\right] \zeta_n^{-1} = S_n\left(\underline{\theta}(n-1), \underline{\xi}(n-1), \delta(n)\right)$$

and at the start of day n, the value to agent j will be

(7) $$V_j(n-1) \equiv V_j\left(n-1; \underline{\theta}(n-1), \underline{\xi}(n-1)\right) = E_{n-1}\left[\sum_{m \geq n} U_j(m, c_j(m))\right]$$

Now suppose that in fact something anomalous happens on day n; agent J tries to grab a holding θ^* of the share by the end of the day. On day n, the equilibrium price p^* arrived at for the share will have to be $S_n(\underline{\theta}(n), \underline{\xi}(n))$, where $\underline{\theta}(n)$, $\underline{\xi}(n)$ have the properties

(8.i) $$\sum_j \theta_j(n) = 1$$

(8.ii) $$\sum_j \xi_j(n) = 0$$

(8.iii) $$\theta_J(n) = \theta^*$$

(8.iv) for each $j = 1, \dots, J-1$, we have

$$U_j(n, c_j(n)) + V_j(n; \underline{\theta}(n), \underline{\xi}(n))$$

$$\geq U_j(n, c_j'(n)) + V_j(n; \underline{\theta}'(n), \underline{\xi}'(n))$$

where $$\theta_j(n-1)\{p^* + \delta_n\} + \xi_j(n-1) = \theta_j(n) p^* + c_j(n) + \xi_j(n)\rho \quad ?$$
$$= \theta_j'(n) p^* + c_j'(n) + \xi_j'(n) \cdot \rho$$

and $\theta_i(n) = \theta_i'(n)$, $\xi_i(n) = \xi_i'(n)$ for every $i \neq j$.

$$c_j(m) = p_j \beta_j^m / \zeta(m)$$

$$\zeta(m) = \gamma_m / \delta_m$$

5). Can we make the solution explicit in some special cases?

A natural choice is to take $U_j(m,x) = \beta_j^m\, U(x)$, where $U'(x) = x^{-R}$. Then we shall have

$$g_j(m) = \left(p_j \beta_j^m / \zeta(m)\right)^{1/R}$$

$$\delta(m) = \sum_j g(m) = \zeta(m)^{-1/R} \sum_j \left(p_j \beta_j^m\right)^{1/R}$$

$$\therefore \quad \zeta_m = \gamma_m\, \delta(m)^{-R} \equiv \left\{\sum_j \left(p_j \beta_j^m\right)^{1/R}\right\}^R \delta(m)^{-R}.$$

Current value of future consumption is

$$\zeta_n^{-1} E_n\left[\sum_{m\geq n} \zeta_m\, g(m)\right] = p_j^{1/R} E_n\left[\sum_{m\geq n} \gamma_m^{1-1/R}\, \delta(m)^{1-R}\, \beta_j^{m/R}\right] / \zeta_n$$

$$= p_j^{1/R}\, \delta(n) \left(\sum_{m\geq n} \gamma_m^{1-1/R}\, \psi^{m-n}\, \beta_j^{m/R}\right)/\gamma_n \qquad \left[\psi \equiv E(Z^{1-R})\right]$$

and ex-dividend price is

$$S_n = \delta(n)\left(\sum_{m>n} \psi^{m-n}\gamma_m\right)/\gamma_n.$$

The case $R=1$ looks most tractable. We have $\psi = 1$, $\gamma_m = \sum_j p_j \beta_j^m$, so the budget constraint (3) to be satisfied becomes

$$\theta_j(n-1)\, \delta_n \sum_{m\geq n} \gamma_m + \xi_j(n-1)\,\gamma_n = \delta_n \sum_{m\geq n} p_j \beta_j^m = \delta_n\, p_j\, \beta_j^n / (1-\beta_j)$$

$$\therefore \quad \theta_j(n-1)\,\delta_n \sum_i \frac{p_i \beta_i^n}{1-\beta_i} + \left(\sum p_i \beta_i^n\right)\xi_j(n-1) = \delta_n\, p_j\, \beta_j^n/(1-\beta_j)$$

Thus

$$\frac{p_j \beta_j^n}{1-\beta_j} = A\,\theta_j(n-1) + B\,\xi_j(n-1), \qquad \begin{cases} A \equiv \sum_i p_i \beta_i^n/(1-\beta_i) \\ B \equiv \sum_i p_i \beta_i^n/\delta_n. \end{cases}$$

Cross-multiplying by $(1-\beta_j)$ and summing over j leads to

$$\delta_n B = A \sum_j (1-\beta_j)\,\theta_j(n-1) + B \sum_j (1-\beta_j)\,\xi_j(n-1)$$

which fixes the ratio A/B, hence for some λ we must have

$$\frac{p_j \beta_j^n}{1-\beta_j} = \lambda\left[\theta_j(n-1)\left\{\delta_n - \sum_i (1-\beta_i)\,\xi_i(n-1)\right\} + \xi_j(n-1)\sum_i (1-\beta_i)\,\theta_i(n-1)\right].$$

Since $\sum p_j = 1$, we can give an explicit expression for λ in terms of $\underline{\theta}(n-1)$, $\underline{\xi}(n-1)$, namely

$$\lambda = \Big[\sum_j (1-\beta_j)\beta_j^{-n}\theta_j(n-1)\cdot\big(\delta_n - \sum_i (1-\beta_i)\xi_i(n-1)\big) + \sum_j (1-\beta_j)\beta_j^{-n}\xi_j(n-1)\sum_i(1-\beta_i)\theta_i(n-1)\Big]^{-1}.$$

Thus the solution p is uniquely determined by $\underline{\theta}(n-1)$, $\underline{\xi}(n-1)$ and δ_n.

The residual value function can also be made more explicit:

$$V_j(n-1) \equiv E_{n-1}\sum_{m\geq n} U_j(m, g(m))$$

$$= E_{n-1}\sum_{m\geq n}\beta_j^m \log(p_j\beta_j^m\delta_m/\gamma_m)$$

$$= \sum_{m\geq n}\beta_j^m\log\left(\frac{p_j\beta_j^m}{\gamma_m}\right) + \frac{\beta_j^n}{1-\beta_j}\log\delta_{n-1} + \frac{\beta_j^n}{(1-\beta_j)^2}E\log Z_1$$

The dependence on $\underline{\theta}(n-1)$, $\underline{\xi}(n-1)$, via p, is all in the first term. The value on day n of unit of food on day $n+1$ will be

$$E_n(\zeta_{n+1}/\zeta_n) = \frac{\gamma_{n+1}}{\gamma_n}E(1/Z) = \frac{\gamma_{n+1}}{\gamma_n}\cdot\left\{\frac{\alpha}{a} + (1-\alpha)a\right\}$$

The expression for the residual value reduces somewhat:

$$\boxed{V_j(n-1) = \frac{\beta_j^n}{1-\beta_j}\log\big(p_j\beta_j^n\delta(n-1)\big) + \frac{\beta_j^n}{(1-\beta_j)^2}\big(\log\beta_j + E\log Z_1\big) - \sum_{m\geq n}\beta_j^m\log\gamma_m}$$

From Bermudan to American by extrapolation (18/5/99)

Suppose we've worked out the price of a Bermudan option where exercise is permitted at times $t_0 = 0 < t_1 < \dots < t_N = T$, and now consider what would happen if we also allowed exercise at $t \in [t_j, t_{j+1}]$. The improvement in payoff as a function of t should be continuous, and zero at the two endpoints.

If we assume that the improvement is of the form

$$a_j \, (t - t_j)(t_{j+1} - t),$$

and if we further assume that if we continue to insert exercise dates into (t_j, t_{j+1}) the gain we make by inserting u into (s,t) will again be

$$a_j \, (t-u)(u-s),$$

then we can deduce the improvement we make by allowing any exercise in $[t_j, t_{j+1}]$.

Fixing $t_j = 0$, $t_{j+1} = 1$ for simplicity of exposition, if we insert at x_1 and then at $x_2 \in (x_1, 1)$, we gain

$$a \{ x_1(1-x_1) + (x_2 - x_1)(1-x_2) \}$$

which is the same as the gain $a\{ x_2(1-x_2) + (x_2-x_1)x_1 \}$ which we get by inserting in the different order. So if we insert at $x_1 < x_2 < \dots < x_n$ in that order, and let $\delta_j \equiv x_j - x_{j-1}$, then the gain we make doesn't depend on the order, and is

$$a \{ x_1(1-x_1) + (x_2-x_1)(1-x_2) + \dots + (x_n - x_{n-1})(1-x_n) \}$$

$$= \{ \delta_1(\delta_2 + \dots + \delta_n) + \delta_2(\delta_3 + \dots + \delta_n) + \dots + \delta_{n-1}(\delta_n + \delta_{n+1}) + \delta_n \delta_{n+1} \}$$

$$= \tfrac{a}{2} \sum \delta_j (1 - \delta_j).$$

In the limit as the mesh $\to 0$, we find that we gain $a/2$. So in particular, if we insert a point in the middle of $[t_j, t_{j+1}]$, the improvement we achieve will be exactly half of the improvement we achieve by allowing exercise at any time in $[t_j, t_{j+1}]$.

If we thought that the value of a varies through the interval, then the overall improvement would be

$$\tfrac{1}{2} \int_0^1 a(u)\,du$$

or, in the original problem,

$$\frac{t_{j+1} - t_j}{2} \int_{t_j}^{t_{j+1}} a(u)\,du$$

$$c_j(m) = p_j \beta_j^m / S(m)$$

<u>Discrete-time multi-agent equilibria again</u> (2/6/99)

1) If we end day $(n-1)$ with agent j holding $\theta_j(n-1)$ shares and a promise of $\xi_j(n-1)$ of food tomorrow $(\sum_j \theta_j(n-1)=1,\ \sum_j \xi_j(n-1)=0)$, we need to derive the equilibrium ex-dividend price S_{n-1} for the share, and the price B_{n-1} for the bond which delivers 1 at time n.

As before, for any $p \in \Delta$ we construct the utility $U(m,c;p)$ and deduce the state-price density

$$\zeta_m = U'(m, \delta(m); p) \qquad (m \geq n)$$

so that

$$S_{n-1} = S(n-1; p) = E_{n-1}\Big[\sum_{m\geq n} \zeta_m\, \delta(m)\Big] / \zeta_{n-1}$$

$$B_{n-1} \equiv B(n-1; p) = E_{n-1}\big[\zeta_n\big] / \zeta_{n-1}$$

and we also require for each j that

$$\theta_j(n-1)\, S(n-1; p) + \xi_j(n-1)\, B(n-1; p) = E_{n-1}\Big[\sum_{m\geq n} \zeta_m\, c_j(m)\Big] / \zeta_{n-1} \qquad \forall j$$

Now the value of ζ_{n-1} is indeterminate initially; the actual value it takes reflects the marginal rate of substitution of good today for good tomorrow, but this should come out in the wash.

2) <u>Logarithmic utility example</u>. For this example, we calculate

$$\begin{cases} S(n-1; p) = \Big(\sum_{m\geq n} \gamma_m\Big) / \zeta_{n-1} = \Big\{\sum_i p_i \beta_i^n/(1-\beta_i)\Big\} / \zeta_{n-1} \\ B(n-1; p) = \gamma_n \{\alpha a^{-1} + (1-\alpha) a\} / \delta(n-1)\, \zeta_{n-1} \\ \zeta_m = \gamma_m / \delta(m) \qquad (m \geq n) \end{cases} \qquad \Big(\gamma_m \equiv \sum_j p_j \beta_j^m\Big)$$

We therefore need to have p chosen to give for each j

$$\boxed{\theta_j(n-1) \sum_i p_i \frac{\beta_i^n}{1-\beta_i} + \frac{\gamma_n \rho}{\delta(n-1)}\, \xi_j(n-1) = p_j \beta_j^n/(1-\beta_j)} \qquad \big[\rho \equiv \alpha a^{-1} + (1-\alpha) a\big]$$

As before, if we write

$$A \equiv \sum_i p_i \beta_i^n/(1-\beta_i), \qquad B \equiv \gamma_n \rho / \delta(n-1)$$

we may cross-multiply by $(1-\beta_j)$ and sum on j to obtain

$$A \sum_j (1-\beta_j)\, \theta_j(n-1) + B \sum_j (1-\beta_j)\, \xi_j(n-1) = B \rho^{-1}\, \delta(n-1)$$

which fixes the ratio of A to B; for some constant λ, we have

$$A = \lambda\left\{ \rho^{-1}\delta(n-1) - \sum_j (1-\beta_j)\xi_j(n-1)\right\}, \qquad B = \lambda \sum_j (1-\beta_j)\,\theta_j(n-1)$$

Using the condition $\sum p_j = 1$ will fix the value of λ:

$$1 = \lambda\left[\left\{\rho^{-1}\delta(n-1) - \sum_j (1-\beta_j)\xi_j(n-1)\right\} \sum_i (1-\beta_i)\,\theta_i(n-1)/\beta_i^n + \sum_j (1-\beta_j)\,\theta_j(n-1) . \sum (1-\beta_i)\xi_i(n-1)/\beta_i^n \right]$$

On the other hand, multiplying all the p_j by the same positive constant leaves to solution unaltered!
<u>Do we necessarily get</u> $p_j \geq 0$ <u>for all</u> j? The expression for residual value given on p 26 is still valid:

$$V_j(n-1) = \frac{\beta_j^n}{1-\beta_j}\log\left(p_j\,\beta_j^n\,\delta(n-1)\right) + \frac{\beta_j^n}{(1-\beta_j)^2}\left\{\log\beta_j + E\log Z_1\right\} - \sum_{m\geq n}\beta_j^m\log\gamma_m$$

We also have the interesting observation that

$$\boxed{\frac{S_{n-1}}{B_{n-1}} = \frac{A}{B} = \frac{\rho^{-1}\delta(n-1) - \sum_j (1-\beta_j)\,\xi_j(n-1)}{\sum_j (1-\beta_j)\,\theta_j(n-1)} = \frac{\sum_i p_i\,\beta_i^n/(1-\beta_i)}{\gamma_n\,\rho}\,\delta(n-1)}$$

3) By foregoing ε of consumption on day $n-1$, agent j loses

$$\varepsilon\,\beta_j^{n-1}/c_j(n-1) \quad + O(\varepsilon^2)$$

of utility. By getting ε more of consumption on day n, agent j gains (to order ε)

$$\varepsilon\,E_{n-1}\left[U_j'(n, c_j(n))\right] = \varepsilon\,E_{n-1}\left(\beta_j^n/c_j(n)\right) = \frac{\varepsilon}{p_j}\,E_{n-1}\left[\zeta_n\right] = \frac{\varepsilon\gamma_n\rho}{p_j\,\delta(n-1)}$$

Thus if b is the equilibrium price of the bond on day $n-1$, we must have

$$\boxed{b = \frac{\gamma_n\,\rho\,c_j(n-1)}{p_j\,\beta_j^{n-1}\,\delta(n-1)}}$$

for each j (each $j < J$ in the case of a big agent shifting prices). It's tempting to write $\zeta_{n-1} = p_j\,\beta_j^{n-1}/c_j(n-1)$ for the common value, in terms of which we have

$$\boxed{b = \gamma_n\,\rho\,/\,\delta(n-1)\,\zeta(n-1)}$$

Consistent with our earlier expression for $B(n-1; p)$.

4) Let's now consider the single-agent optimisation problem:

$$\max\left[\, U_j(n-1, c_j(n-1)) + V_j(n-1; \underline{\theta}(n-1), \xi(n-1))\right]$$

for $\theta_i(n-1)$, $\xi_i(n-1)$ fixed $(i \neq j)$, over choice of $c_j(n-1)$, $\theta_j(n-1)$, $\xi_j(n-1)$ subject to the

$$\frac{\partial v}{\partial \pi} = K^T$$

$$\frac{\partial \pi}{\partial \theta} = \left(\rho^{-1}\delta_{n-1} - \Xi\right) I + (1-\beta)\, \xi^T$$

$$\frac{\partial \pi}{\partial \xi} = \Theta I - (1-\beta)\, \theta^T$$

budget constraint

$$\theta_j(n-2)\,\{\delta(n-1)+S_{n-1}\} + \xi_j(n-2)$$

$$= c_j(n-1) + \theta_j(n-1)\,S_n + \xi_j(n-1)\,B_{n-1}.$$

To keep things simple, we may drop the normalisation condition $\Sigma p_j = 1$ on the weights p_j, since this just multiplies ζ by a constant which cancels out in any expression for a price. We may therefore suppose that

$$\boxed{\frac{p_j\,\beta_j^n}{1-\beta_j} = \theta_j(n-1)\,\{\rho^{-1}\delta(n-1) - \Xi(n-1)\} + \xi_j(n-1)\,\Theta(n-1)}$$

where $\boxed{\Xi(m) \equiv \sum_j (1-\beta_j)\,\xi_j(m), \qquad \Theta(m) \equiv \sum_j (1-\beta_j)\,\theta_j(m).}$

Cross-multiplying by $1-\beta_j$ and summing on j leads to the relation

$$\boxed{\gamma_n = \rho^{-1}\,\delta(n-1)\,\Theta(n-1).}$$

from which we deduce that

$$\boxed{B_{n-1} = \Theta(n-1)/\zeta_{n-1}, \qquad S_{n-1} = \{\rho^{-1}\delta(n-1) - \Xi(n-1)\}/\zeta_{n-1}}$$

5) But is such an analysis correct? The value to the j^{th} agent depends also on the holdings of all others:

$$v_j = \beta_j^{n-1}\log\zeta + \frac{\beta_j^n}{1-\beta_j}\log\left(\frac{\beta_j^n p_j}{1-\beta_j}\right) - \sum_{m\geq n}\beta_j^m\log\gamma_m \quad + \text{constant}$$

Then if we define

$$\boxed{\tau_j \equiv \beta_j^n p_j/(1-\beta_j), \qquad \alpha_{jr} \equiv \sum_{m\geq n}\beta_j^m\,\beta_r^{m-n}\,(1-\beta_r)/\gamma_m}$$

we have

$$\boxed{\begin{aligned}\frac{\partial v_j}{\partial\tau_r} &= \delta_{jr}\cdot\frac{1}{p_j} - \alpha_{jr},\\[1em] \frac{\partial\tau_r}{\partial\theta_k} &= \delta_{rk}\,(\rho^{-1}\delta(n-1) - \Xi(n-1)) + (1-\beta_k)\,\xi_r(n-1),\\[1em] \frac{\partial\tau_r}{\partial\xi_k} &= -\theta_r(n-1)(1-\beta_k) + \delta_{rk}\,\Theta(n-1)\end{aligned}}$$

$(\equiv K_{jr}$ in later notation $)$

$$\begin{pmatrix} r = 1,\dots,J \\ k = 1,\dots,J-1 \\ j = 1,\dots,J-1 \end{pmatrix}$$

(for our large-investor story)

$$\therefore \quad \frac{\partial v}{\partial \theta} = \left(\rho^{-1}\delta_{n-1} - \Xi\right) K^T + (1-\beta)\, \xi^T K^T$$

$$\frac{\partial v}{\partial \xi} = \Theta K^T - (1-\beta)\, \theta^T K^T$$

YES! We are considering a situation where agent J has stepped out of line, and acquired a larger(?) share of the productive asset, so the remaining agents $1, \dots, J-1$ are going to share out the remaining $1-\theta_J$ of the productive asset, and $-\xi_J$ of bonds. Thus we should interpret

$$\frac{\partial v}{\partial \xi} = \left(\frac{\partial v_j}{\partial \xi_i}\right)_{i,j=1,\dots,J-1}, \quad \frac{\partial v}{\partial \theta} = \left(\frac{\partial v_j}{\partial \theta_i}\right)_{i,j=1,\dots,J-1}, \text{ etc}$$

Hence

$$\boxed{\begin{aligned}
\frac{\partial v_j}{\partial \theta_k} &= \frac{1}{p_j}\,\delta_{jk}\,(\rho^{-1}\delta_{n-1} - I) + \frac{\xi_j}{p_j}(1-\beta_k) - \alpha_{jk}\,(\rho^{-1}\delta_{n-1} - I) - (1-\beta_k)\,\alpha_{jr}\,\xi_r \\
\frac{\partial v_j}{\partial \xi_k} &= -\frac{\theta_j}{p_j}(1-\beta_k) + \frac{\delta_{jk}\,('')}{p_j} + (1-\beta_k)\,\alpha_{jr}\,\theta_r - \alpha_{jk}\,('') \\
\frac{\partial v_j}{\partial c_k} &= \delta_{jk}\;\beta_j^{n-1}/c_j
\end{aligned}}$$

6) An equilibrium will have the property that any feasible perturbation which leaves no-one worse off will actually improve no-one. To cast this in linear-algebra terms, consider the matrix

$$\left(\begin{array}{c|cccccc}
\frac{\partial v}{\partial c} & 1 & -1 & 0 & 0 & 0 & 0 \\
\frac{\partial v}{\partial \xi} & 0 & 0 & 1 & -1 & 0 & 0 \\
\frac{\partial v}{\partial \theta} & 0 & 0 & 0 & 0 & 1 & -1
\end{array}\right) \equiv M \equiv \left(z_1\, z_2 \cdots\; z_J\; z_{J+1} \cdots\; z_{J+3} \right)$$

which is $3(J-1)\times(J+3)$ The statement of equilibrium is that

$$x^T M \geqslant 0 \quad\Rightarrow\quad x^T M = 0$$

So if $C \equiv \{x : x^T z_j \geqslant 0 \;\forall j\}$, then $x \in C \Leftrightarrow -x \in C$, so C is in fact a vector-space. If we focus on the subspace generated by the z_i, it must be that

$$0 \in \{\, \Sigma \lambda_i z_i \;:\; \lambda_i > 0 \,\}$$

for if not there is a separating hyperplane and a profitable perturbation. Hence there exist positive $\lambda_1, \dots, \lambda_{J-1}$ such that

$$\left.\begin{array}{ll}
\displaystyle\sum_{j=1}^{J-1} \lambda_j \frac{\partial v_j}{\partial c_k} & \underline{\text{is same for all}}\ k \\[2ex]
\displaystyle\sum_{j=1}^{J-1} \lambda_j \frac{\partial v_j}{\partial \xi_k} & \underline{\text{is same for all}}\ k \\[2ex]
\displaystyle\sum_{j=1}^{J-1} \lambda_j \frac{\partial v_j}{\partial \theta_k} & \underline{\text{is same for all}}\ k
\end{array}\right\}$$

From 'An Introduction to Multivariate Statistical Analysis' T. W. Anderson
J. Wiley, New York, London 1958

THEOREM 4.2.2. *The correlation coefficient in a sample of N from a bivariate normal distribution with correlation ρ is distributed with density*

$$\frac{2^{n-2}(1-\rho^2)^{\frac{1}{2}n}(1-r^2)^{\frac{1}{2}(n-3)}}{(n-2)!\pi}\sum_{\alpha=0}^{\infty}\frac{(2\rho r)^\alpha}{\alpha!}\Gamma^2[\tfrac{1}{2}(n+\alpha)], \tag{34}$$

where $n = N - 1$.

The distribution of r was first found by Fisher (1915). He also gave as another form of the density

$$\frac{(1-\rho^2)^{\frac{1}{2}n}(1-r^2)^{\frac{1}{2}(n-3)}}{\pi(n-2)!}\left[\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{\cos^{-1}(-x)}{\sqrt{1-x^2}}\right\}\bigg|_{x=r\rho}\right]. \tag{35}$$

This is obtained from (28) by letting $a_{11} = ue^{-v}$ and $a_{22} = ue^{v}$.

Hotelling (1953) has made an exhaustive study of the distribution of r. He has recommended the following form, which is derived from (28) by the preceding transformation:

$$\frac{n-1}{\sqrt{2\pi}}\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}(1-\rho^2)^{\frac{1}{2}n}(1-r^2)^{\frac{1}{2}(n-3)} (1-\rho r)^{-n+\frac{1}{2}}F\left(\tfrac{1}{2},\tfrac{1}{2};n+\tfrac{1}{2};\frac{1+\rho r}{2}\right), \tag{36}$$

where

$$F(a,b;c;x) = \sum_{j=0}^{\infty}\frac{\Gamma(a+j)}{\Gamma(a)}\frac{\Gamma(b+j)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+j)}\frac{x^j}{j!} \tag{37}$$

is a hypergeometric function. The series in (36) converges more rapidly than the one in (34). Hotelling discusses methods of integrating the density and also calculates moments of r.

The cumulative distribution of r,

$$\Pr\{r \le r^*\} = F(r^*|N,\rho), \tag{38}$$

has been tabulated by F. N. David (1938) for* $\rho = 0(.1).9$, $N = 3(1)25$, 50, 100, 200, 400, and $r^* = -1(.05)1$. (It should be noted that David's n is our N.) It is clear from the density (34) that $F(r^*|N,\rho) = 1 - F(-r^*|N,-\rho)$ because the density for r, ρ is equal to the density for $-r, -\rho$. These tables can be used for a number of statistical procedures.

First, we consider the problem of using a sample to test the hypothesis

$$H: \rho = \rho_0. \tag{39}$$

* $\rho = 0(.1).9$ means $\rho = 0, 0.1, 0.2, \cdots, 0.9$.

Distribution of sample correlation c/e (17/6/99)

If (X_i, Y_i) are iid $N(\mu, \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$, $i = 1, \dots, N$, then conventionally we'd take as an estimator for ρ the quantity

$$\hat{\rho} \equiv \Sigma (X_i - \bar{X})(Y_i - \bar{Y}) \Big/ \left\{ \Sigma (X_i - \bar{X})^2 \; \Sigma (Y_i - \bar{Y})^2 \right\}^{1/2}.$$

Can we get more information on the distribution of $\hat{\rho}$? Writing $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$, a $2N$-vector, and $P \equiv I - \frac{1}{N} 1 1^T$, we form $\tilde{X}, \tilde{Y}$, defined by $\tilde{X}_i \equiv X_i - \bar{X}$, $\tilde{Y}_i \equiv Y_i - \bar{Y}$ by taking

$$\tilde{X} = PX, \quad \tilde{Y} = PY$$

So $\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \equiv \tilde{Z}$ is $N\left(0, \sigma^2 \begin{pmatrix} P & \rho P \\ \rho P & P \end{pmatrix}\right) \equiv N(0, V)$. Wlog, $\sigma = 1$, so if we are interested in

$$E \exp\left(-\tfrac{1}{2} \alpha \Sigma \tilde{X}_i^2 - \gamma \Sigma \tilde{X}_i \tilde{Y}_i - \tfrac{1}{2} \beta \Sigma \tilde{Y}_i^2 \right)$$

$$= \det (I + QV)^{-1/2}$$

where $Q = \begin{pmatrix} \alpha I & \gamma I \\ \gamma I & \beta I \end{pmatrix}$, we shall need to know about $|I + QV|$.

Now $I + QV = I + \begin{pmatrix} (\alpha + \gamma\rho) P & (\alpha\rho + \gamma) P \\ (\beta\rho + \gamma) P & (\gamma\rho + \beta) P \end{pmatrix} \equiv I + \begin{pmatrix} aP & bP \\ cP & dP \end{pmatrix}$, say

To compute the determinant, we may wlog rotate P to $e_1 e_1^T$, and we now seek

$$\begin{vmatrix} I + aP & bP \\ cP & I + dP \end{vmatrix} = |I + aP| \cdot |I + dP| \begin{vmatrix} I & \frac{b}{1+a} P \\ \frac{c}{1+d} P & I \end{vmatrix} \qquad (I + aP)^{-1} = I - \frac{a}{1+a} P$$

$$= |I + aP| \cdot |I + dP| \cdot \left| I - \frac{bc}{(1+a)(1+d)} P \right|$$

$$= (1+a)^{N-1} (1+d)^{N-1} \left(1 - \frac{bc}{(1+a)(1+d)} \right)^{N-1}$$

$$= \left| (1+a)(1+d) - bc \right|^{N-1}.$$

Hence

$$\boxed{E \exp\left(-\tfrac{1}{2} \alpha \Sigma \tilde{X}_i^2 - \gamma \Sigma \tilde{X}_i \tilde{Y}_i - \tfrac{1}{2} \beta \Sigma \tilde{Y}_i^2 \right) = \left| (1 + \alpha + \gamma\rho)(1 + \beta + \gamma\rho) - (\gamma + \alpha\rho)(\gamma + \beta\rho) \right|^{-(N-1)/2}}$$

This bears the following neat interpretation. If $\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$ are IID $N\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$, $i = 1, \dots, N-1$, then

$$\boxed{\left(\sum_1^N \tilde{X}_i^2, \sum_1^N \tilde{X}_i \tilde{Y}_i, \sum_1^N \tilde{Y}_i^2 \right) \overset{\mathcal{D}}{=} \left(\sum_1^{N-1} \xi_j^2, \sum_1^{N-1} \xi_j \eta_j, \sum_1^{N-1} \eta_j^2 \right)}$$

Discrete-time multiagent equilibria again (21/6/99)

Exploiting the diagonal form of $\partial\sigma/\partial c$, we may wlog have

$$\boxed{\lambda_j = \varsigma_j / p_j^{n-1}}$$

If we now introduce the matrix $K \equiv (K_{ij}) \equiv (\delta_{ij}\, p_j^{-1} - \alpha_{ij})$, we have

$$\frac{\partial v}{\partial \theta} = (\rho^{-1}\delta_{n-1} - \Xi)\, K^T + (1-\beta).(K\xi)^T$$

$$\frac{\partial v}{\partial \xi} = \Theta K^T - (1-\beta).(K\theta)^T$$

and the equilibrium conditions give us

$$\begin{cases} \dfrac{\partial v}{\partial \theta}\lambda = (\rho^{-1}\delta_{n-1} - \Xi)\, K^T\lambda + (1-\beta)\,.(K\xi)^T\lambda \propto 1 \\ \dfrac{\partial v}{\partial \xi}\cdot\lambda = \Theta\, K^T\lambda - (1-\beta)\,(K\theta)^T\lambda \propto 1 \end{cases}$$

Taking the second of these, we find on left-multiplying by θ^T that we must actually have

$$\boxed{\Theta\, K^T\lambda = (1-\beta)\;\theta^T K^T\lambda}$$

Substituting for $K^T\lambda$ in the first of the equilibrium conditions, and using the assumption that not all p_k are the same, we conclude that (provided $\Theta \neq 0$)

$$\boxed{\frac{\rho^{-1}\delta_{n-1} - \Xi}{\Theta}\;\theta^T K^T\lambda + \xi^T K^T\lambda = 0}$$

But $K^T\lambda \propto (1-\beta)$, and if the constant of proportionality were non-zero, we'd have from this that $\rho^{-1}\delta_{n-1} = 0$.... So we conclude that $K^T\lambda = 0$, or again.

$$\boxed{\lambda_j/p_j = \sum_i \lambda_i \alpha_{ij}}$$

Is it worth including a new agent in a coalition? (25/6/99)

1) Suppose we begin at the end of day $n-1$ with agent j holding $\theta_j(n-1)$ shares and $\xi_j(n-1)$ bonds, $j \in A \subset \{1, \dots, J\}$. These agents achieve an equilibrium among themselves, and then consider what would change if another agent were admitted to the group. Assume that all agents have logarithmic utilities:

$$U_j(n, c) = \beta_j^n \log c.$$

Then we seek q_j so that

$$q_j U_j'(m, g(m)) \equiv q_j \beta_j^m / g(m) = \zeta_m$$

and market clearing

$$\sum_j c_j(m) = \theta_\circ \delta(m) + \xi_\circ I_{\{m=n\}} \qquad \left(\begin{array}{l} \theta_\circ = \sum_{j \in A} \theta_j \\ \xi_\circ = \sum_{j \in A} \xi_j \end{array} \right)$$

holds. Thus if $\gamma_m \equiv \sum_{j \in A} q_j \beta_j^m$ as before, we have

$$\boxed{\frac{\gamma_m}{\zeta_m} = \theta_\circ \delta_m + \xi_\circ I_{\{m=n\}}}$$

The equations for the prices S_{n-1} of the (ex-div) share at end of day $n-1$, and the bond B_{n-1} are

$$\begin{aligned} S_{n-1} &= \zeta_{n-1}^{-1} E_{n-1}\Big[\sum_{m \geq n} \zeta_m \delta_m\Big] \\ &= \zeta_{n-1}^{-1} \Big\{ \sum_{m > n} \gamma_m / \theta_\circ + E_{n-1} \frac{\gamma_n \delta_n}{\theta_\circ \delta_n + \xi_\circ} \Big\} \\ &= \frac{1}{\zeta_{n-1}} \Big\{ \sum_{m > n} \gamma_m / \theta_\circ + \gamma_n \varphi \Big\} \end{aligned} \qquad \text{, say, with } \varphi \equiv E_{n-1}\left(\frac{\delta_n}{\theta_\circ \delta_n + \xi_\circ} \right);$$

$$B_{n-1} \equiv \frac{1}{\zeta_{n-1}} E_{n-1} \zeta_n = \frac{1}{\zeta_{n-1}} \gamma_n E_{n-1}\left(\frac{1}{\theta_\circ \delta_n + \xi_\circ} \right) = \frac{1}{\zeta_{n-1}} \gamma_n \psi \text{, say.}$$

Next, the NPV of agent j's consumption is

$$\begin{aligned} \frac{1}{\zeta_{n-1}} E_{n-1}\Big[\sum_{m \geq n} \zeta_m g(m)\Big] &= \frac{1}{\zeta_{n-1}} \frac{q_j \beta_j^n}{1-\beta_j} \\ &= \theta_j(n-1) S_{n-1} + \xi_j(n-1) B_{n-1} \end{aligned}$$

so

$$\boxed{\frac{q_j \beta_j^n}{1-\beta_j} = \theta_j \Big(\sum_{m > n} \gamma_m / \theta_\circ + \gamma_n \varphi \Big) + \xi_j \gamma_n \psi}$$

where we abbreviate $\xi_j(n-1)$ to ξ_j, $\theta_j(n-1)$ to θ_j. As before, if we set

$$A \equiv \sum_{m>n} \gamma_m/\theta_0 + \gamma_n \varphi, \qquad B \equiv \gamma\psi$$

by cross multiplying by $(1-\beta_j)$ and summing on β_j we obtain the ratio A/B

$$\frac{A}{B} = \frac{\psi^{-1} - \Xi}{\Theta}$$

and as the values of q_j only matter up to some positive multiple, we may as well set

$$\boxed{\frac{q_j \beta_j^n}{1-\beta_j} = \theta_j(\psi^{-1} - \Xi) + \xi_j \Theta}$$

which determines the q_j once the θ_j and ξ_j are known. A special case of this is of course the situation on p30.

2) <u>What is the value to agent j of his optimal consumption stream?</u>

It's

$$E_{n-1}\Big[\sum_{m\geq n} \beta_j^m \log g(m)\Big] = E_{n-1}\Big[\sum_{m>n} \beta_j^m \log(q_j\beta_j^m \theta_0 \delta_m/\gamma_m) + \beta_j^n \log(q_j\beta_j^n(\theta_0\delta_n + \xi_0)/\gamma_n)\Big]$$

$$= E_{n-1}\Big[\sum_{m\geq n} \beta_j^m \log(q_j \theta_0 \delta_m/\gamma_m) + \beta_j^n \log\Big(1 + \frac{\xi_0}{\theta_0\delta_n}\Big)\Big] + \text{terms independent of } \theta, \xi.$$

$$= E_{n-1}\Big[\sum_{m\geq n} \beta_j^m \log\Big(\frac{q_j\theta_0}{\gamma_m}\Big) + \beta_j^n \log\Big(1+\frac{\xi_0}{\theta_0\delta_n}\Big)\Big] + \text{const}$$

$$= \sum_{m\geq n} \beta_j^m \log\left(\frac{\theta_j(\psi^{-1}-\Xi) + \xi_j\Theta}{\sum_i \beta_i^{m-n}\{\theta_i(\psi^{-1}-\Xi) + \xi_i\Theta\}(1-\beta_i)}\right) + \frac{\beta_j^n}{1-\beta_j}\log\theta_0 + E_{n-1}\Big(\log\Big(1+\frac{\xi_0}{\theta_0\delta_n}\Big)\Big)\beta_j^n$$

$$+ \text{const}$$

Thus

$$E_{n-1}\Big[\sum_{m\geq n} \beta_j^{m-n} \log g(m)\Big] = \sum_{r\geq 0} \beta_j^r \log\left(\frac{\theta_j(\psi^{-1}-\Xi) + \xi_j\Theta}{\sum_i \beta_i^r\{\theta_i(\psi^{-1}-\Xi)+\xi_i\Theta\}(1-\beta_i)}\right) + \frac{\log\theta_0}{1-\beta_j} + E_{n-1}\log\Big(1+\frac{\xi_0}{\theta_0\delta_n}\Big)$$

$$+ \text{const}.$$

<u>Special case</u>: $\xi_i = 0 \;\forall i$. This way, we get more simply

$$\sum_{r\geq 0} \beta_j^r \log\left(\frac{\theta_j}{\sum_i \beta_i^r \theta_i(1-\beta_i)}\right) + \frac{\log\theta_0}{1-\beta_j} + \text{const} = \sum_{r\geq 0}\beta_j^r \log\left(\frac{\theta_0\,\theta_j}{\sum_i \beta_i^r \theta_i(1-\beta_i)}\right) + \text{const}$$

If we now considered what would happen if we enlarge the coalition to include an agent with discount parameter β and

θ^* shares, the change in agent j's value would be

$$\sum_{r\geq 0} \beta_j^r \log\left[\frac{\theta_0+\theta^*}{\theta_0} \cdot \frac{\sum_{i\in A} \beta_i^r\, \theta_i (1-\beta_i)}{\sum_{i\in A} \beta_i^r\, \theta_i(1-\beta_i) + \beta^r\, \theta^*(1-\beta)}\right]$$

$$= \sum_{r\geq 0} \beta_j^r \log\left[\frac{\theta_0+\theta^*}{\theta_0} \quad \frac{A_r}{A_r+\beta^r\theta^*(1-\beta)}\right]$$

Note that the r^{th} term is non-negative iff

$$A_r(\theta_0+\theta^*) \geq \theta_0\left(A_r+\beta^r\theta^*(1-\beta)\right)$$

$$\Leftrightarrow \quad A_r\,\theta^* \geq \beta^r\,\theta_0\,\theta^*(1-\beta)$$

$$\Leftrightarrow \quad \sum_{i\in A}\left(\beta_i^r(1-\beta_i) - \beta^r(1-\beta)\right)\theta_i \geq 0$$

Thus the sum on r of these terms is zero, so they can't all be non-negative; the decision on admitting the new member is a lot more delicate.

Multiagent equilibria again (19/7/99)

An important observation is that, because of market completeness, once we have decided an agent's consumption stream $(c_j(m))_{m \geq n+1}$, there is a unique holding $(\theta_j(n), \xi_j(n))$ of shares and bonds at the end of day n which will achieve this consumption stream.

1) If we have an equilibrium of the form

$$p_j \beta_j^m / g(m) = \zeta_m, \qquad \zeta_m = \gamma_m / \delta_m \qquad (m \geq n+1)$$

what would be agent j's portfolio at the end of day n to achieve this? How would agent j invest dynamically to finance this consumption? If $w_j(m)$ denotes wealth at end of day m, we have

$$w_j(m) = \frac{1}{\zeta_m} E_m\Big[\sum_{r>m} \zeta_r\, g(r)\Big] = \frac{1}{\zeta_m}\, p_j\, \beta_j^{m+1}/(1-\beta_j).$$

The martingale is

$$M_r = \zeta_r w_j(r) + \sum_{t=n+1}^{r} \zeta_t\, g(t) = \frac{p_j \beta_j^{n+1}}{1-\beta_j}$$

which is constant. The wealth on day m before consumption is

$$w_j(m) + c_j(m) = \frac{p \beta_j^{m+1}}{\zeta_m (1-\beta_j)} + \frac{p_j \beta_j^m}{\zeta_m} = \frac{p \beta_j^m}{\zeta_m(1-\beta_j)} = \frac{\zeta_{m-1}}{\zeta_m} w_j(m-1);$$

what portfolio mix at the end of day $m-1$ would achieve this? To answer this, note that

$$S_r = \frac{1}{\zeta_r} \sum_{t>r} \gamma_t, \quad \delta_r = \gamma_r / \zeta_r$$

so

$$\frac{S_m + \delta_m}{S_{m-1}} = \frac{\zeta_{m-1}}{\zeta_m}$$

and therefore the investment which turns $w_j(m)$ into $w_j(m) + g(m)$ is just to put all wealth into share.

2) Suppose agents enter day n with $(\theta_j(n-1), \xi_j(n-1))$ not all the $\xi_j(n-1)$ zero, and we are after an equilibrium.

In equilibrium, there is Arrow-Debreu pricing given by the state-price density process $(\zeta_m)_{m \geq n}$. If Y is a contingent claim paid at time $m > n$, then the value to agent j of εY would be $\varepsilon E_n\big[Y\, U_j'(m, g(m))\big] + o(\varepsilon)$, and at time n he would have to pay $\varepsilon E_n[\zeta_m Y]$ for this, which would reduce his utility by $\varepsilon U_j'(n, g(n)) E_n[\zeta_m Y] + o(\varepsilon)$. Hence

$$\zeta_m = U_j'(m, g(m)) / U_j'(n, g(n))$$

or if we allow some scalar multiple of ζ, we'd have for $m \geq n$

$$\frac{\zeta_m}{\zeta_n} = \frac{U_j'(m, g(m))}{U_j'(n, g(n))}.$$

Thus we have

$$\zeta_m = p_j \beta_j^m / g(m) \quad , \quad \zeta_m = \gamma_m / \delta_m \qquad (m \geq n)$$

for some parameters p_j which are yet to be determined. The ex-dividend price S_n of the share on day n will be

$$S_n = \frac{1}{\zeta_n} E_n\Big[\sum_{m>n} \zeta_m \delta_m\Big] = \frac{1}{\zeta_n} \sum_{m>n} \gamma_m = \frac{1}{\zeta_n} \sum_i \frac{p_i \beta_i^{n+1}}{1-\beta_i}$$

NPV of all of agent j's consumption from day $n+1$ onwards is

$$\frac{1}{\zeta_n} E_n \sum_{m>n} \zeta_m g(m) = \frac{1}{\zeta_n} \frac{p_j \beta_j^{n+1}}{1-\beta_j}$$

so that the NPV of all of agent j's consumption from day n onwards is

$$\frac{1}{\zeta_n} \frac{p_j \beta_j^n}{1-\beta_j} = \frac{g(n)}{1-\beta_j}$$

The budget constraints are

$$\theta_j(n-1)\{S_n + \delta_n\} + \xi_j(n-1) = \frac{g(n)}{1-\beta_j} = \frac{1}{\zeta_n} \frac{p_j \beta_j^n}{1-\beta_j} \qquad (j = 1, \dots, J)$$

So we shall have

$$\Theta (S_n + \delta_n) + \Xi = \delta_n \qquad \left[\begin{array}{l} \Theta \equiv \sum_j (1-\beta_j)\, \theta_j(n-1) \\ \Xi \equiv \sum_j (1-\beta_j)\, \xi_j(n-1) \end{array}\right]$$

giving the price for the share:

$$\boxed{S_n = \frac{\delta_n - \Xi}{\Theta} - \delta_n}$$

If we impose the normalisation $\boxed{\zeta_n = 1}$ we then obtain the p_j explicitly in terms of the $\theta_j(n-1)$ and $\xi_j(n-1)$:

$$\boxed{p_j = \frac{1-\beta_j}{\beta_j^n} \Big\{ \theta_j(n-1)(S_n + \delta_n) + \xi_j(n-1) \Big\},}$$

and the payoff to agent j is easily calculated to be

$$\boxed{\sum_{m>n} \beta_j^m \log\left(\frac{p_j \beta_j^m}{\gamma_m}\right) + \frac{\beta_j^n}{1-\beta_j} \log \delta_n + \frac{\beta_j^{n+1}}{(1-\beta_j)^2} E \log Z_1 .}$$

At end of day n, agent j holds

$$\boxed{\beta_j S_n^{-1} \{ \theta_j(n-1)(S_n + \delta_n) + \xi_j(n-1) \} \text{ shares.}}$$

<u>Observe:</u> Suppose we take instead the situation where agent J picks a consumption process $(c^*(m))_{m\geq n}$ which he wishes to enjoy. The remaining agents achieve an equilibrium with residual consumption stream

$$\tilde{\delta}_m \equiv \delta_m - c^*(m)$$

so as before we have for some p_j $(j<J)$ that

$$p_j\, u_j'(m, g(m)) = \zeta_m, \qquad \zeta_m = \tilde{\gamma}_m/\tilde{\delta}_m \equiv \sum_{j<J} p_j \beta_j^m / \tilde{\delta}_m,$$

and NPV of all of j's consumption from n onward is $g(n)/(1-\beta_j)$, exactly as here →

So the equilibrium price we arrive at is

$$S_n + \delta_n = (\delta_n - c^*(n) - \tilde{\Xi})/\tilde{\Theta}$$

<u>exactly</u> as before. In order for this all to work, we need the budget constraint also for J:

$$\theta_J(n-1)(S_n+\delta_n) + \xi_J(n-1) = \frac{1}{\zeta_n} E_n\left[\sum_{m\geq n} \zeta_m\, c^*(m)\right]$$

Notice one interesting thing: the equilibrium price S_n depends on $c^*(n)$, but not on future $c^*(m)$!! The only involvement of those is through feasibility or otherwise of the solution.

<u>CARE!!</u> This assumes that agents $1, \ldots, J-1$ don't hold bonds...

3) Suppose that on day n, agent J chooses to consume c^*, but fits in with equilibrium on subsequent days. Thus he is willing to exchange consumption on any later day with any other agent, but not on day n. So he agrees with the market on the relative pricing of consumption at any subsequent node of the tree. So there are constants p_j such that

$$p_j\, U_j'(m, g(m)) = \zeta_m \qquad \text{for all } m \geq n,\ \forall j \ \underline{\text{except}}\ m = n,\ j = J.$$

Hence $\zeta_m = \gamma_m / \delta_m$ for all $m > n$ as before, and for $m = n$ we obtain

$$\zeta_n = \sum_{j<J} p_j \beta_j^n \,/ (\delta_n - c^*)$$

The budget constraints to be satisfied are

$$\theta_j(n-1)\{S_n + \delta_n\} + \xi_j(n-1) = \frac{g_j(n)}{1-\beta_j} = \frac{1}{\zeta_n}\frac{p_j\beta_j^n}{1-\beta_j} \qquad (j \neq J)$$

$$= c^* + \frac{1}{\zeta_n}\frac{p_J\beta_J^{n+1}}{1-\beta_J} \qquad (j = J)$$

So if $\tilde{\Theta} = \sum_{j\neq J}(1-\beta_j)\,\theta_j(n-1)$, $\tilde{\Xi} = \sum_{j\neq J}(1-\beta_j)\,\xi_j(n-1)$, we shall have

$$\tilde{\Theta}(S_n + \delta_n) + \tilde{\Xi} = \delta_n - c^*$$

giving equilibrium price

$$\boxed{S_n + \delta_n = (\delta_n - c^* - \tilde{\Xi}) / \tilde{\Theta}}$$

Imposing $\zeta_n = 1$ allows us to write down the p_j:

$$\boxed{\begin{aligned} p_j &= \{\theta_j(n-1)(S_n+\delta_n) + \xi_j(n-1)\}\,(1-\beta_j)/\beta_j^n \qquad (j<J) \\ p_J &= \{\theta_J(n-1)(S_n+\delta_n) + \xi_J(n-1) - c^*\}\,(1-\beta_J)/\beta_J^n \end{aligned}}$$

Since the NPV of agent j's consumption from $n+1$ onward is $\zeta_n^{-1}\, p_j \beta_j^{n+1}/(1-\beta_j)$, the shares have to be held in these proportions, so

$$\boxed{\begin{aligned} \theta_j(n) &= \lambda\,\beta_j\{\theta_j(n-1)(S_n+\delta_n) + \xi_j(n-1)\} \qquad (j<J) \\ \theta_J(n) &= \lambda\,\beta_J\{\theta_J(n-1)(S_n+\delta_n) + \xi_J(n-1) - c^*\} \end{aligned}}$$

where λ is a normalisation constant to make $\sum \theta_j(n) = 1$

$$\boxed{\lambda = \tilde{\Theta}\{(1-\Theta)(\delta_n - c^* - \tilde{\Xi}) - \tilde{\Theta}(\Xi + \beta_J c^*)\}^{-1}}$$

Leland's working paper "Bond prices, yield spreads, and optimal capital structure with default risk", Nov. 1994, explains how to interpret this assumption in terms of a sinking fund.

The structural approach to credit risk with discontinuous processes (23/7/99)

1) The idea is to develop the analysis of the model of Leland & Toft to a situation where the value of the firm's assets V_t is expressed in the form $V_t = V_0 \exp(X_t)$, where X is a spectrally-negative Lévy process, $E e^{zX_t} = \exp(t\psi(z))$, $Re(z) \geq 0$, and $\psi(1) = r-\delta$, since we're assuming a constant dividend rate of δ, and all calculations are being done in the pricing measure. Let's develop the story in the general situation as far as possible, and then we'll look at special cases of ψ.

We'll set some level V_B such that bankruptcy will be declared when the firm's value hits V_B or goes below it. So if

$$H \equiv \inf \{t : X_t < \log(V_B/V_0)\}$$

then H is the bankruptcy time.

The value of debt with maturity t, face value 1, receiving a coupon stream at constant rate $c(t)$ and recovering a fraction $\rho(t)$ on default will be

$$d(V_0, V_B, t) = c(t)\int_0^t e^{-rs} P(H>s)\,ds + e^{-rt} P(H>t) + \rho(t)\, E\left[V_0 e^{X(H)-rH} : H\leq t\right]$$

2) Assume the firm keeps a constant debt profile $P.\, a e^{-at}$.

Thus there is always debt with face value $P a e^{-at}\,dt$ of maturity t, and the value of the debt is

$$D(V_0, V_B) = P\Big[\int_0^\infty a e^{-at} c(t) \int_0^t e^{-rs} P(H>s)\,ds\,dt + a\int_0^\infty e^{-rt} P(H>t)\,dt\, e^{-at}$$

$$+ \int_0^\infty a e^{-at} \rho(t)\, E\left[V_0 e^{X(H)-rH} : H \leq t\right] dt \Big]$$

$$= P\left\{ \int_0^\infty e^{-rs} P(H>s) \int_s^\infty a e^{-at} c(t)\,dt\,ds + \frac{a}{a+r}\left(1 - E e^{-(a+r)H}\right)\right.$$

$$\left. + V_0 E\left\{ \exp(X(H)-rH)\, \varphi(H)\right\} \right\}$$

where $\phi(s) = \int_s^\infty a e^{-at} . \rho(t)\,dt$.

Without more explicit assumptions on the form of $\rho(\cdot)$ and $c(\cdot)$, there's not a lot we can do. From the sinking-fund interpretation, it seems that we have to assume that c is constant; when the debt is issued, there is no way of knowing when it will be redeemed, it seems. We could use the time-dependence of ρ to model some sort of seniority: if on bankruptcy the older the debt the more senior, then the payout would be only to debt older than a certain age, which would make $\rho(t) = \rho e^{-at}$. To incorporate this, let's assume $\rho(t) = \rho e^{-\varepsilon t}$, where $\varepsilon = 0$, $\varepsilon = a$ are natural choices.

With these assumptions we get

$$\boxed{D(V_0, V_B) = P\left\{ \frac{c+a}{a+r}\left(1 - E\, e^{-(a+r)H}\right) + \frac{a V_0 \rho}{a+\varepsilon} E\left[\exp(X_H - (a+r+\varepsilon)H)\right] \right\}}$$

We can relate ρ to recovery on default by thinking what would happen if $V_0 = V_B +$. Then default is immediate, and the value of the debt should be $(1-\alpha)V_B$, the amount which the bondholders recover. Thus

$$\boxed{(1-\alpha) = a\rho P / (a+\varepsilon)}$$

links α and ρ.

3) So we need to know about $E\, e^{-\lambda H} = P[T_\lambda > H] = P(\underline{X}(T_\lambda) < x)$, where $x = \log(V_B/V_0)$, and $E \exp(X_H - \lambda H)$. By the WH factorisation,

$$\frac{\lambda}{\lambda - \psi(z)} = E\, e^{z\bar{X}(T_\lambda)} \cdot E\, e^{z\underline{X}(T_\lambda)} \equiv \psi_\lambda^+(z) \cdot \psi_\lambda^-(z)$$

For a spectrally negative Lévy process, things simplify: $\bar{X}(T_\lambda)$ has to be exponentially distributed, so

$$\psi_\lambda^+(z) = \frac{\beta^*}{\beta^* - z}$$

for some β^*. Comparing the poles tells us that $\psi(\beta^*) = \lambda$, so providing we can invert ψ (which will have to be numerically for any non-Brownian example), we can do the WH factorisation, discover $\psi_\lambda^-(z)$ and hence by (numerical) inversion of the Laplace transform we get $E\exp(-\lambda H) = P(\underline{X}(T_\lambda) < x)$.

As for $E\exp(X_H - \lambda H)$, let's write $H \equiv H_x$ to emphasise the dependence on $x<0$, and note that

$$E \exp(\theta X(H_x) - \lambda H_x) \cdot E \exp \theta \underline{X}(T_\lambda) = E[e^{\theta \underline{X}(T_\lambda)} ; T_\lambda > H_x]$$

so

$$\psi_\lambda^-(\theta) \cdot \int_{-\infty}^0 \mu e^{\mu x} E \exp(\theta X(H_x) - \lambda H_x)\, dx = \int_{-\infty}^0 \mu e^{\mu x} E\left[e^{\theta \underline{X}(T_\lambda)} : \underline{X}(T_\lambda) < x\right] dx$$

$$= E\left[e^{\theta \underline{X}(T_\lambda)}\left\{1 - e^{\mu \underline{X}(T_\lambda)}\right\}\right]$$

$$= \psi_\lambda^-(\theta) - \psi_\lambda^-(\theta+\mu)$$

$$\therefore \quad \boxed{\int_{-\infty}^0 \mu e^{\mu x} E \exp(\theta X(H_x) - \lambda H_x)\, dx = \frac{\psi_\lambda^-(\theta) - \psi_\lambda^-(\theta+\mu)}{\psi_\lambda^-(\theta)}}$$

which gives us the LT of the thing we're interested in (of course, this formula is not new - Bingham cites it in his survey article).

4) How about the endogenous bankruptcy level? As before, we require the condition

$$\left.\frac{\partial E}{\partial V_0}\right|_{V_0=V_B} = 0, \qquad E = v - D, \quad v \text{ the value of the firm.}$$

Write $V_0 = e^{-x} V_B > V_B$, so that the condition we want is equivalent to

$$\left.\frac{\partial}{\partial x} D(e^{-x}V_B, V_B)\right|_{x=0} = \left.\frac{\partial}{\partial x} v(e^{-x}V_B)\right|_{x=0}$$

Discarding the factor P, the LHS of this equality is

$$0 = \lim_{x\uparrow 0} \left\{ \frac{a+c}{a+r} \frac{E\, 1-e^{-(a+r)H_x}}{-x} - \frac{aV_B\rho}{a+\varepsilon} \frac{1-E\, e^{X(H_x)-(a+r+\varepsilon)H_x}}{-x} \right\} - \frac{aV_B\rho}{a+\varepsilon}.$$

But $-x^{-1}\{1-E\, e^{-(a+r)H_x}\} = -x^{-1} P\left(\underline{X}(T_{a+r}) > x\right)$

$$= -x^{-1}\int_x^0 f(y)\,dy \to f(0)$$

where f is the density of $\underline{X}(T_{a+r})$. But we know that $\int_{-\infty}^0 f(y)\, e^{zy}\,dy = \psi_\lambda^-(z)$ so $f(0) = \lim_{z\to\infty} z\psi_{a+r}^-(z)$. Similarly,

$$\frac{1-E\, e^{X(H_x)-(a+r+\varepsilon)H_x}}{-x} = -\frac{1}{x} \underbrace{E\left(1-e^{X(H_x)-(a+r+\varepsilon)H_x}\right)}$$

this has transform given by the box at foot of previous page

$$\to \lim_{z\to\infty} z\,\psi_{a+r+\varepsilon}^-(1+z) \big/ \psi_{a+r+\varepsilon}^-(1)$$

If we now suppose that the Lévy process is a drifting BM of variance $\sigma^2 t$ plus a compound Poisson process, we have

$$\psi_\lambda^-(z) = \frac{\beta^*-z}{\beta^*} \frac{\lambda}{\lambda - \frac{1}{2}\sigma^2 z^2 - \gamma z - \int (e^{zt}-1)\, \mu(dt)},$$

so $\boxed{z\psi_\lambda^-(z) \to \dfrac{2\lambda}{\sigma^2\beta^*(\lambda)}}$ as $z\to\infty$

This is the key to finding the optimal bankruptcy trigger. We now have explicitly

$$\frac{\partial}{\partial x} D(V_B e^{-x}, V_B)\Big|_{x=0} = P\left\{ \frac{2(a+c)}{\sigma^2 \beta^*(a+r)} - \frac{a V_B p}{a+\varepsilon}\left(1 + \frac{2(a+r+\varepsilon)}{\psi^-_{a+r+\varepsilon}(1)\,\sigma^2 \beta^*(a+r+\varepsilon)}\right)\right\}$$

More simply,

$$v = V_0 + \frac{c\tau P}{r} E(1-e^{-rH}) - \alpha V_0 E\left[e^{-rH + X(H)}\right]$$

so
$$\frac{\partial v}{\partial x}\Big|_{x=0} = -V_B + \frac{c\tau P}{r}\cdot\frac{2\varphi}{\sigma^2\beta^*(r)} + \alpha V_B \frac{2(a+r+\varepsilon)}{\psi^-_{a+r+\varepsilon}(1)\,\sigma^2\beta^*(a+r+\varepsilon)} + \alpha V_B$$

Writing $f(x,\lambda) \equiv E\left[1-e^{-\lambda H_x}\right]$, $g(x,\lambda) \equiv E\left[1-e^{X(H_x)-\lambda H_x}\right]$ for $\lambda > 0 > x$, we know that $f(0,\lambda) = g(0,\lambda) = 0$, and $f'(0,\lambda) = -\lim_{z\to\infty} z\psi^-_\lambda(z)$, $g'(0,\lambda) = -\lim_{z\to\infty} z\psi^-_\lambda(z)/\psi^-_\lambda(1)$, and in terms of these,

$$\begin{cases} D = \frac{P(a+c)}{a+r} f(x, a+r) + (1-\alpha)V_B e^{-x}\{1-g(x,a+r+\varepsilon)\} \\ v = V_B e^{-x} + \frac{c\tau P}{r} f(x,r) - \alpha V_B e^{-x}\{1-g(x,r)\} \end{cases}$$

Hence we obtain for the bankruptcy trigger level

$$\boxed{V_B = \frac{-\frac{P(a+c)}{a+r} f'(0,a+r) + \frac{c\tau P}{r} f'(0,r)}{-\alpha g'(0,r) - (1-\alpha) g'(0, r+a+\varepsilon)}}$$

In the Brownian case, we have $f(x,\lambda) = 1 - e^{x\gamma(\lambda)}$, $g(x,\lambda) = 1 - e^{x + x\gamma(\lambda)}$ where

$$\gamma(\lambda) = \left[(r-\delta-\sigma^2/2) + \{(r-\delta-\sigma^2/2)^2 + 2\sigma^2\lambda\}^{1/2}\right]/\sigma^2 .$$

This is in agreement with Leland's findings.

5) If there are only negative jumps, with intensity λ_0, distributed as $\exp(\gamma)$, then

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + bz - \lambda_0 z/(z+\gamma) \qquad \left(b = r-\delta-\tfrac{1}{2}\sigma^2 + \frac{\lambda_0}{1+\gamma}\right)$$

This gives us

$$\psi^-_\lambda(z) = \frac{\lambda\{\psi^{-1}(\lambda) - z\}}{\psi^{-1}(\lambda)\{\lambda - (\frac{1}{2}\sigma^2 z^2 + bz - \lambda_0 z/(z+\gamma))\}}$$

Assuming wlog that $p_j = \rho_j \theta_j(0)$, agent j's payoff is

$$E \int_0^\infty e^{-\beta_j t} \log\left(\frac{\rho_j \theta_j(0) e^{-\beta_j t}}{\sum_{i<j} \rho_i \theta_i(0) e^{-\beta_i t}} \cdot \varphi_t \delta_t \right) dt$$

Large investor in continuous time (9/8/99)

1) Let's return to the situation of log-utility investors in continuous time. The share produces dividend δ_t and agent J declares at time 0 his intention to consumption stream δ^*_t, leaving $\tilde\delta_t \equiv \delta_t - \delta^*_t$ for agents $1,\dots,J-1$. As before, we shall have for some $p_j > 0$

$$p_j U_j'(t, g(t)) = p_j e^{-\rho_j t}/g(t) = \zeta_t$$

so that $$\boxed{\zeta_t = \tilde\gamma_t / \tilde\delta_t}$$ where $\tilde\gamma_t \equiv \sum_{j<J} p_j e^{-\rho_j t}$.

The share price process that results will be

$$S_t = \frac{1}{\zeta_t} E_t\left[\int_t^\infty \zeta_u \delta_u \, du\right],$$

the npv of j's consumption will be

$$w_j(t) \equiv \frac{1}{\zeta_t} E_t\left[\int_t^\infty \zeta_u g(u)\, du\right] = \frac{p_j e^{-\rho_j t}}{\rho_j \zeta_t} = \frac{g(t)}{\rho_j}.$$

In order to make progress, we'll need to be making some assumptions about δ^*.

2) <u>Suppose that $\tilde\delta_t = \varphi(t)\delta_t$</u> <u>for some deterministic φ</u>. Then we have

$$S_t = \delta_t \frac{\varphi_t}{\tilde\gamma_t} \int_t^\infty \frac{\tilde\gamma_u}{\varphi_u} du = -\delta_t \psi_t / \dot\psi_t \qquad \left(\psi(t) \equiv \int_t^\infty \tilde\gamma_u \varphi_u^{-1} du\right)$$

and

$$w_j(t) = \frac{p_j e^{-\rho_j t}}{\rho_j} \frac{\varphi_t}{\tilde\gamma_t} \cdot \delta_t = \theta_j(t) S_t \qquad \left[\theta_j(t) \equiv -\frac{p_j e^{-\rho_j t}}{\rho_j \tilde\gamma_t} \cdot \varphi_t \cdot \frac{\dot\psi_t}{\psi_t}\right]$$

The wealth equation

$$dw_j(t) = \theta_j(t)\left\{dS_t + \delta_t\, dt\right\} - g(t)\,dt$$

is satisfied, so all the agents in this situation keep all wealth in the share (assuming φ is cts).

Agent j's payoff is

$$E\left[\int_0^\infty e^{-\rho_j t} \log(\rho_j \theta_j(t) S_t)\, dt\right] = \int_0^\infty e^{-\rho_j t} \log \varphi_t \, dt + \text{terms not dependent on } \varphi.$$

At later time t_0, agent J could change to another consumption plan so long as the number of shares $\theta_J(t_0)$ didn't change, equivalently, so long as $\int_{t_0}^\infty \tilde\gamma_u \varphi_u^{-1} du$ is unchanged. By doing this, J should be able to put all the others at a disadvantage.

3) Under what circumstances would the pool of agents $\{1,\dots,J-1\}$ prefer to include J?

Care! It might be that some of the agents were actually better off!

If J were willing to join in the equilibrium with everyone else, then the consumption stream to agent j is

$$\frac{\rho_j\, \theta_j(0)\, e^{-\rho_j t}}{\sum_{k\le J} \rho_k \theta_k(0) e^{-\rho_k t}} \cdot \delta_t$$

in contrast to

$$\frac{\rho_j\, \theta_j(0)\, e^{-\rho_j t}}{\sum_{i< J} \rho_i \theta_i(0) e^{-\rho_i t}} \cdot \sum_{i<J} \theta_i(0) \cdot \delta_t$$

if the pool decided to go it alone. Agent $j \in A \equiv \{1, \dots, J-1\}$ would prefer to go it alone in A rather than be part of $B \equiv \{1, \dots, J\}$ iff

$$\int_0^\infty e^{-\rho_j t} \log\left[f_B(t) / f_A(t) \right] dt > 0 \qquad \left(f_A(t) \equiv \frac{\sum_{i\in A} \rho_i \theta_i(0) e^{-\rho_i t}}{\sum_{i\in A} \theta_i(0)} \right)$$

Is it possible that this should hold for all $j \in A$?

Direct analysis appears hopeless, but there's a piece of general economic theory which applies here. In the equilibrium analysis of the situation for B, we end up with a competitive allocation – there are equilibrium prices, given by the state-price density, and each agent maximises his payoff subject to the budget constraint implied by those prices. The allocation is therefore in the core: there is no blocking coalition. So if we considered all allocations of consumption to agents in $A \subset B$ of the resources of A, it's not possible for all in A to do strictly better – if they did, then all of the allocations to agents in j would have to violate the B-equilibrium budget constraints, because in the B-equilibrium each agent had got the maximal payoff subject to those constraints. Thus the B-equilibrium price of the allocation to $j \in A$ is more than the B-equilibrium price of the resources available to j, and this is a contradiction.

But would the agents $j \in A$ want to include J if he was going to demand a consumption stream $(1-\varphi(t))\delta_t$? He would have to pick $\varphi(t)$ so that each $j \in A$ was indifferent to including him or not:

$$\int_0^\infty e^{-\rho_j t} \log \varphi(t)\, dt \overset{\geq}{=} \frac{1}{\rho_j} \log\left(\sum_{i\in A} \theta_i(0) \right) \qquad \forall j = 1, \dots, J-1$$

and he would have to start from the correct number of shares: after some manipulation, this is the condition

$$\int_0^\infty \tilde{\gamma}_t / \varphi_t \, dt \overset{\leq}{=} 1 . \qquad \left[\tilde{\gamma}_t \equiv \sum_{i<J} \rho_i \theta_i(0) e^{-\rho_i t} \right]$$

Thus agent J would try to max $\int_0^\infty e^{-\rho_J t} \log(1-\varphi_t)\, dt$ subject to these constraints.
The Lagrangian form is

$$\int_0^\infty \left\{ e^{-\rho_J t} \log(1-\varphi_t) + \left(\sum_{i<J} \lambda_i e^{-\rho_i t}\right) \log \varphi_t - \beta\, \tilde{\gamma}_t / \varphi_t \right\} dt$$

So we get the optimality condition

$$-\frac{e^{-\rho_J t}}{1-\varphi_t} + \frac{\sum_{i<J} \lambda_i e^{-\rho_i t}}{\varphi_t} + \frac{\beta \tilde{\gamma}_t}{\varphi_t^2} = 0$$

$$\equiv -\frac{A}{1-\varphi_t} + \frac{B}{\varphi_t} + \frac{C}{\varphi_t^2} \quad , \quad \text{say}$$

$$\begin{cases} A \equiv e^{-\rho_J t} \\ B \equiv \sum_{i<J} \lambda_i e^{-\rho_i t} \\ C \equiv \beta \tilde{\gamma}_t \end{cases}$$

This leads to the quadratic

$$(A+B)\varphi^2 + (C-B)\varphi - C = 0$$

and we want the root in $(0,1)$ of this. This gives

$$\varphi_t = \frac{B-C + \{(B+C)^2 + 4AC\}^{1/2}}{2(A+B)} .$$

Picking the multipliers to match the constraints looks a bit monotonous, but it should be possible to do something numerical.

(Note that φ need not always be monotone; we can choose λ_i, β, $\theta_i(0)$, ρ_i to exhibit non-monotone φ: $\rho_0 = 0.1$, $\rho_1 = 0.3$, $\rho_2 = 1.0$, $\lambda_0 = 0.25$, $\lambda_1 = 0.3$, $\beta = 4$, $\theta_0 = \frac{1}{4} = \theta_2$, $\theta_1 = 0.5$.)

Information conveyed through prices (24/8/99)

1) Suppose we have J agents in a discrete-time model, which for the moment we take to be single-period. The agents can invest in a vector of assets which will give returns $(\delta_1, \dots, \delta_N)$ and they may invest in a bond which delivers 1 for certain. Agent j aims to

$$\max_{\theta,\psi} E\, U_j(\theta\cdot\delta + \psi) \equiv \max_{\theta,\psi} E\ -\exp(-\gamma_j(\theta\cdot\delta+\psi))$$

subject to the budget constraint

$$p\cdot\theta + \xi\psi = w_j .$$

Here, the vector p of prices of the risky assets, and the price ξ of the bond, must be chosen to ensure market clearing. The initial wealth is denoted w_j.

Let's suppose that the agents have different beliefs about the returns; suppose agent j thinks that $\delta \sim N(\mu_j, V_j)$, so the Lagrangian form of his problem is to

$$\min_{\theta,\psi}\ -\gamma_j\psi - \gamma_j\theta\cdot\mu_j + \tfrac{1}{2}\gamma_j^2\theta_j^T V_j\theta_j - \lambda(p\cdot\theta+\xi\psi),$$

so

$$\begin{cases} \gamma_j + \lambda_j\xi = 0 \\ \theta_j = \gamma_j^{-2}V_j^{-1}(\gamma_j\mu_j + \lambda_j p) = \gamma_j^{-1}V_j^{-1}(\mu_j - \xi^{-1}p) \end{cases}$$

Assuming there's just one share of each asset, the market-clearing condition gives us

$$\boxed{\xi^{-1}p = \left(\sum_j \gamma_j^{-1}V_j^{-1}\right)^{-1}\left(-1 + \sum_j \gamma_j^{-1}V_j^{-1}\mu_j\right).}$$

On the other hand, the wealth equation for agent j

$$p\cdot\theta_j + \xi\psi_j = w_j$$

yields
$$\boxed{p\cdot 1 = \sum_j w_j}$$

when we sum, using the market-clearing condition that the net supply of bonds is zero.

So <u>knowing the equilibrium prices is equivalent to knowing</u>

<u>the vector</u> $\left(\sum_j \gamma_j^{-1}V_j^{-1}\right)^{-1}\left(-1 + \sum \gamma_i^{-1}V_i^{-1}\mu_i\right)$ <u>and</u>

<u>the aggregate wealth</u> $\sum_j w_j$,

and the equilibrium is unique.

2) Now we'll consider a multiperiod model where each agent aims to maximise expected utility of terminal wealth, so at each stage we have a situation like the one considered above. We're going to deal with the problem under the assumption that at all times each agent's conditional distribution of the true mean of the dividends about to be observed, and of the other agents' estimators, is MVN. Everyone knows everyone else's covariances, but the actual values of their estimates is private.

Let's suppose that the dividend vector $\delta(n)$ on day n is $N(\mu(n), \Sigma_\delta)$, and that the process $\mu(n)$ is stationary AR(1)

$$\mu(n) = B\,\mu(n-1) + \varepsilon(n)$$

where the $\varepsilon(n)$ are IID $N(0, \Sigma_\varepsilon)$, and the stationary law of $\mu(n)$ is $N(0, \Sigma_\mu)$. We lose no generality in assuming the mean of μ is zero, since the information conveyed is not changed if a constant is added.

The state vector at any time τ is

$$X(\tau) \equiv \begin{pmatrix} \mu(\tau) \\ \mu_1(\tau) \\ \vdots \\ \mu_J(\tau) \end{pmatrix}$$

where $\mu_i(\tau) = E[\,\mu(\tau) \mid \mathcal{Y}^i(\tau)\,]$ and $\mathcal{Y}^i(\tau)$ is the σ-field of information available to agent i at time τ. We have that

$$\left(X(\tau) \mid \mathcal{Y}^j(\tau) \right) \sim N\left(\begin{pmatrix} \mu_j(\tau) \\ \mu_1^j(\tau) \\ \vdots \\ \mu_J^j(\tau) \end{pmatrix}, \begin{pmatrix} V_{00}^j(\tau) & V_{01}^j(\tau) & \cdots & V_{0J}^j(\tau) \\ \vdots & & & \\ V_{J0}^j(\tau) & \cdots & & V_{JJ}^j(\tau) \end{pmatrix} \right)$$

where $\mu_j^i(\tau) \equiv E[\,\mu_j(\tau) \mid \mathcal{Y}^i(\tau)\,]$. Period n unfolds in 3 phases:

<u>First phase, $\tau = n-$ (private phase)</u>

The value of μ is updated $\mu(n) = B\,\mu(n-1) + \varepsilon(n)$ and each agent receives a private signal $z^j(n) = \mu(n) + \eta^j(n)$, where the $\eta^j(n)$ are independent, $\eta^j(n) \sim N(0, \Sigma_j)$;

<u>Second phase, $\tau = n$ (bargaining phase)</u>

The equilibrium prices, equivalently, $\sum_{j=1}^{J} \gamma_j^{-1} V_{00}^j(n-)^{-1} \mu_j(n-)$, are revealed;

<u>Third phase, $\tau = n+$ (resolution phase)</u>

The values of $\delta(n)$ are revealed.

3) How does the updating go? When we enter period n, with information $y^j(n-1+)$, our estimate of $\mu(n)$ will be simply $B\mu_j(n-1+)$, so conditional on $y^j(n-1+)$ the state vector has distribution

$$N\left(\begin{bmatrix} B\mu_j(n-1+) \\ B\mu_1^j(n-1+) \\ \vdots \\ B\mu_J^j(n-1+) \end{bmatrix}, \begin{bmatrix} \Sigma_\varepsilon + BV_{00}^j(n-1+)B^T & BV_{01}^j(n-1+)B^T & \cdots & BV_{0J}^j(n-1+)B^T \\ BV_{10}^j(n-1+)B^T & & \cdots & \cdots \\ \vdots & & & \\ BV_{J0}^j(n-1+)B^T & & \cdots & BV_{JJ}^j(n-1+)B^T \end{bmatrix} \right)$$

$$\equiv N\left(\begin{bmatrix} \mu_j(n--) \\ \mu_1^j(n--) \\ \vdots \\ \mu_J^j(n--) \end{bmatrix}, \begin{bmatrix} V_{00}^j(n--) & \cdots & V_{0J}^j(n--) \\ \vdots & & \vdots \\ V_{J0}^j(n--) & \cdots & V_{JJ}^j(n--) \end{bmatrix} \right)$$

Having seen his signal $z^j(n)$, agent j updates the conditional expectation of μ to

$$\mu_j(n-) = \mu_j(n--) + V_{00}^j(n--)\left(\Sigma_j + V_{00}^j(n--)\right)^{-1}\left(z_j(n) - \mu_j(n--)\right)$$

and of $\mu_i(n--)$ to

$$\mu_i^j(n--) + V_{i0}^j(n--)\left(\Sigma_j + V_{00}^j(n--)\right)^{-1}\left(z_j(n) - \mu_j(n--)\right).$$

However, we require the conditional expectation given $y^j(n-)$ of $\mu_i(n-)$, not of $\mu_i(n--)$! Now conditional on $y^j(n--)$, we have

$$\begin{bmatrix} z^1(n) \\ \vdots \\ z^J(n) \\ \hline \mu(n) \\ \mu_1(n--) \\ \vdots \\ \mu_J(n--) \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_j(n--) \\ \vdots \\ \mu_j(n--) \\ \hline \mu_j(n--) \\ \mu_1^j(n--) \\ \vdots \\ \mu_J^j(n--) \end{bmatrix}, \left[\begin{array}{cccc|cccc} V_{00}^j(n--)+\Sigma_1 & V_{00}^j(n--) & \cdots & V_{00}^j(n--) & V_{00}^j(n--) & V_{01}^j(n--) & \cdots & V_{0J}^j(n--) \\ \vdots & & & \vdots & \vdots & & & \vdots \\ V_{00}^j(n--) & V_{00}^j(n--) & \cdots & V_{00}^j(n--)+\Sigma_J & V_{00}^j(n--) & V_{01}^j(n--) & \cdots & V_{0J}^j(n--) \\ \hline & & & & & & & \\ & \text{(symmetrically)} & & & & V^j(n--) & & \end{array}\right] \right)$$

So if we write $\nu_j(n) \equiv \left(\Sigma_j + V_{00}^j(n--)\right)^{-1}\left(z_j(n) - \mu_j(n--)\right)$ as an abbreviation for the innovation revealed to agent j in his private signal, we shall then have that conditional on $y^j(n-)$

$$\begin{bmatrix} z^1(n) \\ \vdots \\ z^J(n) \\ \hdashline \mu(n) \\ \mu_1(n--) \\ \vdots \\ \mu_J(n--) \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_j(n--) + V y_j(n) \\ \vdots \\ \mu_j(n--) + V y_j(n) \\ \hdashline \mu_j(n--) + V y_j(n) \\ \mu_1^j(n--) + V_{10}^j(n--) y_j(n) \\ \vdots \\ \mu_J^j(n--) + V_{J0}^j(n--) y_j(n) \end{bmatrix}, \ \text{previous} - \begin{bmatrix} V \\ \vdots \\ V \\ \hdashline V \\ V_{10}^j(n--) \\ \vdots \\ V_{J0}^j(n--) \end{bmatrix} (V+\Sigma_j)^{-1} \begin{bmatrix} V \\ \vdots \\ V \\ \hdashline V \\ V_{10}^j(n--) \\ \vdots \\ V_{J0}^j(n--) \end{bmatrix}^T \right)$$

where $V \equiv V_{00}^j(n--)$ for brevity. Thus conditional on $y^j(n-)$, we have that

$$\begin{bmatrix} z^1(n) \\ \vdots \\ z^J(n) \\ \hdashline \mu(n) \\ \mu_1(n--) \\ \vdots \\ \mu_J(n--) \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_j(n--) + V y_j(n) \\ \vdots \\ \mu_j(n--) + V y_j(n) \\ \hdashline \mu_j(n--) + V y_j(n) \\ \mu_1^j(n--) + V_{10}^j(n--) y_j(n) \\ \vdots \\ \mu_J^j(n--) + V_{J0}^j(n--) y_j(n) \end{bmatrix}, \left[\begin{array}{cccc|cccc} \Sigma_1 + M_0 & M_0 & \cdots & M_0 & M_0 & M_1 & \cdots & M_J \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ M_0 & M_0 & \cdots & \Sigma_J + M_0 & M_0 & M_1 & \cdots & M_J \\ \hline M_0^T & M_0^T & \cdots & M_0^T & \tilde{V}_{00}^j(n-) & \tilde{V}_{01}^j(n-) & \cdots & \tilde{V}_{0J}^j(n-) \\ M_1^T & M_1^T & \cdots & M_1^T & \tilde{V}_{10}^j(n-) & \tilde{V}_{11}^j(n-) & \cdots & \tilde{V}_{1J}^j(n-) \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ M_J^T & M_J^T & \cdots & M_J^T & \tilde{V}_{J0}^j(n-) & \tilde{V}_{J1}^j(n-) & \cdots & \tilde{V}_{JJ}^j(n-) \end{array}\right]\right)$$

where $M_i \equiv \Sigma_j (V+\Sigma_j)^{-1} V_{0i}^j(n--)$, $\tilde{V}_{ik}^j(n-) \equiv V_{ik}^j(n--) - V_{i0}^j(n--)(V+\Sigma_j)^{-1} V_{0k}^j(n--)$.

Noticing that $M_0 = M_0^T$, and $\tilde{V}_{00}^j(n-) = M_0$, we deduce that given $y^j(n-)$

$$\begin{bmatrix} z^1(n) - \mu(n) \\ \vdots \\ z^J(n) - \mu(n) \\ \hdashline \mu(n) \\ \mu_1(n--) \\ \vdots \\ \mu_J(n--) \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hdashline \mu_j(n--) + V y_j(n) \\ \mu_1^j(n--) + V_{10}^j(n--) y_j(n) \\ \vdots \\ \mu_J^j(n--) + V_{J0}^j(n--) y_j(n) \end{bmatrix}, \left[\begin{array}{cccc|ccc} \Sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & \cdots & 0 & 0 & & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \Sigma_J & 0 & \cdots & 0 \\ \hline & & 0 & & & \tilde{V}^j_{\cdot\cdot}(n-) & \end{array}\right]\right)$$

But if we recall that

$$\mu_i(n-) = \mu_i(n--) + V_{00}^i(n--)(\Sigma_i + V_{00}^i(n--))^{-1}(z^i(n) - \mu_i(n--))$$

$$= \mu_i(n--) + V_{00}^i(n--)\left(\Sigma_i + V_{00}^i(n--)\right)^{-1}\left\{ z^i(n) - \mu(n) + \mu(n) - \mu_i(n--) \right\},$$

we are able to deduce that given $y^j(n-)$ the expectation of $\mu_i(n-)$ is

$$\boxed{\begin{aligned}\mu_i^j(n-) &= A_i(n)\left\{\mu_i^j(n--) + V_{i0}^j(n--)\,\nu_j(n)\right\} + B_i(n)\left\{\mu_j(n--) + V\,\nu_j(n)\right\}\\ &= A_i(n)\,\mu_i^j(n--) + B_i(n)\,\mu_j(n--) + \left\{A_i(n)\,V_{i0}^j(n--) + B_i(n)\,V_{00}^j(n--)\right\}\nu_j(n)\end{aligned}}$$

where

$$A_i(n) \equiv \tau_i\left(\tau_i + V_{00}^i(n--)\right)^{-1}, \qquad B_i(n) \equiv I - A_i(n).$$

Thus

$$\begin{aligned}\mu_i(n-) - \mu_i^j(n-) = {} & A_i(n)\left\{\mu_i(n--) - \mu_i^j(n--) - V_{i0}^j(n--)\,\nu_j(n)\right\}\\ & + B_i(n)\left\{\mu(n) - \mu_j(n--) - V\,\nu_j(n)\right\}\\ & + V_{00}^i(n--)\left(\tau_i + V_{00}^i(n--)\right)^{-1}\eta^i(n)\end{aligned}$$

so that given $y^j(n-)$ the covariance of $\mu_i(n-)$ and $\mu_k(n-)$ $(i \neq j \neq k)$ will be

$$\boxed{\begin{aligned}V_{ik}^j(n-) \equiv {} & A_i(n)\,\tilde V_{ik}^j(n-)\,A_k(n)^T + A_i(n)\,\tilde V_{i0}^j(n--)\,B_k(n)^T\\ & + B_i(n)\,\tilde V_{0k}(n--)\,A_k(n)^T + B_i(n)\,\tilde V_{00}^j(n--)\,B_k(n)^T\\ & + \delta_{ik}\,V_{00}^i(n-)\left(\tau_i + V_{00}^i(n-)\right)^{-1}\tau_i\left(\tau_i + V_{00}^i(n-)\right)^{-1}V_{00}^i(n-).\end{aligned}}$$

Given $y^j(n-)$, the covariance of $\mu(n)$ is just $\tilde V_{00}^j(n-) = \tau_j\,(V + \tau_j)^{-1}\,V$. As for the covariance of $\mu(n)$ with $\mu_i(n-)$, that's just

$$\tilde V_{00}^j(n-)\,B_i(n)^T + \tilde V_{0i}^j(n-)\,A_i(n)^T \equiv V_{0i}^j(n-).$$

Now we come to the bargaining phase: after equilibrium is achieved, the price vector is known or equivalently, $\sum_j S_j\,\mu_j(n-)$, where $S_j \equiv \gamma_j^{-1}V_j^{-1}$. Denote this vector by ξ. When agent j sees ξ, he updates his mean for $\mu_i(n-)$ to

$$\begin{aligned}\tilde\mu_i^j(n-) &\equiv \mu_i^j(n-) + (V^jS)_i\,(S\cdot VS)^{-1}\left(\xi - \sum_k S_k\,\mu_k^j(n-)\right)\\ &= \mu_i^j(n-) + \left(\sum_p V_{ip}^j(n-)\,S_p^T\right)\left(\sum S_k\,V_{k\ell}^j(n-)\,S_\ell^T\right)^{-1}\left(\xi - \sum_k S_k\,\mu_k^j(n-)\right)\end{aligned}$$

$(i = 0, \ldots, J,\ i \neq j)$.

The covariance matrix updates accordingly: $V_{ik}^j(n-) - (V^jS)_i\,(S\cdot VS)^{-1}\,(V^jS)_k$.

We now hit a snag. Agent k updates his conditional expectations in terms of his innovations $\xi - \sum_i S_i \mu_i^k(n-)$, and so in particular his new estimate of $\mu(n)$ is a linear combination of $\mu_k(n-)$ and his innovation. If we wish to compute

$$E\left[\mu_k(n) \mid y^j(n)\right] \equiv \mu_k^j(n),$$

then we'll need to know $E[\mu_i^k(n-) \mid y^j(n)]$ — that is, what j thinks k thinks i thinks... and that is not in the formulation at the moment...

4) Another approach. We've set the problem up so that $\mu(n)$ is a stationary AR(1) process so that we may eventually get a time-invariant form of the solution. So let's go for that straight away!

We look for a solution

$$\boxed{\begin{aligned} \xi(n) &= \sum_{r\geq 1} D(r)\,\delta(n-r) + \sum_{r\geq 0}\sum_j C^j(r)\, z^j(n-r) \\ \mu_j(n-) &= \sum_{r\geq 1}\left\{ D^j(r)\,\delta(n-r) + \Gamma^j(r)\,\xi(n-r)\right\} + \sum_{r\geq 0} Z^j(r)\, z^j(n-r)\end{aligned}}$$

We shall require that $\xi(n) = \sum_i S_i \mu_i(n-)$ and also that $\mu_j(n-) = E[\mu(n) \mid y^j(n-)]$. We introduce the notation

$$\Theta_{rp} \equiv E\left[\mu(n-r)\,\mu(n-p)^T\right] = B^{(p-r)^+} \mathcal{V}_\mu \left(B^{(r-p)^+}\right)^T$$

in order to express the conditions in a more concise way.

For the expression above to be $E(\mu(n) \mid y^j(n-))$, we need three lots of conditions satisfied:

(i) for all $p \geq 1$,

$$\begin{aligned} E\left[\mu(n)\,\delta(n-p)^T\right] &= B^p \mathcal{V}_\mu \\ &= \sum_{r\geq 1}\left\{ D^j(r)\, E\left[\delta(n-r)\,\delta(n-p)^T\right] + \Gamma^j(r)\, E\left[\xi(n-r)\,\delta(n-p)^T\right]\right\} \\ &\quad + \sum_{r\geq 0} Z^j(r)\, E\left[z^j(n-r)\,\delta(n-p)^T\right] \end{aligned}$$

$$= \sum_{r\geq 1}\Big\{ D^j(r)\left(\Theta_{rp} + \delta_{rp}\mathcal{V}_\delta\right) + \Gamma^j(r)\Big(\sum_{t\geq 1} D(t)\left(\Theta_{r+t,p} + \delta_{r+t,p}\mathcal{V}_\delta\right)$$

$$+ \sum_{t\geq 0}\sum_i C^i(t)\,\Theta_{r+t,p}\Big)\Big\} + \sum_{r\geq 0} Z^j(r)\,\Theta_{rp}$$

(ii) $(\forall p\geq 0)$

$$\begin{aligned} E\left[\mu(n)\, z^j(n-p)^T\right] &= B^p \mathcal{V}_\mu \\ &= \sum_{r\geq 1} D^j(r)\,\Theta_{rp} + \sum_{r\geq 0} Z^j(r)\left(\Theta_{rp} + \delta_{rp}\mathcal{V}_j\right) + \end{aligned}$$

$$+ \sum_{r\geq 1} \Gamma^j(r) \Big\{ \sum_{t\geq 1} D(t)\, \Theta_{r+t,p} + \sum_{t\geq 0} \sum_i C^i(t) \big(\Theta_{r+t,p} + \delta_{ij}\, \delta_{r+t,p}\, \mathcal{Z}_j \big) \Big\}$$

(iii) $(p \geq 1)$

$$E\left[\mu(n)\, \xi(n-p)^T\right] = \sum_{t\geq 1} B^{p+t} \mathcal{Z}_\mu D(t)^T + \sum_{t\geq 0} \sum_i B^{p+t} \mathcal{Z}_\mu C^i(t)^T$$

$$= \sum_{r\geq 1} \Big\{ D^j(r)\, E\left[\delta(n-r)\, \xi(n-p)^T\right] + \Gamma^j(r)\, E\left(\xi(n-r)\, \xi(n-p)^T\right) \Big\}$$

$$+ \sum_{r\geq 0} Z^j(r)\, E\left[\zeta^j(n-r)\, \xi(n-p)^T\right]$$

where

$$E\left[\delta(n-r)\, \xi(n-p)^T\right] = \sum_{t\geq 1} \left(\Theta_{r,p+t} + \delta_{r,p+t} \mathcal{Z}_\mu \right) D(t)^T + \sum_{t\geq 0} \sum_i \Theta_{r,p+t}\, C^i(t)^T,$$

$$E\left[\zeta^j(n-r)\, \xi(n-p)^T\right] = \sum_{t\geq 1} \Theta_{r,p+t}\, D(t)^T + \sum_{t\geq 0} \sum_i \left(\Theta_{r,p+t} + \delta_{r,p+t}\, \delta_{ij}\, \mathcal{Z}_j \right) C^i(t)^T,$$

$$E\left[\xi(n-r)\, \xi(n-p)^T\right] = \sum_{t\geq 1} D(t)\, E\left[\delta(n-t-r)\, \xi(n-p)^T\right] + \sum_{t\geq 0} \sum_i C^i(t)\, E\left[\zeta^i(n-r-t)\, \xi(n-p)^T\right].$$

As for the condition $\xi(n) = \sum S_j\, \mu_j(n-)$, this gives us

$$\sum_{r\geq 1} D(r)\, \delta(n-r) + \sum_{r\geq 0} \sum_i C^i(r)\, \zeta^i(n-r)$$

$$= \sum_{r\geq 1} \Big\{ \sum_j S_j D^j(r) + \sum_{p=1}^{r-1} \sum_j S_j \Gamma^j(p)\, D(r-p) \Big\}\, \delta(n-r)$$

$$+ \sum_{r\geq 0} \Big\{ \sum_j S_j Z^j(r) + \sum_j \sum_i \sum_{p=1}^{r} S_i \Gamma^i(p)\, C^j(r-p) \Big\}\, \zeta^j(n-r)$$

so that

$$\boxed{\begin{aligned} D(r) &= \sum_j S_j D^j(r) + \sum_{p=1}^{r-1} \sum_j S_j \Gamma^j(p)\, D(r-p) \\ C^j(r) &= S_j Z^j(r) + \sum_i \sum_{p=1}^{r} S_i \Gamma^i(p)\, C^j(r-p) \end{aligned}}$$

Introducing the transforms $\hat{D}(x) = \sum_{r\geq 1} x^r D(r)$, etc, we have simply

$$\boxed{\begin{aligned} \hat{D}(x) &= \sum_j S_j \left(\hat{D}^j(x) + \hat{\Gamma}^j(x)\, \hat{D}(x) \right) \\ \hat{C}^j(x) &= S_j \hat{Z}^j(x) + \sum_i S_i \hat{\Gamma}^i(x)\, \hat{C}^j(x) \end{aligned}}$$

so that once $\hat{D}^j$, $\hat{\Gamma}^j$, $\hat{Z}^j$ are known, the D's and C^j's can in principle be obtained.

If we don't have $B = \beta I$, the underlined term won't simplify to this product form in general

5) <u>Simplifying assumption</u>: $B = \beta I$ <u>for some</u> $0 < \beta < 1$.

This turns out to be a key simplification later, not just notationally simpler. We're going to transform the equations for c/e, so we need to know

$$\sum_{p\geq 1} x^p H_{rp} = \sum_{p\geq 1} \beta^{|r-p|} x^p \mathbb{1}_\mu = \left(x\frac{x^r-\beta^r}{x-\beta} + x^r \frac{\beta x}{1-\beta x}\right)\mathbb{1}_\mu ,$$

$$\sum_{p\geq 0} x^p H_{rp} = \left(\frac{x^{r+1}-\beta^{r+1}}{x-\beta} + x^r \frac{\beta x}{1-\beta x}\right)\mathbb{1}_\mu .$$

After some calculations, we obtain

(i)

$$x\beta(1-x\beta)^{-1}\mathbb{1}_\mu = \frac{x(1-\beta^2)}{(x-\beta)(1-\beta x)}\left\{ \hat{D}^j(x) + \underline{\hat{\Gamma}^j(x)\,\hat{D}(x)} + \sum_i \hat{\Gamma}^j(x)\,\hat{c}^i(x) + \hat{Z}^j(x)\right\}\mathbb{1}_\mu$$

$$- \frac{x}{x-\beta}\left\{ \hat{D}^j(\beta) + \hat{\Gamma}^j(\beta)\Big(\hat{D}(\beta) + \sum_i \hat{c}^i(\beta)\Big) + \hat{Z}^j(\beta)\right\}\mathbb{1}_\mu$$

$$+ \left\{ \hat{D}^j(x) + \hat{\Gamma}^j(x)\,\hat{D}(x)\right\}\mathbb{1}_\delta \qquad \text{for each } j;$$

(ii)

$$(1-\beta x)^{-1}\mathbb{1}_\mu = \frac{x(1-\beta^2)}{(x-\beta)(1-\beta x)}\left\{ \hat{D}^j(x) + \hat{\Gamma}^j(x)\hat{D}(x) + \hat{\Gamma}^j(x)\sum_i \hat{c}^i(x) + \hat{Z}^j(x)\right\}\mathbb{1}_\mu$$

$$- \frac{\beta}{x-\beta}\left\{ \hat{D}^j(\beta) + \hat{\Gamma}^j(\beta)\Big(\hat{D}(\beta) + \sum_i \hat{c}^i(\beta)\Big) + \hat{Z}^j(\beta)\right\}\mathbb{1}_\mu$$

$$+ \left\{ \hat{Z}^j(x) + \hat{\Gamma}^j(x)\,\hat{c}^j(x)\right\}\mathbb{1}_j$$

Interesting questions

1) After hearing the talk on robust hedging, Christian Hipp asked whether you could get better bounds by requiring the martingales to be restricted, say to the case of cts with vol bdd above + below?

2) Mark Broadie + Alan White both asked whether we could do the relaxed investor instead by rebalancing when the proportion of wealth in the risky asset had moved by a certain amount.

Why so long to fill this notebook? Quite a lot of time spent on Credit things esp Leland & Toft, and RISK presentations, on Bermudan pricing + associated programming, and on Ben.