Stochastic versions of some models of Arrow & Kurz (15/8/2000)

(i) Arrow + Kurz look at a two-sector (government, private sector) model of an economy, where the rate of production of the single good at time t is

$$Y_t = F(K_p(t), K_g(t), L_t T_t) = C_t + I_p(t) + I_g(t) \qquad (1)$$

where $K_p(t)$ is the amount of private capital at time t, $K_g(t)$ is the amount of government capital at time t, L_t is the population size at time t, T_t is the labour-augmenting technology factor at time t; and $I_p(t)$ is rate of investment in private capital, $I_g(t)$ the rate of investment in public capital, and C_t the rate of consumption. The function F is increasing in all arguments, homogeneous of degree 1. K_p and K_g evolve as

$$\dot{K}_p = I_p - \delta_p K_p, \qquad \dot{K}_g = I_g - \delta_g K_g \qquad (2)$$

where $\delta_p, \delta_g \geq 0$ are constant rates of capital depreciation. The government's goal is to

$$\max E \int_0^\infty e^{-\rho_g t} U(\tilde{c}(t), \tilde{K}_g(t)) L_t \, dt \qquad (3)$$

where $\tilde{c}_t = C_t / L_t$, $\tilde{K}_g(t) = K_g(t)/L_t$ are per-capita rates of consumption, levels of govt capital, and the private sector wishes to

$$\max E \int_0^\infty e^{-\rho_p t} L_t u(\tilde{c}_t, \tilde{K}_g(t)) \, dt \qquad (4)$$

where U is homogeneous of degree $1-R$, u is homogeneous of degree $1-S$.

(ii) Assuming capital can be instantaneously + costlessly moved between the two sectors, A+K solve the deterministic version of this problem for the government. The government must now choose various taxes so that the private sector by its own optimisation ends up with the government's optimal solution!

Firstly, let's analyse the government's problem when we suppose

$$dL_t = L_t(\mu_L dt + dZ_t^L)$$
$$dT_t = T_t(\mu_T dt + dZ_t^T) \qquad (5)$$

where $dZ^a dZ^b = v_{ab} dt$ for any superscripts a, b. All the v's are assumed constant. If we let $\eta_t \equiv L_t T_t$, we have

$$d\eta_t = \eta_t \left[dZ_t^T + dZ_t^L + (\mu_L + \mu_T + v_{LT}) dt \right] \qquad (6)$$

and hence

$$d\eta_t^{-1} = \eta_t^{-1} \left[-dZ_t^T - dZ_t^L + (v_{TT} + v_{LT} + v_{LL} - \mu_L - \mu_T) dt \right]. \qquad (7)$$

Thus if lower case letters correspond to the upper-case variable divided by η, we shall have

$$y = f(k_p, k_g) \equiv F(k_p, k_g, 1) \tag{8}$$

$$dk_g = i_g\,dt - k_g\left(dZ^T + dZ^L + \gamma_g\,dt\right), \qquad \gamma_g \equiv \mu_L + \mu_T - v_{TT} - v_{LT} - v_{LL} + \delta_g,$$

$$dk_p = i_p\,dt - k_p\left(dZ^T + dL^T + \gamma_p\,dt\right), \qquad \gamma_p = \mu_L + \mu_T - v_{TT} - v_{LT} - v_{LL} + \delta_p.$$

The payoff which the government aims to maximise is therefore

$$E\int_0^\infty e^{-\rho_g t}\, U(c_t, k_g(t))\, T_t^{1-R} L_t\, dt$$

$$= E\int_0^\infty e^{-\lambda t} e^{M_t - \frac{1}{2}\langle M\rangle_t}\, U(c_t, k_g(t))\, dt \qquad \left[\begin{array}{l}\lambda \equiv \rho_g + \frac{R}{2}(1-R)v_{TT} - (1-R)v_{LT} - \mu_L - (1-R)\mu_T \\ M \equiv Z^L + (1-R)Z^T\end{array}\right]$$

$$= \tilde{E}\int_0^\infty e^{-\lambda t}\, U(c_t, k_g(t))\, dt$$

where $\tilde{P}$ is the measure under which

$$\left.\begin{array}{l} dZ^L = d\tilde{Z}^L + \left(v_{LL} + (1-R)v_{LT}\right)dt \\ dZ^T = d\tilde{Z}^T + \left(v_{LT} + (1-R)v_{TT}\right)dt \end{array}\right\} \tag{9}$$

The dynamics of k_g, k_p now can be expressed as

$$\left.\begin{array}{l} dk_g = i_g\,dt - k_g\left(d\tilde{Z}^T + d\tilde{Z}^L + \left(\gamma_g + v_{LL} + (1-R)v_{LT} + v_{LT} + (1-R)v_{TT}\right)dt\right) \\ dk_p = i_p\,dt - k_p\left(d\tilde{Z}^T + d\tilde{Z}^L + \left(\gamma_p + v_{LL} + (2-R)v_{LT} + (1-R)v_{TT}\right)dt\right) \end{array}\right\} \tag{10}$$

or for short

$$\boxed{\begin{array}{l} dk_g = i_g\,dt - k_g\left(d\tilde{Z}^T + d\tilde{Z}^L + \tilde{\gamma}_g\,dt\right) \\ dk_p = i_p\,dt - k_p\left(d\tilde{Z}^T + d\tilde{Z}^L + \tilde{\gamma}_p\,dt\right) \end{array}} \tag{11}$$

and the government's problem is to

$$\boxed{\max\ \tilde{E}\int_0^\infty e^{-\lambda t}\, U(c_t, k_g(t))\, dt} \tag{12}$$

with

$$\boxed{\begin{array}{rl} dk_t &= \left\{f(k_p(t), k_g(t)) - \tilde{\gamma}_g k_g(t) - \tilde{\gamma}_p k_p(t) - c_t\right\}dt - k_t\left(d\tilde{Z}^T_t + d\tilde{Z}^L_t\right) \\ &= \left\{\varphi(k_t, k_g(t)) - c_t\right\}dt - k_t\left(d\tilde{Z}^T_t + d\tilde{Z}^L_t\right) \end{array}} \tag{13}$$

The value function $V(k)$ for this problem will satisfy the HJB equation $(\sigma^2 = v_{TT} + 2v_{LT} + v_{LL})$

$$\boxed{\sup_{c,\,k_g}\left[U(c, k_g) - \lambda V + \tfrac{1}{2}\sigma^2 k^2 V'' + (\varphi(k, k_g) - c)V'\right] = 0.} \tag{14}$$

A_p matches definition at (1), p179 of AHK

X: Include human capital?

The optimal solution will be $c = c^*(k)$, $k_g = k_g^*(k)$, so that k_t under optimal control is an autonomous diffusion; there appear to be no interesting examples where there is a closed-form solution, but Peter has efficient numerical procedures for solving the HJB equations.

(iii) Next, we understand the problem of the private sector, and how taxes work on that. By a similar analysis to the above, the private sector's problem is to

$$\max \hat{E} \int_0^\infty e^{-\lambda_p t} u(c_t, k_g(t))\, dt \qquad \left(\lambda_p \equiv \rho_p - \mu_L - (1-S)\mu_T - (1-S)v_{TL} + \tfrac{1}{2}S(1-S)v_{TT} \right)$$

where

$$\begin{aligned} dZ_t^L &= d\hat{Z}_t^L + (v_{LL} + (1-S)v_{LT})\,dt \\ dZ_t^T &= d\hat{Z}_t^T + (v_{LT} + (1-S)v_{TT})\,dt \end{aligned} \tag{15}$$

expresses the change of measure to $\hat{P}$.

Since $aF(k_p, k_g, \ell) = F(ak_p, ak_g, a\ell)$, we have $F = k_p F_p + k_g F_g + \ell F_\ell$. This means that we can apportion output

$$Y = K_p F_p + K_g r_g + W$$

to private capital, govt capital + wages. Here, r_g would be equal to F_g if the govt could appropriate all the output of its capital (e.g. toll roads), but typically would be less than F_g. One natural assumption is that $r_g = 0$, so the govt can't appropriate any of its share of returns.

Notice that $F_p(K_p, K_g, L) = f_p(k_p, k_g)$.

Now we introduce taxes $1-\beta_W$ on wages, $1-\beta_c$ on consumption, $1-\beta_k$ on dividends from private capital, $1-\beta_r$ on interest payments on govt debt, and $1-\beta_s$ on savings. We think $0 \leq \beta_\cdot \leq 1$, and that $\beta_\cdot$ is the proportion left after the tax. Let D_t denote the amount of govt debt at time t, X_t the value of the private sector's assets, so

$$X_t = K_p(t) + D_t .$$

The total income of the private sector, $\overset{r_tD_t+}{\angle} K_p F_p + W$, gets taxed in various ways, and the residue is split between consumption, private investment I_p, and the rest, which goes into govt. bonds. We suppose private investment is a form of saving which gets taxed. Thus

$$\dot{D} = \beta_s \left[\beta_k (K_p F_p - \beta_s^{-1} I_p) + \beta_r r D + \beta_W W - \beta_c^{-1} C \right] \qquad \text{(What about govt investment?)}$$

$$\dot{K}_p = I_p - \delta_p K_p \tag{16}$$

Note that we're supposing by this that the private sector investment I_p is exempt from corporation tax $1-\beta_k$. If we had $\beta_s^{-1} I_p \leq K_p F_p$, then this corresponds to the usual practice of deducting the reinvestment from profits before tax; therefore we interpret the situation where $\beta_s^{-1} I_p > K_p F_p$ as a govt incentive to invest in private capital if the required investment exceeded the profits of

? Are two large pores <u>really</u> needed?

$$\begin{cases} w = f(k_p, k_g) - r_p k_p - r_g k_g \\ f(k_p, k_g) = i_p + i_g + c \end{cases}$$

$$\int_0^\infty \psi \, dx = \psi_\infty x_\infty - \psi_0 x_0 - \int_0^\infty x \, d\psi - [x, \psi]_\infty$$

the private sector: this seems quite plausible. In any case, different conventions on what taxes operate on what can easily be built in.

Notice that we must have

$$r\beta_r = \beta_k F_p \tag{17}$$

if money in shares and bonds can be freely moved: the rate of return on govt. debt must equal the rate of return on private capital.

(iv) The dynamics of X are given by

$$\dot{X} = \dot{K}_p + \dot{D} = (1-\beta_k)I_p - \delta_p K_p + r\beta_r\beta_s X + \beta_s\beta_w W - \beta_s\beta_c^{-1} C \tag{18}$$

and so the dynamics of $x \equiv \eta^{-1}X$ are given by

$$\boxed{dx = (1-\beta_k)i_p - \delta_p k_p + r\beta_r\beta_s x + \beta_s\beta_W w - \beta_s\beta_c^{-1}c + x(-d\hat{Z} + \hat{\mu}\,dt)} \tag{19}$$

where

$$d\eta^{-1} = \eta^{-1}\left[-d\hat{Z} + \hat{\mu}\,dt\right] = \eta^{-1}\left[-d\hat{Z}^L - d\hat{Z}^T + \hat{\mu}\,dt\right], \qquad \hat{\mu} = v_{TT} - \mu_L - \mu_T - (1-S)(v_{LT} + v_{TT})$$

Now we are considering a situation where the govt. has chosen a desired trajectory for k_g and therefore for i_g, and has a trajectory it wants followed for c, which should (by manipulation of taxes + debt) be optimal for the private sector. We have

$$w = f(k_p, k_g) - k_p f_p - r_g k_g, \qquad f(k_p, k_g) = i_p + i_g + c$$

which allows us to eliminate i_p from the equations, and express all in terms of c, k_p. Also,

$$\boxed{dk_p = i_p\,dt + k_p\{-d\hat{Z} + (\hat{\mu} - \delta_p)\,dt\}} \tag{20}$$

Introduce Lagrangian semimartingales $d\psi = a\,d\hat{Z} + b\,dt$, $d\theta = \tilde{a}\,d\hat{Z} + \tilde{b}\,dt$ to absorb dynamics of k_p and x; assuming that all contributions at ∞ vanish, as do the means of stochastic integrals wrt $\hat{Z}$, and writing $d\langle \hat{Z}\rangle_t = v\,dt$, the Lagrangian to be maximized is

$$\hat{E}\Big[\int_0^\infty e^{-\lambda_p t}u(c,k_g)\,dt + \int_0^\infty \psi\{(\hat{\mu} + r\beta_r\beta_s)x - \delta_p k_p + \beta_s\beta_w w - \beta_s\beta_c^{-1}c + (1-\beta_k)i_p\}\,dt + \psi_0 x_0$$
$$+ \int_0^\infty x(b - av)\,dt + \int_0^\infty \theta(i_p + k_p(\hat{\mu} - \delta_p))\,dt + \theta_0 k_p(0) + \int_0^\infty k_p(\tilde{b} - \tilde{a}v)\,dt\Big]$$

$$= \psi_0 x_0 + \theta_0 k_p(0) + \hat{E}\Big[\int_0^\infty\{e^{-\lambda_p t}u(c,k_g) + \psi\big((\hat{\mu} + r\beta_r\beta_s)x - \delta_p k_p + \beta_s\beta_w(f(k_p,k_g) - k_p f_p - r_g k_g) - \beta_s\beta_c^{-1}c + (1-\beta_k)(f - i_g - c)\big)$$
$$+ x(b - av) + \theta(f - i_g - c + k_p(\hat{\mu} - \delta_p)) + k_p(\tilde{b} - \tilde{a}v)\}\,dt\Big]$$

$$d\psi = a\,d\hat{Z} + b\,dt = \left(-\lambda_p \varphi\,dt + \varphi\left[\alpha\,d\hat{Z} + \beta\,dt\right]\right) e^{-\lambda_p t} \quad \therefore \quad b = \psi(\beta - \lambda_p), \quad a = \psi\,\alpha$$

Max over $c \geq 0$:

$$\boxed{e^{-\lambda_p t} u_c = \psi\left(1-\beta_R+\beta_s\beta_c^{-1}\right)+\theta} \tag{21}$$

Max over $x \geq 0$:

$$\boxed{\psi\left(\hat{\mu}+r\beta_r\beta_s\right)+b-a v \;\leq 0} \tag{22}$$

$(r\beta_r = \beta_R\beta_c??)$

Max over k_p:

$$\boxed{\psi\left\{(1-\beta_R)f_p-\delta_p+\beta_s\beta_w k_p f_{pp}\right\}+\theta\left(f_p+\hat{\mu}-\delta_p\right)+\tilde{b}-\tilde{a}v=0} \tag{23}$$

(V) It is now clear that we should instead work with $\varphi \equiv e^{\lambda_p t}\psi$, $\xi \equiv e^{\lambda_p t}\theta$, and postulate the exponential forms

$$d\varphi = \varphi\left\{\alpha\, d\hat{Z} + \beta\, dt\right\}, \qquad d\xi = \xi\left\{\tilde{\alpha}\, d\hat{Z} + \tilde{\beta}\, dt\right\}$$

for then

$$\boxed{\begin{aligned} &u_c = \varphi\left\{1-\beta_R+\beta_s\beta_c^{-1}\right\}+\xi \\ &\hat{\mu}+r\beta_r\beta_s+\beta-\lambda_p-\alpha v \leq 0 \\ &\varphi\left[(1-\beta_R)f_p-\delta_p+\beta_s\beta_w k_p f_{pp}\right]+\xi\left[f_p+\hat{\mu}-\delta_p+\tilde{\beta}-\lambda_p-\tilde{\alpha}v\right]=0 \end{aligned}} \tag{24}$$

This suggests that we should be looking for solutions where everything is a function of k_t, the total capital, because this is what happens for the government's optimal solution

(VI) <u>How does this compare with what A+K do?</u> (22/9/2000)

The process $X \equiv D + K_p$ is what they denote A^M (material assets); they introduce a process A^H (human assets) which solves

$$\dot{A}^H = r\beta_r\beta_s A^H - \beta_s\beta_w W + \delta_p K_p - (1-\beta_R) I_p$$

so that $A \equiv A^M + A^H$ solves the DE

$$\dot{A} = r\beta_r\beta_s A - \beta_s\beta_c^{-1} C$$

so that $x \equiv \eta^{-1} A$ now solves

$$dx = \left(r\beta_r\beta_s x - \beta_s\beta_c^{-1} c\right)dt + x\left(-d\hat{Z} + \hat{\mu}\, dt\right).$$

Introduce the Lagrange multiplier process $\psi = \exp(-\lambda_p t)\,\varphi$, where

$$\varphi^{-1} d\varphi = \alpha\, d\hat{Z} + \beta\, dt$$

and by the old Pontryagin trick the Lagrangian to be maximized is

$$\hat{E}\int_0^\infty e^{-\rho_p t}\{u(c,k_g) + \varphi(r\beta_r\beta_s x - \beta_s\beta_c^{-1}c + \hat{\mu}x) + x\varphi(b-\lambda_p) - xa\varphi v\}dt \quad + \psi_0 x_0$$

Maximising over c tells us that

$$\boxed{u_c = \varphi\beta_s\beta_c^{-1}} \qquad (25)$$

and maxing over $x \geq 0$ gives

$$\boxed{r\beta_r\beta_s + \hat{\mu} + b - \lambda_p - av \leq 0} \qquad (26)$$

State-price density? Since the private sector's objective is to maximise

$$E\int_0^\infty e^{-\rho_p t} L_t\, u(\tilde{c}_t, \tilde{k}_g(t))dt = \hat{E}\int_0^\infty e^{-\rho_p t} L_t\, u(\tilde{c}_t, \tilde{k}_g(t))\,\Lambda_t\, dt$$

where $\Lambda_t \equiv dP/d\hat{P}\,|_{\mathcal{F}_t}$, we can by considering consumption of infinitesimal amount δ at time t deduce that its impact on the payoff will be

$$\hat{E}\left[e^{-\rho_p t} L_t \Lambda_t\, u_c(\tilde{c}_t, \tilde{k}_g(t))\,\frac{\delta}{L_t}\right]$$

$$= \delta\,\hat{E}\left[e^{-\rho_p t}\Lambda_t\, u_c(\tilde{c}_t, \tilde{k}_g(t))\right]$$

$$= \delta\,\hat{E}\left[e^{-\rho_p t}\Lambda_t T_t^{-S}\, u_c(c_t, k_g(t))\right]$$

Hence the state-price density process for the private sector's valuation is

$$\boxed{\zeta_t = e^{-\rho_p t}\Lambda_t T_t^{-S}\, u_c(c_t, k_g(t))} \qquad 27$$

In the case of zero noise, we can follow this through and confirm that the condition on tax rates Prop VII.4(a), p 167, do indeed result.

BUT IS THIS ANALYSIS CORRECT? A change of consumption now affects A not only through $-\beta_s\beta_c^{-1}C\,dt$, but also it has implications for investment now, which knocks on into wage income in the future...

... but not if you consider an infinitesimal private investor.

$$\hat{\psi}_\lambda^-(z) = \frac{\lambda}{\lambda - \psi(z)} \frac{\beta^*(\lambda) - z}{\beta^*(\lambda)}$$

Some Lévy process calculations for the credit-risk paper (16/10/2000)

In the paper with Bianca, we need to justify the 'smooth pasting' condition for the optimal bankruptcy level. This looks tough in general, but for our specific case where the Lévy exponent is

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + bz - \frac{az}{c+z}$$

we can make certain things a lot more explicit, because the downward jumps are exponential, and therefore at the first crossing of $-x<0$ we either pass through continuously or we make an exp(c) jump over the level.

But first, using the fluctuation identity

$$\int_0^\infty \mu e^{-\mu x} E^x\left(e^{\theta X(H_0) - \lambda H_0}\right) dx = \frac{\mu}{\mu-\theta}\left\{1 - \frac{\psi_\lambda^-(\mu)}{\psi_\lambda^-(\theta)}\right\}$$

and letting $\theta \to \infty$, we obtain

$$\boxed{\int_0^\infty \mu e^{-\mu x} E^x\left(e^{-\lambda H_0} : X(H_0) = 0\right) dx = \frac{\sigma^2 \beta^*(\lambda)}{2\lambda} \cdot \mu \psi_\lambda^-(\mu)}$$

Since $\mu\psi_\lambda^-(\mu) = \mu E^0 e^{\mu \underline{X}(T_\lambda)} = \int_0^\infty \mu e^{-\mu x} P^0(-\underline{X}(T_\lambda) \in dx)$, we conclude that

$$E^x\left[e^{-\lambda H_0} : X(H_0) = 0\right] = \frac{\sigma^2 \beta^*(\lambda)}{2\lambda} \cdot P^0\left(-\underline{X}(T_\lambda) \in dx\right)$$

so

$$\boxed{\frac{\partial}{\partial x} E^x\left(e^{-\lambda H_0}\right) = -\frac{2\lambda}{\sigma^2 \beta^*(\lambda)} \cdot E^x\left[e^{-\lambda H_0} ; X(H_0) = 0\right]}$$

We have

$$\int_0^\infty \mu e^{-\mu x} E^x\left[e^{\theta X(H_0) - \lambda H_0} : X(H_0) < 0\right] dx = \frac{\mu}{\mu-\theta}\left\{1 - \frac{\psi_\lambda^-(\mu)}{\psi_\lambda^-(\theta)}\right\} - \frac{\sigma^2\beta^*(\lambda)}{2\lambda} \mu \psi_\lambda^-(\mu)$$

$$= \frac{2\mu a c}{(c+\theta)\left\{\mu^2\sigma^2(c+\beta) + \mu(c+\beta)(\sigma^2 c + 2b + \sigma^2\beta) + c\left(\sigma^2\beta c + 2b(c+\beta) - 2a + \sigma^2\beta^2\right)\right\}}$$

Notice the interpretation that given $X(H_0)<0$, the law of $-X(H_0)$ is exp(c)

Arrow-Kurz model with random effects and taxes (23/11/00)

Refer back to the notes on pages 1-6. The optimal solution to the government's problem has the property that k^*_t is an autonomous diffusion, and $k_g = k_g^*(k^*_t)$, $c_t = c^*(k^*_t)$ under optimal control. This means that the interpretation (11) of the dynamics can't be correct, because the q.v. won't match up; we have to think of $i_g\,dt$ more generally as the differential of a semimartingale. To understand how this fits together, let's look at the non-scaled quantities first.

We have

$$Y = I_g + I_p + C \tag{28}$$

$$= K_p F_p + K_g r_g + W$$

and the government's cashflow will be

$$I_g = r_g K_g + \dot D - rD + (1-\beta_r) rD + (1-\beta_w) W + (1-\beta_c)\beta_c^{-1} C + (1-\beta_k)(K_p F_p - \beta_s^{-1} I_p)$$
$$+ (1-\beta_s)\beta_s^{-1}(\dot D + I_p)$$

$$= r_g K_g + \beta_s^{-1}\dot D - r\beta_r D + (1-\beta_w)W + (1-\beta_c)\beta_c^{-1} C + (1-\beta_k)K_p F_p + (\beta_k-\beta_s)\beta_s^{-1} I_p \tag{29}$$

Now (28), (29) are two simultaneous linear equations for (I_p, I_g) which we may solve to eliminate I_p, I_g from the development. We get

$$\boxed{\begin{aligned}\beta_k\beta_s^{-1} I_g &= \beta_s^{-1}(\beta_k-\beta_s)(Y-C) + r_g K_g + \beta_s^{-1}\dot D - r\beta_r D + (1-\beta_w)W + (1-\beta_c)\beta_c^{-1} C + (1-\beta_k)K_p F_p \\ &= (\beta_k\beta_s^{-1}-\beta_w)Y + \beta_w r_g K_g + \beta_s^{-1}\dot D - r\beta_r D - (\beta_s^{-1}\beta_k - \beta_c^{-1})C + (\beta_w-\beta_k)K_p F_p\end{aligned}} \tag{30}$$

Therefore the dynamics (2) for K_g can be expressed without reference to I_g as

$$\boxed{dK_g = \beta_k^{-1} dD + \beta_s\beta_k^{-1}\Big((\beta_k\beta_s^{-1}-\beta_w)Y + \beta_w r_g K_g - r\beta_r D + (\beta_w-\beta_k)K_p F_p\Big)dt - \Big(1-\frac{\beta_s}{\beta_k\beta_c}\Big)C\,dt - \delta_g K_g\,dt} \tag{31}$$

From

$$\boxed{dK = \{Y - C - \delta_p K_p - \delta_g K_g\}\,dt} \tag{32}$$

we can deduce $dK_p = dK - dK_g$ if we want. The dynamics of k will still be given by (13), so we may write

$$\boxed{dk^*_t = \mu(k^*_t)\,dt - \sigma k^*_t\,d\tilde Z_t,} \tag{33}$$

where

$$\boxed{\begin{aligned} \mu^*(k) &= f(k_p^*(k), k_g^*(k)) - \tilde{\gamma}_g\, k_g^*(k) - \tilde{\gamma}_p\, k_p^*(k) - c^*(k) \\ \sigma\, d\tilde{Z} &= d\tilde{Z}^T + d\tilde{Z}^L \end{aligned}} \qquad (34)$$

From this, we can find the dynamics of $k_g \equiv K_g/\eta$:

$$\boxed{\begin{aligned} dk_g = \beta_k^{-1} \frac{dD}{\eta} - \sigma k_g\, d\tilde{Z} &+ \beta_s \beta_k^{-1}\Big[(\beta_k \beta_s^{-1} - \beta_w) y + \beta_w \tau_g k_g - r\beta_r D/\eta + (\beta_w - \beta_k) k_p f_p \Big] dt \\ &- \tilde{\gamma}_g k_g\, dt - \Big(1 - \frac{\beta_s}{\beta_c \beta_k}\Big) c\, dt - \sigma \beta_k^{-1} \frac{dD}{\eta}\, d\tilde{Z} \end{aligned}} \qquad (35)$$

The idea now is to choose tax rates as functions of k, and debt as a function of k, so that along an optimally-controlled trajectory (35) is correct.

Notice immediately from (35) that when we have noise in the system, a non-zero debt is essential to achieving government's goals!

A very simple model of a firm (29/11/2000)

1) A firm gets set up with shareholder and bondholder capital, and the bonds attract a coupon δ up to the moment of bankruptcy. Bankruptcy happens at rate $\mu(k)$ when the capital value is k, and a fraction α of firm value is lost on default. The shareholders choose the consumption rate c to maximise the net present value of all their cashflows up to bankruptcy. We therefore have the dynamics

$$\dot{k} = f(k) - (c+\delta)$$

where f is the production function, and the value function V must satisfy

$$\int_0^{t\wedge\tau} e^{-rs} c_s\, ds + e^{-rt} V(k_t) I_{\{t\le\tau\}} + e^{-r\tau}(1-\alpha)V(k_\tau) I_{\{t\ge\tau\}} \quad \text{is a supermg}$$

and a mg under optimal control. The HJB for this is therefore

$$\sup_c \; c - rV + V'(f(k) - c - \delta) - \mu(k)\alpha V = 0$$

So we get no consumption while $k < k^*$, $V'(k^*) = 1$, and consumption at rate $f(k^*) - \delta$ once we get up to k^*; initially, the firm is building its capital stock before dividends begin to be paid. In the region $(0, k^*)$, we have

$$(f(k)-\delta)V'(k) = (r + \alpha\mu(k))V(k)$$

so $V(k) = V(k^*)\exp\left(-\int_k^{k^*}\varphi(s)\,ds\right)$, $\qquad \varphi(s) \equiv \dfrac{r+\alpha\mu(s)}{f(s)-\delta}$

and the condition $V'(k^*) = 1$ gives us $1 = \varphi(k^*)V(k^*)$. We therefore choose k^* to maximise

$$\frac{1}{\varphi(k^*)} \exp\left(-\int_k^{k^*}\varphi(s)\,ds\right)$$

Calculus tells us to look for y to solve $\varphi'(y) = -\varphi(y)^2$, or $\psi'(y) = 1$, where $\psi \equiv 1/\varphi$

2) It appears hard to find good choices for f and μ. Doing this on the road, one example one could take is where

$$\mu(k) = \mu, \text{ constant} \quad \text{and} \quad f(k) = A\left(1 - e^{-\theta k}\right).$$

This gives us

$$k_t = \frac{1}{\theta}\log\left[\frac{b\,e^{\theta(A-\delta)t} + A}{A-\delta}\right] \qquad \left[b = (A-\delta)e^{\theta k_0} - A\right]$$

for the good constant b chosen to match k_0, and $\dfrac{V'(k)}{V(k)} = \dfrac{r+\alpha\mu}{f(k)-\delta} = (r+\alpha\mu)\dfrac{dt}{dk}$

so that

$$V(k_t) = B e^{(r+\alpha\mu)t} \qquad \text{for some } B$$

and we have

$$k^* = -\frac{1}{\theta} \log\left(\frac{r+\alpha\mu}{A\theta}\right)$$

$$V(k) = \text{const.}\left((A-\delta)e^{\theta k} - A\right)^{(r+\alpha\mu)/\theta(A-\delta)}$$

for some constant determined by the condition $V(k^*)\varphi(k^*) = 1$; $V(k^*) = \frac{A-\delta}{r+\alpha\mu} - \frac{1}{\theta}$, and

$$V(k) = V(k^*)\left\{\frac{(A-\delta)e^{\theta k} - A}{(A-\delta)e^{\theta k^*} - A}\right\}^{(r+\alpha\mu)/\theta(A-\delta)}$$

The time when k^* is attained will be

$$t^* = \frac{1}{\theta(A-\delta)} \log\left[\frac{(A-\delta)e^{\theta k^*} - A}{(A-\delta)e^{\theta k_0} - A}\right]$$

which depends (of course) on k_0. Concerning the NPV of all payments to bond-holders, we have this is

$$\int_0^{t^*}\left(\delta + (1-\alpha)\mu V(k_t)\right)e^{-(r+\mu)t}\,dt + \int_{t^*}^{\infty}\left(\delta + (1-\alpha)\mu V(k^*)\right)e^{-(r+\mu)t}\,dt$$

$$= \int_0^{t^*}\left(\delta + (1-\alpha)\mu B e^{(r+\alpha\mu)t}\right)e^{-(r+\mu)t}\,dt + \frac{\delta + (1-\alpha)\mu V(k^*)}{r+\mu} e^{-(r+\mu)t^*}$$

$$= \frac{\delta}{r+\mu}\left(1 - e^{-(r+\mu)t^*}\right) + B\left(1 - e^{-(1-\alpha)\mu t^*}\right) + \frac{\delta + (1-\alpha)\mu V(k^*)}{r+\mu} e^{-(r+\mu)t^*}$$

$$= \frac{\delta}{r+\mu} + V(k^*)\left[e^{-(r+\alpha\mu)t^*}\left(1 - e^{-(1-\alpha)\mu t^*}\right) + \frac{(1-\alpha)\mu}{r+\mu} e^{-(r+\mu)t^*}\right]$$

$$= \frac{\delta}{r+\mu} + V(k^*)\left[e^{-(r+\alpha\mu)t^*} - \frac{r+\alpha\mu}{r+\mu} e^{-(r+\mu)t^*}\right]$$

3) If we assume to make life easier that we are going to start the firm with exactly the optimal capital k^*, then this simplifies to

$$\frac{\delta}{r+\mu} + \frac{\mu(1-\alpha)}{r+\mu} V(k^*)$$

Note $S - C = K$ on conversion boundary

$$\Rightarrow \frac{\partial(S-C)}{\partial m} + \eta' \frac{\partial(S-C)}{\partial V} = 0 \quad \text{on conversion bdry}$$

$$\therefore \quad \frac{\partial}{\partial V}(S-C) = 0 \qquad (\text{else } \eta' = -K)$$

More on convertibles (24/1/01)

1) Suppose a firm is set up with N shares, and M convertible bonds, each of which attracts coupons at rate $\rho\,dt$ until conversion (or surrender etc). Let V_t be value of firm's assets at time t, and suppose there are m_t live bonds at time t. We have

(1) $$dV_t = \sigma V_t\, dW_t + (r-\delta) V_t\, dt - K\, dm_t$$

where K is the amount paid on conversion by the bondholder for the share. The cashflow δV to the shareholders + bondholders reflects the usual convention that shareholders are forbidden to sell the firm's assets to pay coupons.

Let $C(m,V)$ (resp $S(m,V)$) denote the value of a convertible if $m_t = m$, $V_t = V$ (resp, share). Then we must have the following (assuming no losses on default/bankruptcy)

(2) $$\mathcal{L}C + \rho = 0$$

(3) $$\mathcal{L}S + \frac{\delta V - m\rho}{N+M-m} = 0$$

(4) $$mC + (N+M-m)S = V$$

(5) $$C + K = S \qquad \text{when conversion takes place}$$

(6) $$\frac{\partial C}{\partial m} = K\frac{\partial C}{\partial V}, \quad \frac{\partial S}{\partial m} = K\frac{\partial S}{\partial V} \qquad \text{when conversion takes place}$$

Here, of course, $\mathcal{L} = \frac{1}{2}\sigma^2 V^2 \frac{\partial^2}{\partial V^2} + (r-\delta)V\frac{\partial}{\partial V} - r$, and the equation $\mathcal{L}f = 0$ has solutions V^θ, where θ is a root of

$$\tfrac{1}{2}\sigma^2\theta(\theta-1) + (r-\delta)\theta - r = 0$$

Clearly, the roots are of opposite sign, $-\alpha < 0$ and $\beta > 1$. For each m we can solve (2) explicitly:

$$C(m,V) = \rho/r + a(m)V^{-\alpha} + b(m)V^{\beta}.$$

For boundedness near zero, it looks like we want $a(m)=0$, but if this were the case then as $V \downarrow 0$ $C \to \rho/r$ and so for small V (using (4)) S will go negative! This cannot of course happen; the shareholders would declare bankruptcy if V got too low + just walk away. So there have to be two boundaries where things happen, $\xi(\cdot) < \eta(m)$. The lower is the bankruptcy boundary, where $S=0$, the upper is the conversion boundary where agents holding bonds convert them.

<u>How do we determine ξ and η</u>?

Using (4) and (5) we can say that at conversion

$$C = \frac{\eta + mK}{N+M} - K$$

$$= \frac{\rho}{r} + a\eta^{-\alpha} + b\eta^{\beta}$$

so once η has been settled, the solution C has only one indeterminate a, since b is given in terms of a as

$$b\eta^{\beta} = \frac{\eta + mK}{N+M} - K - \rho/r - a\eta^{-\alpha}$$

Rearranging (4) gives us

$$(N+M-m)S = V - mC$$

$$= V - m\left[\rho/r + aV^{-\alpha} + \left(\frac{V}{\eta}\right)^{\beta}\left\{\frac{\eta+mK}{N+M} - K - \rho/r - a\eta^{-\alpha}\right\}\right]$$

$$= ma\eta^{-\alpha}\left\{\left(\frac{V}{\eta}\right)^{\beta} - \left(\frac{V}{\eta}\right)^{-\alpha}\right\} + \text{terms not involving } a$$

The shareholders will pick a so as to maximise the value of shares. Now for any $V<\eta$, the term $\{\cdot\}$ will be negative, so the shareholders want to make a as low as possible, i.e. as large negative. As we push a down, we are adding multiples of the function

$$V \mapsto \left(\frac{V}{\eta}\right)^{-\alpha} - \left(\frac{V}{\eta}\right)^{\beta} \equiv \psi(\eta)$$

onto S. For $0<V<\eta$, this function is positive, decreasing, and also convex. We can't add too large a multiple of this f^n, else S would never vanish in $(0,\eta)$ and the value of the share would go to $+\infty$ as $V\downarrow 0$, which is absurd: we must have a zero of S at some point $\xi \in (0,\eta)$. Suppose ξ had been chosen optimally, yet $\frac{\partial S}{\partial V} > 0$ at ξ. By adding a tiny multiple of ψ to S, we can increase S everywhere but also ensure that S has a zero in $(0,\eta)$, a little to the left of ξ. This contradicts the supposed optimality of ξ, so we must have $\partial S/\partial V \leq 0$ at optimal ξ. A negative gradient would violate $S \geq 0$, so the conclusion must be that

$$\boxed{S = 0, \quad \frac{\partial S}{\partial V} = 0 \quad \text{at the bankruptcy boundary } \xi.}$$

Expressed in terms of C, we therefore have

$$\boxed{\begin{gathered} mC(m,\xi(m)) = \xi(m), \quad m\frac{\partial C}{\partial V}(m,\xi(m)) = 1, \quad \frac{\partial C}{\partial m}(m,\eta(m)) = K\frac{\partial C}{\partial V}(m,\eta(m)) \\ C(m,\eta(m)) = \frac{\eta(m)+mK}{N+M} - K \end{gathered}}$$

Can also be shown that $\xi'(0) = \mu / \sigma p$ — Jon reckons this isn't correct...

In terms of $\xi = \xi(m)$ therefore,

$$\boxed{C = \frac{\rho}{r}\left(1 - \frac{\beta}{\beta+\alpha}\left(\frac{V}{\xi}\right)^{-\alpha} - \frac{\alpha}{\beta+\alpha}\left(\frac{V}{\xi}\right)^{\beta}\right) + \frac{\xi(\beta-1)}{m(\beta+\alpha)}\left(\frac{V}{\xi}\right)^{-\alpha} + \frac{\xi(\alpha+1)}{m(\beta+\alpha)}\left(\frac{V}{\xi}\right)^{\beta}}$$

2) Suppose now we change the problem to incorporate losses on default; a fraction $(1-p)$ of the value of the firm gets wiped out if the firm defaults + the shareholders jump out. The dynamics are as before, except that the total value equation (4) is no longer correct. We will have bankruptcy boundary ξ and conversion boundary η, and these are specified by the conditions

$$(7i) \qquad S = 0 \quad , \quad \frac{\partial S}{\partial V} = 0 \quad \text{on } \xi$$

$$(7ii) \qquad m\,C(m,\xi) = p\xi \quad , \quad C(m,\eta) + K = S(m,\eta)$$

$$(7iii) \qquad \frac{\partial C}{\partial m} = K\frac{\partial C}{\partial V} \quad , \quad \frac{\partial S}{\partial m} = K\frac{\partial S}{\partial V} \qquad \text{on } \eta$$

Conditions (7i) fix the constants in S (once ξ is chosen), conditions (7ii) fix the constants in C (once η is chosen) and (7iii) gives a pair of differential equations for the unknown functions ξ, η.

<u>What about boundary conditions?</u> As $m \downarrow 0$, we should have that the bankruptcy boundary will go down to 0, and the conversion boundary will go to

$$\boxed{\frac{\beta}{\beta-1}(N+M)\left(\frac{\rho}{r}+K\right)}$$

Why? If we have an infinitesimal number of live convertibles left, and the value of the firm's assets is x, then when we compare immediate conversion with conversion when we get to $x+\varepsilon$, the NPV of immediate conversion is

$$-K + \frac{x}{N+M}$$

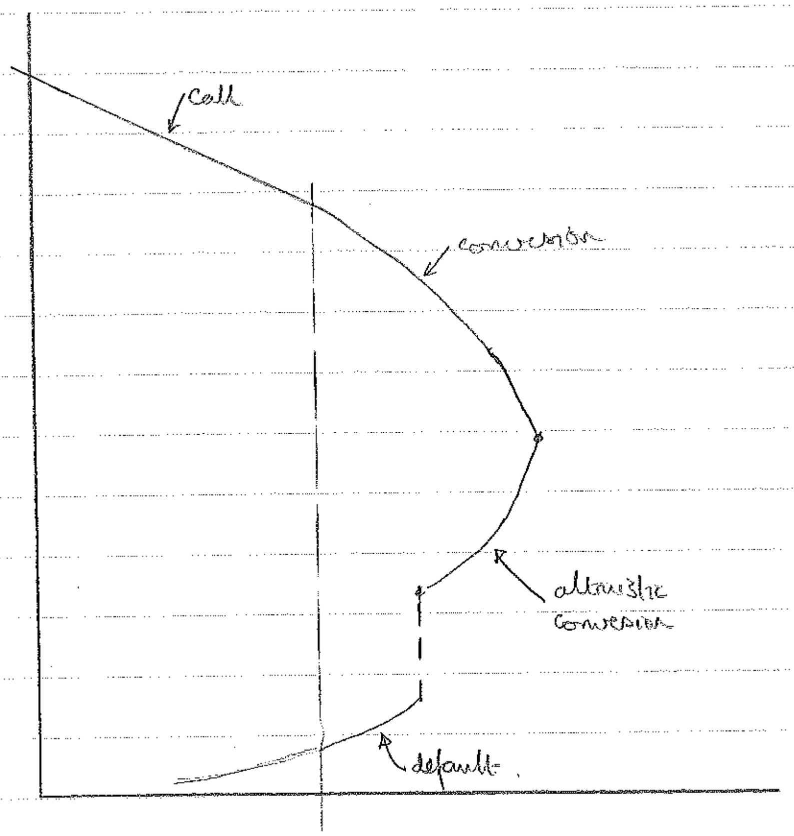
and the NPV of waiting will be

$$E^x\left(\int_0^{H_{x+\varepsilon}} \rho e^{-rs}\,ds\right) + E^x\left(e^{-rH_{x+\varepsilon}}\right)\left\{-K + \frac{x+\varepsilon}{N+M}\right\} = \frac{\rho}{r}\left\{1-\left(\frac{x}{x+\varepsilon}\right)^{\beta}\right\} + \left(\frac{x}{x+\varepsilon}\right)^{\beta}\left(-K + \frac{x+\varepsilon}{N+M}\right)$$

So we prefer immediate exercise if

$$\left(\frac{x}{N+M} - K\right)\left\{1-\left(\frac{x}{x+\varepsilon}\right)^{\beta}\right\} - \frac{\rho}{r}\left\{1-\left(\frac{x}{x+\varepsilon}\right)^{\beta}\right\} - \frac{\varepsilon}{N+M}\left(\frac{x}{x+\varepsilon}\right)^{\beta} > 0$$

Letting $\varepsilon \downarrow 0$ in this gives the desired boundary condition. If we include losses on default, this


call
conversion
alternative
conversion
default

Should make no difference, since for infinitesimal amounts of live convertibles default is not going to occur.

3) Do we ever find that the bankruptcy + conversion boundaries meet? If so, this puts an upper bound on the number of convertibles the firm can issue.

4) Would there ever be altruistic conversion of bonds when the share price was low, as a way to save the firm's value? If so, along ξ, η we'd have

$$S - C = K$$

so that

$$\frac{\partial}{\partial m}(S-C) + \eta' \frac{\partial}{\partial V}(S-C) = 0 = \frac{\partial}{\partial m}(S-C) + \xi' \frac{\partial}{\partial V}(S-C)$$

so we deduce that generically

$$\frac{\partial}{\partial V}(S-C) = 0 \quad \text{along } \xi, \eta$$

This determines ξ, η (we hope - 4 equations and 4 unknowns), and then $\frac{\partial C}{\partial m} = K \frac{\partial C}{\partial V}$ on ξ, η tells us all the rest.

5) Call provision? If the issue can be called, surrender price K^*, then if we suppose the firm calls when there are m live bonds, firm value $= V$, and if m' are surrendered, then the remainder prefer conversion if

$$\frac{V - m'K^* + (m-m')K}{N+M-m'} - K \geq K^*$$

i.e. $\quad V + mK \geq (N+M)(K^* + K)$

For indifference, we'd pick

$$\boxed{V = (N+M)(K^* + K) - mK}$$

[In practice, issues are often called late so that all bondholders see that conversion is better ... use the calling boundary for some $K' > K^*$!]

Beware !! This is too simple-minded to be correct: the entire solution must be recomputed. The calling boundary is as above, and we call when m falls to m^*, and V gets to $(N+M)(K^*+K) - m^*K = \eta(m^*)$. The share is then worth $(\eta(m^*) + m^*K)/(N+M)$, and at bankruptcy $S(m^*, \xi(m^*)) = 0 = \frac{\partial S}{\partial V}(m^*, \xi(m^*))$. These conditions determine $\xi(m^*)$ and $S(m^*, \cdot)$ in terms of m^*. But we also get

$$C(m^*, \eta(m^*)) = K^*, \qquad C(m^*, \xi(m^*)) = p\,\xi(m^*)$$

which determines $C(m^*, \cdot)$, and the derivative condition

$$\frac{\partial}{\partial V}(S-C) = 0 \quad \text{at } \eta$$

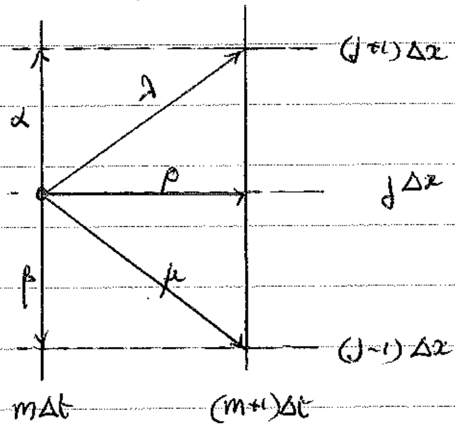
should now give us what m^* is. This done, we solve the DEs out to the right.

Some remarks on the probabilistic interpretation of some finite-difference schemes (26/1/01)

Suppose that X is some one-dimensional diffusion

$$dX_t = \sigma(t, X_t)\, dW_t + b(t, X_t)\, dt$$

with nice coefficients, and we aim to approximate the bivariate process $(T_t, X_t) \equiv (t, X_t)$ by a Markov chain (τ_t, x_t) on the grid $(\Delta t . \mathbb{Z}) \times (\Delta x . \mathbb{Z})$, with jump intensities as shown (the parameters $\alpha, \beta, \lambda, \mu, \rho$ will in general depend on the position in the grid)



For a test function φ,

$$\varphi(\tau_t, x_t) - \int_0^t Q\varphi(\tau_s, x_s)\, ds \quad \text{is a martingale,}$$

where

$$\begin{aligned} Q\varphi(m\Delta t, j\Delta x) = {} & \lambda\{\varphi((m+1)\Delta t, (j+1)\Delta x) - \varphi(m\Delta t, j\Delta x)\} \\ & + \mu\{\varphi((m+1)\Delta t, (j-1)\Delta x) - \varphi(m\Delta t, j\Delta x)\} \\ & + \rho\{\varphi((m+1)\Delta t, j\Delta x) - \varphi(m\Delta t, j\Delta x)\} \\ & + \alpha\{\varphi(m\Delta t, (j+1)\Delta x) - \varphi(m\Delta t, j\Delta x)\} \\ & + \beta\{\varphi(m\Delta t, (j-1)\Delta x) - \varphi(m\Delta t, j\Delta x)\} \end{aligned}$$

$$\begin{aligned} = {} & (\lambda+\mu+\rho)\varphi_t \Delta t + \tfrac{1}{2}(\lambda+\mu+\rho)\varphi_{tt}(\Delta t)^2 + \Delta t \Delta x(\lambda-\mu)\varphi_{tx} \\ & + (\lambda-\mu+\alpha-\beta)\varphi_x \Delta x + \tfrac{1}{2}(\Delta x)^2(\lambda+\mu+\alpha+\beta)\varphi_{xx} + \text{higher order terms} \end{aligned}$$

So what we'd be aiming to get would be as $\Delta x, \Delta t \to 0$ in some appropriate fashion

$$\boxed{\begin{aligned} (\lambda+\mu+\rho)\Delta t &\longrightarrow 1 \\ (\lambda-\mu)\Delta t \Delta x &\longrightarrow 0 \\ (\lambda-\mu+\alpha-\beta)\Delta x &\longrightarrow b(t,x) \\ (\lambda+\mu+\alpha+\beta)(\Delta x)^2 &\longrightarrow \sigma(t,x)^2 \end{aligned}}$$

Explicit scheme: this has $\alpha = \beta = 0$, and easily $\Delta t\, \sigma(t,x)^2/(\Delta x)^2$ is bdd above by 1 asymptotically. This means that small Δx forces small Δt, so lots of time steps.

Fully implicit scheme: $\lambda = \mu = 0$. Now there are no constraints on the relative magnitudes of $\Delta x, \Delta t$, but since the law of the step in the spatial variable is now two-sided geometric, we are going to need Δt quite small in order to approximate the step distribution well in any case!

$$\omega_p = \lambda_p + \hat{\gamma}\ ; \quad b = (1-\theta)\,\Phi\left(\frac{\theta}{1-\theta}\right) - \hat{\gamma} - q$$

$$\omega = \omega_p - \Phi\left(\frac{\theta}{1-\theta}\right) + \frac{\theta}{1-\theta}\,\Phi'\left(\frac{\theta}{1-\theta}\right)$$

Arrow + Kurz model; a simple special case (30/1/01)

If we assume that the production function is homogeneous of degree 1, and the utility is homogeneous of degree $1-R$, then we shall have

$$f(k_p, k_g) = k_p \,\Phi(k_g/k_p), \qquad U(c, k_g) = c^{1-R}\,\Psi(k_g/c)$$

for some functions φ, ψ. The HJB equation

$$\sup_{c, k_g} \; -\lambda_g V(k) + V'(k)\{f(k_g, k_p) - \tilde{\gamma} k - c\} + \tfrac{1}{2}\sigma^2 k^2 V''(k) + U(c, k_g) = 0$$

has a solution of the form

$$V(k) = a\,k^{1-R}, \qquad c = q k, \qquad k_g = \theta k$$

provided (a, q, θ) solve the non-linear equations:

$$\boxed{\begin{aligned} &-a\lambda_g + a(1-R)\left\{(1-\theta)\,\Phi\!\left(\tfrac{\theta}{1-\theta}\right) - \tilde{\gamma} - q\right\} - \tfrac{R\sigma^2}{2}(1-R)\,a + q^{1-R}\,\Psi(\theta/q) = 0 \\ &q^{-R}\left[(1-R)\,\Psi\!\left(\tfrac{\theta}{q}\right) - \tfrac{\theta}{q}\,\Psi'(\theta/q)\right] = a(1-R) \\ &q^{-R}\,\Psi'(\theta/q) = a(1-R)\left\{\Phi\!\left(\tfrac{\theta}{1-\theta}\right) - \tfrac{1}{1-\theta}\,\Phi'\!\left(\tfrac{\theta}{1-\theta}\right)\right\} \end{aligned}}$$

The optimally-controlled process k^* is a log-Brownian motion.

When considering the private sector's optimisation problem, assuming that the government sets fiscal + debt policy so that we get the feedback form

$$dk_g(t) = \{H_t + h_t c_t\}dt + G_t\, d\hat{Z}_t$$

we find we seek multiplier processes $e^{-\lambda_p t}\psi_t$, $e^{-\lambda_p t}\varphi_t$ where along the optimal trajectory

$$\begin{cases} u_c = \varphi - \psi h \\ \mu_\varphi - \sigma^2\sigma_\varphi = \varphi\,(\omega_p - f_p) \\ u_g + \mu_\psi = \varphi(f_p - f_g) + \lambda_p \psi \end{cases} \qquad \begin{pmatrix} d\varphi = \mu_\varphi dt + \sigma_\varphi\, d\hat{Z} \\ d\psi = \mu_\psi dt + \sigma_\psi\, d\hat{Z} \end{pmatrix}$$

If we have $dk^*_t = b(k^*_t)dt - k^*_t\, d\hat{Z}_t$ $\quad (b(x) \equiv f(\bar{k}_p(x), \bar{k}_g(x)) - \hat{\gamma}x - \bar{c}(x))$, and we seek $\varphi_t = \varphi(k^*)$, $\psi_t = \psi(k^*)$, we have in this special case that $b(k) = b.k$ for some const b and the DE for φ becomes

$$\tfrac{1}{2}\sigma^2 x^2 \varphi''(x) + bx\varphi'(x) + \sigma^2 x\varphi'(x) = \varphi(x)\left(\omega_p - f_p(x - \bar{k}_g(x), \bar{k}_g(x))\right)$$

$$\equiv \omega\,\varphi(x)$$

which can be solved; $\varphi(x)$ is a linear combination of two powers of x. The DE for ψ can also be solved.

Of course, $x \mapsto f(x) \equiv x^{(\sigma^2-2r)/2\sigma^2} K_\nu\left(\frac{\sqrt{8\alpha x}}{\sigma}\right)$ is another solution

Pricing Asian options: some notes (5/2/01)

1) The basic problem is to compute the expectation

$$E\left(K - \int_0^T S_u\,du\right)^+ \equiv E\,(K - Z_T)^+$$

where $S_t = \exp\{\sigma W_t + (r - \sigma^2/2)t\}$, K, T are arbitrary positive constants. Now if we perform a Laplace transform in K, we get

$$\int_0^\infty e^{-\alpha K}\, E\,(K - Z_T)^+\, dK = \alpha^{-2}\, E \exp(-\alpha Z_T)$$

and next we may Laplace transform in T to give us

$$E\left[\int_0^\infty \lambda e^{-\lambda t} \exp\left(-\alpha \int_0^t S_u\,du\right)dt \;\Big|\; S_0 = s\right] \equiv f(s)$$

In the usual fashion, we obtain an ODE satisfied by f:

$$\boxed{(-\lambda - \alpha S)f + \tfrac{1}{2}\sigma^2 S^2 f'' + rSf' + \lambda = 0} \qquad (*)$$

together with the boundary conditions

$$f(0) = 1, \quad f(\infty) = 0, \quad f \text{ is decreasing.}$$

Closed-form solution of the ODE looks difficult. However, the homogeneous equation is solved by

$$x \mapsto x^{(\sigma^2 - 2r)/2\sigma^2}\, I_\nu\left(\frac{\sqrt{8\alpha x}}{\sigma}\right) \equiv f_0(x) \qquad \left[\nu = \sqrt{(\sigma^2 - 2r)^2 + 8\sigma^2\lambda}\,/\sigma^2\right]$$

and we can seek a series solution $f(x) = \sum_{n \geq 0} a_n x^n$, which has a nice recursive form to the solution:

$$\boxed{\left(\tfrac{1}{2}\sigma^2 n^2 + (r - \sigma^2/2)n - \lambda\right) a_n = \alpha a_{n-1}, \quad a_0 = 1}$$

We can alternatively express this as

$$(n - \rho_+)(n + \rho_-)\, a_n = \frac{2\alpha}{\sigma^2}\, a_{n-1}$$

where $\boxed{\rho_+ = \frac{\nu}{2} + \frac{\sigma^2 - 2r}{2\sigma^2}, \quad \rho_- = \frac{\nu}{2} - \frac{\sigma^2 - 2r}{2\sigma^2}}$ Hence

$$\boxed{a_n = \left(\prod_{k=1}^n \frac{1}{(k - \rho_+)(k + \rho_-)}\right) \cdot \left(\frac{2\alpha}{\sigma^2}\right)^n = \frac{\Gamma(1+\rho_-)\,\Gamma(1-\rho_+)}{\Gamma(n+1+\rho_-)\,\Gamma(n+1-\rho_+)} \left(\frac{2\alpha}{\sigma^2}\right)^n}$$

Note that $I_\nu(x) \sim e^{x}/\sqrt{2\pi x}$ if $Re(\nu+\frac{1}{2})>0$

$$K_\nu(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}$$ as $x \to \infty$

$$I_\nu(x) \sim (\tfrac{1}{2}x)^\nu / \Gamma(\nu+1)$$

$$K_\nu(x) \sim (\tfrac{1}{2}x)^{-\nu}\,\Gamma(\nu)/2 \quad \text{if } \nu>0$$ $x \downarrow 0$

Hence $f_1(x) \sim x^{(\sigma^2-2r)/2\sigma^2}\left(\frac{2\alpha x}{\sigma^2}\right)^{-\nu/2}\frac{\Gamma(\nu)}{2}$ $(x\downarrow 0)$

$$f_0(x) \sim x^{(\sigma^2-2r)/2\sigma^2}\left(\frac{2\alpha x}{\sigma^2}\right)^{\nu/2} / \Gamma(\nu+1) \qquad (x\downarrow 0)$$

and

$$f_0(x) \sim x^{(\sigma^2-2r)/2\sigma^2}\exp\left(\frac{\sqrt{8\alpha x}}{\sigma}\right)\left(2\pi\,\frac{\sqrt{8\alpha x}}{\sigma}\right)^{-\frac{1}{2}} \qquad (x\to\infty)$$

$$f_1(x) \sim x^{(\sigma^2-2r)/2\sigma^2}\exp\left(-\frac{\sqrt{8\alpha x}}{\sigma}\right)\left(\frac{2}{\pi}\,\frac{\sqrt{8\alpha x}}{\sigma}\right)^{-\frac{1}{2}}$$

This is a generalised hypergeometric function. The solution to the ODE which we seek is of the form

$$f(x) + \theta f_0(x)$$

where θ is chosen to keep the solution bounded at ∞ ($f_0(0)=0$, it can be seen). So we need to have behaviour at infinity of generalised hypergeometric functions.

2) We can also write out the solution to the inhomogeneous DE in terms of $f_0(\cdot)$, $f_1(\cdot)$. Using Maple, I get the general solution to be

$$\boxed{F(x) = f_0(x)\int_x^\infty \lambda f_1(y)\, y^{-2+2r/\sigma^2}\frac{4\,dy}{\sigma^2} + f_1(x)\int_0^x \lambda f_0(y)\, y^{-2+2r/\sigma^2}\frac{4\,dy}{\sigma^2} + C_0 f_0(x) + C_1 f_1(x)}$$

Now

$$\int_0^x y^{-2+2r/\sigma^2} f_0(y)\,dy \sim \frac{1}{2^\nu \Gamma(\nu+1)}\int_0^x \left(\frac{8\alpha y}{\sigma^2}\right)^{\nu/2} y^{-(\sigma^2-2r)/\sigma^2}\cdot\frac{dy}{y}\cdot y^{(\sigma^2-2r)/2\sigma^2} \quad (x\downarrow 0)$$

$$= \left(\frac{2\alpha}{\sigma^2}\right)^{\nu/2}\frac{1}{\Gamma(\nu+1)}\int_0^x y^{\nu/2-(\sigma^2-2r)/2\sigma^2}\frac{dy}{y}$$

$$= \left(\frac{2\alpha}{\sigma^2}\right)^{\nu/2}\frac{2}{\Gamma(\nu+1)}\frac{x^{\nu/2-(\sigma^2-2r)/2\sigma^2}}{(\nu-(\sigma^2-2r)/\sigma^2)}$$

Hence

$$f_1(x)\int_0^x f_0(y)\, y^{-2+2r/\sigma^2}\,dy \longrightarrow \frac{1}{\nu(\nu-(\sigma^2-2r)/\sigma^2)} \quad (x\downarrow 0)$$

Similarly,

$$f_0(x)\int_x^\infty f_1(y)\, y^{-2+2r/\sigma^2}\,dy \to \frac{1}{\nu(\nu+(\sigma^2-2r)/\sigma^2)} \quad (x\downarrow 0)$$

Combining these, the first two terms in F converge to 1 as $x\to 0$. This tells us that for the solution we seek, $C_1 = 0$.

Studying the asymptotics at ∞, we get

$$\left.\begin{aligned} f_1(x)\int_0^x y^{-2+2r/\sigma^2} f_0(y)\,dy &\sim \sigma^2/8\alpha x \\ f_0(x)\int_x^\infty y^{-2+2r/\sigma^2} f_1(y)\,dy &\sim \sigma^2/8\alpha x \end{aligned}\right\}$$

The conclusion therefore is that $\boxed{C_0 = C_1 = 0.}$

3) Parisian options   This is essentially a reprise of Chesney-Jeanblanc-Yor, done via excursions. For the moment, let's just do down-and-in Parisian options. If we have to compute

$$E^x\left[f(X_t);\ t \geq \tau\right]$$

where $X_t = W_t + \mu t$ is a drifting BM, and $\tau \equiv \inf\{t;\ t - g_t > D\}$, where $D>0$ is fixed, and $g_t \equiv \sup\{u \leq t: X_u > b\}$, then we may by change-of-measure reduce to the case of $\mu = 0$. The best thing then is to LT in t, and compute

$$E^x\left[f(X_T);\ T \geq \tau\right]$$

where $T \sim \exp(\lambda)$ independent of W. Wlog let's take $b=0$. The essential is to solve for $x=0$, as the $x \neq 0$ cases can be expressed in terms of this. Since X_T has a bilateral $\exp(\theta)$ dist^n $(\theta = \sqrt{2\lambda})$, we may equally well compute

$$P^0\left(X_T \in dy\ ;\ \tau > T\right)$$

Rate of negative excursions of duration $> D$ with no mark before D

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \cdot P^{-\varepsilon}\left[H_0 > D\right] \cdot e^{-\lambda D}$$

$$= e^{-\lambda D} / \sqrt{2\pi D}.$$

Rate of positive excursions with mark when excursion is in dy

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \left\{ \tfrac{1}{2}\theta\, e^{-\theta(y-\varepsilon)} - \tfrac{1}{2}\theta\, e^{-\theta(y+\varepsilon)} \right\} dy$$

$$= \tfrac{1}{2}\theta^2 e^{-\theta y}\, dy \quad \equiv p_+(y)\, dy, \text{ say}$$

Rate of negative excursions with mark when excursion is in dy, before time D

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^D \frac{\lambda e^{-\lambda s}}{\sqrt{2\pi s}} \left\{ e^{-(\varepsilon+y)^2/2s} - e^{-(\varepsilon-y)^2/2s} \right\} ds\ dy$$

$$= \left(\int_0^D \lambda e^{-\lambda s - y^2/2s} \frac{|y|}{\sqrt{2\pi s^3}}\, ds \right) dy \quad \equiv p_-(y)\, dy, \text{ say}$$

$$= \lambda \left\{ e^{-\theta y}\, \Phi\left(\frac{y - \theta D}{\sqrt{D}}\right) + e^{\theta y}\, \Phi\left(\frac{y + \theta D}{\sqrt{D}}\right) \right\}$$

if it helps. Another simplification of sorts is

$$\int_{-\infty}^0 p_-(y)\, dy = \theta \left\{ \Phi(\theta\sqrt{D}) - \tfrac{1}{2} \right\}$$

Yes, $\int_0^{z} e^{-t^2/2}\,dt = \sum_{m\geq 0} \frac{(-1)^m}{m!} \int_0^{z} \left(\frac{t^2}{2}\right)^m dt = \sum_{m\geq 0} \frac{(-1)^m}{m!} \frac{z^{2m+1}}{2m+1} \cdot 2^{-m}$!

The example on p18 has us looking at $F(2\alpha z/\sigma^2)$, where $p=1$, $q=2$, $\alpha_1 = 1 = \rho_1 = \rho_2$, and $\beta_1 = 1$, $\mu_1 = 1-\rho_+$, $\mu_2 = 1+\rho_-$.
Thus $\kappa = 2$, $h = 1$, $\theta = \rho_+ - \rho_- - \frac{1}{2} = \frac{1}{2} - 2r/\sigma^2$, $A_0 = (2\pi)^{-\frac{1}{2}}\, 2^{2r/\sigma^2}$

Assembling the DE solution gives an expression for the solution at $s=1$:

$$\Gamma(1+\rho_-)\Gamma(1+\rho_+) \sum_{n\geq 0} \left\{ \frac{1}{\Gamma(n+1+\rho_-)\Gamma(n+1-\rho_+)} - \left(\frac{2\alpha}{\sigma}\right)^{\rho_+} \frac{1}{n!\,\Gamma(\nu+n+1)} \right\} \left(\frac{2\alpha}{\sigma^2}\right)^n$$

$$\left[A_0 = (2\pi)^{-\frac{1}{2}}\, 2^{2r/\sigma^2} \right]$$

$$B = -\left(\frac{2\alpha}{\sigma^2}\right)^{\frac{1}{2}-r/\sigma^2} \quad 2 \cdot \Gamma(1+\rho_-)\,\Gamma(1-\rho_+)$$

Altogether then,

$$\boxed{\begin{aligned} P(X_\tau \in dy,\ T<\tau) &= \frac{p_+(y) + p_-(y)}{\theta\,\Phi(\theta\sqrt{D}) + e^{-\lambda D}/\sqrt{2\pi D}}\,dy \\ &= \frac{e^{-\theta y} I_{\{y>0\}} + \left\{ e^{-\theta y}\,\Phi\!\left(\frac{y-\theta D}{\sqrt{D}}\right) + e^{\theta y}\,\Phi\!\left(\frac{y+\theta D}{\sqrt{D}}\right)\right\} I_{\{y<0\}}}{\theta\,\Phi(\theta\sqrt{D}) + e^{-\lambda D}/\sqrt{2\pi D}}\ .\ \lambda dy \end{aligned}}$$

Crucial to the numerical inversion of the LT will be an efficient computational method for $\Phi(z)$ for complex arguments z. Maple appears to have such a thing.

Maybe it's worth remarking that the answers here and in CJY look different.

4) Back to the asymptotics of generalised hypergeometric functions. There's a paper of E.M. Wright 'The asymptotic expansion of the generalised hypergeometric function' PLMS 46, 389–408, 1940 where he considers the function $F(z) = \sum_{n\geq 0} f(n) z^n/n!$, where

$$f(n) = \prod_{r=1}^{p} \Gamma(\beta_r + \alpha_r n) \Big/ \prod_{r=1}^{q} \Gamma(\mu_r + \rho_r n)$$

Write $\kappa = 1 + \rho_1 + \dots + \rho_q - \alpha_1 - \dots - \alpha_p$. The α_i, ρ_i are all assumed positive, and κ must be >0 too.

Set

$$h \equiv \prod_{r=1}^{p} \alpha_r^{\alpha_r} \Big/ \prod_{r=1}^{q} \rho_r^{\rho_r}, \qquad \theta \equiv \sum_{r=1}^{p} \beta_r - \sum_{r=1}^{q} \mu_r + \tfrac{1}{2}(q-p)$$

If $\zeta = \arg(z) \in (-\pi, \pi]$, and we write

$$Z = \kappa\,(h|z|)^{1/\kappa}\, e^{i\zeta/\kappa}$$

then in the notation of the paper, Thm 1 applies:

"If $\kappa > 0$ and $|\zeta| \leq \tfrac{1}{2}\pi(\kappa\wedge 2) - \varepsilon$, then $F(z) \sim I(Z)$"

where $I(Z) = Z^\theta e^Z \left\{ \sum_{m=0}^{M-1} A_m Z^{-m} + O(Z^{-M}) \right\}$, and the constants A_r are determined by a procedure detailed in the paper, but the main point of which is that we have an explicit form for A_0:

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)}\, \kappa^{\frac{1}{2}-\theta} \prod_{r=1}^{p} \alpha_r^{\beta_r - \frac{1}{2}} \prod_{r=1}^{q} \rho_r^{-\mu_r + \frac{1}{2}}$$

(Wright's paper switches the limits on the products, but this must be an error)

Assembling these, in order that for our f, $f(x) + B f_0(x)$ remains bounded at infinity, we must have

$$\boxed{B = -\left(\frac{8\alpha}{\sigma^2}\right)^{\frac{1}{2} - r/\sigma^2} 2^{2r/\sigma^2}\, \Gamma(1+\rho_-)\,\Gamma(1-\rho_+)}$$

5) For the numerics of our desired solution, if the argument is very large then $f(x) + B f_0(x)$ is the difference of two extremely big numbers, so it's liable to be difficult to compute accurately. In view of the asymptotic behaviour at infinity of I_ν, it makes sense to consider instead

the function $g(x) = (x/2)^{-\nu} e^{-x} \Gamma(\nu) I_\nu(x) \equiv e^{-x} \sum_{k\geq 0} \Gamma(\nu)(x/2)^{2k}/k!\,\Gamma(k+\nu+1)$. This has the boundary conditions at $x=0$:

$$g(0) = 1, \quad g'(0) = -1.$$

Since I_ν solves

$$f'' + \tfrac{1}{x} f' - (1+\nu^2/x^2) f = 0$$

it's easy to work out that g must solve

$$g'' + \left(2 + \frac{2\nu+1}{x}\right) g' + \frac{2\nu+1}{x}\, g = 0$$

If we try for a power-series solution of this thing then we have

$$g(x) = \sum_{n\geq 0} a_n x^n$$

where a_n satisfy the recursion $a_0 = 1$, $a_1 = -1$,

$$a_{n+2}(n+2)(n+1) + 2(n+1)a_{n+1} + (2\nu+1)(n+2)a_{n+2} + (2\nu+1)a_{n+1} = 0$$

whence

$$\boxed{a_n = \frac{(-2)^n}{n!}\,\frac{\Gamma(n+\nu+\frac12)}{\Gamma(n+2\nu+1)}}$$

Correspondingly, for the solution f, if we take

$$F(z) \equiv \sum_{n\geq 0} c_n z^n/n!\,, \qquad c_n \equiv \frac{\Gamma(n+1)\,\Gamma(1+\rho_-)\,\Gamma(1-\rho_+)}{\Gamma(n+1+\rho_-)\,\Gamma(n+1-\rho_+)}$$

then by Wright's result

$$F(z) \sim (2\sqrt{z})^{\frac12 - 2r/\sigma^2} \exp(2\sqrt{z})\, A_0 \qquad \left[A_0 = (2\pi)^{-\frac12}\, 2^{2r/\sigma^2}\right]$$

Now we found that the solution f can be expressed as $f(x) = F(2\alpha x/\sigma^2)$, and has initial conditions

$$\boxed{f(0) = 1, \quad f'(0) = \alpha/(r-\lambda)}$$

Removing the exponential growth at infinity therefore suggests that we ought to consider the function h, where

$$h(b\sqrt{x}) = f(x)\exp(+b\sqrt{x}) \qquad b \equiv \sqrt{8\alpha/\sigma^2}$$

since this should have polynomial growth at infinity. Reworking the DE for f gives that

$$\boxed{\frac{\sigma^2 t^2}{8} h''(t) + \left\{\frac{\sigma^2 t}{4} - \frac{\sigma^2}{8} + \frac{r}{2}\right\} t h'(t) + \left\{-\frac{rt}{2} - \lambda + \frac{\sigma^2 t}{8}\right\} h(t) + \lambda e^{+t} = 0}$$

Making a power series solution $h(t) = \sum_{n \geq 0} \theta_n t^n$ to this gives the recursion

$$\boxed{\theta_n \left\{ \frac{\sigma^2}{8} n(n-1) + \left(\frac{r}{2} - \frac{\sigma^2}{8}\right) n - \lambda \right\} - \theta_{n-1} \left\{ \frac{\sigma^2}{4}(n-1) + \frac{r}{2} - \frac{\sigma^2}{8} \right\} + \frac{\lambda}{n!} = 0}$$

starting from $\theta_0 = 1$. This gives the correct initial conditions for f, it can be confirmed. While it is not obvious what the general term of the θ sequence will be, the recursive computation of the θ-values should be very quick.

However, for large n we have approximately that

$$\theta_n = \frac{2}{n} \theta_{n-1}$$

so $h(t)$ is looking like e^{2t} for big n, which is not at all what we're after!

More thoughts on Monte Carlo (22/2/01)

1) Recall the situation on p47 of WN XVIII; we've got a process Z and we have the American-style problem of computing

$$Y_0 = \sup_\tau E(Z_\tau)$$

where τ denotes a generic stopping time with values in $[0,T]$. We saw that

$$Y_0 = \inf_{X \in \mathcal{M}_0} \sup_\sigma E(Z_\sigma - X_\sigma) = \inf_{X \in \mathcal{M}_0} E\Big[\sup_{s \le T} (Z_s - X_s)\Big]$$

where σ here denotes *any* random time with values in $[0,T]$.

2) Let's suppose we have a one-dimensional Brownian world, with $Z_t = \varphi(t, W_t)$ for some nice enough function φ. Since we can represent $X \in \mathcal{M}_0$ as $\int_0^\cdot H_s\, dW_s$, we can cast this dual problem in an optimal control form, writing

$$x_t = Z_t - \int_0^t H_u\, dW_u, \qquad \bar{x}_t = \sup_{s \le t} x_s$$

and letting

$$V(t, w, x, \xi) = \inf_H E\Big[\sup_{t \le s \le T} \Big(Z_s - \int_0^s H_u\, dW_u\Big) \vee \xi \,\Big|\, W_t = w,\ x_t = x,\ \bar{x}_t = \xi\Big]$$

Then $V(t, W_t, x_t, \bar{x}_t)$ is a submartingale, and a martingale with optimal control, so we get

$$\dot{V} + \tfrac{1}{2}\Big\{V_{ww} + 2V_{wx}(\varphi' - H) + V_{xx}(\varphi' - H)^2\Big\} + (\dot{\varphi} + \tfrac{1}{2}\varphi'')V_x \ge 0$$

and minimising

$$\boxed{\dot{V} + \tfrac{1}{2}\big(V_{ww} - V_{wx}^2 / V_{xx}\big) + (\dot{\varphi} + \tfrac{1}{2}\varphi'')V_x = 0}$$

Like all HJB things, easy to write down, hard to do much with. We expect $V_{xx} > 0$ for the minimisation to be well posed.

3) Let's consider one classical example, where $\varphi(t, S_t) = (K - e^{\sigma X_t})^+$, $X_t = W_t + (r - \sigma^2/2)t/\sigma$.
If $K = e^{k\sigma}$, and L denotes the local time of X at k, then

$$Z_t = Z_0 - \int_0^t I_{\{X_u \le k\}} e^{\sigma X_u}(\sigma dW_u + \mu du) + \int L_t^x\, f''(x)\, dx$$

where $\mu = r - \tfrac{1}{2}\sigma^2$, and $f''(x)\,dx = -e^{\sigma x} I_{\{x < k\}}\sigma^2 dx + \sigma e^{\sigma k}\delta_k(dx)$

$$\boxed{Z_t - Z_0 \doteq -\int_0^t I_{\{X_u \le k\}} e^{\sigma X_u} r\, du + \sigma e^{\sigma k} L_t}$$

One natural choice for the martingale in the dual formulation would be simply the martingale part of Z. This might not be a great bound though.

Looks like simulation is the simplest way. I simulated this using Scilab, with parameters $\sigma = 0.4$, $r = 0.06$, $K = 100$, $T = 0.5$, nstep = 50, nsim = 5000

S_0	True value	Upper bound by simulation
85	18.04	19.07 ± 0.09
100	9.94	10.57 ± 0.12
115	5.12	5.38 ± 0.10

These figures were obtained using only the martingale part of Z. If we now allow a linear combination of the martingale part of Z, the European put payoff (once it goes in-the-money) and the share once in the money, we get (with nstep = 40)

S_0	True	nsim = 200	nsim = 5000
85	18.04	18.14 ± .077	18.21 ± 0.017
100	9.94	10.16 ± 0.26	10.15 ± 0.052
115	5.12	4.85 ± 0.34	5.23 ± 0.07

These results look quite encouraging. What about a more challenging example?

4) Take the case of a min-put on two assets, as studied by Peter Hartley.

First examples all use $K=100$, $T=0.5$, $r=0.06$, $\sigma_1 = \sigma_2 = 0.6$

$S_1(0)$	$S_2(0)$	True	nsim = 200	nsim = 5000
80	80	37.30	37.71 ± 0.83	37.80 ± 0.16
80	100	32.08	30.90 ± 0.65	33.22 ± 0.16
80	120	29.14	29.06 ± 0.54	29.99 ± 0.12
100	100	25.06	25.58 ± 0.63	25.15 ± 0.12
100	120	20.91	21.33 ± 0.53	21.14 ± 0.12
120	120	15.92	16.18 ± 0.64	16.06 ± 0.13

Next with $\sigma_1 = 0.4$, $\sigma_2 = 0.8$, other figures as above, we get

$S_1(0)$	$S_2(0)$	True	nsim = 200	nsim = 5000
80	80	38.01	37.82 ± 0.90	38.52 ± 0.14
80	100	32.23	31.43 ± 0.72	32.86 ± 0.16
80	120	28.54	29.29 ± 0.70	29.15 ± 0.14
100	80	33.34	33.67 ± 0.55	33.82 ± 0.11
100	100	25.81	26.63 ± 0.67	26.14 ± 0.13
100	120	20.75	20.99 ± 0.72	21.01 ± 0.14
120	80	31.21	31.78 ± 0.39	31.63 ± 0.07
120	100	22.77	22.70 ± 0.54	23.16 ± 0.12
120	120	16.98	16.38 ± 0.76	17.44 ± 0.15

$$\frac{2r-\sigma^2}{2r} = \frac{2r}{2r+\sigma}$$

5) Let's note the stochastic integral representations of some of the martingales which might be useful here

(a) The BS formula for a put is

$$BSput(\sigma, r, T, S_0, K) = Ke^{-rT}\Phi(a) - S_0\Phi(a-\sigma\sqrt{T}) \qquad \left[a \equiv \frac{1}{\sigma\sqrt{T}}\left\{\log K/S_0 - rT + \sigma^2 T/2\right\}\right]$$

leading to martingale

$$M_t = e^{-rt}\, BSput(\sigma, r, T-t, S_t, K)$$

and hence

$$\boxed{dM_t = -e^{-rt}\,\Phi(d_\tau)\, dS_t} \qquad \text{where } d_\tau = \frac{1}{\sigma\sqrt{\tau}}\left(\log\frac{K}{S_0} - r\tau - \sigma^2\tau/2\right)$$

(b) If $X_t \equiv W_t + ct$, $c \equiv (r - \sigma^2/2)/\sigma$, $\underline{X}_t \equiv \inf_{s\le t} X_s$, we have for $x<0$

$$P^0(\underline{X}_t \in dx)/dx = \frac{2e^{-(x-ct)^2/2t}}{\sqrt{2\pi t}} + 2ce^{2cx}\,\Phi\left(\frac{x+ct}{\sqrt{t}}\right) \equiv \varphi(t,x) \quad \text{, say}$$

and thus

$$e^{-r\tau} E\left[h(\underline{X}_T) \mid X_t = x, \underline{X}_t = y\right] = E\left[h(y \wedge \underline{X}_{t,T}) \mid X_t = x\right] e^{-r\tau}$$

$$= \int_{-\infty}^{0} h(y \wedge (z+x))\,\varphi(\tau, z)\,dz\; e^{-r\tau} \qquad (\tau \equiv T-t,\ h(x) = (K - e^{\sigma x})^+)$$

$$\equiv \psi(\tau, x, y) \text{, say.}$$

For a lookback option, where the payoff is $h(x) = (K - e^{\sigma x})^+$, we have that $\psi(T-t, X_t, \underline{X}_t)\, e^{-rt}$ is a martingale, with differential

$$\psi_x(\tau, X_t, \underline{X}_t)\, e^{-rt}\, dW_t$$

So we need to evaluate ψ_x. But this is just

$$\psi_x(\tau, x, y) = e^{-r\tau}\int_{-\infty}^{0\wedge(y-x)} h'(z+x)\,\varphi(\tau, z)\,dz$$

$$= e^{-r\tau}\int_{-\infty}^{\theta} -\sigma e^{\sigma(z+x)}\,\varphi(\tau, z)\,dz \qquad \text{where } \theta = 0 \wedge (y-x) \wedge (\sigma^{-1}\log K - x)$$

$$= -e^{-r\tau}\sigma e^{\sigma x}\Bigg[2e^{r\tau}\,\Phi\left(\frac{\theta - (c+\sigma)\tau}{\sqrt{\tau}}\right) + \frac{2r-\sigma^2}{2r}e^{2r\theta/\sigma}\,\Phi\left(\frac{\theta + c\tau}{\sqrt{\tau}}\right)$$

$$- \frac{2r-\sigma^2}{2r}\; e^{r\tau}\;\Phi\left(\frac{\theta - (\sigma+c)\tau}{\sqrt{\tau}}\right)\Bigg]$$

✓

6) In the Asian option example studied by Longstaff + Schwartz, the payoff of the option is $(A_T - K)^+$, where $A_t = (\delta A_0 + \int_0^t S_u\,du)/(\delta + t)$ is the average price over $[-\delta, t]$. The European version of this would require us to find explicitly the martingale

$$M_t = e^{-rT} E_t (A_T - K)^+ = \frac{e^{-rT}}{\delta + T} E_t \Big(\int_t^T S_u\,du - \underbrace{(K(\delta+T) - \delta A_0 - \int_0^t S_u\,du)}_{\equiv k_t \text{, say.}} \Big)^+$$

One approximation which is known to work quite well for the Asian option is to replace the mean of the share by a lognormal variable with the same first two moments. Writing $\tau = T - t$, we have

$$E_t \int_t^T S_u\,du = \underbrace{S_t (e^{r\tau} - 1)/r}_{\equiv \hat{S}_t}, \qquad E_t\Big[\big(\int_t^T S_u\,du\big)^2\Big] = \underbrace{\frac{2S_t^2\left(r e^{(2r+\sigma^2)\tau} - (2r+\sigma^2)e^{r\tau} + r + \sigma^2\right)}{r(r+\sigma^2)(2r+\sigma^2)}}_{\equiv \hat{S}_t^2 e^{v_t}}$$

where v_t is the variance of the log of the logGaussian variable. The martingale M_t is thus approx.

$$\left.\begin{array}{ll} \dfrac{e^{-rT}}{\delta+T}\left[\hat{S}_t \bar{\Phi}(a_t - \sqrt{v_t}) - k_t \bar{\Phi}(a_t)\right] & \text{if } k_t > 0 \\[2ex] \dfrac{e^{-rT}}{\delta+T}\left(\hat{S}_t + k_t^-\right) & \text{if } k_t \le 0 \end{array}\right\} \qquad a_t \equiv \frac{\log(k_t/\hat{S}_t) + \frac{1}{2}v_t}{\sqrt{v_t}}$$

The differential of this Itô process has all its martingale part in the term $\dfrac{e^{-rT}}{\delta+T}\bar{\Phi}(a_t - \sqrt{v_t})\,d\hat{S}_t$

Discretizing the convertible bond question (14/3/01)

1) Return to the problem of p12, with the notation of that section. In order to deal with issues such as altruistic conversion, calling the issue at a low level, etc, we can simply discretise everything and solve by dynamic programming.

How will this work, in more detail? Fix some $\Delta m > 0$, to be the quantum of conversion, and now we discretize V onto the grid $\Delta m . \mathbb{Z}^+$ à la Rogers-Stapleton. Decisions only take place when V moves across to neighbouring points on the grid; at such a time, the order of decision process is

(i) Shareholders decide whether to call, declare bankruptcy or continue

(ii) If called, bondholders decide whether to surrender or convert

If shareholders decide to continue, bondholders decide whether to convert or not

2) We need to compute a number of quantities for transitions through the grid. If v denotes the current grid point, v_+ the grid point above, v_- the grid point below, and if $\tau \equiv \inf\{t: V_t = v_+ \text{ or } v_-\}$, we need to compute:

(i) $$P^v(V_\tau = v_+) = \frac{v^\theta - v_-^\theta}{v_+^\theta - v_-^\theta} \equiv p_+ \qquad \theta \equiv -\frac{2}{\sigma^2}(r-\delta-\sigma^2/2)$$

$$P^v(V_\tau = v_-) = \frac{v_+^\theta - v^\theta}{v_+^\theta - v_-^\theta} \equiv p_-$$

(ii) $$E^v\left[e^{-r\tau} : V_\tau = v_+\right] = \left\{ \left(\frac{v}{v_-}\right)^\beta - \left(\frac{v}{v_-}\right)^{-\alpha} \right\} \Big/ \left\{ \left(\frac{v_+}{v_-}\right)^\beta - \left(\frac{v_+}{v_-}\right)^{-\alpha} \right\} \equiv \tilde{p}_+$$

$$E^v\left[e^{-r\tau} : V_\tau = v_-\right] = \left\{ \left(\frac{v}{v_+}\right)^\beta - \left(\frac{v}{v_+}\right)^{-\alpha} \right\} \Big/ \left\{ \left(\frac{v_-}{v_+}\right)^\beta - \left(\frac{v_-}{v_+}\right)^{-\alpha} \right\} \equiv \tilde{p}_-$$

(iii) $$E^v\left[\int_0^\tau e^{-rs} ds : V_\tau = v_\pm\right] = \frac{1}{r}\left\{ P^v(V_\tau = v_\pm) - E^v\left[e^{-r\tau} ; V_\tau = v_\pm\right] \right\} \equiv \varphi_\pm$$

(iv) $$E^v\left[\int_0^\tau e^{-rs} V_s \, ds : V_\tau = v_+\right] = E^v\left[\int_0^\tau e^{-ru} V_u \frac{s(V_u) - s(v_-)}{s(v_+) - s(v_-)} du\right] \equiv \psi_+$$

Thus if we define

$$f(v) = E^v\left[\int_0^\tau e^{-rs} V_s \, ds ; V_\tau = v_+\right]$$

we find that f solves

$$\boxed{\tfrac{1}{2}\sigma^2 v^2 f''(v) + (r-\delta) v f'(v) - r f(v) + \frac{v^{1+\theta} - v_-^\theta v}{v_+^\theta - v_-^\theta} = 0 \qquad , f(v_+) = f(v_-) = 0}$$

Best to let Maple and fortran handle this? No, not necessary!

Value of <u>all</u> convertibles after converting one is

$$\Delta m \left(S_{j-1, k+1} - K - B_{jk} \right) + (j-1) \Delta m \; B_{j-1, k+1}$$

$$= j \Delta m \; B_{j, k+1} + \Delta m \left\{ S_{j-1, k+1} - K - B_{jk} - B_{j-1, k+1} \right\}$$

so value per original bond is thus $/ \, j \Delta m$

<u>Calling</u> : <u>if</u> $K^* > -K + (k + jK) \Delta m / n \equiv k_c$, bondholders surrender and shares become worth

$$\left(k \Delta m - j \Delta m \, K^* \right) / (n - j \Delta m)$$

<u>else</u> bondholders convert, and shares become worth

$$\left(k \Delta m + j \Delta m \, K \right) / n$$

As $k \uparrow k_c$, we get limit $K^* + K$

as $k \downarrow k_c$, we get limit $K^* + K$

There is a particular solution

$$-\frac{v^{1+\theta}}{(\sigma^2\theta-\delta)(v_+^\theta-v_-^\theta)} - \frac{v\,v_-^\theta}{\delta(v_+^\theta-v_-^\theta)} \equiv f_0(v)$$

so for the solution we want we shall solve for A, B

$$f(v) \equiv A v^{-\alpha} + B v^{\beta} + f_0(v) = 0 \quad \text{at } v = v_\pm$$

3) Now suppose that the solutions S, B are defined at gridpoints

$$S_{jk} \equiv S(j\Delta m, k\Delta m), \quad B_{jk} \equiv B(j\Delta m, k\Delta m) \qquad \left\{ \begin{array}{l} j = 0, 1, \dots, n_m \\ k = 0, 2, \dots, n_v \end{array} \right\}$$

We already know that

$$S_{0k} = \frac{k\Delta m}{n}, \quad B_{0k} = S_{0k} - K \qquad \forall k$$

and we have more generally that at $(j\Delta m, k\Delta m)$ if there was no calling

$$B_{jk} = \max\left\{ \rho(\varphi_+ + \varphi_-) + \tilde{p}_+ B_{j,k+1} + \tilde{p}_- B_{j,k-1},\ B_{j-1,k+1} + \{S_{j-1,k+1} - K - B_{jk} - S_{j-1,k+1}\}/j \right\}$$

and if the issue had just been called we have

$$B_{jk} = \max\{ K^*, -K + (k + jK)\Delta m/n \}$$

On the other hand, if the shareholders have just declared bankruptcy, the bonds are each worth

$$p\, k\Delta m / j\Delta m = pk/j .$$

The shareholders have to choose the best of the three options:

$$S_{jk} = \max\left[\frac{\delta(\psi_+ + \psi_-) - j\Delta m\, \rho(\varphi_+ + \varphi_-)}{n - j\Delta m} + \tilde{p}_+ S_{j k+1} + \tilde{p}_- S_{j k-1},\ 0,\ \min\left(\frac{(k - jK^*)\Delta m}{n - j\Delta m}, \frac{(k + jK)\Delta m}{n} \right) \right]$$

This may appear to be independent of what's happening with the bond, but it's not; if $k^* = k^*(j)$ is the value at which bond holders convert a little, the above relation only holds for $k < k^*$; $S_{jk^*} = S_{j-1, k^*+1}$

A little story on liquidity (18/3/01)

1) Just as a very simple-minded first story, suppose that we have J agents who are going to hold positions $\theta_j(t)$ in some financial derivative in zero net supply. At time T, the value of the derivative will be σB_T and agent j tries to maximise

$$E\, U_j(w_j(T))$$

where $U_j(x) = -\exp(-\gamma_j x)$, and $w_j(t)$ is the running value of agent j's wealth. Agent j sees a signal

$$\xi_j(t) = \sigma B_t + \sigma_j W_j(t)$$

where the W_j are independent standard BMs. Then it's easy to confirm that conditional on $(\xi_j(u);\, u \le t)$

$$B_t \sim N\left(a_j \xi_j(t),\ v_j(t)\right) \qquad a_j = \sigma\,(\sigma^2+\sigma_j^2)^{-1},\ v_j(t) = \sigma_j^2 t/(\sigma^2+\sigma_j^2) \checkmark$$

If the price at time t of the asset were p_t, then each agent would try to achieve

$$\max_\theta E\, U_j(\sigma\theta B_T - p_t\theta) = \max_\theta E - \exp\{-\gamma_j(\sigma\theta B_T - p_t\theta)\}$$

$$= -\min_\theta \exp\left(\gamma_j p_t\theta - \gamma_j\sigma\theta\,\xi_j(t)\, a_j + \tfrac{1}{2}\theta^2\gamma_j^2 V_j(t)\right)$$

achieved when

$$\theta = \left(\sigma a_j \xi_j(t) - p_t\right)/\gamma_j V_j(t) \qquad \left[V_j(t) = \sigma^2(T-t) + v_j(t)\sigma^2\right]$$

Market clearing $\Rightarrow$

$$\boxed{p_t = \frac{\sum_j \sigma a_j \xi_j(t)/\gamma_j V_j(t)}{\sum_j 1/\gamma_j V_j(t)} \xrightarrow[(T\to\infty)]{} \frac{\sum_j \sigma a_j \xi_j(t)/\gamma_j}{\sum_j \gamma_j^{-1}}}$$

2) How would things change in a situation where there were transactions costs, so an ask price $\bar{p}_t >$ bid price p_t ?

Agent j with $\theta = \theta_j(t)$ units of the derivative would have a reason to increase to $\theta+\varepsilon$ provided

$$E\, U_j(\sigma(\theta+\varepsilon)B_T - \varepsilon\bar{p}_t) > E\, U_j(\sigma\theta B_T)$$

i.e.

$$\exp\left[-\gamma_j\left(-\varepsilon\bar{p}_t + \sigma(\theta+\varepsilon)a_j\xi_j(t)\right) + \tfrac{1}{2}(\theta+\varepsilon)^2\gamma_j^2 V_j(t)\right] < \exp\left[-\gamma_j\sigma\theta a_j\xi_j(t) + \tfrac{1}{2}\theta^2\gamma_j^2 V_j(t)\right]$$

i.e.

$$\bar{p}_t - \sigma a_j\xi_j(t) < -\gamma_j V_j(t)\,(\theta + \varepsilon/2)$$

Hence agent j will do no trading provided

$$\boxed{\underline{p}_t \leq \sigma a_j \xi_j(t) - \gamma_j V_j(t) \theta_j(t) \leq \bar{p}_t}$$

This must hold at all times for all j. Assuming that the market-makers hold zero inventory, we also have at all times a cleared market. Now buying can only happen when for some j the right-hand inequality holds with equality and for some j' the left-hand inequality holds with equality

3) A simple special case: $J = 2$, $\bar{p}_t - \underline{p}_t \leq \delta \ \forall t$. In this case, if agent 1 holds θ_t at time t, we get

$$\underline{p}_t \leq \sigma a_1 \xi_1(t) - \gamma_1 \theta_t V_1(t) \leq \bar{p}_t$$

$$\underline{p}_t \leq \sigma a_2 \xi_2(t) + \gamma_2 \theta_t V_2(t) \leq \bar{p}_t$$

We only get trading when

$$\sigma a_1 \xi_1(t) - \sigma a_2 \xi_2(t) - \theta_t(\gamma_1 V_1(t) + \gamma_2 V_2(t)) = \delta \text{ or } -\delta$$

and the trading is just the local time trading required to keep the process in the interval.

Simplification: Let's suppose $\gamma_i \sigma^2 T \to c_i$ as $T \to \infty$, so that the reckoning is infinitely distant, and the γ_i are getting small correspondingly. Then we have

$$\begin{cases} \underline{p}_t \leq \sigma a_1 \xi_1(t) - c_1 \theta_t \leq \bar{p}_t \\ \underline{p}_t \leq \sigma a_2 \xi_2(t) + c_2 \theta_t \leq \bar{p}_t \end{cases}$$

with trading when $\sigma(a_1 \xi_1(t) - a_2 \xi_2(t)) - (c_1 + c_2)\theta_t = \delta$ or $-\delta$.

$$\equiv M_t - L_t^+ + L_t^- \equiv X_t$$

where M is a martingale, $d\langle M\rangle_t / dt = \sigma^4 (\sigma_1^2 + \sigma_2^2) / (\sigma^2 + \sigma_1^2)(\sigma^2 + \sigma_2^2) \equiv \Sigma^2$, say.
Now X is a Brownian motion with volatility Σ, constrained by reflection to remain in $[-\delta, \delta]$.
We have

$$(X_t + \delta)^2 - \Sigma^2 t + 4\delta L_t^+ \quad \text{is a martingale,}$$

so that

$$\boxed{E L_t^+ \sim \Sigma^2 t / 4\delta}$$

gives the rate of growth of L^+. The same rate holds for L^-. Notice also that if $\xi_i(0) = \theta_i(0) = 0$

$$\boxed{\theta_t = \frac{L_t^+ - L_t^-}{c_1 + c_2}}$$

What happens to the market-maker? Every time there is a trade, he gains δ times the volume of the trade, so

$$\boxed{\text{rate of growth of market-maker's wealth} = \frac{\not{Z}^2}{2(c_1+c_2)}}$$

What about agent 1? His trading balance at time t will be

$$-\int_0^t \frac{\bar{p}_s\, dL_s^+}{c_1+c_2} + \int_0^t \frac{\underline{p}_s\, dL_s^-}{c_1+c_2}$$

$$= \frac{1}{c_1+c_2}\left[-\int_0^t \{\sigma a_1 \xi_1(s) - c_1\theta(s)\}\, dL_s^+ + \int_0^t (\sigma a_1 \xi_1(s) - c_1\theta(s))\, dL_s^- \right]$$

$$\doteq \frac{1}{c_1+c_2}\left[-\sigma a_1 \xi_1(t) L_t^+ + \sigma a_1 \xi_1(t) L_t^- + c_1\int_0^t \theta_s\, (dL_s^+ - dL_s^-) \right]$$

$$= \frac{1}{c_1+c_2}\left\{ -\sigma a_1 \xi_1(t)(L_t^+ - L_t^-) + \frac{c_1}{c_1+c_2}\,\frac{(L_t^+-L_t^-)^2}{2} \right\}$$

Now

$$E\left[\sigma a_1 \xi_1(t) \mid M_u : u \le t\right] = \sigma^2 a_1 \cdot \frac{(a_1-a_2)\sigma^2 + a_1\sigma_1^2}{\not{Z}^2} M_t ,$$ so agent 1's trading balance at time t will be worth on average

$$E\left[-\frac{\sigma^2 a_1\{(a_1-a_2)\sigma^2 + a_1\sigma_1^2\}}{(c_1+c_2)\not{Z}^2} M_t (L_t^+ - L_t^-) + \frac{c_1}{(c_1+c_2)^2}\,\frac{(L_t^+-L_t^-)^2}{2} \right]$$

$$= \left(\frac{c_1}{2(c_1+c_2)^2} - \frac{\sigma^2 a_1\{(a_1-a_2)\sigma^2 + a_1\sigma_1^2\}}{2(c_1+c_2)\not{Z}^2} \right) E(L_t^+ - L_t^-)^2 - \frac{\sigma^2 a_1\{(a_1-a_2)\sigma^2 + a_1\sigma_1^2\}}{(c_1+c_2)\not{Z}^2} E X_t (L_t^+ - L_t^-)$$

It's clear that $E X_t(L_t^+ - L_t^-) = O(\sqrt{t})$, so for the linear growth term, we get that the rate of growth of agent 1's trading balance is on average

$$\frac{c_1 \not{Z}^2}{2(c_1+c_2)^2} - \frac{\sigma^2 a_1\{(a_1-a_2)\sigma^2 + a_1\sigma_1^2\}}{2(c_1+c_2)} = \frac{1}{c_1+c_2}\left\{ \frac{c_1\not{Z}^2}{2(c_1+c_2)} - \frac{\sigma^4\sigma_2^2}{2(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)} \right\}$$

Likewise, 2's trading balance grows at average rate

$$\frac{1}{c_1+c_2}\left\{ \frac{c_2\not{Z}^2}{2(c_1+c_2)} - \frac{\sigma^4\sigma_1^2}{2(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)} \right\}$$

$$\bar{p}_t = Z_t + \left\{ \frac{2\rho - 1}{2} + \frac{c_2 - c_1}{2(c_2 + c_1)} \right\} M_t + \frac{1}{2} |X_t| - \frac{c_2 - c_1}{2(c_2 + c_1)} X_t$$

We can express this somewhat more simply; the mean rate of growth of agent 1's trading account is

$$\boxed{\frac{\Xi^2}{c_1+c_2}\left\{\frac{c_1}{2(c_1+c_2)} - \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right\}}$$

with a corresponding expression for agent 2. This looks very surprising - if σ_2 is bigger, agent 1 does less well ?!! But if you realise that the QV of $a_2\xi_2(\cdot)$ is $\sigma^2 dt/(\sigma^2+\sigma_2^2)$ then a big σ_2 means that $a_2\xi_2$ is moving relatively little, and so the buys and sells of agent 2 will be much more nearly of the same value.

How do we characterize the dynamics of the prices?

If $X_i = \sigma a_i \xi_i$, $M \equiv X_1 - X_2$ then the joint distribution of (M, X_1) is the same as the joint distribution of

$$(M,\ \rho M + Z)$$

where $\rho = \sigma_2^2/(\sigma_1^2+\sigma_2^2)$ and the variance of Z is

$$\frac{\sigma^4}{(\sigma_1^2+\sigma_2^2)(\sigma^2+\sigma_1^2)(\sigma^2+\sigma_2^2)}\left\{\sigma_1^2\sigma_2^2 + \sigma^2(\sigma_1^2+\sigma_2^2)\right\} \equiv \sigma_Z^2$$

with Z independent of M. We could define

$$\bar{p}_t = \max\{\sigma a_1 \xi_1(t) - c_1\theta_t\ ,\ \sigma a_2\xi_2(t) + c_2\theta_t\}$$

$$\underline{p}_t = \min\{\sigma a_1 \xi_1(t) - c_1\theta_t\ ,\ \sigma a_2\xi_2(t) + c_2\theta_t\}$$

Thus in distribution

$$\boxed{\begin{aligned}\bar{p}_t &= Z_t + \frac{2\rho-1}{2}M_t + \frac{c_2-c_1}{2(c_2+c_1)}(L_t^+ - L_t^-) + \tfrac{1}{2}|X_t| \\ \underline{p}_t &= \bar{p}_t - |X_t|\end{aligned}}$$

This would be one possibility

Arrow-Kurz questions: deterministic case (23/4/01)

1) This is essentially taking the situation of pp 1-6, or of Peter's notes, with all volatilities set to zero. Thus $\dot{\eta}_t = \eta_t (\mu_L + \mu_T)$ and the renormalized capital and govt capital processes k, k_g will solve

$$(1) \quad \begin{cases} \dot{k} = f(k_p, k_g) - \gamma k - c \\ \dot{k}_g = f(k_p, k_g) - \gamma k_g - c - \beta_k^{-1} \left\{ \beta_s \left[(\beta_k - \beta_w) k_p f_p + \beta_w f - \beta_c^{-1} c + \beta_k f_p \Delta \right] - \dot{\Delta} + (\gamma - \delta)\Delta \right\} \end{cases}$$

where $\Delta \equiv \eta^{-1} D$, $\gamma \equiv \mu_T + \mu_L + \delta$. The private sector is free to choose c and Δ, and the question is when a particular trajectory is controllable, and with what instruments. Private sector's objective is to max

$$\int_0^\infty e^{-\lambda t} u(c_t, k_g(t))\, dt$$

over choice of (c, Δ). Government may choose tax rates, and the debt trajectory Δ, but for controllability the chosen Δ would also have to be optimal for the private sector.

2) Let's suppose that govt plans to make tax rates f^n of (k_p, k_g) only, so that we may write

$$(2) \quad \begin{cases} \dot{k} = \Phi(k, k_g, c) \\ \dot{k}_g = \Psi(k, k_g, c, \Delta, \dot{\Delta}) \end{cases}$$

Now if we have found a choice (c, Δ) which is optimal for the private sector, and we perturb this to $(c + \varepsilon\xi, \Delta + \varepsilon\zeta)$, the first-order change in the objective should be zero. If we suppose that the change perturbs k to $k + \varepsilon\varphi$, k_g to $k_g + \varepsilon\psi$, then we should have

$$(3) \quad \begin{cases} \dot{\varphi} = \Phi_1 \varphi + \Phi_2 \psi + \Phi_3 \xi \\ \dot{\psi} = \Psi_1 \varphi + \Psi_2 \psi + \Psi_3 \xi + \Psi_4 \zeta + \Psi_5 \dot{\zeta} \end{cases}$$

Change in payoff is to first order

$$\int_0^\infty e^{-\lambda t} \left\{ u_c \cdot \xi + u_g \cdot \psi \right\} dt$$

which we need to express solely in terms of (ξ, ζ). If we write $z_t \equiv \begin{pmatrix} \varphi_t \\ \psi_t \end{pmatrix}$ then (2) has the form

$$(4) \quad \dot{z}_t = A(t) z_t + x_t \quad , \quad x_t = \begin{pmatrix} \Phi_3 \xi \\ \Psi_3 \xi + \Psi_4 \zeta + \Psi_5 \dot{\zeta} \end{pmatrix}$$

where all of the derivatives of Φ, Ψ are known, because they are evaluated along the desired trajectory. Thus $A(\cdot)$ is a known matrix-valued f^n of time. If we solve

$$(5) \quad \boxed{\dot{B}(t) + B(t) A(t) = 0, \quad B(0) = I}$$

then we have

$$\frac{d}{dt}\left(B(t) z_t\right) = B(t)\, x_t$$

so that $$z_t = B_t^{-1}\left\{ z_0 + \int_0^t B_s x_s \, ds \right\}$$

and the first-order change in the payoff will be

$$\int_0^\infty e^{-\lambda t}\left\{ u_c(c_t, k_g(t)).\, \xi_t + u_g(c_t, k_g(t))\,(0,1)\, B_t^{-1}\left\{ z_0 + \int_0^t B_s x_s\, ds \right\}\right\} dt$$

If we suppose (quite reasonably) that $z_0 = 0$, we get a simpler expression for the first-order change in payoff:

$$\int_0^\infty e^{-\lambda t} u_c\, \xi_t\, dt + \int_0^\infty \left\{ \int_s^\infty e^{-\lambda t} u_g(c_t, k_g(t))\,(0,1) B_t^{-1}\, dt \right\} B_s x_s\, ds$$

If we pick out the piece of this which depends on ξ, we shall find

$$\boxed{0 = e^{-\lambda t} u_c(c_t, k_g(t)) + \int_t^\infty e^{-\lambda s} u_g(c_s, k_g(s))\,(0,1)\, B_s^{-1} B_t \binom{-1}{-1}\, ds}$$

using the explicit forms $\Phi_3 = \Phi_3 = -1$. Similarly, if we pick out the terms involving ζ we find

$$\int_0^\infty \left\{ g_1(t)\, \zeta_t + g_2(t)\, \dot{\zeta}_t \right\} dt$$

where

$$\begin{cases} g_1(t) = -\int_t^\infty e^{-\lambda s} u_g(c_s, k_g(s))\,(0,1)\, B_s^{-1} B_t \binom{0}{1} \left(\beta_s f_p + \beta_k^{-1}(\gamma - \delta)\right)_t ds \\[2ex] g_2(t) = \beta_k^{-1} \int_t^\infty e^{-\lambda s} u_g(c_s, k_g(s))\,(0,1)\, B_s^{-1} B_t \binom{0}{1}\, ds \end{cases}$$

and for this to be independent of ζ we shall have to have

$$\boxed{g_1(t) = \dot{g}_2(t)}$$

3) <u>General formulation</u> We have a system controlled by two players, A and B, evolving as

$$\dot{x}_t = \Phi(t, x_t, a_t, b_t)$$

where (a_t) is the action chosen by player A, b_t the action chosen by player B. First B declares his action b, then A chooses a so as to

$$\max \int_0^T \varphi(t, x_t, a_t)\, dt$$

If $A(t) = D_x \Phi(t, x_t, a_t)$ and we have $\dot{B}_t = A(t) B_t$, $B_0 = I$, then

$$\frac{d}{dt}\left(B_t^{-1} \xi_t\right) = B_t^{-1} \theta_t \quad \therefore \quad \xi_t = B_t\left[\xi_0 + \int_0^t B_s^{-1} \theta_s \, ds\right] \qquad \theta_t = D_a \Phi(t, x_t, a_t) \alpha_t$$

More generally, suppose that player B chooses the evolution Φ from some allowed set, so

$$\dot{x}_t = \Phi(t, x_t, a_t)$$

and player A wishes to maximise

$$\int_0^\infty \varphi(t, x_t, a_t)\, dt.$$

When is a particular trajectory (a_t) optimal for player A? If A perturbs the trajectory a to $a + \varepsilon\alpha$, then x changes to $x + \varepsilon\xi$ and

$$\dot{\xi} = D_x\Phi \cdot \xi + D_a\Phi \cdot \alpha$$

so that the first-order change in objective must be

$$0 = \int_0^\infty \left(D_x\varphi \cdot \xi + D_a\varphi \cdot \alpha \right) dt$$

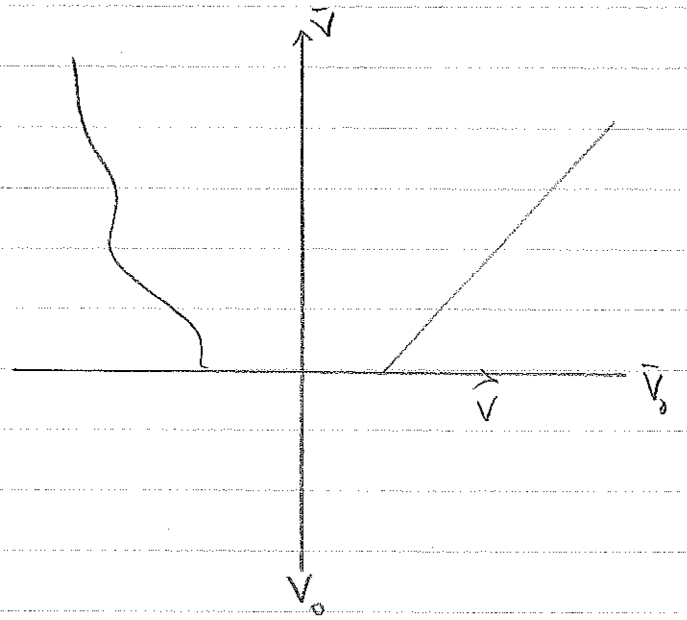
If $(F_{st})_{0 \le s \le t}$ denotes the flow map of Φ, we conclude that for a to be optimal for A, with corresponding trajectory x, it's necessary that

$$\boxed{\begin{aligned} 0 &= D_a\varphi(t, x_t, a_t) + \int_t^\infty D_x\varphi(s, x_s, a_s)\, DF_{ts}(x_t)\, D_a\Phi(t, x_t, a_t)\, ds \\ 0 &= \dot{x}_t - \Phi(t, x_t, a_t) \end{aligned}}$$

In our example, x is the 3-vector $(k, k_g, \Delta)^T$, a is the 2-vector $(c, \tilde{c})^T$ where

$$\dot{x} \equiv \begin{pmatrix} \dot{k} \\ \dot{k}_g \\ \dot{\Delta} \end{pmatrix} = \begin{bmatrix} f(k-k_g, k_g) - \gamma k - c \\ f(k-k_g, k_g) - \gamma k_g - c - \beta_k^{-1}\left(\beta_s\left((\beta_k - \beta_w) f_b k_p + \beta_w f - \beta_c^{-1} c + \beta_k f_b \Delta \right) - \tilde{c} + (\gamma - \delta)\Delta \right) \\ \tilde{c} \end{bmatrix}$$

$$\equiv \Phi(t, x_t, a_t)$$


$\tilde{V}$
$\hat{V}$
$\bar{V}_0$
V_0

Transforming the convertible bond problem (9/5/01)

1) Returning to the problem and notation of p 12–16, let's introduce a function φ, and define $v_t \equiv \varphi(m_t) V_t$, so that we shall have

$$dv_t = v_t \left(\sigma dW_t + (r-\delta)dt\right) + \left\{\varphi'(m_t) V_t - K\varphi(m_t)\right\} dm_t$$

so that if

$$\varphi'(m)\eta(m) = K\varphi(m) \quad , \text{ that is, } \quad \boxed{\varphi(m) = \exp\left\{\int_0^m K\eta(t)^{-1}\, dt\right\}}$$

then v is a standard log-Brownian motion, nothing fancy happening at the max.

Now consider an asset which generates cashflow at rate $p(v_t, \bar{v}_t)$ where $\bar{v}_t = (\sup_{u\le t} v_u) \vee \bar{v}_0$, and such that when it reaches boundary $h(\bar{v})$ gets a rebate $R(\bar{v})$.

The asset is exchanged for A^* when $\bar{v}$ first hits v^*.

We shall have in our examples $p(v,\bar{v}) = c_0(\bar{v}) + c_1(\bar{v}).v$ If $f(v,\bar{v})$ is the value of this asset, then we shall have the valuation equations

$$\boxed{\begin{aligned}
&\tfrac{1}{2}\sigma^2 v^2 \frac{\partial^2 f}{\partial v^2} + (r-\delta)v\frac{\partial f}{\partial v} - rf + c_0(\bar{v}) + c_1(\bar{v})v = 0\\
&f(h(\bar{v}),\bar{v}) = R(\bar{v})\\
&f(v^*, v^*) = A^*\\
&\frac{\partial f}{\partial \bar{v}} = 0 \quad \text{at} \quad v = \bar{v}
\end{aligned}}$$

The solution takes the form

$$f(v,\bar{v}) = A(\bar{v})v^{-\alpha} + B(\bar{v})v^{\beta} + c_0(\bar{v})/r + c_1(\bar{v})v/\delta$$

The quantities m and $\bar{v}$ are related via

$$\boxed{\bar{v} = \varphi(m)\eta(m) = \eta(0)\exp\left\{\int_0^m \frac{K+\eta'(t)}{\eta(t)}\, dt\right\}}$$

which is a 1-1 correspondence provided $\eta' < -K$. Let's write $m = \Phi(\bar{v})$. We also found earlier that $\eta(0) = \beta n(K+p/r)/(\beta-1)$.

Suppose that the function h has been fixed, the boundary condition at h also given. By using this, we may eliminate $A(\cdot)$ and obtain from this a first-order linear ODE for $B(\cdot)$: the boundary condition is

$$\eta_0^{\beta} B(\eta_0) = -K + \eta_0/n - p/r = (K+p/r)/(\beta-1)$$

and

$$\left\{\bar{v}^{\alpha+\beta} - h(\bar{v})^{\alpha+\beta}\right\} B'(\bar{v}) - (\alpha+\beta)\, h(\bar{v})^{\alpha+\beta}\, \frac{h'}{h}(\bar{v})\, B(\bar{v})$$

$$+ \left(h(\bar{v})^{\alpha} R(\bar{v})\right)' - \frac{\rho}{r}\left(h(\bar{v})^{\alpha}\right)' = 0$$

Thus if we set

$$H(t) \equiv \int_t^{\eta_0} \frac{(\alpha+\beta)\, h(s)^{\alpha+\beta-1}\, h'(s)\, ds}{s^{\alpha+\beta} - h(s)^{\alpha+\beta}}$$

then

$$\frac{\partial}{\partial v}\left[e^{H(\bar{v})} B(\bar{v})\right] = e^{H(\bar{v})}\, \frac{\frac{\rho}{r}\left(h(\bar{v})^{\alpha}\right)' - \left(h(\bar{v})^{\alpha} R(\bar{v})\right)'}{\bar{v}^{\alpha+\beta} - h(\bar{v})^{\alpha+\beta}}$$

so

$$B(\bar{v}) = e^{-H(\bar{v})}\left\{B(\eta_0) - \int_{\bar{v}}^{\eta_0} e^{H(s)}\, \frac{\frac{\rho}{r}\left(h(s)^{\alpha}\right)' - \left(h(s)^{\alpha} R(s)\right)'}{s^{\alpha+\beta} - h(s)^{\alpha+\beta}}\, ds\right\}$$

2) This appears to be little better. We can transform the original formulation if we make use of the notation $\xi(m) = \theta(m)\,\eta(m)$. We have

$$\boxed{S = \frac{V - mp/r}{n-m} - \left(\frac{V}{\xi}\right)^{-\alpha} \frac{(\beta-1)\xi r - \beta m p}{r(\alpha+\beta)(n-m)} - \left(\frac{V}{\xi}\right)^{\beta} \frac{(\alpha+1)\xi r - \alpha m p}{r(\alpha+\beta)(n-m)}}$$

Now

$$\frac{\partial S}{\partial m} = -\frac{p/r}{n-m} + \frac{V - mp/r}{(n-m)^2} - \left(\frac{V}{\xi}\right)^{-\alpha}\left[\frac{-\beta p}{r(\alpha+\beta)(n-m)} + \frac{(\beta-1)\xi r - \beta m p}{r(\alpha+\beta)(n-m)^2} + \left\{\frac{\xi(\beta-1)r + \alpha(\beta-1)\xi r - \alpha\beta m p}{r(\alpha+\beta)(n-m)\xi}\right\}\xi'\right]$$

$$-\left(\frac{V}{\xi}\right)^{\beta}\left[\frac{-\alpha p}{r(\alpha+\beta)(n-m)} + \frac{(\alpha+1)\xi r - \alpha m p}{r(\alpha+\beta)(n-m)^2} + \left\{\frac{\xi(\alpha+1)r - \beta(\alpha+1)r\xi + \alpha\beta m p}{r(\alpha+\beta)(n-m)\xi}\right\}\xi'\right]$$

and

$$\frac{\partial S}{\partial V}(m, \eta(m)) = \frac{1}{n-m} + \frac{\alpha}{\eta}\,\theta^{\alpha}\, \frac{(\beta-1)r\theta\eta - \beta m p}{r(\alpha+\beta)(n-m)} - \frac{\beta}{\eta}\,\theta^{-\beta}\, \frac{(\alpha+1)\theta\eta r - \alpha m p}{r(\alpha+\beta)(n-m)}$$

We have the boundary condition at η

$$r(n-m)(\alpha+\beta)\left\{\frac{\partial S}{\partial m} - K\frac{\partial S}{\partial V}\right\}(m, \eta(m)) = 0$$

which becomes

$$(\alpha+1)(\beta-1) = 2\bar{r}/\sigma^2 \quad , \quad \alpha\beta = 2r/\sigma^2$$

$$0 = -\rho(\beta+\alpha) + \frac{\beta+\alpha}{n-m}(r\eta - m\rho) - \theta^{\alpha}\left[-\beta\rho + \frac{(\beta-1)\theta\eta r - \beta m\rho}{n-m} + \left\{(\beta-1)r + \alpha(\beta-1)r - \frac{\alpha\beta m\rho}{\xi}\right\}\xi'\right]$$

$$- \theta^{-\beta}\left[-\alpha\rho + \frac{(\alpha+1)r\theta\eta - \alpha m\rho}{n-m} + \left\{(\alpha+1)r - \beta(\alpha+1)r + \frac{\alpha\beta m\rho}{\xi}\right\}\xi'\right]$$

$$- K\left(r(\beta+\alpha) + \frac{\alpha}{\eta}\theta^{\alpha}\left((\beta-1)\theta\eta r - \beta m\rho\right) - \frac{\beta}{\eta}\theta^{-\beta}\left((\alpha+1)\theta\eta r - \alpha m\rho\right)\right)$$

Hence

$$\xi'\cdot\left\{\theta^{\alpha}\left((\beta-1)r + \alpha(\beta-1)r - \frac{\alpha\beta m\rho}{\xi}\right) + \theta^{-\beta}\left((\alpha+1)r - \beta(\alpha+1)r + \frac{\alpha\beta m\rho}{\xi}\right)\right\}$$

$$= -\rho(\beta+\alpha) + \frac{\beta+\alpha}{n-m}(r\eta - m\rho) - \theta^{\alpha}\left(\frac{(\beta-1)\theta\eta r - \beta m\rho}{n-m} - \beta\rho + \frac{K\alpha}{\eta}\left((\beta-1)\theta\eta r - \beta m\rho\right)\right)$$

$$- \theta^{-\beta}\left(\frac{(\alpha+1)r\theta\eta - \alpha m\rho}{n-m} - \alpha\rho - \frac{K\beta}{\eta}\left((\alpha+1)\theta\eta r - \alpha m\rho\right)\right) - rK(\alpha+\beta)$$

So

$$\boxed{\begin{aligned} &\xi'\left(\theta^{\alpha} - \theta^{-\beta}\right)\left\{(\alpha+1)(\beta-1)r - \alpha\beta m\rho/\xi\right\} \\ &= (\alpha+\beta)\left\{\frac{r\eta - m\rho}{n-m} - \rho - rK\right\} - \theta^{\alpha}\left\{-\beta\rho + \frac{(\beta-1)\theta\eta r - \beta m\rho}{n-m}\left(1 + \frac{\alpha K(n-m)}{\eta}\right)\right\} \\ &\qquad - \theta^{-\beta}\left\{-\alpha\rho + \frac{(\alpha+1)\theta\eta r - \alpha m\rho}{n-m}\left(1 - \frac{\beta K(n-m)}{\eta}\right)\right\} \end{aligned}}$$

$$= (1-\theta^{\alpha})\left\{-\beta\rho + \frac{(\beta-1)\theta\eta r - \beta m\rho}{n-m}\left(1 + \frac{\alpha K(n-m)}{\eta}\right)\right\}$$

$$+ (1-\theta^{-\beta})\left\{-\alpha\rho + \frac{(\alpha+1)\theta\eta r - \alpha m\rho}{n-m}\left(1 - \frac{\beta K(n-m)}{\eta}\right)\right\}$$

$$+ (\alpha+\beta)(1-\theta)\left\{\frac{\eta r}{n-m} - Kr\right\}$$

This leads to an expression for ξ' in terms of θ and η. However, the boundary condition $Y(m, \xi(m)) = -\rho\,\xi(m)/m$ gives us an expression for η in terms of θ:

$$\boxed{\eta = \frac{n\rho(\alpha+\beta) - \theta^{-\alpha}\left(\beta K r(n-m) + \beta n\rho\right) - \theta^{\beta}\left(\alpha K r(n-m) + \alpha n\rho\right)}{r(\alpha+\beta)\theta - r(\beta-1)\theta^{-\alpha} - r(\alpha+1)\theta^{\beta} + \rho\theta r(\alpha+\beta)(n-m)/m}}$$

$$= \frac{(\beta\theta^{-\alpha} + \alpha\theta^{\beta})\left(Kr(n-m) + n\rho\right) - n\rho(\alpha+\beta)}{r\left\{(\alpha+1)\theta^{\beta} + (\beta-1)\theta^{-\alpha} - (\alpha+\beta)\theta - (\alpha+\beta)\theta\rho(n-m)/m\right\}} \equiv \frac{F(m,\theta)}{G(m,\theta)}$$

Writing

$$S(m,V) = \frac{V - mp/r}{n-m} + a(m)\, V^{-\alpha} + b(m)\, V^{\beta}$$

we expect that $\boxed{b(m) = \frac{\alpha m p - (\alpha+1) r \xi(m)}{(\alpha+\beta) r (n-m)} \cdot \xi(m)^{-\beta} < 0}$

if $\xi'(0) = \frac{\alpha p}{r(\alpha+1)}$

Now let's consider the numerator and denominator. We have

$$F(m,\theta) = (\beta\theta^{-\alpha} + \alpha\theta^{\beta})\,(Kr(n-m) + n\rho) - (\alpha+\beta)n\rho$$

and this is a convex f^n of θ, minimised at $\theta = 1$; so $F(m,\theta) \geq 0$, $= 0$ only when $\theta = 1$, $m = n$.
As for G, we have that

$$(\beta-1)\,\theta^{-\alpha} + (\alpha+1)\,\theta^{\beta} - (\alpha+\beta)\,\theta$$

is a convex function minimised at $\theta = 1$, so G is positive for θ near 0, negative for θ near 1; there is a unique $\theta^*(m)$ at which G vanishes, and by comparing with what happens if we ignore the θ^β term, we learn that

$$0 \leq \frac{\theta^*(m)}{\bar\theta(m)} \leq 1 \qquad\qquad \bar\theta(m) \equiv \left\{ \frac{(\beta-1)m}{(\alpha+\beta)(n\rho + m(1-\rho))} \right\}^{1/1+\alpha}$$

with the ratio tending to 1 as $m \downarrow 0$. Thus η is C^∞ and positive in the open set

$$\{ (m,\theta) : 0 < \theta < \theta^*(m),\ 0 < m < n \}$$

<u>What do we think happens near zero?</u>

$$\boxed{\lim_{m\to 0} \eta(m) = \eta_0 = \frac{\beta}{\beta-1}\left(Kn + n\rho/r\right), \qquad \frac{\xi(m)}{m} \to \xi'(0) = \frac{\alpha\rho}{r(\alpha+1)}}$$

by using the facts that $A(n)$, $b(m) \to 0$ as $m \downarrow 0$; also,

$$\boxed{\begin{aligned} &\xi(m) - \frac{\alpha\rho}{r(\alpha+1)}\,m = o(m^\beta) \\ &\eta(m) - \eta(0) \sim -\frac{\beta K r m + (\alpha+\beta)n\rho\,\theta^\alpha}{r\,(\beta-1)} \end{aligned}} \qquad \begin{aligned} &\text{since } b(m) \to 0 \\ &(m \downarrow 0) \end{aligned}$$

So we get linear behaviour if $\alpha \geq 1$, but an infinite gradient at 0 if $\alpha < 1$. Thus what we think happens is that

$$\boxed{\theta'(0) = \xi'(0)/\eta_0 = \frac{\alpha(\beta-1)}{\beta(\alpha+1)}\ \frac{\rho}{n(Kr+\rho)}}$$

If now we fix some C such that

$$\frac{\alpha\rho}{r(\alpha+1)\eta_0} < C < \frac{\beta}{\beta-1}\ \frac{\alpha\rho}{r(\alpha+1)\eta_0}$$

and restrict attention to paths $\theta(\cdot)$ which satisfy for some $\varepsilon > 0$

$$\varepsilon m \leq \theta(m) \leq Cm$$

in some nhd of 0, then the conjectured asymptotics do indeed hold. If we can fix these two constants and find some other constant such that

$$|\Phi(m,\theta) - \Phi(m',\theta)| \leq \tilde{c}\,|m - m'| \qquad \text{for } 0 < m, m' \leq 1,\quad 0 < \theta \leq 1,\quad \varepsilon m \leq \theta \leq Cm,$$

then the ODE $\theta' = \Phi(m,\theta)$ has a unique solution with $\varepsilon m \leq \theta(m) \leq Cm$ for all small enough

m. The trick is to consider the inverse function $\tau_x \equiv \inf\{m : \theta(m) > x\}$, which solves

$$\tau_x' = \frac{1}{\Phi(\tau_x, x)}$$

Notes on a result of Bertoin & Le Jan (23/6/01)

Bertoin & Le Jan (Ann. Prob 20, 538–548) show that for suitable Markov processes (Hunt processes with a recurrent point 0) a given law μ which does not charge 0 can be embedded by using a fluctuating additive functional which grows away from 0, and decreases only at zero. Here's how to understand this in the case of a irreducible finite Mkv chain, using the WH factorisation story of BRW.

We use the WH factorization

$$V^{-1}Q\begin{pmatrix} I \\ \Pi_+ \end{pmatrix} = V^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I \\ \Pi_+ \end{pmatrix} = \begin{pmatrix} I \\ \Pi_+ \end{pmatrix} G_+$$

The problem considered by Bertoin & Le Jan is a kind of inverse problem, where Π_+ (a row vector) is given, and we have to find V to make this happen. We get

$$Q\begin{pmatrix} I \\ \mu \end{pmatrix} = V\begin{pmatrix} I \\ \mu \end{pmatrix} G_+$$

Case 1: μ is a subprobability. Left multiplying by the invariant law m, we discover that

$$mV\begin{pmatrix} I \\ \mu \end{pmatrix} = 0 \qquad [v_0 \equiv -1, \text{ let's suppose wlog}]$$

since G_+ is non singular. Hence

$$m_i v_i = m_0 \mu_i \qquad (i \in E^+)$$

Case 2: μ is a true probability. In this case, the matrix $A + B\mu$ is a conservative Q-matrix, with invariant measure θ (say). Again left-multiplying by m leads us to conclude that

$$mV\begin{pmatrix} I \\ \mu \end{pmatrix} = c\,\theta V^0 \qquad \text{for some } c, \text{ where } V^0 \text{ is restriction of } V \text{ to } E^+ \times E^+$$

so

$$\boxed{m_i v_i = m_0 \mu_i + c\,\theta_i v_i} \qquad \forall i \in E^+$$

Thus $$v_i = \frac{m_0 \mu_i}{m_i - c\theta_i}$$

and by taking $c \geq 0$ we make the additive functional grow much faster, so can get shorter mean equalisation times.

Regularising utilities. (29/6/01)

Suppose that C is a closed convex cone*, and $U : C \to \mathbb{R} \cup \{-\infty\}$ is a concave function on C, finite-valued on $\operatorname{int}(C)$, increasing in the partial order of C. Let C^* be the dual cone. Assume for now that C has non-empty interior, also C^*.

Proposition For any $\varepsilon > 0$ we can find $U_\varepsilon, U^\varepsilon : C \to \mathbb{R} \cup \{-\infty\}$ such that

$$U(x) - \varepsilon \le U_\varepsilon(x) \le U(x) \le U^\varepsilon(x) \le U(x) + \varepsilon$$

and such that $U_\varepsilon, U^\varepsilon$ are strictly concave, differentiable, and satisfy the Inada condition at 0: for any $x \in C$

$$\lim_{\lambda \downarrow 0} \frac{\partial}{\partial \lambda} U_\varepsilon(\lambda x) = +\infty = \lim_{\lambda \downarrow 0} \frac{\partial}{\partial \lambda} U^\varepsilon(\lambda x)$$

Proof Suppose that $\{x_1, \dots, x_d\} \subseteq C$ is a basis, $\{y_1, \dots, y_d\} \subseteq C^*$ is a basis, and consider the functions

$$\tilde{u}(y) + \frac{\varepsilon}{2} \exp\Big(-\sum_{j=1}^d \sqrt{x_j \cdot y}\Big)$$

$$\tilde{u}(y) + \frac{\varepsilon}{2} \exp\Big(-\sum_{j=1}^d \sqrt{x_j \cdot y}\Big) - \frac{\varepsilon}{2}$$

which sandwich $\tilde{u}$ within $\varepsilon/2$, are strictly decreasing in y, and are strictly convex. The dual functions, $u_\pm$, say, are therefore differentiable in $\operatorname{int}(C)$, and sandwich U to within $\varepsilon/2$, and are increasing. They may fail to be strictly concave, but if we then use

$$u_+(x) + \frac{\varepsilon}{2}\Big\{1 - \exp\Big(-\sum_{j=1}^d \sqrt{y_j \cdot x}\Big)\Big\}$$

$$u_-(x) - \frac{\varepsilon}{2} \exp\Big(-\sum_{j=1}^d \sqrt{y_j \cdot x}\Big)$$

we have even the strict concavity too, also the Inada condition. $\square$

If the interior of C is empty, then C lies in some lower-dimensional subspace, wlog $C \subseteq \mathbb{R}^m$, $m < d$, and C has non empty interior in $\mathbb{R}^m$. Just carry out the above regularisation in $\mathbb{R}^m$.

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* $C \subseteq \mathbb{R}^d$

The duality template in action again (29/6/01)

(i) Sasha Stoikov (?), a student of Thaleia, asks about the following problem. If we have the usual dynamics

$$dX = rX\,dt + \theta(\sigma dW + (\rho - r)dt) - c\,dt \equiv (rX - c)\,dt + dZ$$

and the objective

$$\sup E \int_0^T \{U(c_s) + V(Z_s)\}\,ds$$

what is the dual problem? With multiplier process $d\xi = \xi(\alpha dW + \beta dt)$ for X and $d\eta = \eta(a dW + b dt)$ for Z we can derive the dual form

$$\min E\left[\int_0^T \{\tilde{U}(\xi_s) + \tilde{V}(\eta_s b_s)\}\,ds + X_0\xi_0 - Z_0\eta_0\right]$$

with dual feasible conditions $r + \beta \le 0$, $(\xi - \eta)(r - \rho)\sigma^{-1} = \alpha\xi - a\eta$, $\xi_T \ge 0 \ge \eta_T$

(ii) The motivation for this is that investors get 'feelgood' from their shares doing well. However the above formulation (Stoikov's) gets this wrong: we should have $V(\theta_t)$ not $V(Z_t)$! Taking the problem

$$\sup E\left[\int_0^T \{U(t, c_t) + V(t, \theta_t)\}\,dt + U(T, X_T)\right]$$

we get by the usual methods the dual problem

$$\inf E\left[\int_0^T \{\tilde{U}(t, \xi_t) + \tilde{V}(t, \xi_t(r_t - \rho_t - \alpha_t\sigma_t))\}\,dt + \tilde{U}(T, \xi_T) + X_0\xi_0\right]$$

$$\text{where } \begin{cases} d\xi_t = \xi_t(\alpha_t dW_t + \beta_t dt) \\ \beta_t + r_t \le 0 \\ r_t - \rho_t - \alpha_t\sigma_t \ge 0 \end{cases}$$

(iii) Simplifying to U, V not time dependent, $U(T, \cdot) \equiv 0$, r, ρ, σ positive constants, and setting

$$f(t, \xi) \equiv \inf E_t\left[\int_t^T (\tilde{U}(\xi_s) + \tilde{V}(\xi_s(r - \rho - \alpha_s\sigma)))\,ds \,\Big|\, \xi_t = \xi\right]$$

we get $\left(a \equiv \frac{r - \rho}{\sigma}\right)$

$$\inf_{\alpha \le a}\left[\dot{f} - r\xi f_\xi + \tfrac{1}{2}\alpha^2\xi^2 f_{\xi\xi} + \tilde{U}(\xi) + \tilde{V}(\xi(r - \rho - \alpha\sigma))\right] = 0$$

as the HJB equation to tackle.

(iv) Next an example of Detemple & Karatzas of some sort of Habit formation. We have

$$dX_t = r_t X_t\,dt + \theta_t(\sigma_t\,dW_t + (\rho_t - r_t)\,dt) - c_t\,dt, \qquad X_0 = x,$$
$$dz_t = -\alpha z_t\,dt + \delta c_t\,dt + h\,dW_t$$

with objective

$$\sup E\left[\int_0^T U(s, c_s - z_s)\,ds + U(T, X_T)\right]$$

(It's a bit weird, but there it is.) By multiplying throughout by $e^{\alpha t}$ we can and shall suppose $\alpha = 0$ (D+K also have an endowment stream, but let's suppose that has been taken into initial wealth in the usual way). If we take multiplier processes (recycling the symbol α)

$$d\xi = \xi(\alpha\,dW + \beta\,dt) \qquad \text{for } X$$
$$d\eta = \lambda\,dW + \gamma\,dt \qquad \text{for } z$$

and using the template we have Lagrangian

$$\sup E\Big[\int_0^T U(t, c_t - z_t)\,dt + U(T, X_T) - \Big(X_T\xi_T - X_0\xi_0 - \int_0^T \xi_t X_t \beta_t\,dt - \int_0^T \xi_t\alpha_t\theta_t\sigma_t\,dt\Big)$$
$$+ \int_0^T \xi_t(r_t X_t + \theta_t(\rho_t - r_t) - c_t)\,dt - \Big(z_T\eta_T - z_0\eta_0 - \int_0^T z_t\gamma_t\,dt - \int_0^T \lambda_t h_t\,dt\Big) + \int_0^T \eta_t\,\delta c_t\,dt\Big]$$

$$= \sup E\Big[\int_0^T \{U(t, c_t - z_t) - (\xi_t - \delta\eta_t)c_t + z_t\gamma_t\}\,dt + \tilde U(T, \xi_T) + X_0\xi_0 + z_0\eta_0 + \int_0^T \xi X(\beta + r)\,dt$$
$$+ \int_0^T \theta\xi(\rho - r + \alpha\sigma)\,dt + \int_0^T \lambda h\,dt\Big]$$

If we set $c - z \equiv y$, and max over z, c, we find the maxed value of $\tilde U(t, \gamma_t)$ and the complementary slackness condition $\xi - \delta\eta - \gamma \geq 0$; so Lagr form gives us the dual problem

$$\boxed{\begin{aligned} &\inf E\left[\int_0^T \tilde U(t, \gamma_t)\,dt + \tilde U(T, \xi_T) + X_0\xi_0 + \eta_0 z_0 + \int_0^T \lambda_t h_t\,dt\right] \\ &\text{s.t.} \quad \alpha_t = \sigma_t^{-1}(r_t - \rho_t) \\ &\qquad\quad \beta_t + r_t \leq 0 \\ &\qquad\quad \xi_t \geq \gamma_t + \delta\eta_t, \quad \text{with equality when } c_t > 0 \\ &\qquad\quad \eta_T = 0 \end{aligned}}$$

We also have $\xi_t = \xi_0 \zeta_t \exp(-\int_0^t \varepsilon_s\,ds)$, where ζ is the standard state-price density process, $\varepsilon \equiv -(\beta + r)$. Clearly, for optimal behaviour we'll want $\varepsilon \equiv 0$.

Moreover, since $\eta_t - \int_0^t \gamma_s\,ds$ is a martingale, and $\eta_T = 0$, we must have the following expression for η:

$$\boxed{\eta_t = -E_t\left[\int_t^T \gamma_u\,du\right]}$$

Now the development proceeds as in the original Detemple-Karatzas account; we have

$$c_t^* - z_t^* = I(t, \gamma_t)$$

so putting this into the habit-formation equation

$$dz^*_t = \delta\{z^*_t + I(t,\gamma_t)\}\,dt + h_t\,dW_t$$

so that $\quad z^*_t = e^{\delta t}\Big[z_0 + \delta\int_0^t e^{-\delta s} I(s,\gamma_s)\,ds + \int_0^t e^{-\delta s} h_s\,dW_s\Big]$

$$\equiv e^{\delta t}\Big[z_0 + \int_0^t \delta G_s\,ds + M_t\Big], \quad \text{say} \qquad \left[\begin{array}{l} G_t \equiv e^{-\delta t} I(t,\gamma_t) \\ M_t = \int_0^t e^{-\delta s} h_s\,dW_s \end{array}\right]$$

and $\quad c^*_t = z^*_t + I(t,\gamma_t),$

expressing c^*, z^* in terms of γ. The complementary slackness condition for c gives us

$$\xi_0\,\zeta_t \geq \gamma_t + \delta\eta_t \equiv \gamma_t - \delta E_t\Big[\int_t^T \gamma_u\,du\Big]$$

with equality everywhere that $c_t > 0$. Working this inequality à la Gronwall gives the bound

$$\boxed{\gamma_t \leq \xi_0\zeta_t + \delta\,E_t\Big[\int_t^T e^{\delta(s-t)}\,\xi_0\zeta_s\,ds\Big] \equiv \Gamma_t}$$

say. Moreover, if c is positive in $[\tau,T]$, it must be that this inequality is an equality in $[\tau,T]$. Now the fact that $c^* \geq 0$ forces

$$e^{-\delta t} c^*_t = z_0 + \delta\int_0^t G_s\,ds + M_t + G_t \geq 0$$

so $\quad G_t \geq -z_0 - \int_0^t \delta G_s\,ds - M_t \equiv Y_t$

One key observation is that from the derivation of $\gamma_t \leq \Gamma_t$ we see that if $\gamma_t = \Gamma_t$ for some t, then $\gamma_u = \Gamma_u$ for $u \geq t$. So there exists some $\tau = \inf\{t : \gamma_t = \Gamma_t\}$, and $c_t = 0$ for $t < \tau$. Thus we have prior to τ that

$(*)\qquad G_t = -z_0 - \int_0^t \delta G_s\,ds - M_t$

which we can solve for G, and hence deduce γ_t, for if G^*_t is the solution to $(*)$, we let γ be defined by $\gamma_t = U'(t, e^{\delta t} G^*_t)$ until the first time that this reaches Γ_t, after which $\gamma = \Gamma$.

(v) An alternative (more realistic?) model for habit formation would be to use the dynamics

$$\begin{cases} dX = rX\,dt + \theta(\sigma\,dW + (\mu - r)\,dt) - c\,dt \\ dz = -\alpha' z\,dt + \delta c\,dt \end{cases}$$

with objective

$$\max \quad E\Big[\int_0^T U(s, c_s/z_s)\,ds\Big]$$

As before, we may simplify by assuming that $\alpha' = 0$, and introduce multiplier processes $d\xi = \xi\,(\alpha\,dW + \beta\,dt)$, $d\eta = \eta\,b\,dt$ for X, z respectively. Using the template, we can work the Lagrangian form of the problem round to something we can deal with. The first step is rework the problem in terms of $x_t \equiv X_t/z_t$, solving

$$dx_t = (rx - c/z)dt + \tilde{\theta}(\sigma dW + (\rho - r)dt) - x\,\delta\, c/z\, dt$$

So if we introduce the process $\tilde{c}_t \equiv c_t/z_t$, the problem is to

$$\max E \int_0^T U(s, \tilde{c}_s)\,ds$$

subj to $\quad dx_t = rx_t\,dt + \tilde{\theta}_t\{\sigma dW_t + (\rho - r)dt\} - \tilde{c}_t(1 + \delta x_t)dt$. With Lagrangian process $d\xi = \xi(\alpha dW + \beta dt)$ we form the Lagrangian problem

$$\sup E\Big[\int_0^T U(s, \tilde{c}_s)\,ds - \Big(x_T\xi_T - x_0\xi_0 - \int_0^T x_t\xi_t\beta_t\,dt - \int_0^T \alpha_t\xi_t\tilde{\theta}_t\sigma_t\,dt\Big)$$
$$+ \int_0^T \xi_t\big(rx_t + \tilde{\theta}_t(\rho - r) - \tilde{c}_t(1 + \delta x_t)\big)\,dt\Big]$$

$$= \sup E\Big[\int_0^T \tilde{U}(s, \xi_t(1 + \delta x_t))\,ds + \int_0^T \xi_t x_t(\beta_t + r_t)\,dt\Big] + x_0\xi_0$$

$$\begin{bmatrix} x_T \geq 0 \\ \rho - r + \alpha\sigma = 0 \end{bmatrix}$$

$$= E\Big[\int_0^T \tilde{U}(s, \xi_s)\,ds + x_0\xi_0\Big]$$

$$\beta + r \leq 0$$

Analysis of callable convertible bonds (9/7/01)

1) We refer back to the model on pp 12–15, with a call provision. The calling boundary will be $V^*(m) \equiv n(K+K^*) - mK$, and since we know/expect in the no-calling situation that

$$\eta(0) = \frac{\beta}{\beta-1} n(K+\rho/r)$$

we shall suppose that

$$K^* < \frac{K+\beta\rho/r}{\beta-1}$$

to make the calling possibility a reality. We have that

$$S(m,V) = \frac{V-m\rho/r}{n-m} + a(m)V^{-\alpha} + b(m)V^{\beta}$$

with the boundary conditions $S = \frac{\partial S}{\partial V} = 0$ at $V = \xi(m)$. This determines $a(m)$, $b(m)$ in terms of $\xi(m)$ as

$$a(m) = \frac{\beta m\rho - (\beta-1)r\xi(m)}{(\beta+\alpha)r(n-m)}\xi(m)^{\alpha}, \qquad b(m) = \frac{\alpha m\rho - (\alpha+1)r\xi(m)}{(\beta+\alpha)r(n-m)}\xi(m)^{-\beta}.$$

In the region $0 \le m \le m^*$ we have the further condition $S(m, V^*(m)) = K+K^*$ to fix $\xi(m)$. If we write $\xi(m) = \theta(m)V^*(m)$ in that region, we can solve explicitly for m in terms of θ:

$$m = \frac{nr\theta(K+K^*)\{\theta^{\alpha}(\beta-1) + \theta^{-\beta}(\alpha+1)\}}{(\alpha+\beta)(rK^*-\rho) + rK(\beta-1)\theta^{\alpha+1} + rK(\alpha+1)\theta^{-\beta} + \rho\beta\theta^{\alpha} + \rho\alpha\theta^{-\beta}}$$

Maybe better is to write the equation to be solved,

$$S(m, V^*(m)) = K + K^*$$

in the form

$$(\beta+\alpha)(n-m)r(K+K^*) = r(\beta+\alpha)\left(n(K+K^*) - mK - m\rho/r\right)$$
$$+ \left(\beta m\rho - (\beta-1)r\theta V^*\right)\theta^{\alpha} + \left(\alpha m\rho - (\alpha+1)r\theta V^*\right)\theta^{-\beta}$$

or again

$$(1) \quad \boxed{m(\beta+\alpha)(\rho - rK^*) = \left(\beta m\rho - (\beta-1)r\theta V^*\right)\theta^{\alpha} + \left(\alpha m\rho - (\alpha+1)r\theta V^*\right)\theta^{-\beta} \equiv f(\theta).}$$

If we consider the RHS as a function of θ, then as $\theta \downarrow 0$ it increases to $+\infty$,* and as $\theta \uparrow 1$ it goes to

$$m(\alpha+\beta)\left(\rho - \frac{rV^*}{m}\right) < m(\beta+\alpha)(\rho - rK^*)$$

Thus there is at least one root for $\theta \in (0,1)$. In fact, we can prove that there is only one root, as follows. Differentiating f, we find that

* Provided $\alpha m\rho - (\alpha+1)r\theta V^* > 0$ when $\theta = 0$, which it is.

$$f'(\theta) = (\theta^{\alpha} - \theta^{-\beta}) \left[\frac{\alpha\beta m\rho}{\theta} - (\alpha+1)(\beta-1) r V^* \right]$$

$$f''(\theta) = (\alpha\theta^{\alpha-1} + \beta\theta^{-1-\beta}) \left[\frac{\alpha\beta m\rho}{\theta} - (\alpha+1)(\beta-1) r V^* \right] + \frac{\theta^{-\beta} - \theta^{\alpha}}{\theta^2} \alpha\beta m\rho$$

If we wish to solve $f(\theta) = x$, where $x > f(1)$, then there certainly is a root; and if there were more than one root, there would have to be some θ_1 such that $f'(\theta_1) = 0$, and $f''(\theta_1) < 0$. But from the above expressions for f', f'' we can see that this is impossible. Thus there exists a unique root, and so a unique value for $\xi(m)$.

It is not hard to see that $\theta(m)$ must be increasing in m, since we are trying to solve

$$(\beta+\alpha)(\rho - rK^*) = \left(\beta\rho - (\beta-1) r\theta \frac{V^*}{m}\right)\theta^{\alpha} + \left(\alpha\rho - (\alpha+1) r\theta \frac{V^*}{m}\right)\theta^{-\beta}$$

and V^*/m is decreasing with m. So if we slightly raise m, with θ fixed, RHS gets raised, so we have to increase θ to restore equality. (?)

2) Could it be that we always have $\xi(m)/m \geq \frac{\alpha\rho}{r(\alpha+1)}$? By looking at the boxed equation and substituting in this value for ξ, we see that this holds iff

$$\left\{\beta\rho - (\beta-1) r \cdot \frac{\alpha\rho}{r(\alpha+1)}\right\} \cdot \left(\frac{m\alpha\rho}{r(\alpha+1)\{n(K+K^*) - mK\}}\right)^{\alpha} \geq (\beta+\alpha)(\rho - rK^*)$$

$$\Leftrightarrow \quad \frac{\rho}{\alpha+1}\left(\frac{m\alpha\rho}{r(\alpha+1)\{n(K+K^*) - mK\}}\right)^{\alpha} \geq (\rho - rK^*)$$

$$\Leftrightarrow \quad \rho - rK^* \leq 0$$

So the conjecture is true iff $\boxed{K^* \geq \rho/r}$ (so K^* has to be big enough).

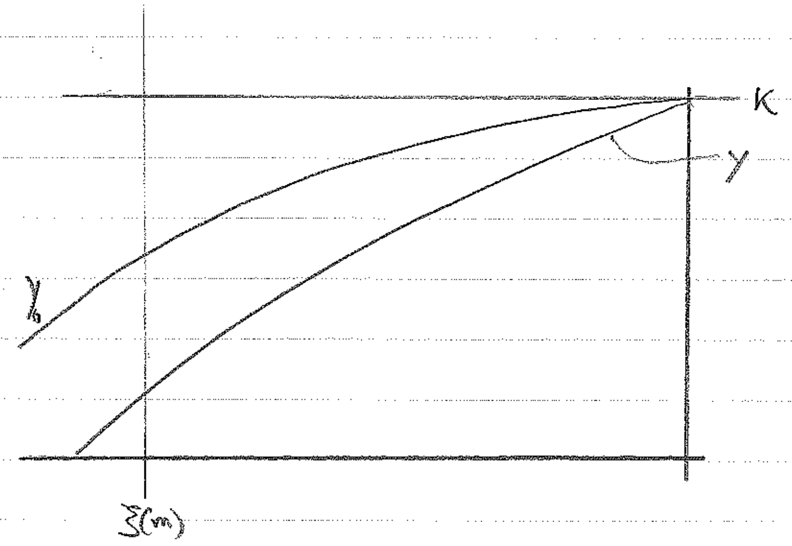
Notice that since $f'(\theta) < 0$ for $\theta < \frac{\alpha\beta m\rho}{(\alpha+1)(\beta-1) r V^*(m)}$

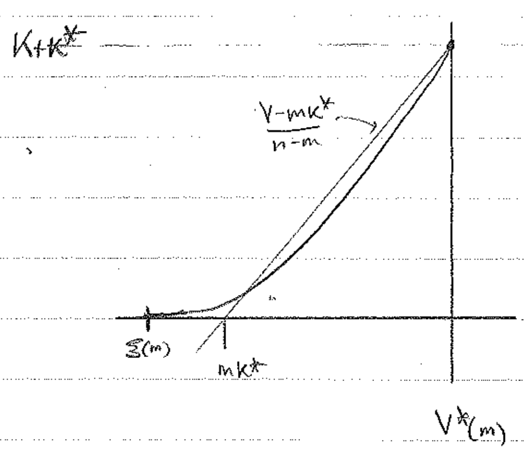
> 0 for $\theta >$ ——— ,

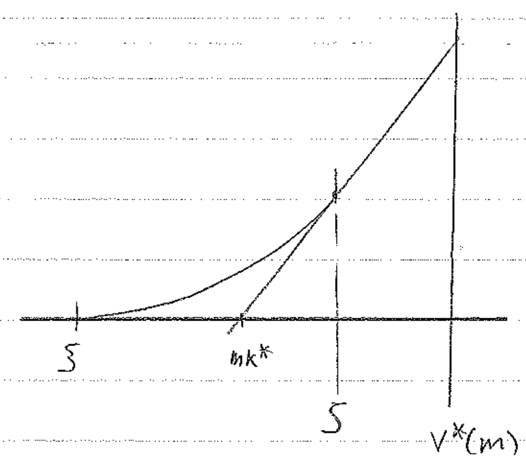
it must be that the value of θ we want is always less than $\alpha\beta m\rho / (\alpha+1)(\beta-1) rV^*(m)$, so

$$\boxed{\xi(m) \leq \frac{\alpha\beta m\rho}{r(\alpha+1)(\beta-1)}}$$

3) This determines the boundary ξ, but how do we decide the critical value m^*? This is actually quite simple. Let's consider $Y \equiv S - B$, which we know must have the form


K
y
γ_0
$\xi(m)$


K+K*
$\frac{V-mK^*}{n-m}$
$\xi(m)$
mK*
$V^*(m)$


ξ
mK*
ξ
$V^*(m)$

$$Y(m,V) = \frac{V - np/r}{n-m} + A(m)\, V^{-\alpha} + B(m)\, V^{\beta}$$

Now if we construct the Solution $Y_0(m,V)$ which satisfies the boundary conditions

$$Y_0(m, V^*(m)) = K, \quad \frac{\partial Y_0}{\partial V}(m, V^*(m)) = 0,$$

we would have the existing expressions for A, B

$$A = \frac{\beta np + \beta r K(n-m) - (\beta-1) r V^*}{r(\alpha+\beta)(n-m)} (V^*)^{\alpha}$$

$$B = \frac{\alpha np + \alpha r K(n-m) - (\alpha+1) r V^*}{r(\alpha+\beta)(n-m)} (V^*)^{-\beta}$$

but this would not match the boundary condition $Y(m, \xi(m)) = -p\,\xi(m)/m$ at $\xi(m)$. So the difference between Y and Y_0 must be expressible as

$$Y(m,V) = Y_0(m,V) + \lambda(m)\left\{\left(\frac{V}{V^*(m)}\right)^{\beta} - \left(\frac{V}{V^*(m)}\right)^{-\alpha}\right\} \le Y_0(m,V)$$

with $\lambda(m) \geq 0$ (since $Y(m,V) \le K$ always). The value m^* will be the smallest value of m for which

$$Y_0(m, \xi(m)) \le -p\,\xi(m)/m,$$

if this should happen.

[Alternatively, we have for the problem without calling a unique value of $\eta(m)$ for a given $\xi(m)$; just keep going until $\eta(m) \le V^*(m)$? No, because the paths of ξ won't be the same]

4) There is, however, one more possibility which Jon's DP calculations show up. That is that for small values of m, there may be calling at a value below $V^*(m)$. The reason for this is that if we construct S as at (1) above, the slope $\frac{\partial S}{\partial V}$ at $V^*(m)$ may exceed $\frac{1}{n-m}$, and since we know that always $S \geq \min\left\{\frac{V-mK^*}{n-m}, \frac{V+mK}{n}\right\}$, this cannot happen. What we should rather do is look for ξ, ζ, $\xi < mK^* < \zeta$ such that $S = \frac{\partial S}{\partial V} = 0$ at ξ, and $S = (V-mK^*)/(n-m)$, $\partial S/\partial V = 1/n-m$ at ζ

If we select ξ and then choose $a(m)$, $b(m)$ to make $S = 0 = \partial S/\partial V$ at ξ, then the place ζ where $\partial S/\partial V = 1/n-m$ must be given by solving

$$-\alpha\, a(m)\, \zeta^{-\alpha-1} + \beta\, b(m)\, \zeta^{\beta-1} = 0$$

or equivalently

$$\boxed{\zeta^{\alpha+\beta} = \frac{\alpha\left(\beta mp - (\beta-1) r \xi\right)}{\beta\left(\alpha mp - (\alpha+1) r\xi\right)} \cdot \xi^{\alpha+\beta}}$$

(This can only be satisfied if $\xi < \frac{\alpha mp}{r(1+\alpha)}$ or $\xi > \frac{\beta mp}{r(\beta-1)}$)

$$\frac{\xi}{m} < \frac{\alpha\rho}{(\alpha+1)r} \quad \text{or} \quad \frac{\xi}{m} > \frac{\beta\rho}{(\beta-1)r}$$

At ξ, we want S to be equal to $(\xi - mK^*)/(n-m)$, which is to say

$$0 = \frac{m(K^* - p/r)}{n-m} + a(m)\,\xi^{-\alpha} + b(m)\,\xi^{\beta}$$

and this gets reworked to

$$(2) \qquad \boxed{m(\beta+\alpha)(p - rK^*) = \left[(c\xi)^{-\alpha} + (c\xi)^{\beta}\right] \left|\alpha mp - (\alpha+1)r\xi\right|^{\alpha/\alpha+\beta} \cdot \left|\beta mp - (\beta-1)r\xi\right|^{\beta/\alpha+\beta}} \qquad \left[c = \left(\frac{\alpha}{\beta}\right)^{1/\alpha+\beta}\right]$$

We need for this either $\xi < p\alpha m/r(\alpha+1)$, or $\xi > \beta mp/r(\beta-1)$; since we have to have $p > rK^*$ for any possibility of a root, and since we are also looking for a root $\xi < mK^*$, this means we are in effect only looking for $\xi < \alpha mp/r(\alpha+1) \wedge mK^*$. Now if we call the function of ξ on the RHS of $\square$ $f(\xi)$, it's clear that $f(\xi) \to \infty$ $(\xi \downarrow 0)$, $f(\xi) \to 0$ $(\xi \uparrow \alpha mp/r(\alpha+1))$, so there is a root in $(0, \alpha mp/r(\alpha+1))$. Maple plots show that the function isn't monotone in general, and we can have that the root is in $(mK^*, \alpha mp/r(\alpha+1))$, so not one for us. Also, the root (when it exists) can be very close to $\alpha mp/r(\alpha+1)$.

5) To sum up then: suppose we want to match $S(m,V) = \frac{V - mp/r}{n-m} + a(m)V^{-\alpha} + b(m)V^{\beta}$ to the line $(V - mK^*)/(n-m)$ smoothly at some point V_0. To match the slope,

$$S(m,V) = \frac{V - mp/r}{n-m} + c\left[\beta\left(\frac{V}{V_0}\right)^{-\alpha} + \alpha\left(\frac{V}{V_0}\right)^{\beta}\right] \equiv \frac{V - mp/r}{n-m} + c\,g(V/V_0)$$

where g is a convex function. To match the value, we find that c must solve

$$\frac{m(K^* - p/r)}{n-m} + c(\beta+\alpha) = 0$$

So if $K^* \geq p/r$, it's impossible that we have smooth fit to 0 and to $V \mapsto \frac{V - mK^*}{n-m}$, and we just decide using (1).

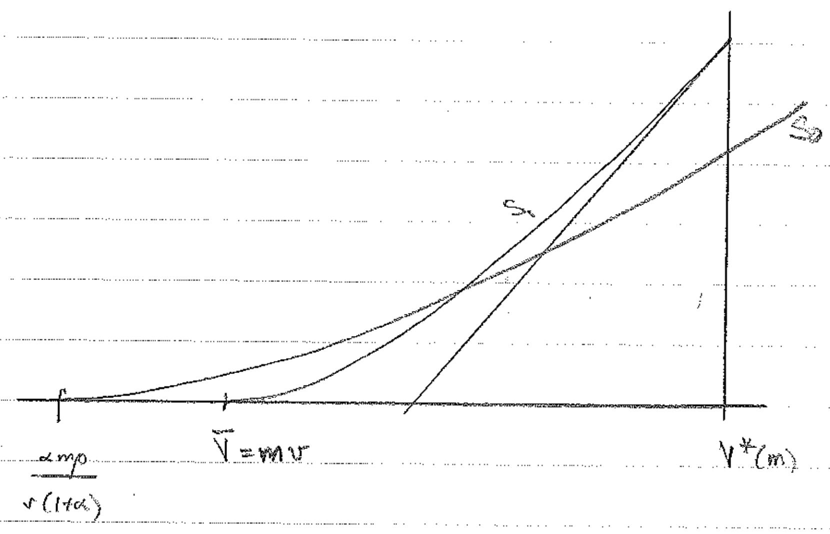
On the other hand, if $K^* < p/r$, it may be possible, and everything depends on

$$\inf_{V \leq V_0} \left\{ \frac{V - mp/r}{n-m} + \frac{m(p/r - K^*)}{(n-m)(\beta+\alpha)}\, g(V/V_0) \right\}$$

when $V_0 = V^*(m)$. If this inf is positive, we just slide V_0 down from $V^*(m)$ until the inf is zero, and we find that ξ is determined by (2). On the other hand, if the inf is < 0, we can't achieve smooth fit at ξ and $\bar{\xi}$, and we revert to (1) to find ξ.

PS (20/8/01) Jon points out that in the case $K^* < p/r$ we do have something more, namely that V_0 is linear in m, and the point where S hits 0 also depends linearly on m. To see this, suppose $\tilde{V}$ is the place where $S = \frac{\partial S}{\partial V} = 0$, V_0 is the place where S smooth pastes onto $(V - mK^*)/(n-m)$. We have from the smooth pasting at V_0 that

$$S(m,V) = \frac{V - mp/r}{n-m} + m\frac{p/r - K^*}{(n-m)(\alpha+\beta)}\left\{\beta\left(\frac{V}{V_0}\right)^{-\alpha} + \alpha\left(\frac{V}{V_0}\right)^{\beta}\right\}$$



S
S
$\bar{V} = mv$
$\frac{\alpha m p}{r(1+\alpha)}$
$V^*(m)$

As for $\bar{V}$ we must have

$$0 = \bar{V} - m\rho/r + m\frac{(\rho/r - K^*)}{\alpha+\beta}\left[\beta\left(\frac{\bar{V}}{V_0}\right)^{-\alpha} + \alpha\left(\frac{\bar{V}}{V_0}\right)^{\beta}\right]$$

$$0 = \bar{V} + m\frac{\rho/r - K^*}{\alpha+\beta}\left\{\left(\frac{\bar{V}}{V_0}\right)^{\beta} - \left(\frac{\bar{V}}{V_0}\right)^{-\alpha}\right\}\alpha\beta$$

So if $\bar{V} = mv$, $\bar{V}/V_0 = \theta$, we must solve the equations

$$\begin{cases} 0 = v - \rho/r + \dfrac{\rho/r - K^*}{\alpha+\beta}\left[\beta\theta^{-\alpha} + \alpha\theta^{\beta}\right] \\ 0 = v + \dfrac{\rho/r - K^*}{\alpha+\beta}\alpha\beta\left[\theta^{\beta} - \theta^{-\alpha}\right] \end{cases}$$

for (v, θ). Eliminating v gives an equation for θ:

$$\boxed{\rho/r = \frac{\rho/r - K^*}{\alpha+\beta}\left[-\alpha(\beta-1)\theta^{\beta} + \beta(\alpha+1)\theta^{-\alpha}\right]}$$

Now the RHS is monotone decr in θ, tends to $+\infty$ as $\theta \downarrow 0$, tends to $\rho/r - K^*$ as $\theta \uparrow 1$, so $\exists!$ root $\theta^* \in (0,1)$ and hence unique $v^* > 0$.

From this, we can deduce that in fact $\boxed{v < \alpha\rho/r(1+\alpha)}$ (3/9/01)

Pf. Taking the argument from p3 of WN XX, we see that if we smooth paste to 0 at $\alpha m\rho/r(1+\alpha)$, the solution $S_0(m, V^*(m)) < K + K^*$ iff

$$\rho/r - K^* > \theta^{\alpha}\frac{\rho}{r(1+\alpha)}$$

$$\text{iff} \quad \theta^{-\alpha}\left(\frac{\rho}{r} - K^*\right)\frac{\beta(\alpha+1)}{\alpha+\beta} = \frac{\rho}{r} + \frac{\alpha(\beta-1)\theta^{\beta}}{\alpha+\beta}(\rho/r - K^*) > \frac{\beta}{\alpha+\beta}\,\frac{\rho}{r}$$

which is clearly true. Now suppose we've pushed m up until V_0 reaches V^*, and that at this place $\bar{V} = mv > \alpha m\rho/r(1+\alpha)$. If S_0 is the sol^n smooth pasted to 0 at $\alpha m\rho/r(1+\alpha)$ and S_1 is the other, there is some $x \in (\bar{V}, V^*)$ where the two cross, and for some $c < 0$

$$S_1 = S_0 + c\left\{\left(\frac{V}{x}\right)^{-\alpha} - \left(\frac{V}{x}\right)^{\beta}\right\}$$

But then $\dfrac{\partial S_1}{\partial V} > 0$ at $\bar{V}$ $\times\!\!\!\!\times$.

Interesting questions.

1) DGH reports that Neil O'Connell spoke about the following. You take n independent BMs, and select non-optional times $T_0 = 0 \le T_1 < T_2 < \dots < T_n = 1$ so as to maximise $\sum\{B_n(T_n) - B_n(T_{n-1})\} \equiv \xi$. Then ξ has the same law as the top e-value of a Hermitian matrix with Gaussian entries.

2) Nizar Touzi was looking at various optimal investment/consumption problems with a utility $U: [0,\infty)^d \to \mathbb{R}$ which may fail to be smooth, or strictly concave. However, it is possible to replace U by some regularised U_n, $U \le U_n^* \le U + \frac{1}{n}$, which is strictly concave and differentiable. This needs some results on the Legendre transform from Rockafeller (Ch 26). Thm 26.3 states that a closed proper convex function is essentially strictly convex iff its conjugate is essentially smooth.

closed: $\{(x,y): y \ge f(x)\}$ is closed subset of $\mathbb{R}^d \times \mathbb{R}$

proper: $f(x) > -\infty$ everywhere

essentially strictly convex: f is strictly convex on every convex subset of $\{x: \partial f(x) \neq \emptyset\}$, where $\partial f(x)$
$= \{y: f(z) \ge f(x) + y\cdot(z-x) \;\; \forall z\}$

essentially smooth: if $C = \text{int}(\text{dom } f)$, then C is non-empty, f is differentiable throughout C and $|\nabla f(x)| \to \infty$ as $x \to \bar{x} \in \partial C$.

Given U, let $\tilde{U}(y) \equiv \sup_x [U(x) - (x,y)]$, which is convex, decreasing. If $f(x)$ is defined by $f(x) \equiv \exp(-\sum_1^d \sqrt{x_i})$, then f is convex, so if we consider

$\tilde{U}(y) + \frac{1}{2n} f(y) \equiv \tilde{U}_n(y)$, then $\tilde{U}_n$ is essentially strictly convex, so $U_n(x) \equiv \inf\{\tilde{U}_n(y) + (x,y)\}$ is essentially smooth, and $U_n(x) + \frac{1}{2n}(1 - f(x))$ is essentially smooth strictly concave, and $U \le U_n^* \equiv U_n + \frac{1}{2n}(1-f) \le U + \frac{1}{n}$.

3) A question of Mark Davis. If ρ is a random time such that for all $\delta > 0$, for all martingales M (in some suitable space)

$$E\, M_{\rho+\delta} - M_\rho = 0$$

Is ρ necessarily a stopping time?