More on the BESQ market (3/5/02)

(i) In the example studied in WN XX, p48, the aggregate consumption Δ_t can be expressed as

$$\Delta_t = e^{-\beta t} Z(A_t)$$

where $A_t \equiv \sigma^2(e^{\beta t}-1)/4\beta$, Z is a BESQ$(4\alpha/\sigma^2)$ and $d\Delta = \sigma\sqrt{\Delta}\,dW + (\alpha-\beta\Delta)\,dt$. Expressions for bond prices follow quite easily, but for the share price we need to compute

$$E\int_0^\infty e^{-\rho t}\,\delta_j(t)\,\Delta_t^{-R}\,dt.$$

We can approach this via Laplace transforms, since for a BESQ^d process Z we know

$$E^z \exp(-\lambda Z_t) = (1+2\lambda t)^{-d/2}\exp\{-\lambda z/(1+2\lambda t)\},$$

so that for independent BESQ^{d_j} processes Z^j, $j=1,2$, we have

$$E\left[Z_t^2 \exp(-\lambda_1 Z_t^1 - \lambda_2 Z_t^2)\right] = (1+2\lambda_1 t)^{-d_1/2}(1+2\lambda_2 t)^{-d_2/2} e^{-\left(\frac{\lambda_1 z_1}{1+2\lambda_1 t}+\frac{\lambda_2 z_2}{1+2\lambda_2 t}\right)}\left\{\frac{d_2 t}{(1+2\lambda_2 t)} + \frac{z_2}{(1+2\lambda_2 t)^2}\right\}$$

So in particular

$$E\left[Z_t^2 e^{-\lambda(Z_t^1+Z_t^2)}\right] = (1+2\lambda t)^{-(d_1+d_2)/2}\exp\left[-\lambda(z_1+z_2)/(1+2\lambda t)\right]\left\{\frac{d_2 t}{1+2\lambda t} + \frac{z_2}{(1+2\lambda t)^2}\right\}.$$

and so we get an expression

$$E\left[(Z_t^1+Z_t^2)^{-R} Z_t^2\right] = \int_0^\infty \frac{\lambda^{R-1}}{\Gamma(R)}(1+2\lambda t)^{-(d_1+d_2)/2}\exp\left\{-\frac{\lambda(z_1+z_2)}{1+2\lambda t}\right\}\left\{d_2 t + \frac{z_2}{(1+2\lambda t)}\right\}\frac{d\lambda}{1+2\lambda t}$$

which will be finite if $R < 1+(d_1+d_2)/2$. By the substitution $1-x = 1/(1+2\lambda t)$ this becomes

$$\int_0^1 \left(\frac{x}{2t(1-x)}\right)^{R-1}(1-x)^{(d_1+d_2)/2}\exp(-x/2t)\,(d_2 t + z_2(1-x)^2)\,\frac{dx}{2t\,\Gamma(R)(1-x)^2}$$

$$= \sum_{m\geq 0}\left(-\frac{1}{2t}\right)^m \frac{1}{m!}\frac{(2t)^{-R}}{\Gamma(R)}\left\{d_2 t\, B\!\left(m+R, \tfrac{d_1+d_2}{2}-R\right) + z_2\, B\!\left(m+R, \tfrac{d_1+d_2}{2}-R+2\right)\right\}$$

$$\left(B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right)$$

(ii) For the pricing of share j, we write $\Delta_t = \delta_j(t) + \Delta^j(t)$, a sum of independent CIR processes, so that the price at time 0 is

$$E\int_0^\infty e^{-\rho t}\left(e^{-\beta t} z_j(A_t)\right).\left(e^{-\beta t} Z(A_t)\right)^{-R} dt$$

where z_j is a BESQ^{d_j} process, Z is a BESQ^{d} process.

We have alternatively the expression

$$\mathrm{Price} = \int_0^\infty e^{-\rho t - \beta(1-R)t} \left(\int_0^\infty \frac{\lambda^{R-1}}{\Gamma(R)} E\{ z_j(A_t) e^{-\lambda A_t} \} d\lambda \right) dt$$

$$= \int_0^\infty e^{-\rho t - \beta(1-R)t} \left(\int_0^\infty \frac{\lambda^{R-1}}{\Gamma(R)} \frac{\exp\left(-\frac{\lambda Z_0}{1+2\lambda A_t}\right)}{(1+2\lambda A_t)^{\alpha}} \left\{ \alpha_j A_t + \frac{z_j(0)}{(1+2\lambda A_t)} \right\} \frac{d\lambda}{1+2\lambda A_t} \right) dt$$

subs $\lambda = v/A_t$, $A_t = u$, so $t = \frac{1}{\beta} \log(1 + 4\beta u/\sigma^2)$

$$dt = \frac{4\,du/\sigma^2}{1+4\beta u/\sigma^2}$$

$$= \int_0^\infty \left(1 + \frac{4\beta u}{\sigma^2}\right)^{R-1-\rho/\beta} \left[\int_0^\infty \frac{v^{R-1}}{\Gamma(R)} \frac{\exp\left\{-\frac{v Z_0/u}{1+2v}\right\}}{(1+2v)^{\alpha+1}} \left\{ \alpha_j u + \frac{z_j(0)}{1+2v} \right\} dv \right] u^{-R} \frac{4}{\sigma^2} \frac{du}{(1+4\beta u/\sigma^2)}$$

Substitute $\frac{2v}{1+2v} = w$, $u = 1/s$

$$= \int_0^\infty \left(s + \frac{4\beta}{\sigma^2}\right)^{R-2-\rho/\beta} s^{\rho/\beta} \frac{4}{\sigma^2} \left[\frac{2^{-R}}{\Gamma(R)} \int_0^1 \left(\frac{w}{1-w}\right)^{R-1} (1-w)^{\alpha-1} e^{-Z_0 ws/2} \left\{ \frac{\alpha_j}{s} + z_j(0)(1-w) \right\} dw \right] ds$$

Two-Sector stochastic growth examples (10/5/02)

We take the two-sector stochastic growth problem again, and try to build solutions where the value f$^{\underline{n}}$ is $V(k) = k^{1-S}/(1-S)$ and $U(c, k_g) = k_g^{1-R_g}\, \xi^{\nu}/\nu$, where $\xi \equiv c/k_g$. For this to be a utility, we must have either $0 < \nu < 1-R_g < 1$ or $1-R_g < \nu < 0$. Once V, U are chosen, there remains only the choice of Φ.

Example 1: $\Phi(k) = Ak^B - \mu k$. We have in this case

$$\hat{U}(k) = k^{1-S}\left[\frac{\rho_g}{1-S} + \tfrac{1}{2}\sigma^2 S + \mu\right] - Ak^{B-S} \equiv k^{1-S}\left[q - Ak^{B-1}\right] \qquad \left(q \equiv \frac{\rho_g}{1-S} + \tfrac{1}{2}\sigma^2 S + \mu\right)$$

Solving for ξ leads to the equation

$$\xi^{(\nu+R_g-1)/R_g} = (\nu\hat{U})\, k^{-S(1-R_g)/R_g}$$

which is only possible if $\nu\hat{U}$ remains positive always. Since Φ must be concave and goes to 0 at 0, we have to have $\boxed{A>0,\ B\in(0,1)}$ so for $\nu\hat{U}$ to be always positive, the only possibility is if $\boxed{q\le 0 \text{ and } \nu<0.}$ Since we expect that $\mu>0$, this would necessitate $\boxed{S>1}$ and since $\nu<0$ we shall have to have also $\boxed{\omega \equiv \nu+R_g-1>0}$ for U to be a utility, and then

$$\xi = (\nu\hat{U})^{R_g/(\nu+R_g-1)}\; k^{S(R_g-1)/(\nu+R_g-1)}$$

$$= k^{(R_g-S)/\omega}\left[\nu q - \nu A k^{B-1}\right]^{R_g/\omega}$$

$$k_g = k^{(\nu+S-1)/\omega}\left[\nu q - \nu A k^{B-1}\right]^{(\nu-1)/\omega}$$

$$c = k\left[\nu q - \nu A k^{B-1}\right]$$

$$f_p - f_g = -\frac{\omega}{\nu}\,\xi$$

$$f_p = \gamma_g - \mu - (S-1)q + ASk^{B-1} \qquad \boxed{\therefore\ \gamma_g - \mu - (S-1)q \ge 0 \text{ is needed}}$$

$$f_g = \gamma_g - \mu - (S-1)q + ASk^{B-1} + \frac{\omega}{\nu}\, k^{(R_g-S)/\omega}\left[\nu q - \nu A k^{B-1}\right]^{R_g/\omega}$$

To have any hope of $f_p \ge 0$, $f_g \ge 0$, we must then have

$$\boxed{q = 0, \quad \gamma_g \ge \mu, \quad R_g = S + (1-B)(1-\nu)}$$

The inequality $1-R_g < \nu$ also forces us to have $\boxed{S > B(1-\nu)}$ We obtain for f_g

$$\boxed{f_g = \gamma_g - \mu + \left\{ AS + \frac{\omega}{\nu}(-\nu A)^{R_g/\omega} \right\} k^{B-1}}$$

and we shall need the coefficient $\{\cdot\}$ to be non-negative (which can certainly be done for small A). We find that

$$\boxed{k_g = (-\nu A)^{(\nu-1)/\omega}\, k}$$

so would therefore really want $\boxed{-\nu A > 1}$. The production function along the optimal trajectory is

$$\boxed{f = \Phi + \gamma_g k + c = (1-\nu)A k^B + (\gamma_g - \mu)k}$$

<u>SO</u> the situation we are considering is one where we are given $\lambda_g > 0$, $\sigma > 0$, $\nu \equiv -n < 0$, $R_g > 1$, $\alpha \in (0,1)$, $B \in (0,1)$, and we are seeking a solution of the form $V(k) = k^{1-S}/(1-S)$, $k_g = \varphi k$ for some $\varphi \in (0,1)$. If we set

$$\boxed{\begin{aligned} S &= R_g - (1-B)(1-\nu) \\ \mu &= -\tfrac{1}{2}\sigma^2 S - \lambda_g/(1-S) \end{aligned}}$$

we shall certainly need $\boxed{S > 1,\ \mu \le \gamma_g}$, we get $\hat{U} = -A k^{B-S}$, $\varphi = (nA)^{1-R_g/\omega}$ so must have for $0 < \varphi < 1$ that $\boxed{nA > 1}$, and get $c = nA k^B$, $\xi = (nA)^{R_g/\omega} k^{B-1}$,

$$f_p - f_g = \frac{\omega}{n}\xi = \frac{\omega}{n}(nA)^{R_g/\omega} k^{B-1}$$

If we had $f(k_p, k_g) = A_f (k_p^{\alpha} k_g^{(1-\alpha)})^B + (\gamma_g - \mu)k$, then for consistency of f_g, $f_p - f_g$ we will require

$$\begin{cases} A_f B \alpha (1-\varphi)^{\alpha B - 1} \varphi^{(1-\alpha)B} = AS \\[2ex] A_f B (1-\varphi)^{\alpha B} \varphi^{(1-\alpha)B} \dfrac{\varphi + \alpha - 1}{\varphi(1-\varphi)} = \dfrac{\omega}{n}(nA)^{R_g/\omega} = \dfrac{\omega}{n}\cdot\dfrac{1}{\varphi}\cdot nA = \dfrac{\omega A}{\varphi} \end{cases}$$

Taking the ratio gives the condition

$$\boxed{\frac{\omega\alpha}{\alpha + \varphi - 1} = S}$$

so we have φ uniquely determined by this equation (since $S > \omega$, as is readily checked); this tells us what A must be. This requires an <u>exact</u> value of A_f to work.

Nevertheless, the form of the solution (with a more complicated f?) will still be OK without this.

$S = P_y$ is Economically silly – it leads us to conclude that $F_L = 0$ . . .

Example 2: $\Phi(k) = \dfrac{Ak}{(k+B)^{\theta}} - \mu k$ where $A, B > 0$ and $\theta \in (0,1)$, or $B > 0$, $A < 0$, $\theta < 0$.

Then we deduce (with $V(k) = k^{1-S}/(1-S)$ as before) that

$$\hat{U}(k) = \left(\frac{\lambda_g}{1-S} + \tfrac{1}{2}\sigma^2 S + \mu\right)k^{1-S} - \frac{Ak^{1-S}}{(B+k)^{\theta}} \equiv k^{1-S}\left[q - \frac{A}{(B+k)^{\theta}}\right] \qquad \left(q = \frac{\lambda_g}{1-S} + \tfrac{1}{2}\sigma^2 S + \mu\right)$$

$$\xi = \left[\nu q - \frac{\nu A}{(B+k)^{\theta}}\right]^{1+R_g/\omega} k^{(R_g - S)/\omega} \qquad (\omega = \nu + R_g - 1)$$

$$k_g = k^{(\nu+S-1)/\omega}\left[\nu q - \frac{\nu A}{(B+k)^{\theta}}\right]^{-(1-\nu)/\omega}$$

$$c = k\left[\nu q - \frac{\nu A}{(B+k)^{\theta}}\right]$$

From this, we see that non-negativity of c requires $\boxed{\nu q \geq 0, \quad \nu q \geq \nu A/B^{\theta}}$ The equality

$$f_p - f_g = -\frac{\omega}{\nu}\xi$$

requires ω and ν to be of opposite sign, but we get that anyway from concavity of U. Finally,

$$f = \frac{A(1-\nu)k}{(B+k)^{\theta}} + (\gamma_g - \mu + \nu q)k$$

so we certainly shall need

$$\boxed{\gamma_g - \mu + \nu q \geq 0, \qquad \frac{A(1-\nu)}{B^{\theta}} + \gamma_g - \mu + \nu q \geq 0}$$

with either $A > 0$, $\theta \in (0,1)$ or $A < 0$, $\theta < 0$. We also have

$$f_p = \frac{AS}{(B+k)^{\theta}} + (1-S)q + \gamma_g - \mu$$

Look at the form of k_g. In order that $0 \leq k_g \leq k$ always, we shall have to have

$$\boxed{\text{if } \nu q > \nu A/B^{\theta}, \quad S = R_g}$$

$$\text{if } \nu q = \nu A/B^{\theta}, \quad S + \nu > 1 + R_g$$

First looks like a better condition to concentrate on, since we then have

$$k_g/k = \left[\nu q - \frac{\nu A}{(B+k)^{\theta}}\right]^{(\nu-1)/\omega}$$

If $\nu > 0$, need $\boxed{\nu q \leq 1, \quad \nu q - \nu A/B^{\theta} \leq 1.}$ If $\nu < 0$, need $\boxed{\nu q \geq 1, \quad \nu q - \frac{\nu A}{B^{\theta}} \geq 1}$

Both examples have a common structure: we do not in fact need to make the form of Φ explicit until much later in the calculation, for we have with $V(k) = A_V k^{1-S}/(1-S)$ that

$$\hat{U}(k) = \left(\frac{\lambda_g}{1-S} + \tfrac{1}{2}\sigma^2 S\right) A_V k^{1-S} - A_V k^{-S}\,\Phi(k)$$

$$\equiv V'(k)\left[Qk - \Phi(k)\right] \qquad \boxed{Q \equiv \frac{\lambda_g}{1-S} + \tfrac{1}{2}\sigma^2 S}$$

$$\equiv V'(k)\,H(k)/\nu \qquad \text{where } \boxed{H(k) \equiv \nu Q k - \nu \Phi(k)}$$

Hence immediately

$$\boxed{\begin{aligned} c &= H \\ \xi &= H^{R_g/\omega}(V')^{1/\omega} \\ k_g &= H^{(\nu-1)/\omega}(V')^{-1/\omega} \\ f_p &= \gamma_g + \frac{S}{k}\Phi + (1-S)Q = \gamma_g + Q - \frac{S}{k\nu} H \\ f_p - f_g &= -\frac{\omega}{\nu}\xi \\ f &= (\gamma_g + Q)k + \left(1 - \tfrac{1}{\nu}\right)H \end{aligned}}$$

$\omega = \nu + R_g - 1$

Concavity of U forces $\omega\nu < 0$. We need to check $f \geq 0$, $f_p \geq 0$, $f_g \geq 0$, $f - k_p f_p - k_g f_g \geq 0$, as well as the tangent inequality. Since consumption is in any case non-negative we must choose Φ, ν such that $\boxed{H \geq 0;}$ for checking the condition $f - k_p f_p - k_g f_g \geq 0$, note that

$$f - k_p f_p - k_g f_g = f - k f_p + k_g(f_p - f_g)$$

$$= \left(1 - \tfrac{1}{\nu}\right)H + \frac{S}{\nu}H - \frac{\omega}{\nu}\xi \cdot k_g$$

$$= \frac{H}{\nu}(S - R_g)$$

after some simplifications. We resist the temptation to look for solutions with $S = R_g$, as this implies $f_L = 0$, which isn't too appealing economically. So since $H \geq 0$, we need also

$$\boxed{(S - R_g)/\nu > 0}$$

And now need only check $f_g \geq 0$, $f \geq 0$ and the tangent inequality. We can rework the tangent inequality to

$$\left(1 - \tfrac{1}{\nu}\right)H(k) \leq \left(1 - \tfrac{1}{\nu}\right)H(z) - (k - z)\frac{S}{\nu z}H(z) + \frac{\omega}{\nu}\left[k_g(k)\,\xi(z) - H(z)\right]$$

Taking the RHS − LHS and differentiating wrto k, we get after some rearrangement that the derivative is

$$\left(1-\frac{1}{\nu}\right) H'(k) \left\{ \frac{\xi(z)}{\xi(k)} - 1 \right\} + \frac{S\xi(z)}{\nu} \left\{ \frac{k_g(k)}{k} - \frac{k_g(z)}{z} \right\}$$

If we now ensure that this is ≥ 0 for $k \geq z$, and ≤ 0 for $k \leq z$, we have the tangent inequality.

Don't forget also that we want $0 \leq k_g \leq k$.

Some remarks on a result of Christo Borell (4/6/02)

1) Christo Borell has a preprint 'On risk aversion and optimal terminal wealth' in which he considers the situation of finding

$$V_T(x) \equiv V(x) = \sup E\left[U(X_T) \mid X_0 = x \right]$$

for a standard single-asset market, where the utility U has increasing relative risk aversion:

$$R(x) \equiv \frac{-xU''(x)}{U'(x)} \quad \text{increases, equivalently,} \quad U'(e^t) = \exp(\varphi(t)) \text{ for some concave } \varphi.$$

What he proves amounts in effect to showing that if we define

$$R_T(x) \equiv -x V_T''(x) / V_T'(x)$$

then R_T is also increasing. The result can be generalised and the proof simplified in the following way.

2) Take a standard complete Brownian market with bounded stochastic coefficients:

$$dX_t = r_t X_t \, dt + \theta_t \{ \sigma_t \, dW_t + (\mu_t - r_t 1) \, dt \} \qquad \begin{pmatrix} W \text{ is } BM(\mathbb{R}^d) \\ \sigma \text{ is } d\times d. \end{pmatrix}$$

where $r, \sigma, \sigma^{-1}, \mu$ are all bounded, and we next define the state-price density process

$$dH_t = H_t \left(-r_t \, dt - (\sigma_t^{-1}(\mu_t - r_t), dW_t) \right), \qquad H_0 = 1$$

as usual. Then the standard result is the duality result

$$V_T(x) = \inf_\lambda E\left\{ \tilde{U}(\lambda H_T) + \lambda x \right\}$$

so that the dual function of V_T is $\tilde{V}_T(\lambda) \equiv E\, \tilde{U}(\lambda H_T)$. The coefficient of relative risk aversion is easily seen to satisfy

$$R(I(\lambda)) = \frac{-\tilde{U}'(\lambda)}{\lambda \tilde{U}''(\lambda)} \qquad \tilde{U}(\lambda) \equiv \sup (U(x) - \lambda x)$$

by simple calculus, so the hypothesis on R is equivalent to

$$-\tilde{U}'(e^y) = \exp\{\psi(y)\} \qquad \text{for some concave } \psi$$

The result will follow if we can prove that $\log(-\tilde{V}_T'(e^y))$ is concave, that is

$$y \mapsto \log\left(-E\left[H_T \tilde{U}'(e^y H_T) \right] \right) = \log E\left[H_T \exp\{ \psi(y + \log H_T) \} \right] \quad \text{is concave.}$$

Equivalently, we must prove for any $0 \le \alpha = 1-\beta \le 1$, for any y, z, that

$$E\left[H_T \exp\{\psi(\alpha y + \beta z + \log H_T)\}\right] \geq E\left[H_T \exp \psi(y + \log H_T)\right]^\alpha . E\left[H_T \exp \psi(z + \log H_T)\right]^\beta$$

To finish, we appear to need Prekopa's inequality, which says the following.

If $F, f, g : \mathbb{R}^d \to (0,\infty)$ and $\forall\, x, y \in \mathbb{R}^d$

$$F(\alpha x + \beta y) \geq f(x)^\alpha \, g(y)^\beta$$

then $\int F(z)\,dz \geq (\int f(x)\,dx)^\alpha \left(\int g(y)\,dy\right)^\beta$.

Still depth here!

Two-sector growth problems with discretionary labour (24/6/02)

(i) If we return to the Arrow-Kurz problem, but now allow the population to vary the proportion $\pi \in (0,1)$ of effort they devote to production, then the output changes to

$$F(K_p(t), K_g(t), \pi_t L_t T_t)$$

where all the variables have their usual interpretations. The objective of the government is

$$\max E_g \int_0^\infty e^{-\gamma t} U(c_t, k_g(t), \pi_t)\, dt$$

where under $\mathbb{P}_g$ $\quad dk = k(dz^0_g - dz^L_g) + \Phi(k)\,dt$ under optimal control, $\Phi(k) = f(k_p, k_g, \pi) - \gamma_g k - c$.
The value f^n V and optimal choices must satisfy

$$\begin{cases} 0 = U - \lambda_g V + \tfrac{1}{2}\sigma^2 k^2 V'' + \Phi V' \\ \Phi = f - \gamma_g k - c \\ U_c = V' \\ U_g = (F_p - F_g) V' \\ U_\pi = -F_L V' \end{cases}$$

whence

$$F_p = \frac{\hat{U}'}{V'} + \gamma_g + \Phi'$$

and these are also (in effect) sufficient.

(ii) How would we generate solutions? We need homogeneity of U in the first two variables:

$$U(c, k_g, \pi) = k_g^{1-R_g} h(\xi, \pi) \qquad (\xi \equiv c/k_g)$$

so that $U_c = k_g^{-R_g} h_\xi$, $\quad U_g = k_g^{-R_g}[(1-R_g)h - \xi h_\xi]$, $\quad U_\pi = k_g^{1-R_g} h_\pi$. We also have a consistency condition:

$$\begin{aligned} f = f(k) &\equiv F(k_p^*(k), k_g^*(k), \pi^*(k)) \\ &= \Phi(k) + \gamma_g k + c^*(k) \\ &= k_p F_p + k_g F_g + \pi F_L \\ &= k F_p - (F_p - F_g) k_g + \pi F_L \end{aligned}$$

Multiplying by V' gives us the equations

$$V'(\Phi + \cancel{\gamma_g} k + c) = k(\hat{U}' + \cancel{\gamma_g} V' + V'\Phi') - k_g U_g - \pi U_\pi$$

$H_1(\xi) = \xi^2/2$ still, gauss

Notice also that $k_g U_g = (1-R_g) k_g^{1-R_g} h_\xi - cU_c$, so we can cancel $cU_c \equiv cV'$ both sides to conclude

$$V'\Phi = k(\hat{u}' + V'\Phi') - (1-R_g) k_g^{1-R_g} h - \pi U_\pi$$

$$\therefore \quad (V' + kV'')\Phi = k\hat{u}' + k(\Phi V')' - (1-R_g) k_g^{1-R_g} h - \pi U_\pi$$

so
$$\boxed{\left(1 + \frac{kV''}{V'}\right)(\lambda_g V - \hat{u} - \tfrac{1}{2}\sigma^2 k^2 V'') = k(\lambda_g V - \tfrac{1}{2}\sigma^2 k^2 V'')' - k_g^{1-R_g}\left[(1-R_g)h + \pi h_\pi\right]}$$

Using the fact that $\hat{u} = k_g^{1-R_g} h$, we can rearrange this to give a little more simply

$$\boxed{\left(\frac{\lambda_g V - \frac{1}{2}\sigma^2 k^2 V''}{kV'}\right)' = \frac{k_g^{1-R_g}}{k^2 V'}\left[\pi h_\pi - \left(\frac{kV''}{V'} + R_g\right)h\right]} \qquad (*)$$

An easy deduction is that if $V(k) = A_V^{\nu-1} k^{1-S}/(1-S)$, the LHS is identically zero, which can only happen if

$$\pi h_\pi + (S - R_g) h = 0$$

so if h were of product form, we would have to have that π were <u>constant</u>, or proportional to a power of π.

<u>So to generate solutions</u>, we could:

<u>ASSUME FORM of</u> V, h, k_g (plus the various constants):

<u>then</u> from $V' = U_c$ we discover value of h_ξ along the path

from (*) discover value of $\pi h_\pi - \left(\frac{kV''}{V'} + R_g\right)h$ along the path

We would require that these two pieces of information allowed us to determine (ξ, π).

(iii) <u>Specialising</u>: <u>V is CRRA</u>, $k^{1-S}/(1-S)$.

(a) If we had $h(\xi,\pi) = H_1(\xi) H_2(\pi)$, we would then have for all k

$$\boxed{\pi H_2'(\pi) = (R_g - S) H_2(\pi)}$$

For example, if $H_2(\pi) = (1-\pi)^{\kappa}$, $H_1(\xi) = \xi^\nu/\nu$, we have the condition determining π to be

$$\frac{\pi \kappa}{1-\pi} = S - R_g$$

If $\nu < 0$, we must have $S > 1$, and $\kappa < 0$, so deduce $R_g > S > 1$
If $\nu > 0$, we must have $0 < S < 1$ and $\kappa > 0$, so have $1 > S > R_g > 0$
and $\pi = \dfrac{S - R_g}{\kappa + S - R_g}$

More generally, if π^* is the optimal value of π, if $H_2(\pi^*) = \Theta^{1-\nu}$, if $V'(k) = A_V^{\nu-1} k^{1-S}/(1-S)$ and $k_g/k = \varphi^{(1-\nu)/\omega}$, then a little further calculation leads to expressions

$$C = A_V \Theta\, k^B \varphi(k) \qquad [(1-\nu)B = S + \omega]$$

$$U = \textcircled{H} A_\nu^{\nu} \; k^{R-S} \varphi / \nu$$
$$\Phi = Qk - c/\nu \qquad \left(Q \equiv \frac{\lambda_g}{1-S} + \tfrac{1}{2}\sigma^2 S \right)$$
$$c = \textcircled{H} A_\nu \, \varphi k^R$$
$$f = (Q + \gamma_g)k + (1 - 1/\nu)c$$
$$F_p - F_g = \frac{c}{k_g} \cdot \frac{\omega}{\nu}$$
$$F_p = Q + \gamma_g - cS/k\nu$$
$$F_L = \frac{-H_2'(\pi^*)c}{\nu H_2(\pi^*)}$$

We still need to check the tangent inequality, and consistency. The analysis of the tangent inequality goes through without change, and the assumptions $\nu < 0$,

φ is increasing, ξ is decreasing

will as before guarantee the tangent inequality. We get

$$F_g = Q + \gamma_g + \frac{k^{R-1}}{-\nu} \left[S\varphi + \omega \, \varphi^{-R_g/\omega} \right] A_\nu$$

which can be made ≥ 0 by suitable choice of parameters. If we assume $\nu < 0$, we will have $H_2 > 0$, $H_2' > 0$, $H_2'' > 0$, so $F_L \geq 0$, and $F_p \geq 0$ as $F_p \geq F_g$.

(b) Another general form which we could try would be

$$U(c, k_g, \pi) = k_g^{1-R_g} H(\xi(1-\pi)) \qquad , \qquad V(k) \propto k^{1-S}/(1-S)$$

If we assume the form of k_g, when we consider $V' = U_c = k_g^{-R_g}(1-\pi) H'(\xi(1-\pi))$ we will obtain $(1-\pi) H'(\xi(1-\pi))$ as a function of k. Moreover, the condition

$$\pi h_\pi \equiv -\pi \xi \; H'(\xi(1-\pi)) = (R_g - S) \; H(\xi(1-\pi))$$

will give us

$$\frac{\pi}{(1-\pi)(S - R_g)} = \frac{H(x)}{x H'(x)} \qquad x \equiv \xi(1-\pi)$$

so provided $x \mapsto x H'(x)/H(x)$ is strictly monotone we could from this work out for a given π what x and thus ξ should be, and hence we may hope to get $\xi(k), \pi(k)$.

Are there any good examples here? It might be most fruitful to propose $x(k) \equiv \xi(k)(1-\pi(k))$ and try to deduce H. We have

$$\begin{cases} (1-\pi) \; H'(x) = V'(k) \, k_g^{-R_g} \equiv \Phi_0(k), \text{ say} \\ -\pi\xi \; H'(x) = (R_g - S) \; H(x) \end{cases}$$

Keeping BM away from a level (8/8/02)

(i) Suppose we have a process

$$X_t = \theta W_t + bt$$

and some level $a > 0$. b is a constant, and θ is constant, but to be chosen. Thus $a>0$ fixed, $T>0$ fixed. Let $H_a(X) \equiv \inf\{t : X_t > a\}$; we want to make $P[H_a(X) \le T]$ as small as we can by choice of θ.

The taboo transition density is

$$_a p_t(0,y) = \left\{ e^{-y^2/2t} - e^{-(2a-y)^2/2t} \right\} / \sqrt{2\pi t}$$

and introducing a drift μ changes this to

$$\left\{ e^{-(y-\mu t)^2/2t} - e^{2\mu a}\, e^{-(y-2a-\mu t)^2/2t} \right\} / \sqrt{2\pi t}$$

so

$$P^{\mu}(H_a(W) > t) = \Phi\left(\frac{a-\mu t}{\sqrt{t}}\right) - e^{2\mu a}\, \Phi\left(\frac{-a-\mu t}{\sqrt{t}}\right).$$

(ii) By Brownian scaling,

$$X_t \overset{\mathcal{D}}{=} W_{\theta^2 t} + bt = W_{\theta^2 t} + \frac{b}{\theta^2}\cdot \theta^2 t \equiv \tilde{X}_{\theta^2 t}$$

and so

$$P\left[H_a(X) > T\right] = P\left[H_a(\tilde{X}) > \theta^2 T\right] = P^{b\theta^{-2}}\left[H_a(W) > \theta^2 T\right]$$

$$= P\left(Z \le \frac{a-bT}{\theta\sqrt{T}}\right) - e^{2ab/\theta^2}\, P\left(Z \le \frac{-a-bT}{\theta\sqrt{T}}\right).$$

(iii) How if $b \sim N(\beta, v)$?

$$P\left(Z \le \frac{a-bT}{\theta\sqrt{T}}\right) = P\left(\theta\sqrt{T}Z \le a - \beta T + \sqrt{v}\,Z'T\right)$$

$$= P\left(\theta\sqrt{T}Z + \sqrt{v}\,TZ' \le a - \beta T\right)$$

$$= P\left(Z \le \frac{a-\beta T}{(\theta^2 T + vT^2)^{1/2}}\right).$$

For the other term, we get

$$\int_{-\infty}^{\infty} \exp\left[-\frac{(b-\beta)^2}{2v} + \frac{2ab}{\theta^2}\right] P\left(Z \le \frac{-a-bT}{\theta\sqrt{T}}\right) db / \sqrt{2\pi v}$$

$$= \int_{-\infty}^{\infty} \exp\left[-\frac{(b-\beta-2av/\theta^2)^2}{2v} - \frac{\beta^2}{2v} + \frac{(\beta+2av/\theta^2)^2}{2v}\right] P\left[Z \le \frac{-a-bT}{\theta\sqrt{T}}\right] db/\sqrt{2\pi v}$$

$$= \exp\left[\frac{2a\beta}{\theta^2} + \frac{2a^2 v}{\theta^4}\right] P\left[Z \le \frac{-a-(\beta+2av/\theta^2)T}{(\theta^2 T + vT^2)^{1/2}}\right] = \exp\left\{-\frac{(a-\beta T)^2}{2V}\right\} F_1\left(\frac{+a+(\beta+2av/\theta^2)T}{\sqrt{V}}\right)$$

Differentiating wrto θ gives us

$$\boxed{-\frac{e^{-\Lambda^2/2V}}{\sqrt{2\pi V}}\cdot\frac{2a\,(2vT+\theta^2)}{\theta^3}+e^{2a\tilde\beta/\theta^2}\,\Phi\left(\frac{-\Lambda-2\tilde\beta T}{\sqrt V}\right)\left\{\frac{4a\tilde\beta}{\theta^3}+\frac{4a^2v}{\theta^5}\right\}}$$

$(\Lambda\equiv a-\beta T,\ \tilde\beta\equiv\beta+av/\theta^2,\ V\equiv\theta^2T+vT^2)$

When is this zero? Setting $y\equiv(\Lambda+2\tilde\beta T)/\sqrt V$, the condition is

$$\boxed{\frac{\theta^2\,(2vT+\theta^2)}{2\,(\beta\theta^2+2av)}=\sqrt{2\pi V}\;e^{y^2/2}\,\Phi(-y)}$$

Notice that

$$F_1(y)\equiv e^{y^2/2}\,\Phi(-y)=\frac{1}{\sqrt{2\pi}}\int_y^\infty e^{-t^2/2}\,dt\;e^{y^2/2}=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-v^2/2-yv}\,dv$$

and good expansions are available for this

Clearly, $yF_1(y)$ increases with y, from 0 to $\frac{1}{\sqrt{2\pi}}$, $F_1(0)=\frac12$

Special case: $v=0$

Writing $\theta^2=z$, the equation to solve is

$$z/2\beta=\sqrt{2\pi zT}\;F_1\left(\frac{a+\beta T}{\sqrt{zT}}\right),$$

equivalently,

$$\frac{1}{\sqrt{2\pi}}\;\frac{a+\beta T}{2\beta T}=yF_1(y)\qquad\qquad \left(y\equiv(a+\beta T)/\sqrt{zT}\right)$$

$\boxed{\text{This has a solution provided } a<\beta T.}$

General case: $v\geq0$

Again, with $\theta^2=z$, $y=\{a+\beta T+2avT/z\}/\sqrt V$, $V=zT+vT^2$, we get the equality to be satisfied is

$$\frac{z\,(2vT+z)}{2\,(\beta z+2av)}=\sqrt V\,\sqrt{2\pi}\;F_1(y)$$

$$\Leftrightarrow\quad\frac{(a+\beta T+2avT/z)}{zT+vT^2}\cdot\frac{z\,(2vT+z)}{2(\beta z+2av)}=\sqrt{2\pi}\;yF_1(y),$$

which is clearly decreasing in z from 1 to 0

so we want

$$\sqrt{2\pi}\;y\,F_1(y)=\frac{2avT+z\,(a+\beta T)}{z+vT}\cdot\frac{z+2vT}{2av+\beta z}\cdot\frac{1}{2T}$$

As $z \to \infty$, this rational function of z tends to $\frac{a+\beta T}{2\beta T}$

As $z \to 0$, ———————————————— 1.

As a consequence, there should always be a root!

(iv) If $\bar{X}_t \equiv \sup_{0 \le s \le t} X_s$, we have really the task of choosing θ to minimise

$$E(\bar{X}_T \vee a) = a + \int_a^\infty P(\bar{X}_T > x)\, dx$$

We know $P(\bar{X}_T > x) = P\left(Z > \frac{x - bT}{\theta\sqrt{T}}\right) + e^{2bx/\theta^2} P\left(Z > \frac{x+bT}{\theta\sqrt{T}}\right)$, hence

$$\boxed{\int_a^\infty P(\bar{X}_T > x)\, dx = \left(bT - a + \frac{\theta^2}{2b}\right)\bar{\Phi}\left(\frac{a-bT}{\theta\sqrt{T}}\right) + \theta\sqrt{T}\, \frac{e^{-(a-bT)^2/2\theta^2 T}}{\sqrt{2\pi}} - \frac{\theta^2}{2b} e^{2ab/\theta^2} \bar{\Phi}\left(\frac{a+bT}{\theta\sqrt{T}}\right)}$$

The derivative of this wrt θ will be

$$\frac{\theta}{b}\left[\bar{\Phi}\left(\frac{a-bT}{\theta\sqrt{T}}\right) + e^{2ab/\theta^2}\bar{\Phi}\left(\frac{a+bT}{\theta\sqrt{T}}\right)\left(\frac{2ab}{\theta^2} - 1\right)\right]$$

If we allow that $b \sim N(\beta, v)$, we've got $\quad (V \equiv \theta^2 T + vT^2)$

$$P(\bar{X}_T > x) = \bar{\Phi}\left(\frac{x - \beta T}{\sqrt{V}}\right) + \exp\left\{\frac{2x\beta}{\theta^2} + \frac{2x^2 v}{\theta^4}\right\}\bar{\Phi}\left(\frac{x + (\beta + 2xv/\theta^2)T}{\sqrt{V}}\right)$$

but integrating this isn't so nice

(V) <u>But let's think again (7/10/02):</u> The optimal hedging strategy for an American put option does not in fact depend on the max value attained to date ... So maybe we just need to choose so as to max $E(e^{-\alpha \bar{Z}_T})$? What α?

(Vi) $$\int_a^\infty \gamma e^{\gamma x} P\left(Z > \frac{x-\beta}{v}\right) dx = e^{\gamma\beta + \frac{1}{2}\gamma^2 v^2}\bar{\Phi}\left(\frac{a - \beta - \gamma v^2}{v}\right) - e^{\gamma a}\bar{\Phi}\left(\frac{a-\beta}{v}\right)$$

for later reference.

$$= \int \left(e^{\gamma y} - e^{+\gamma a}\right)^+ P\left(N(\beta, v^2) \in dy\right)$$

$$\gamma - v_{0L} = \delta - \mu_0 \quad = \delta + \mu_L + \mu_T - v_{LL}$$

$$W_i = 1 - \gamma_i - R_i \,, \qquad i = p, g$$

$$Q = \frac{\lambda_g}{1 - s_g} + \frac{1}{2}\sigma^2 S_g$$

Good solutions of stochastic 2-sector growth models (20/8/02)

1) Let's take up the problem with the notation of our preprint: the conditions of Theorem 2 are

$$0 = \psi\left[\beta_k F_p - \gamma - \lambda_p + v_{0L}(1-\beta_k)\right] + \psi'\left[\Phi + \beta_k \sigma^2 k + (2-\beta_k)(\gamma_g - \gamma)k\right] + \tfrac{1}{2}\sigma^2 k^2 \psi'' \qquad (PS1)$$

$$u_c = \beta_c^{-1}\psi \qquad (PS2)$$

$$u_\pi = -\beta_W F_L \psi \qquad (PS3)$$

$$0 = \psi(r\beta_r + \mu_0 - \lambda_p) + \psi'\left[\Phi + 2(\gamma_g - \gamma)k\right] + \tfrac{1}{2}\sigma^2 k^2 \psi'' \qquad (PS4)$$

Introduce the notation

$$\boxed{J(k) \equiv -\tfrac{1}{2}\sigma^2 k^2 \frac{\psi''(k)}{\psi(k)} - \frac{\psi'(k)}{\psi(k)}\left[\Phi(k) + 2(\gamma_g - \gamma)k\right]}$$

In terms of this, we have

$$\boxed{\begin{aligned} r\beta_r &= \lambda_p - \mu_0 + J(k) \\ \beta_k &= \frac{\delta + \lambda_p - \mu_0 + J(k)}{F_p - v_{0L} + (v_{00} - v_{0L})\, k\psi'(k)/\psi(k)} \end{aligned}}$$

Thus once the optimal (government) solution is known, and the multiplier function ψ is specified, the tax rates can be derived from this.

2) Special case: $U(c, k_g, \pi) = k_g^{\omega_g} c^{\nu_g} H_g(\pi)/\nu_g$, $u(c, k_g, \pi) = k_g^{\omega_p} c^{\nu_p} H_p(\pi)/\nu_p$

We get

$$c = A k^B \varphi, \quad k_g = k\, \varphi^{(1-\nu_g)/\omega_g}, \quad \Phi = Qk - c/\nu_g,$$

$$F_p = \gamma_g + Q - S_g c/\nu_g k$$

and hence

$$u_c = A^{\nu_p - 1} \mathcal{H}_p\, k^{-S_p} \varphi^{\alpha} \qquad \left[\begin{aligned} \alpha &= \nu_p - 1 + \omega_p(1-\nu_g)/\omega_g \\ -S_p &= \omega_p - B(1-\nu_p) \end{aligned}\right]$$

where $\mathcal{H}_i = H_i(\pi^*)$.

We may take β_c and β_W to be constant, by selecting

$$\boxed{\psi = \beta_c u_c = \beta_c A^{\nu_p - 1} \mathcal{H}_p\, k^{-S_p} \varphi^{\alpha}}$$

and then

$$\boxed{\beta_W = \frac{\nu_g \zeta_p}{\nu_p \zeta_g} \cdot \frac{1}{\beta_c}} \qquad \zeta_i \equiv H_i'(\pi^*)/H_i(\pi^*)$$

$\left(n_g \equiv -\nu_g\right)$

When ψ is determined from φ in this way, we get

$$J = -\tfrac{1}{2}\sigma^2\left[S_p(S_p+1) + \alpha(\alpha-1)\left(\frac{k\varphi'}{\varphi}\right)^2 - 2\alpha S_p \frac{k\varphi'}{\varphi} + \alpha\frac{k^2\varphi''}{\varphi}\right]$$

$$-\left(-S_p + \alpha\frac{k\varphi'}{\varphi}\right)\left[Q + 2(\gamma_g-\gamma) - \frac{c}{k\gamma_g}\right]$$

<u>Example 1</u>: <u>$\varphi(k) = \varphi_0(1+ak)^{\varepsilon}$</u>

A little calculation leads us to

$$J = \left(S_p - \alpha\varepsilon\frac{ak}{1+ak}\right)\left[Q + 2(\gamma_g-\gamma) + \frac{A\varphi_0}{n_g}k^{\beta-1}(1+ak)^{\varepsilon} - \tfrac{1}{2}\sigma^2\left(S_p + 1 - \alpha\varepsilon\frac{ak}{1+ak}\right)\right] - \tfrac{1}{2}\sigma^2\alpha\varepsilon\frac{ak}{(1+ak)^2}$$

which is like $A\varphi_0 S_p k^{\beta-1}/n_g$ as $k\to 0$, and converges to a finite constant at infinity.

The denominator in the expression for β_p is

$$F_p - v_{0L} + (v_{00} - v_{0L})\, k\psi'/\psi$$

$$= \gamma_g + Q - v_{0L} + (v_{00} - v_{0L})\left\{-S_p + \alpha\varepsilon\frac{ak}{1+ak}\right\} - \frac{S_g}{\gamma_g}A\varphi_0 k^{\beta-1}(1+ak)^{\varepsilon}$$

which is positive near zero. We certainly don't want it to go negative ever, which will be ok provided

$$\gamma_g + Q - v_{0L} + S_p(v_{0L} - v_{00}) + \alpha\varepsilon(v_{00} - v_{0L})\wedge 0 \geq 0$$

<u>Example 2</u>: <u>$\varphi(k) = \varphi_0(1+ak^{\varepsilon})$</u>

Again a few calculations get us to

$$J = -\tfrac{1}{2}\sigma^2\left[S_p(1+S_p) + \alpha\varepsilon(\varepsilon-1-2S_p)T + \alpha(\alpha-1)T^2\right]$$

$$+(S_p - \alpha\varepsilon T)\left[Q + 2(\gamma_g-\gamma) + \frac{A\varphi_0}{n_g}k^{\beta-1}(1+ak^{\varepsilon})\right], \qquad \left(T \equiv \frac{ak^{\varepsilon}}{1+ak^{\varepsilon}}\right)$$

$$F_p = \gamma_g + Q - v_{0L} + \frac{S_g A}{n_g}\varphi_0 k^{\beta-1}(1+ak^{\varepsilon}) + (v_{00} - v_{0L})(-S_p + \alpha\varepsilon T).$$

The case of <u>constant</u> φ is a special case of both examples.

Loss of efficiency in the Merton problem due to uncertain drift (13/9/02)

1) Suppose we consider the situation where the investor selects a fixed portfolio proportion vector θ and invests in a constant structure market for time t: wealth evolves as

$$dx_t = x_t \ \theta^T(\sigma dW + (\mu - r)dt) + r x_t \, dt$$

So if $U(x) = x^{1-R}/(1-R)$ we shall have

$$E\, U(x_T) = U(x_0) \exp\left[(1-R)T \left\{ r + \theta^T(\mu - r) - \tfrac{1}{2}\theta^T R \Sigma \theta \right\} \right] \qquad (\Sigma \equiv \sigma\sigma^T).$$

Now suppose that we do not see μ, but rather we observe an estimate $\hat{\mu}$, and suppose that $\mathcal{L}(\mu | \hat{\mu}) = N(\hat{\mu}, V)$, with V known. We could try two approaches to the choice of θ, the first being to use the Merton rule $\theta = (R\Sigma)^{-1}(\hat{\mu} - r)$ with the point estimate $\hat{\mu}$, the second being to hold θ fixed, average over the conditional law of μ, then optimise over θ.

Approach 1

$$E\, U(x_T) = U(x_0) \exp\left[(1-R)T \left\{ r + \tfrac{1}{2}(\hat{\mu} - r)^T (R\Sigma)^{-1} \left[R\Sigma + (1-R)TV \right] (R\Sigma)^{-1}(\hat{\mu} - r) \right\} \right]$$

after we simplify;

$$= U(x_0) \exp\left[(1-R)T \left\{ r + \tfrac{1}{2}(\hat{\mu} - r)^T (R\Sigma)^{-1} A (R\Sigma)^{-1}(\hat{\mu} - r) \right] \right.$$

where $A \equiv R\Sigma + (1-R)TV$

Approach 2 For fixed θ, we obtain expected utility of

$$U(x_0) \exp\left[(1-R)T \left\{ r + \theta^T(\hat{\mu} - r) - \tfrac{1}{2}\theta^T B \theta \right\} \right], \qquad B \equiv R\Sigma + (R-1)TV$$

which is optimised by choosing $\theta = B^{-1}(\hat{\mu} - r)$, giving value

$$U(x_0) \exp\left[(1-R)T \left\{ r + \tfrac{1}{2}(\hat{\mu} - r)^T B^{-1} (\hat{\mu} - r) \right\} \right]$$

We shall of course need to make the

ASSUMPTION : $R > 1$, A is positive-definite

[NB : if A fails to be positive-definite, approach 1 gives $-\infty$ payoff ...]

What is the efficiency of the dumb rule? That is, find $\lambda_T \in (0,1)$ such that the smart guy starting with λ_T does as well as the dumb one with initial wealth 1. We find λ_T solves

$$\frac{1}{T}\log \lambda_T + \tfrac{1}{2}(\hat{\mu} - r)^T B^{-1}(\hat{\mu} - r) = \tfrac{1}{2}(\hat{\mu} - r)^T (R\Sigma)^{-1} A (R\Sigma)^{-1}(\hat{\mu} - r)$$

$$\boxed{\therefore \ -\frac{1}{T}\log \lambda_T = \tfrac{1}{2}(\hat{\mu} - r)^T B^{-1} C^2 (\hat{\mu} - r)} \qquad C \equiv (R-1)TV(R\Sigma)^{-1}$$

2) Assuming that Σ is known, and taking a prior $\mu \sim N(0, M\Sigma)$ for μ, we observe the evolution of log price Y up til time T_0, and have $Y_{T_0} = \mu T_0 + \varepsilon$, where $\varepsilon \sim N(0, \Sigma T_0)$. Given the value of $y \equiv Y(T_0)$, the posterior distn of μ is again normal, with

mean $\frac{M}{1+T_0 M} y$, covariance $\frac{M}{1+MT_0} \Sigma$

Thus we have

$$B = b\Sigma, \qquad b = R + \frac{(R-1)TM}{1+MT_0}$$

$$C = c\Sigma, \qquad c = \frac{(R-1)T}{R} \frac{M}{1+MT_0}$$

and efficiency is given by

$$\log \lambda_T = -\frac{1}{2} \cdot \frac{c^2 T}{b} \cdot (\hat{\mu}-r)^T \Sigma (\hat{\mu}-r)$$

3) I computed the values for this, + for loss of efficiency, using the R dataset EuStockMarkets ... the loss of efficiency was always negligible (not surprising, in view of the conclusions of the relaxed investor) but the change in portfolio could be substantial ...

One would naturally next turn to the consumption version of the problem, but this isn't going to go down in closed form (cf relaxed investor again).

Solutions to 2-sector stochastic growth problems (17/9/02)

When seeking explicit solutions, we take the govt's felicity in the form

$$U(c, k_g, \pi) = k_g^{1-R_g} \; h(\xi, \pi) \qquad (\xi \equiv c/k_g)$$

and try various forms of h. THROUGHOUT, assume $V(k) = A_g \, k^{1-S}/(1-S)$, so that we must have

$$\pi h_\pi = (R_g - S) \, h$$

along the optimal trajectory, and the optimality criterion gives us

$$k_g^{-R_g} h_\xi = V' = A_g \, k^{-S}.$$

Let's now explore various general forms for h.

1) General form $h(\xi, \pi) = h_1(\xi) \, h_2(\pi)$, with k_g, π, h_1 and h_2 assumed known.

Here the equations to be satisfied are

$$\begin{cases} \pi \, h_2'(\pi) = (R_g - S) \, h_2(\pi) \\ k_g^{-R_g} \, h_1'(\xi) h_2(\pi) = A_g \, k^{-S} \end{cases}$$

and the first determines π (there may be ambiguity if h_2 solves the ODE in an interval), then the second determines $\xi(k)$, hence $c(k)$, and the entire solution.

Example: $h_1(\xi) = \xi^\nu$, dealt with in the paper

2) General form $h(\xi, \pi) = H(\xi \, g(\pi))$, where k_g, π, g are assumed given.

This time, the equations to be satisfied take the form

$$\begin{cases} g(\pi) \, H' = k_g^{R_g} A_g \, k^{-S} \\ \xi g'(\pi) H' = \dfrac{H}{\pi} (R_g - S) \end{cases}$$

so from the first we find $H' \equiv H'(\tau(k)) \equiv H'(\xi(k) \, g(\pi(k)))$ explicitly as a function of k. Next

$$\frac{\tau H'}{H} = \frac{\xi g(\pi) H'}{H} = \frac{g}{g'} \quad \frac{R_g - S}{\pi}$$

is exhibited as a known function of k, and so H/τ is known, as is its derivative

$$\frac{d}{dk}\left(\frac{H}{\tau} \right) = \frac{\tau'}{\tau^2}\left[\tau H' - H \right] = \frac{\tau'}{\tau} \cdot \frac{H}{\tau} \cdot \left[\frac{\tau H'}{H} - 1 \right]$$

This tells us τ (to within a multiplicative constant) hence $\xi(k)$, and thus $c(k)$.

For F_g to remain $\geqslant 0$ near $k = 0$, it turns out that we shall need $\theta R_g < S$, which is inconsistent with $\theta \geqslant 1$, $R_g > S > 1$

Example Suppose we have positive R_g, S, and $\theta \geq 1$ such that $R_g > S > 1$, and $\theta R_g < S+1$. We seek a solution having the form

$$\pi(k) = \frac{1}{1+ak}, \quad k_g = \frac{b\,k^{\theta}}{(1+ak)^{1/R_g}}, \quad g(\pi) = 1-\pi.$$

Because of the assumed inequalities, $k_g/k \sim k^{\theta-1-1/R_g} \to 0$ as $k\to\infty$, for $b>0$ small enough we shall have $k_g/k \leq 1 \;\forall k$. We suppose $a>0$, of course.

Now

$$H'(\tau) = \frac{A_g}{a} b^{R_g} k^{\theta R_g - S - 1} \qquad \text{(which is decreasing with } k\text{)}$$

and $$\frac{\tau H'}{H} = \frac{1-\pi}{\pi}(S-R_g) = ak(S-R_g) \quad \therefore \quad \frac{H}{\tau} = -\frac{A_g\, b^{R_g}}{a^2(R_g-S)}\cdot k^{\theta R_g - S - 2}$$

so that

$$\frac{d}{dk}\left[\log\frac{H}{\tau}\right] = \frac{\theta R_g - S - 2}{k} = -\frac{\tau'}{\tau}\{1+ak(R_g-S)\}$$

This gives us $$\tau(k) \propto \left(\frac{k}{1+a(R_g-S)k}\right)^{S+2-\theta R_g}$$ which increases with k.

Therefore H' is a decreasing function, and so H is concave. We deduce the form of c:

$$\boxed{c(k) = \text{const.}\, \frac{k^{\theta+S+1-\theta R_g}\,(1+ak)^{1-1/R_g}}{(1+a(R_g-S)k)^{S+2-\theta R_g}}}\,.$$

3) General form $h(\xi,\pi) = h_1(\xi) + h_2(\pi)$, where π and k_g will be taken as given, as well as one of h_1, h_2. We have

$$\begin{cases} h_1'(\xi) = A_g\, k_g^{R_g}\, k^{-S} \\ \pi h_2'(\pi) = (R_g - S)(h_1 + h_2) \end{cases}$$

If we assume that h_2 is known, then

$$h_1 = -h_2 - \frac{\pi h_2'(\pi)}{S-R_g}$$

is known, so we can differentiate to get

$$\xi'(k)\, h_1'(\xi(k)) = \xi'(k)\, A_g\, k_g^{+R_g} k^{-S} = \pi'\left[-h_2' - \frac{h_2' + \pi h_2''}{S-R_g}\right]$$

from which we deduce ξ (up to an additive constant) and thence h_1

Alternatively, if h_1 is supposed known, we immediately recover ξ from the first equation, and from the second we get that

$$\frac{1}{\pi(k)} \frac{d}{dk}\left[\pi(k)^{S-R_g} h_2(\pi(k)) \right] = (R_g - S)\, \pi(k)^{S-R_g-1} h_1(\xi(k))$$

from which we can in principle obtain h_1.

$BESQ^{\eta}(x)$ transition semigroup: it is

$$q^{\nu}_t(x,y) = \frac{1}{2t} e^{-(x+y)/2t} \sum_{k \geq 0} \frac{\left(\frac{x}{2t}\right)^{k} \left(\frac{y}{2t}\right)^{k+\nu}}{k!\, \Gamma(k+\nu+1)} \quad \text{, in fact!}$$

For finiteness of the integral (dy) for every k, we require $\nu + 1 - R > -1$, that is,

$$\boxed{\frac{2A}{\sigma^2} > R - 1}$$

Some calculations of certain functionals of certain diffusions related to squared Bessel processes (15/10/02)

In the work with John Aquilina, we have a total dividend process Δ solving

$$d\Delta = \sigma\sqrt{\Delta}\, dW + (A - \beta\Delta)\, dt$$

and then we can express the value of the market portfolio in terms of this:

$$\Sigma_t = E_t\left[\int_t^\infty \zeta_u\, \Delta_u\, du\right] \Big/ \zeta_t \qquad\qquad \zeta_t \equiv e^{-\rho t}\Delta_t^{-R}$$

so that $\Sigma_t = f(\Delta_t)$, where

$$f(x) = x^R\, E^x\left[\int_0^\infty e^{-\rho t}\, \Delta_t^{1-R}\, dt\right]$$

$$= x^R\, E^x \int_0^\infty e^{-\rho t}\left(e^{-\beta t}\, Y(A_t)\right)^{1-R} dt$$

$$\begin{cases} A_t \equiv \frac{\lambda}{\beta}\left(e^{\beta t}-1\right) \\ \lambda = \sigma^2/4, \quad \eta = 4A/\sigma^2, \\ Y \text{ is } BESQ^\eta(x) \end{cases}$$

$$= x^R\, E^x \int_0^\infty e^{-\rho t - \beta(1-R)t}\, Y(A_t)^{1-R}\, dt$$

$$A_t = a, \quad da = \lambda e^{\beta t}\, dt, \quad e^{\beta t} = 1 + \beta a/\lambda.$$

$$\theta = (\rho + \beta(2-R))/\beta$$

$$= x^R\, E^x \int_0^\infty \left(1 + \frac{\beta a}{\lambda}\right)^{-\theta} Y(a)^{1-R}\, \frac{da}{\lambda}$$

$$(\nu = \eta/2 - 1)$$

(!)

$$= x^R \int_0^\infty \left(1+\frac{\beta a}{\lambda}\right)^{-\theta} \sum_{k\geq 0} \int_0^\infty \frac{(2a)^{1-R}}{2a}\, e^{-(x+y)/2a}\, \frac{(x/2a)^k (y/2a)^{k+\nu+1-R}}{k!\, \Gamma(k+\nu+1)}\, dy\, \frac{da}{\lambda}$$

$$= x^R \int_0^\infty \left(1+\beta a/\lambda\right)^{-\theta} \sum_{k\geq 0} \frac{(x/2a)^k\, e^{-x/2a}\, (2a)^{1-R}\, \Gamma(k+\nu+2-R)}{k!\, \Gamma(k+\nu+1)}\, \frac{da}{\lambda}$$

$$= x^2 \int_0^\infty \left(1 + \frac{\beta x}{2v\lambda}\right)^{-\theta} \sum_{k\geq 0} \frac{v^{k+R-3}\, e^{-v}\, \Gamma(k+\nu+2-R)}{k!\, \Gamma(k+\nu+1)}\, \frac{dv}{2\lambda} \qquad \left[\begin{aligned} v &= x/2a \\ da &= -\frac{x\, dv}{2v^2}\end{aligned}\right]$$

$\sim x^2 \cdot$ const as $x \downarrow 0$ provided $R > 2$.

If $R = 2$, the behaviour is determined by the $k=0$ term in the sum:

$$\int_0^\infty \left(1 + \beta x/2v\lambda\right)^{-\theta} e^{-v}\, \frac{dv}{v} \sim \text{const} \cdot \log 1/x \qquad (x \downarrow 0)$$

If $R < 2$, then once again the term $k=0$ dominates, and we have $(\varepsilon \equiv 2-R)$

$$\int_0^\infty \left(1 + \frac{\beta}{2\lambda u}\right)^{-\theta} (xu)^{-1-\varepsilon}\, e^{-xu}\, x\, du \sim x^{-\varepsilon} \int_0^\infty \left(1 + \frac{\beta}{2\lambda u}\right)^{-\theta} u^{-1-\varepsilon}\, du$$

which is OK, since $\theta - \varepsilon = \rho/\beta > 0$.

Representative agent market with default: first case (25/10/02)

Suppose we have representative agent seeking to max $E\int_0^\infty e^{-\rho t}\, U(c_t)\, dt$, where $U(x) = -\exp(-\gamma x)$. The output of the economy is modelled as $\Delta_t = \sigma W_t + \mu t$, but what we propose is that the representative agent can abandon the economy at a time of his choosing, and then continue for ever at subsistence consumption $c=0$. Clearly the issue is to choose the critical level b at which the agent shuts down the economy. Let $\tau = \inf\{t : \Delta_t \le b\}$ and compute the value of the agent's objective, namely

$$V_b(x) = E^x\left[-\int_0^\tau e^{-\rho t}\, e^{-\gamma \Delta_t}\, dt - e^{-\rho\tau}/\rho \right]$$

Now $E^x[e^{-\rho\tau}] = e^{-\beta(x-b)}$ for $x \ge b$, where $\alpha > 0 > -\beta$ are roots of the quadratic

$$p(x) \equiv \tfrac{1}{2}\sigma^2 x^2 + \mu x - \rho = 0$$

Moreover, if we set $h(x) \equiv E^x\left[\int_0^\tau e^{-\rho t}\, e^{-\gamma\Delta_t}\, dt\right]$ then $\mathcal{G}h - \rho h + e^{-\gamma x} = 0$, so we get

$$\boxed{h(x) = A\left[e^{-\gamma x} - e^{-\gamma b - \beta(x-b)}\right], \qquad A = 1/(\rho + \mu\gamma - \tfrac{1}{2}\sigma^2\gamma^2)} = -1/p(-\gamma) = \frac{+2}{\sigma^2(\alpha+\gamma)(\beta-\gamma)}$$

Notice $A > 0 \Leftrightarrow \gamma < \beta$, so h is always > 0 for $x > b$. We therefore have

$$V_b(x) = -A\left[e^{-\gamma x} - e^{-\gamma b - \beta(x-b)}\right] - \frac{1}{\rho}\, e^{-\beta(x-b)}$$

Optimising this over b leads to

$$\boxed{b^* = \frac{1}{\gamma}\log\left[A\rho(\beta-\gamma)/\beta\right]} = \frac{1}{\gamma}\log\left[\frac{2\rho}{\sigma^2\beta(\alpha+\gamma)}\right] = \frac{1}{\gamma}\log\left[\frac{\rho}{\rho + \frac{1}{2}\sigma^2\gamma\beta}\right]$$

What is the share then worth? We have to calculate

$$e^{\gamma x}\, E^x\left[\int_0^\tau e^{-\rho t - \gamma\Delta_t}\, \Delta_t\, dt\right]$$

$$= \frac{\{x - b e^{(\gamma-\beta)(x-b)}\}(\rho + \mu\gamma - \frac{1}{2}\sigma^2\gamma^2) + (\sigma^2\gamma - \mu)\left(e^{(\gamma-\beta)(x-b)} - 1\right)}{(\rho + \mu\gamma - \frac{1}{2}\sigma^2\gamma^2)^2}$$

We get $\left.\frac{\partial S}{\partial x}\right|_{x=b} = A^2(\gamma-\beta)^2\left[\tfrac{1}{2}\sigma^2 + b(\tfrac{1}{2}\sigma^2\gamma + \tfrac{1}{2}\sigma^2\beta - \mu)\right]$, which is not generally zero

However, $\left.\frac{\partial V}{\partial x}\right|_{x=b} = 0$, and this makes good sense

<u>Note</u>: $v_{io} = v_{oi} = v_{ii}$

Transmission of information through a linear-Gaussian multi-agent model (28/10/02)

We follow He + Wang (RFS $\underset{\sim}{8}$, 919-972) in supposing there is some system with J agents and an underlying 'true' value Π with a Gaussian distribution. The agents at any moment have estimates $\hat{\Pi}^j$ of Π, where $\hat{\Pi}^j$ is the conditional mean of Π given what agent j currently knows, and we suppose

$$\begin{pmatrix} \Pi \\ \hat{\Pi} \end{pmatrix} \sim N(0, V)$$

Agents acquire information by three mechanisms:

(i) Receipt of private signal $S^j = \Pi + \varepsilon_j$ (ε is Gaussian, indept of everything)

(ii) Receipt of public signal $S = \Pi + \eta$ (η is Gaussian, indept of all)

(iii) Common revelation of $\lambda^T \hat{\Pi}$ for some known fixed $\lambda \in \mathbb{R}^J$.

<u>How do estimates get updated for each of these three?</u>

(i) Agent i sees a world with

$$\begin{pmatrix} \Pi \\ \hat{\Pi}^i \\ S^i \end{pmatrix} \sim N\left(0, \begin{pmatrix} v_{00} & v_{0i} & v_{00} \\ v_{i0} & v_{ii} & v_{i0} \\ v_{00} & v_{i0} & v_{00} + \sigma_\varepsilon^2 \end{pmatrix}\right) \equiv N\left(0, \begin{pmatrix} a & b^T \\ b & D \end{pmatrix}\right)$$

Conditional on $\hat{\Pi}^i$ and S^i, new estimate of Π is $b^T D^{-1} \begin{pmatrix} \hat{\Pi}^i \\ S^i \end{pmatrix} \equiv \hat{\Pi}^i_+$ and we have

$$\begin{pmatrix} \Pi \\ \hat{\Pi}^i_+ \end{pmatrix} \sim N\left(0, \begin{pmatrix} 1 & 0 \\ 0 & b^T D^{-1} \end{pmatrix} \begin{pmatrix} a & b^T \\ b & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D^{-1} b \end{pmatrix}\right) = N\left(0, \begin{pmatrix} a & b^T D^{-1} b \\ b^T D^{-1} b & b^T D^{-1} b \end{pmatrix}\right)$$

What is the correlation of $\hat{\Pi}^i_+, \hat{\Pi}^j_+$? We need $(i \neq j)$

$$E\left[\hat{\Pi}^i_+ \;\hat{\Pi}^j_+\right] = b_i^T D_i^{-1} E \begin{pmatrix} \hat{\Pi}^i \\ \Pi + \varepsilon_i \end{pmatrix} (\hat{\Pi}^j \;\; \Pi + \varepsilon_j) \,.\, D_j^{-1} b_j$$

$$= b_i^T D_i^{-1} \begin{pmatrix} v_{ij} & v_{i0} \\ v_{j0} & v_{00} + \mathrm{cov}(\varepsilon_i, \varepsilon_j) \end{pmatrix} D_j^{-1} b_j$$

where we denote by subscript i, j the corresponding b, D for the two agents. This determines completely how the estimates update with this information.

(ii) The updating of each $\hat{\Pi}^i$ to $\hat{\Pi}^i_+$ goes exactly as before, and the <u>only</u> difference will be in updating the covariances of different agents' estimates. This is accounted for by a different covariance matrix for the noises, so this is now trivial.

(iii) For the final case, agent i now works with

$$\begin{pmatrix} \pi \\ \hat{\pi}^i \\ \lambda\cdot\hat{\pi} \end{pmatrix} \sim N\left(0, \begin{pmatrix} v_{00} & v_{0i} & \sum_j v_{0j}\lambda_j \\ v_{i0} & v_{ii} & \sum_j v_{ij}\lambda_j \\ \sum_j v_{0j}\lambda_j & \sum_j v_{ij}\lambda_j & \sum_j\sum_k \lambda_j v_{jk}\lambda_k \end{pmatrix}\right) \equiv N\left(0, \begin{pmatrix} a & \beta_i^T \\ \beta_i & \Delta_i \end{pmatrix}\right)$$

Any. This gives an updated estimate of

$$\hat{\pi}^i_+ = \beta_i^T \Delta_i^{-1} \begin{pmatrix} \hat{\pi}^i \\ \lambda\cdot\hat{\pi} \end{pmatrix}$$

with revised covariance

$$\mathrm{Cov}\begin{pmatrix} \pi \\ \hat{\pi}^i_+ \end{pmatrix} = \begin{pmatrix} a & \beta_i^T\Delta_i^{-1}\beta_i \\ \beta_i^T\Delta_i^{-1}\beta_i & \beta_i^T\Delta_i^{-1}\beta_i \end{pmatrix}$$

and an expression for

$$\mathrm{cov}\left(\hat{\pi}^i_+, \hat{\pi}^j_+\right) = \beta_i^T\Delta_i^{-1} \begin{pmatrix} v_{ij} & \sum_k v_{ik}\lambda_k \\ \sum_k \lambda_k v_{kj} & \sum\sum \lambda_k v_{kl}\lambda_l \end{pmatrix} \Delta_j^{-1} \beta_j$$

Phil Dybvig suggests:

Allow an endowment process $h \cdot X_t$, and a dividend process DX_t. The effect of this is change β to $\beta + D$ (so this doesn't alter the structure of the equations) and to add a term $+ h^T X_t dt$ to the wealth equation.

Linear-Gaussian-exponential utility maximisation (4/11/02)

1) Suppose that we have a linear Gaussian process X in $\mathbb{R}^d$, obeying

$$dX = A\,dW + BX\,dt$$

where A is a $d \times n$ matrix, W is a standard BM ($\mathbb{R}^n$), $n \le d$, and B is a $d\times d$ matrix. We consider an agent whose wealth w_t at time t obeys

$$dw_t = -c_t\,dt + r w_t\,dt + \theta_t^T(\alpha\,dX_t + \beta X_t\,dt) + h^T X_t\,dt$$

where r is fixed, and α is a $k\times d$ matrix, β is a $k \times d$ matrix, and θ is a k-vector.
The consumption rate c and portfolio weights θ are to be chosen in such a way as to maximise the utility

$$E\int_0^\infty -\exp\{-\rho t - \gamma c_t\}\,dt$$

subject to a transversality condition.

2) We conjecture that the value function has the form

$$V(w,x) \equiv \sup E\left[\int_0^\infty -\exp(-\rho t - \gamma c_t)\,dt \;\middle|\; w_0 = w,\ X_0 = x\right]$$

$$= -\exp\{-\lambda w - \tfrac{1}{2}x^T Q x - \varphi\}$$

for some $\lambda > 0$, and some $d\times d$ symmetric matrix Q. Note that this formulation allows the possibility of a term linear in X in the value f^n, by extending X to $\tilde{X} = \binom{X}{1}$. The current formulation is a bit more compact.

Setting

$$\xi_t = \lambda w_t + \tfrac{1}{2}X_t^T Q X_t + \varphi$$

we have

$$d\xi_t = (X^TQ + \lambda\theta^T\alpha)A\,dW + \left[X^TQBX + \tfrac{1}{2}\mathrm{tr}\,A^TQA + \lambda(-c + rw + \theta^T(\alpha B + \beta)X)\right]dt + \lambda h^T X_t\,dt$$

and the martingale principle of optimal control:

$$Y_t \equiv -\int_0^t e^{-\rho s - \gamma c_s}\,ds + e^{-\rho t}V(w_t, X_t) \quad \text{is a supermartingale etc}$$

Now we have

$$\boxed{e^{\rho t}\,dY_t \doteq -e^{-\gamma c}\,dt + e^{-\xi}\Big\{X^TQBX + \tfrac{1}{2}\mathrm{tr}\,A^TQA - \lambda c + \lambda r w + \lambda\theta^T(\alpha B + \beta)X + \rho + \lambda h^T X - \tfrac{1}{2}(X^TQ + \lambda\theta^T\alpha)AA^T(QX + \lambda\alpha^T\theta)\Big\}dt}$$

(here, e_N is the unit vector of the final component)

Maximising over c tells us that

$$\gamma e^{-\gamma c} = \lambda e^{-\xi}$$

so
$$\boxed{c = \frac{1}{\gamma}\left(\xi + \log(\gamma/\lambda)\right)}$$

The drift therefore has the form

$$e^{-\xi}\Big[-\frac{\lambda}{\gamma} + \lambda r w - \frac{\lambda}{\gamma}\left(\xi + \log\frac{\gamma}{\lambda}\right) + \tfrac{1}{2}\operatorname{tr} A^TQA + X^TQBX + \lambda\,\theta^T(\alpha B+\beta)X + \rho + \lambda h^T X$$
$$- \tfrac{1}{2}(X^TQ + \lambda\theta^T\alpha)AA^T(QX + \lambda\alpha^T\theta)\Big]$$

and to eliminate dependence on w, we find we must have

$$\boxed{\lambda = r\gamma}$$

and the thing to be maximised depends on θ as

$$-\tfrac{1}{2}(X^TQ + \lambda\theta^T\alpha)AA^T(QX + \lambda\alpha^T\theta) + \lambda\,\theta^T(\alpha B+\beta)X$$

and optimising this over θ will lead us to

$$\boxed{\theta = \Gamma X \equiv \lambda^{-1}(\alpha AA^T\alpha^T)^{-1}\left\{\alpha B + \beta - \alpha AA^TQ\right\}X.}$$

Returning this to the quantity to be optimised gives us

$$-r - r\left(\tfrac{1}{2}X^TQX + \varphi - \log r\right) + \tfrac{1}{2}\operatorname{tr} A^TQA + X^TQBX + \lambda X^T\Gamma^T(\alpha B+\beta)X + \rho + \lambda h^TX$$
$$-\tfrac{1}{2}X^T(Q + \lambda\Gamma^T\alpha)AA^T(Q + \lambda\alpha^T\Gamma)X$$

$$\equiv \tfrac{1}{2}X^T\Big\{-rQ - (Q+\lambda\Gamma^T\alpha)AA^T(Q+\lambda\alpha^T\Gamma) + QB + B^TQ + \lambda\Gamma^T(\alpha B+\beta) + \lambda(B^T\alpha^T + \beta^T)\Gamma^T$$
$$+\lambda(he_N^T + e_N h^T)\Big\}X$$
$$- r - r\varphi + r\log r + \tfrac{1}{2}\operatorname{tr} A^TQA + \rho$$

We must choose Q to make the quadratic terms vanish, then finally select φ to make the constant terms vanish. The equation for Q will be

$$\boxed{\begin{gathered} 0 = -rQ + QB + B^TQ + (Z-Y)^TM^{-1}(Z-Y) - QAA^TQ + \lambda(he_N^T + e_N h^T) \\ Z \equiv \alpha B + \beta, \quad Y \equiv \alpha AA^TQ, \quad M \equiv \alpha AA^T\alpha^T \end{gathered}}$$

<u>Notice that</u> Q <u>is independent of the preferences</u> (ρ, γ) <u>of the particular agent</u>!

Linear-Gaussian processes: the filtering story (7/11/02)

1) Suppose there is some underlying linear-Gaussian process

$$dx = a\,dW + bx\,dt$$

where we assume wlog that aa^T is non-singular (otherwise certain components of x are lin. combinations of others, and can be dropped from the specification).

An agent observes a signal Y, where

$$dY = \eta\,dW + Mx\,dt:$$

How does he filter the underlying process x from the observations?

2) Let's suppose wlog that $\eta\eta^T$ is non-singular. How? If it were not, then there is some matrix h such that $h \neq 0$, $h\eta = 0$, and so the observer sees $hMx\,dt$ with no noise, that is he sees hMx. If we multiply Y by some non-singular square matrix so as to make η look like $\left(\frac{\eta_0}{0}\right)$, where η_0 has full row rank, we could simply replace the final rows of Y by the linear functionals of X which are known from observation of Y, and this way we obtain an observation process giving exactly the same information, but with $\eta\eta^T$ non-singular.

3) The innovations martingale is ν,

$$d\nu = dY - M\hat{x}\,dt = \eta\,dW + M(x-\hat{x})\,dt$$

we shall have

$$d\hat{x} = H\,d\nu + b\hat{x}\,dt$$

for some non-random but possibly time-dependent matrix H which we must identify. To do this, try the old trick:

$$d(\hat{x}Y^T) = d\hat{x}\,.\,Y^T + \hat{x}\,dY^T + H\,dY\,dY^T$$

$$= \{ b\hat{x}Y^T + \hat{x}\hat{x}^T M^T + H\eta\eta^T \}\,dt + d\,(\mathcal{Y}\text{-mart})$$

On the other hand,

$$d(xY^T) = dx\,.\,Y^T + x\,dY^T + a\,dW\,dY^T$$

$$= \{ bxY^T + xx^T M^T + a\eta^T \}\,dt + d\,(\mathcal{F}\text{-mart})$$

Projecting onto the $\mathcal{Y}$-filtration, we get

$$b\hat{x}Y^T + \hat{x}\hat{x}^T M^T + H\eta\eta^T = b\hat{x}Y^T + (xx^T)^\wedge M^T + a\eta^T$$

$$\dot{V} = a\left(I - \eta^T(\eta\eta^T)^{-1}\eta\right)a^T + \left(b - a\eta^T(\eta\eta^T)^{-1}M\right)V + V\left(b - a\eta^T(\eta\eta^T)^{-1}M\right)^T - V M^T(\eta\eta^T)^{-1} M V$$

which rearranges to

$$\boxed{\begin{aligned} H\eta\eta^T &= \left((xx^T)^\wedge - \hat{x}\hat{x}^T \right) M^T + a\eta^T \\ &\equiv V_t M^T + a\eta^T \end{aligned}}$$

where V_t is the conditional co-variance of x_t given $\mathcal{Y}_t$, a deterministic matrix in this linear Gaussian situation.

4) How to characterise the covariance V_t ?

We have by substituting that

$$d(x-\hat{x}) = (a - H\eta)\,dW + (b - HM)(x-\hat{x})\,dt$$

and if $z \equiv x - \hat{x}$ we get

$$d\,zz^T = \left\{ (b-HM)zz^T + zz^T(b-HM)^T + (a-H\eta)(a-H\eta)^T dt \right\} + d(\text{mg})$$

Taking expectations throughout gives us

$$\boxed{\frac{d}{dt}V_t = (b-HM)V_t + V_t(b-HM)^T + (a-H\eta)(a-H\eta)^T}$$

More simply, we get

$$\boxed{\dot{V}_t = bV_t + V_t b^T + aa^T - H(\eta\eta^T)\,H^T} \qquad H \equiv (V_t M^T + a\eta^T)(\eta\eta^T)^{-1}$$

5) Maybe of use later:

If $N = I + \alpha\, xx^T + \gamma(xy^T + yx^T) + \beta\, yy^T$, then $N^{-1} = I + a xx^T + c(xy^T + yx^T) + b yy^T$

where

$$c = \frac{(\alpha\beta - \gamma^2)x\cdot y - \gamma}{\xi_1\xi_2 - \eta_1\eta_2}$$

$$a = -(\alpha + \eta_1 c)/\xi_1, \quad b = -(\beta + \eta_2 c)/\xi_2$$

$$\left(\begin{aligned} \xi_1 &= 1 + \gamma\, x\cdot y + \alpha|x|^2 \\ \xi_2 &= 1 + \gamma\, x\cdot y + \beta|y|^2 \\ \eta_1 &= \alpha\, x\cdot y + \gamma|y|^2 \\ \eta_2 &= \beta\, x\cdot y + \gamma|x|^2 \end{aligned} \right)$$

When $\gamma = 0$, $\psi(\eta, 0, \beta, v) = v \left\{ \frac{e^{-(\eta-\beta)^2/2v^2}}{\sqrt{2\pi}} - \frac{\eta-\beta}{v} \bar{\Phi}\left(\frac{\eta-\beta}{v}\right) \right\}$

Generally, $E\left(\xi - e^{\bar{X}_{t,\tau}}\right)^+ = I_{\{\xi>0\}} \left[(\xi-1)^+ + \psi\left(-\log(\xi \wedge 1), -1, \mu\tau, \sigma\sqrt{\tau}\right) + \psi\left(-\log(\xi\wedge 1), \frac{2\mu}{\sigma^2} - 1, -\mu\tau, \sigma\sqrt{\tau}\right) \right]$

MC hedging of American options: use of lookbacks? (18/11/02)

Suppose we have reached time $t \in (0,T)$ and have observed $\overline{Z_t - M_t} \equiv \sup_{s \le t}(Z_s - M_s) = \bar{a}$, where

$$Z_t = (e^{-rt}K - S_0 \exp(\sigma W_t - (\tfrac{1}{2}\sigma^2+\delta)t))^+ = e^{-rt}(K-S_t)^+$$

is the payoff of a single American put. If the hedging martingale were now turned off, the thing we'd be concerned to control would be

$$\sup_{u \in [t,T]} (Z_u - M_t - \bar{a})^+ = \sup_{u\in[t,T]} (Z_u - a)^+, \quad \text{say}, \quad a \equiv \bar{a} + M_t \ge Z_t.$$

$$= \sup_{u\in[t,T]} \left(\left(Ke^{-ru} - \tilde{S}_t e^{\sigma(W_u - W_t) - \mu(u-t)}\right)^+ - a\right)^+ \qquad \tilde{S}_t \equiv e^{-rt} S_t$$

$$= \sup_{u\in[t,T]} \left(Ke^{-ru} - \tilde{S}_t e^{\sigma(W_u-W_t) - \mu(u-t)} - a\right)^+$$

Hedging this will not be possible in closed form, but we could try to hedge instead

$$\Gamma(S_t) = \sup_{u\in[t,T]} \left(Ke^{-rT} - a - \tilde{S}_t e^{\sigma(W_u - W_t) - \mu(u-t)}\right)^+ \qquad \tilde{S}_t \equiv e^{-rt} S_t$$

$$= \left(Ke^{-rT} - a - \tilde{S}_t e^{-\bar{X}_{t,T}}\right)^+$$

where we set $\bar{X}_{t,T} = \sup_{u\in[t,T]} \{ \sigma(W_t - W_u) + \mu(u-t) \}$ $\quad (\mu \equiv \delta + \tfrac{1}{2}\sigma^2)$

This requires us to compute the expectation $\left(\xi \equiv (Ke^{-rT} - a)/\tilde{S}_t \right)$

$$E\left(\xi - e^{-\bar{X}_{t,T}}\right)^+$$

$$= \int_{(-\log\xi)^+}^{\infty} (\xi - e^{-x})\, P(\bar{X}_{t,T} \in dx) = \int_{(-\log \xi)^+}^{\infty} e^{-y} P(\bar{X}_{t,T} > y)\, dy \; + (\xi - 1)^+$$

Introducing

$$\psi(\eta, \gamma, \beta, v) \equiv \int_\eta^\infty e^{\gamma x} P\left(Z > \frac{x-\beta}{v}\right) dx = \int_\eta^\infty e^{\gamma x}\, \bar{\Phi}\left(\frac{x-\beta}{v}\right) dx$$

$$= \gamma^{-1}\left[e^{\gamma\beta + \gamma^2 v^2/2}\, \bar{\Phi}\left(\frac{\eta-\beta}{v} - \gamma v\right) - e^{\gamma\eta}\, \bar{\Phi}\left(\frac{\eta-\beta}{v}\right)\right]$$

and using expressions on p 15, ASSUMING $\xi \le 1$, and $\xi > 0$

$$\boxed{E\left(\xi - e^{-\bar{X}_{t,T}}\right)^+ = \psi\left((-\log\xi)^+, -1, \mu\tau, \sigma\sqrt{\tau}\right) + \psi\left((-\log\xi)^+, \frac{2\mu}{\sigma^2} - 1, -\mu\tau, \sigma\sqrt{\tau}\right)} \qquad (\tau \equiv T-t)$$

Differentiating Γ with respect to S_t gives us

$$e^{-rt}\,\mathbb{E}\left(\xi - e^{-\bar{X}_{t,T}}\right)^{+} - \frac{Ke^{-rT}-a}{S_t}\,P\left(\bar{X}_{t,T} > -\log\xi\right)$$

$$= \left\{\psi\left(-\log\xi, -1, \mu\tau, \sigma\sqrt{\tau}\right) + \psi\left(-\log\xi, \tfrac{2\mu}{\sigma^2}-1, -\mu\tau, \sigma\sqrt{\tau}\right)\right\} e^{-rt}$$

$$- \frac{Ke^{-rT}-a}{S_t}\left\{\bar{\Phi}\left(-\frac{\log\xi+\mu\tau}{\sigma\sqrt{\tau}}\right) + \xi^{-2\mu/\sigma^2}\,\bar{\Phi}\left(\frac{-\log\xi+\mu\tau}{\sigma\sqrt{\tau}}\right)\right\}.$$

Another discretisation of drifting BM (27/11/02)

1) Suppose we consider $X_t \equiv \sigma W_t + \mu t$ running in $[-\delta, \delta]$, and we let $\tau_\pm \equiv \inf\{t : X_t = \pm\delta\}$, with $\tau_0 \equiv \inf\{t : L_t > T_\lambda\}$, where $\lambda > 0$, $T_\lambda \sim \exp(\lambda)$. We now stop the process at the time

$$\tau = \tau_+ \wedge \tau_0 \wedge \tau_-$$

and thereby construct a trinomial embedding. What are the exit probabilities for this? Suppose there is killing at a constant rate r; what are the answers then?

2) The ODE

$$\tfrac{1}{2}\sigma^2 f'' + \mu f' - r = 0 \qquad \text{has roots} \qquad \left(-\mu \pm \sqrt{\mu^2 + 2r\sigma^2}\right)/\sigma^2 = \alpha, \beta,$$

where we will take $\alpha > 0 > \beta$.

Rate of upward unkilled excursions getting to δ

$$\equiv \lambda_+ = \lim \frac{1}{2\varepsilon} \frac{e^{\alpha\varepsilon} - e^{\beta\varepsilon}}{e^{\alpha\delta} - e^{\beta\delta}} = \frac{1}{2} \frac{\alpha - \beta}{e^{\alpha\delta} - e^{\beta\delta}}$$

Rate of downward unkilled excursions getting to $-\delta$ $\equiv \lambda_- = \lim \frac{1}{2\varepsilon} \frac{e^{-\alpha\varepsilon} - e^{-\beta\varepsilon}}{e^{-\alpha\delta} - e^{-\beta\delta}}$

$$= \frac{1}{2} \frac{\beta - \alpha}{e^{-\alpha\delta} - e^{-\beta\delta}}$$

Rate of upward excns killed before getting to $\{0, \delta\}$

$$\equiv \lambda_{+0} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \left\{ 1 - E^\varepsilon e^{-r(H_0 \wedge H_\delta)} \right\}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \left\{ 1 - \frac{(1 - e^{\beta\delta}) e^{\alpha\varepsilon} - (1 - e^{\alpha\delta}) e^{\beta\varepsilon}}{e^{\alpha\delta} - e^{\beta\delta}} \right\}$$

$$= \frac{1}{2} \frac{-(1 - e^{\beta\delta})\alpha + (1 - e^{\alpha\delta})\beta}{e^{\alpha\delta} - e^{\beta\delta}}$$

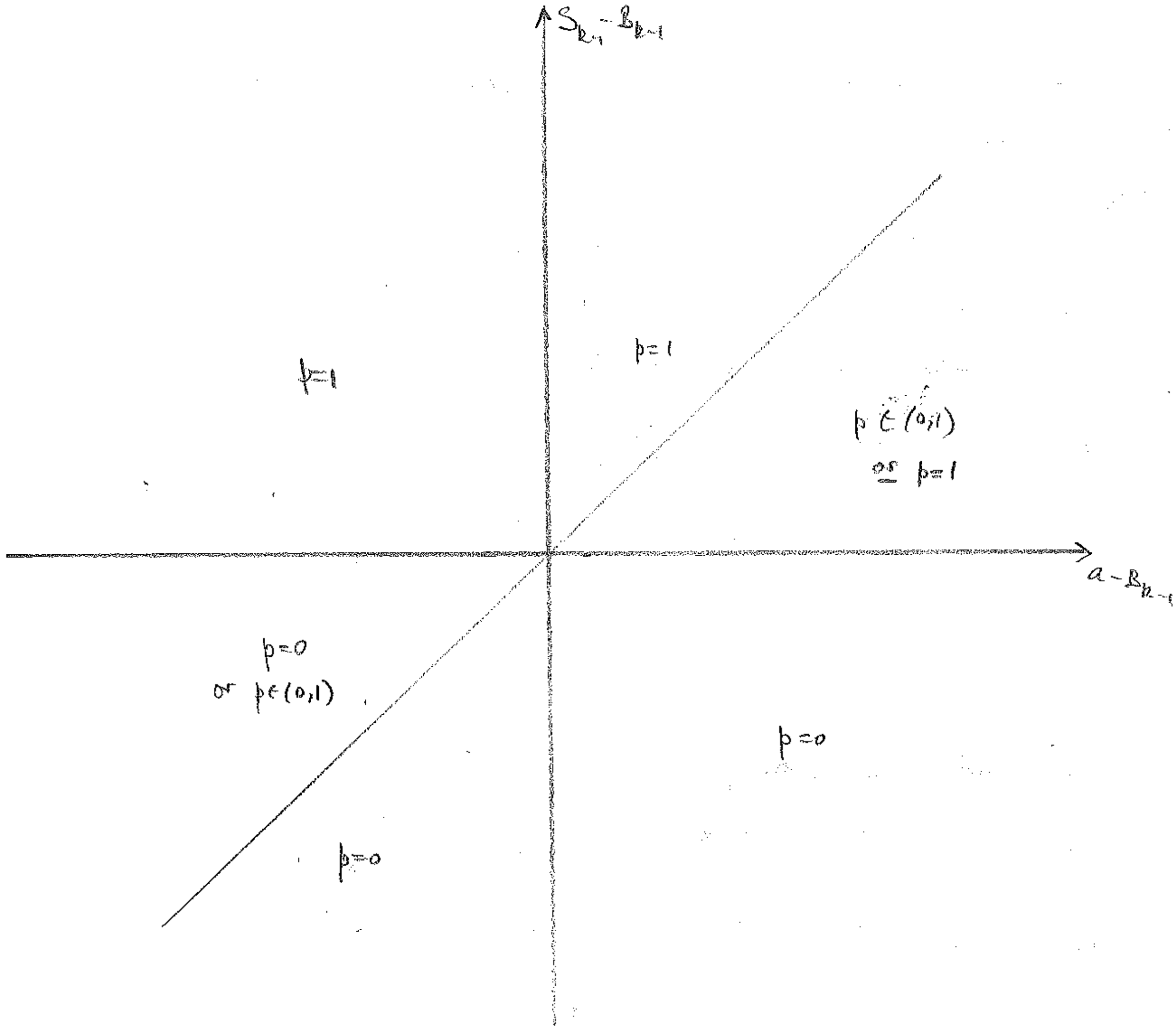
Rate of downward excns killed before getting to $\{0, -\delta\}$

$$\equiv \lambda_{-0} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \left\{ 1 - \frac{(1 - e^{-\beta\delta}) e^{-\alpha\varepsilon} - (1 - e^{-\alpha\delta}) e^{-\beta\varepsilon}}{e^{-\alpha\delta} - e^{-\beta\delta}} \right\}$$

$$= \frac{1}{2} \frac{\alpha(1 - e^{-\beta\delta}) - \beta(1 - e^{-\alpha\delta})}{e^{-\alpha\delta} - e^{-\beta\delta}}$$

If $\frac{1}{2}\theta$ were the intensity of killing at 0 in this scale, then

$$E\left[e^{-r\tau} : X_\tau = \delta\right] = \lambda_+ / (\lambda_+ + \lambda_- + \theta + \lambda_{+0} + \lambda_{-0})$$



$S_{k-1} - B_{k-1}$
$a - B_{k-1}$
$p=1$
$p=1$
$p \in (0,1)$
or $p=1$
$p=0$
or $p\in(0,1)$
$p=0$
$p=0$

Randomised behaviour of bondholders (9/12/02)

Suppose we have the discretised convertible bond question, and that we are at the stage where there are $k\Delta m$ bonds still live. Each bond holder offers his bond for conversion with prob^y p, independently of all the rest. If none get offered, then each is worth

$$a \equiv \frac{\rho}{r} E^v(1-e^{-\rho\tau_1}) + E^v\left[e^{-\rho\tau_1} B(m, V(\tau_1))\right]$$

If we now consider a single bondholder, we can discover the equilibrium value of $p \equiv 1-q$. If the bondholder certainly offers bond for conversion, the value of his bond is

$$\sum_{r=0}^{k-1} \binom{k-1}{r} p^r q^{k-1-r} \left\{ \frac{1}{r+1} S_{k-1} + \frac{r}{r+1} B_{k-1} \right\}$$

$$= B_{k-1} + (S_{k-1} - B_{k-1}) \sum_{r=0}^{k-1} \binom{k-1}{r} \frac{p^r q^{k-1-r}}{r+1}$$

$$= B_{k-1} + \frac{(S_{k-1} - B_{k-1})}{kp}(1-q^k) \qquad = \begin{cases} S_{k-1} & \text{if } p=0 \\ B_{k-1} + \frac{1}{k}(S_{k-1}-B_{k-1}) & \text{if } p=1 \end{cases}$$

If the bondholder doesn't offer bond for conversion, it's worth

$$q^{k-1} a + (1-q^{k-1}) B_{k-1} \qquad = \begin{cases} a & \text{if } p=0 \\ B_{k-1} & \text{if } p=1 \end{cases}$$

So if some $q \in (0,1)$ is an equilibrium value, we must have

$$\boxed{\frac{a - B_{k-1}}{S_{k-1} - B_{k-1}} = \frac{1}{k}\sum_{i=0}^{k-1} q^{-i} \geq 1} \qquad (*)$$

<u>So</u>: If $a - B_{k-1} > 0 > S_{k-1} - B_{k-1}$ an equilibrium is $p=0$

$a - B_{k-1} < 0 < S_{k-1} - B_{k-1}$ an equilibrium is $p=1$

If $a - B_{k-1}$, $S_{k-1} - B_{k-1}$ both >0, an equilibrium is $p=1$ if $a-B_{k-1} < S_{k-1} - B_{k-1}$

an equilibrium from $(*)$ if $a - B_{k-1} \geq S_{k-1} - B_{k-1}$

If $a - B_{k-1}$, $S_{k-1} - B_{k-1}$ both <0, an equilibrium is $p=0$ if $|a - B_{k-1}| < |S_{k-1} - B_{k-1}|$

from $(*)$ otherwise

However, if $S_{k-1} \geq B_{k-1}$, then $p=1$ is <u>also</u> an equilibrium

if $a \geq S_{k-1}$, then $p=0$ is <u>also</u> an equilibrium

$$\alpha\beta = 2r/\sigma^2$$

$$(\alpha+1)(\beta-1) = 2\delta/\sigma^2$$

Let $\psi_i(x) = \varphi_i(\log x)$, so that

$$\left.\begin{aligned} \psi_0'(x) &= \frac{2r}{\sigma^2}\left(x^{\beta} - x^{-\alpha}\right) x^{-1} \\ \psi_1'(x) &= \frac{2\delta}{\sigma^2}\left(x^{\beta} - x^{-\alpha}\right) x^{-2} \end{aligned}\right\} \Rightarrow \frac{\psi_0'(x)}{\psi_1'(x)} = \frac{r}{\delta}\, x$$

Convertible bonds again (30/1/03)

It appears that if we define functions

$$\varphi_0(t) \equiv \int_0^t \alpha\beta\,(e^{\beta u} - e^{-\alpha u})\,du = \alpha e^{\beta t} + \beta e^{-\alpha t} - \alpha - \beta$$

$$\varphi_1(t) = \int_0^t (\alpha+1)(\beta-1)\,(e^{\beta u} - e^{-\alpha u})\,e^{-u}\,du = (\alpha+1)\,e^{(\beta-1)t} + (\beta-1)\,e^{-(\alpha+1)t} - \alpha - \beta$$

then these may play a rôle in the study. Notice firstly that both are convex and non-negative, zero at zero. In the no-calling case, we shall have

(1) $$\boxed{S(m,V) = \frac{m\rho'\,\varphi_0(\log V/\xi) \quad - rV\varphi_1(\log V/\xi)}{r(\alpha+\beta)(n-m)}}$$

and

(2) $$\boxed{B(m,V) - S(m,V) = \frac{rV\varphi_1(\log V/\eta) - (m\rho' + (n-m)\rho)\,\varphi_0(\log V/\eta)}{r(n-m)(\alpha+\beta)}}$$

and

(3) $$\boxed{\xi(m) = \frac{(n - m\tau)\,\varphi_0(\log\theta)\,\rho/r}{\varphi_1(\log\theta) - (\alpha+\beta)\,\rho\,(n-m)/m}}$$ $\theta \equiv \theta(m) \equiv \xi(m)/\eta(m)$

Observe that for $t>0$, $\varphi_1(-t)/\varphi_0(-t)$ increases, $\varphi_0(-t)$ increases, so

$$t \mapsto \frac{(n-m\tau)\;\varphi_0(-t)\,\rho/r}{\varphi_1(-t) - (\alpha+\beta)\,\rho\,(n-m)/m}$$ decreases (while >0)

Differentiating the expression for S and using $\frac{\partial S}{\partial m} = 0$ at $V=\eta$, we derive

(4) $$\boxed{\frac{\partial\xi}{\partial m} = \frac{\theta\xi\,\{n\rho'\,\psi_0(1/\theta) - r\eta\,\psi_1(1/\theta)\}}{\psi_0'(1/\theta)\,(n-m)\,(m\rho' - \delta\xi)}}$$

and differentiating the expression for ξ in terms of θ gives

(5) $$\boxed{\frac{d}{dm}\log\xi = -\frac{\tau}{n-m\tau} - \frac{(\alpha+\beta)\,p\,n/m}{m\psi_1(\theta) - p(\alpha+\beta)(n-m)} + \frac{d\theta}{dm}\left[\frac{\psi_0'(\theta)}{\psi_0(\theta)} - \frac{\psi_1'(\theta)}{\psi_1(\theta) - p(\alpha+\beta)(n-m)/m}\right]}$$

(6) $$= \frac{n\rho'\,\theta\,\psi_0(1/\theta) - r\xi\,\psi_1(1/\theta)}{(n-m)\,\psi_0'(1/\theta)\,(m\rho' - \delta\xi)}$$ from (4)

(PTO)

$\gamma = \alpha + \beta$

$$
\begin{aligned}
&\text{pp}[\text{m\_},\ \theta\_] = \Big(\theta\ (\beta - \gamma\,\theta^{\alpha} + \alpha\,\theta^{\alpha+\beta})\ \Big(n\ (\alpha - \gamma\,\theta^{\beta} + \beta\,\theta^{\alpha+\beta})\\
&\quad ((m-n)\ p\,\gamma\,\theta^{1+\alpha} + m\ (-1 + \beta - \gamma\,\theta^{1+\alpha} + (1+\alpha)\ \theta^{\alpha+\beta}))^{2}\\
&\quad (-1 + \tau)\ (n - m\,\tau) +\\
&\quad (-m + n)\ (1 + \alpha)\ (-1 + \beta)\ \theta\ (-1 + \theta^{\alpha+\beta})\ (\beta - \gamma\,\theta^{\alpha} + \alpha\,\theta^{\alpha+\beta}\\
&\quad (-n + m\,\tau)\ (n^{2}\ p\,\gamma\,\theta^{1+\alpha} - 2\,m\,n\,p\,\gamma\,\theta^{1+\alpha}\ \tau +\\
&\quad\ m^{2}\ (-1 + \beta - \gamma\,\theta^{1+\alpha} + p\,\gamma\,\theta^{1+\alpha} + \theta^{\alpha+\beta} + \alpha\,\theta^{\alpha+\beta})\ \tau) +\\
&\quad ((m-n)\ p\,\gamma\,\theta^{1+\alpha} + m\ (-1 + \beta - \gamma\,\theta^{1+\alpha} + (1+\alpha)\ \theta^{\alpha+\beta}))\\
&\quad (m\ (\beta - \gamma\,\theta^{\alpha} + \alpha\,\theta^{\alpha+\beta})\ ((1+\alpha)\ \theta - \gamma\,\theta^{\beta} + (-1 + \beta)\ \theta^{1+\alpha+\beta}\\
&\quad\ (n - m\,\tau)^{2} + (m - n)\ \alpha\,\beta\ (-1 + \theta^{\alpha+\beta})\\
&\quad\ (-1 + \tau)\ (n^{2}\ p\,\gamma\,\theta^{1+\alpha} - 2\,m\,n\,p\,\gamma\,\theta^{1+\alpha}\ \tau +\\
&\quad\ m^{2}\ (-1 + \beta - \gamma\,\theta^{1+\alpha} + p\,\gamma\,\theta^{1+\alpha} + \theta^{\alpha+\beta} + \alpha\,\theta^{\alpha+\beta})\ \tau)))\Big)\Big);
\end{aligned}
$$

$$
\begin{aligned}
&\text{qq}[\text{m\_},\ \theta\_] = \Big(m\ (-m + n)\ (-1 + \theta^{\alpha+\beta})^{2}\ (-n\,p\,\alpha\,\beta\,\gamma\,\theta^{1+\alpha} +\\
&\quad m\ (-\beta^{2} + \beta\ (1 + \gamma\,\theta^{\alpha} + \alpha\,\gamma\,\theta^{\alpha} - \alpha\,\gamma\,\theta^{1+\alpha} + p\,\alpha\,\gamma\,\theta^{1+\alpha}) +\\
&\quad\ (1 + \alpha)\ \theta^{\alpha}\ (-\gamma + \alpha\,\theta^{\beta})))\\
&\quad (n - m\,\tau)\ (-\alpha\,\beta\ ((m-n)\ p\,\gamma\,\theta^{1+\alpha} +\\
&\quad\ m\ (-1 + \beta - \gamma\,\theta^{1+\alpha} + (1+\alpha)\ \theta^{\alpha+\beta}))\ (-1 + \tau) -\\
&\quad (1 + \alpha)\ (-1 + \beta)\ \theta\ (\beta - \gamma\,\theta^{\alpha} + \alpha\,\theta^{\alpha+\beta})\ (n - m\,\tau))\Big);
\end{aligned}
$$

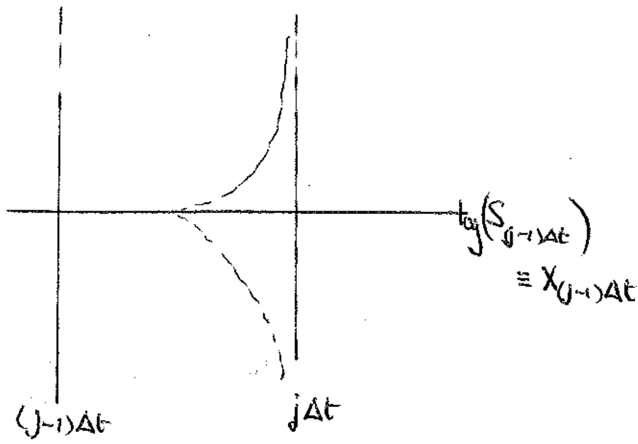
Solve $\dfrac{d\theta}{dt} = pp(m(t), \theta(t))$

$\dfrac{dm}{dt} = qq(m(t), \theta(t))$

and this will describe the solution

An approach to liquidity modelling via order book analysis (19/2/03)

(i) At times $j\Delta t$, IID random demands ξ_j for a single asset arrive at a market. We shall assume that there is a density $\varphi'(\cdot)$ of limit orders, working in log price, centred at the current spot $X_{(j-1)\Delta t} \equiv \log S_{(j-1)\Delta t}$. (This is a very rough assumption).



In addition, there are certain hedging demands h_j arriving at the market. The jump

$\Delta X = X_{j\Delta t} - X_{(j-1)\Delta t}$

in the log-price at time $j\Delta t$ must therefore satisfy

$$\int_0^{\Delta X} \varphi'(y)\,dy \equiv \varphi(\Delta X) = \xi_j + h_j$$

Let's suppose that the hedger wants to have a portfolio $H(t, X_t)$ at the times $t = j\Delta t$. This therefore means we have the condition

$$\boxed{\varphi(\Delta X) = \xi_j + H(j\Delta t, X_{(j-1)\Delta t} + \Delta X) - H((j-1)\Delta t, X_{(j-1)\Delta t})} \qquad (*)$$

so that the jump ΔX is a function of t, ξ_j and $X_{j\Delta t - \Delta t}$. The average price paid for a share when this jump occurred is

$$\frac{\int_0^{\Delta X} \varphi'(y)\, e^y\, dy}{\varphi(\Delta X)} \cdot S_{j\Delta t - \Delta t}$$

So we deduce that the hedger's bank account increases by an amount

$$-\frac{h_j}{\xi_j + h_j} \int_0^{\Delta X} \varphi'(y)\, e^y\, dy \; S_{(j-1)\Delta t} = -h_j \frac{\int_0^{\Delta X} e^y \varphi'(y)\,dy}{\varphi(\Delta X)} S_{(j-1)\Delta t}$$

Assuming φ, H are smooth enough, and $\xi_j \sim N(\tilde{\mu}\Delta t, \tilde{\sigma}^2 \Delta t)$, we can expand $(*)$ to obtain

$$\varphi'(0)\Delta X + \tfrac{1}{2}\varphi''(0)\Delta X^2 = (\tilde{\mu}\Delta t + \tilde{\sigma}\Delta W) + \dot{H}(t, X_t)dt + H'(t, X_{t-})\Delta X_t + \tfrac{1}{2} H''(t, X_{t-})\Delta X_t^2$$

so that

$$\{\varphi'(0) - H'(t, X_{t-})\}\Delta X_t = (\tilde{\mu}\Delta t + \tilde{\sigma} dW) + \dot{H}(t, X_t)dt + \tfrac{1}{2}\{H''(t, X_{t-}) - \varphi''(0)\}\Delta X^2$$

If we simply supposed that H was a continuous semimartingale, we'd get

$$d(S_t H_t + C_t) = H_t \partial S_t = H_t dS_t + \frac{1}{2} dH_t dS_t$$

so that passing to the limit would give (formally)

$$\boxed{dX = \frac{\tilde\sigma}{\varphi'(0) - H'(t,X)}\, dW + \frac{\frac{1}{2}\tilde\sigma^2}{(\varphi'(0) - H'(t,X))^3}\left\{ H''(t,X) - \varphi''(0) \right\} dt + \frac{\tilde\mu + \dot H(t,X)}{\varphi'(0) - H'(t,X)}}$$

If C_t denotes the cash balances at time t ($r=0$, let's assume for simplicity) we get

$$\boxed{\begin{aligned} dC_t &= -\left(\dot H(t,X_t)\,dt + H'(t,X_t)\,dX_t + \tfrac{1}{2} H''(t,X_t)\,d\langle X\rangle_t \right)\left(1 + \tfrac{1}{2}\,dX_t\right) S_t \\ &= -\left\{ H'(t,X_t)\,dX_t + \dot H(t,X_t)\,dt + \tfrac{1}{2}\left(H''(t,X_t) + H'(t,X_t)\right) d\langle X\rangle_t \right\} S_t \end{aligned}}$$

(note that to relevant order $\int_0^x e^y \varphi'(y)\,dy \Big/ \int_0^x \varphi'(y)\,dy = 1 + \frac{1}{2}x$!!)

(ii) Here's something which might be useful; certainly it's a bit surprising!

$$\boxed{d\left(S_t H(t,X_t) + C_t \right) = H_t\, \partial S_t \equiv H_t\, dS_t + \tfrac{1}{2} S_t H_t'\, d\langle X\rangle_t}$$

Yes - Stratonovich integral!! Another possibly useful observation is that

$$\boxed{d\left(\varphi'(0) X_t - H(t,X_t) \right) = \tilde\sigma\, dW_t + \left\{ \tilde\mu - \tfrac{1}{2}\left(\frac{\tilde\sigma}{\varphi'(0) - H'(t,X_t)} \right)^2 \varphi''(0) \right\} dt}$$

When $\varphi''(0) = 0$, this is particularly simple:

$$\varphi'(0) X_t - H(t,X_t) = \tilde\sigma W_t + \tilde\mu t + \text{const}.$$

Effect of limited liability on a simple exchange economy (20/2/03)

(i) Let's consider a very simple single-period model, as in my notes on the SFM course; there are N risky assets, and at time 1 they give random rewards $S_1 \sim N(\mu, V)$, $S_1 = (S_1^1, \dots, S_1^N)^T$. The time-0 prices are denoted S_0, to be determined by equilibrium. Adjoin a riskless asset, asset 0, worth 1 at both times, to S, to make $\bar{S}_t \equiv (S_t^0, S_t^1, \dots, S_t^N)^T$, and denote means and variances by $\bar{\mu}$, $\bar{V}$.

Suppose there are J agents, all with CARA utility, γ_j is the coeff of ARA of agent j, and that agent j enters the market with endowment $\bar{\alpha}_j = [0; \alpha_j]$ (initial cash balance makes no difference to the optimisation problem.) Suppose the prices $\bar{S}_0$ are announced; agent j's goal is now to

$$\min_{\bar{\theta}} E \exp(-\gamma_j \bar{\theta}\cdot\bar{S}_1) \qquad \text{s.t. } \bar{\theta}\cdot\bar{S}_0 = \bar{\alpha}_j\cdot\bar{S}_0$$

which in Lagrangian form becomes

$$\min \quad \tfrac{1}{2}\gamma_j^2 \, \bar{\theta}\cdot\bar{V}\bar{\theta} - \gamma_j \bar{\theta}(\bar{\mu} - \lambda\bar{S}_0)$$

solved when

$$\gamma_j \bar{V}\bar{\theta} = \bar{\mu} - \lambda\bar{S}_0$$

Looking at the 0^{th} component tells us that $\lambda = 1$ to get both sides equal 0, and then we deduce

$$\theta_j^* = \gamma_j^{-1}\theta_M \equiv \gamma_j^{-1} V^{-1}(\mu - S_0)$$

When do markets clear? We need

$$A \equiv \sum_j \alpha_j = \sum_j \theta_j^* = \Gamma^{-1} V^{-1}(\mu - S_0) \qquad (\Gamma^{-1} = \sum \gamma_j^{-1})$$

so $\boxed{S_0 = \mu - \Gamma V A.}$ $\qquad \theta_M = \Gamma A.$

The negative of the maximised utility of the agent is

$$\boxed{\begin{aligned} &\exp\left[-\gamma_j \alpha_j\cdot S_0 - \tfrac{1}{2}\Gamma^2 A\cdot VA\right] \\ &= \exp\left[-\gamma_j \alpha_j\cdot(\mu - \Gamma VA) - \tfrac{1}{2}\Gamma^2 A\cdot VA\right]\end{aligned}}$$

(ii) In a limited-liability market, prices S_0 cannot be an equilibrium if $S_0^i < 0$ for some i, because if this were so, anyone holding asset i physically would simply discard it, optimise with a more generous budget constraint, and do better. So $S_0^i < 0$ is impossible in an equilibrium with a positive net supply of asset i.

For an asset with zero equilibrium price, anyone can hold any amount, so what we shall have is

$$\theta_j^* = \gamma_j^{-1}\,\theta_M = \gamma_j^{-1}\,V^{-1}(\mu - S_0)$$

and

$$\sum \theta_j^* = \Gamma^{-1}V^{-1}(\mu - S_0) = A - \gamma$$

where $0 \leq \gamma \leq A$. Thus

$$\boxed{S_0 = \mu - \Gamma V(A - \gamma) = \hat{S}_0 + \Gamma V \gamma}$$

where $\hat{S}_0$ is the equilibrium prices without limited liability. We seek some γ such that $\gamma_i > 0 \Rightarrow S_0^i = 0$. However, if $\gamma_i = A_i$, there is none of asset i left in the economy, and it now doesn't matter what S_0^i is.

This is a linear complementarity problem, or can be expressed as a quadratic program: if

$$x \equiv \begin{pmatrix} S_0 \\ \gamma \end{pmatrix}, \quad \text{we want} \quad x \geq 0, \quad \gamma \leq A$$

and

$$S_0 - \Gamma V \gamma = (I \quad -\Gamma V)\,x = \mu - \Gamma V A$$

and as objective to minimize $\frac{1}{2}x^T Q x$, where $Q \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ is symmetric, invertible, evalues ± 1.

Effect of random endowments. (18/3/03)

If agent j has random endowment ξ_j, zero mean, $E(\xi S_1) = v_j$, and suppose that agent j believes $S_1 \sim N(\mu_j, V_j)$, then we shall find that at price p, agent j would optimally choose to invest in the portfolio

$$\theta_j = \gamma_j^{-1}V_j^{-1}\{\mu_j - (1+r)p - \gamma_j v_j\}$$

so market clearing gives

$$\boxed{p = \frac{1}{1+r}\Big(\sum_j \gamma_j^{-1}V_j^{-1}\Big)^{-1}\Big\{\sum_j \gamma_j^{-1}V_j^{-1}\mu_j - A - \sum V_j^{-1}v_j\Big\}}$$

If beliefs are homogeneous, this simplifies to

$$\boxed{p = \frac{1}{1+r}\Big\{\mu - \Gamma V A - \Gamma\Big(\sum v_j\Big)\Big\}}$$

which in effect alters μ a bit.

$$\psi_0(x) = \alpha x^{\beta} + \beta x^{-\alpha} - \alpha - \beta$$

$$\psi_1(x) = (\alpha+1) x^{\beta-1} + (\beta-1) x^{-\alpha-1} - \alpha - \beta$$

$$\frac{\psi_0(1-y)}{\alpha\beta(\alpha+\beta)} = \frac{1}{2}y^2 + \frac{\alpha-\beta+3}{6}y^3 + \left(\beta^2 - 6\beta - \alpha\beta + 11 + 6\alpha + \alpha^2\right)\frac{y^4}{4!} + \cdots$$

$$\frac{\psi_1(1-y)}{(\alpha+1)(\beta-1)(\alpha+\beta)} = \frac{1}{2}y^2 + \frac{\alpha-\beta+5}{6}y^3 + \left(\beta^2 - 9\beta - \alpha\beta + 9\alpha + 26 + \alpha^2\right)\frac{y^4}{4!} + \cdots$$

Near $m=0$, we get

$$\xi(m) = \frac{\alpha m p'}{r(1+\alpha)} + o(m^{\beta})$$

$$\eta(m) = \frac{\beta n p}{r(\beta-1)} - \frac{\beta m r_c p}{r(\beta-1)} - m^{\alpha}\,\frac{np}{r}\,\frac{(\alpha+\beta)}{\beta-1}\left\{\cancel{1-\frac{p(1-r_c)\alpha}{1+\alpha}}\right\}\left(\frac{\alpha(\beta-1)(1-r_c)}{\beta n(1+\alpha)}\right)^{\alpha} + o(m^{\alpha})$$

198

the second by looking at $Y(m, \xi(m))$ as $m \to 0$

Some asymptotics for the convertible bond story (3/3/03)

Returning to the situation p35, we have

$$\xi(m) = \frac{(n - m\tau)\, n\psi_0(\theta)\, p/r}{\psi_1(\theta) - p(\alpha+\beta)(n-m)/m}$$

and we know that as $m \equiv n - z \uparrow n$ we shall have $\xi(m) \uparrow np'/\delta$, and $\theta(m) \equiv 1 - \varepsilon(m) \uparrow 1$. Rewriting ξ in terms of z, ε, we get

$$\xi = \frac{(n-z)\{(1-\tau)n + \tau z\}\, \psi_0(1-\varepsilon)\, p/r}{(n-z)\,\psi_1(1-\varepsilon) - p(\alpha+\beta) z}$$

$$= \frac{(n-z)\{(1-\tau)n + \tau z\}\, \frac{\psi_0(1-\varepsilon)}{\varepsilon^2} \cdot p/r}{(n-z)\,\frac{\psi_1(1-\varepsilon)}{\varepsilon^2} - p(\alpha+\beta)\, z/\varepsilon^2}$$

$$\to \frac{n^2(1-\tau)\,\frac{1}{2}\alpha\beta(\alpha+\beta)\, p/r}{n\cdot\frac{1}{2}(\alpha+1)(\beta-1)(\alpha+\beta) - p\lambda(\alpha+\beta)} \qquad \text{where} \quad \lambda = \lim z/\varepsilon^2$$

$$= \frac{n^2 p'\, \alpha\beta/r}{n(\alpha+1)(\beta-1) - 2p\lambda}$$

$$= \frac{n^2 p'}{n\delta - \sigma^2 p\lambda} = \frac{np'}{\delta}$$

So $\boxed{\lambda = \lim_{z \downarrow 0} z/\varepsilon^2 = 0}$ i.e. $\boxed{\lim_{m \uparrow n} \frac{n-m}{(1-\theta(m))^2} = 0}$

Question with Amand; discrete-time version (7/3/03)

1) The dynamics of the underlying process are

$$x_{t+1} = Ax_t + H\varepsilon_{t+1}$$

and the observation process is

$$y_{t+1} = Cx_{t+1} + J\varepsilon_{t+1}$$

where we'll assume that the ε_n are IID $N(0,I)$. What is the filtering story? If $(X_t \mid y_t) \sim N(\hat{x}_t, V_t)$ then

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} \Big| \mathcal{Y}_t \sim N\left(\begin{pmatrix} A\hat{x}_t \\ CA\hat{x}_t \end{pmatrix}, \begin{pmatrix} AV_tA^T + HH^T & AV_tA^TC^T + H(CH+J)^T \\ \cdots & CAV_tA^TC^T + (CH+J)(CH+J)^T \end{pmatrix} \right)$$

Therefore

$$\boxed{\hat{x}_{t+1} = A\hat{x}_t + \Gamma_t(y_{t+1} - CA\hat{x}_t)}$$

where $\Gamma_t \equiv \{AV_tA^TC^T + H(CH+J)^T\}\{CAV_tA^TC^T + (CH+J)(CH+J)^T\}^{-1}$, and the new conditional covariance is

$$\boxed{V_{t+1} = AV_tA^T + HH^T - \Gamma(AV_tA^TC^T + H(CH+J)^T)}$$

2) How about the optimal consumption/investment side of things? This time, let's suppose that the return takes the form

$$K\varepsilon_{t+1} + GX_t$$

for the risky assets, so that the wealth equation is

$$w_{t+1} = r(w_t - c_t) + \theta_t \cdot (GX_t + K\varepsilon_{t+1})$$

The objective is to obtain

$$V(w,x) = \min E\left[\sum_{n\geq 0} \exp(-\gamma c_n - \rho n) \mid x_0 = x, w_0 = w\right]$$

and we conjecture that the value function has the form

$$V(w,x) = \exp\left[-\lambda w - \tfrac{1}{2}x^TQx - \varphi\right]$$

If this were the case, the Bellman equation would give

$$V(w,x) = \exp\left[-\lambda w - \tfrac{1}{2}x^TQx - \varphi\right]$$

$$= \min_{c,\theta} \left[e^{-\gamma c} + e^{-\rho} E\left\{ \exp(-\lambda w_{t+1} - \tfrac{1}{2} x_{t+1}^T Q x_{t+1} - \varphi) \,\middle|\, w_t = w, x_t = x \right\} \right]$$

$$= \min_{c,\theta} \left[e^{-\gamma c} + e^{-\rho - \lambda r(w-c) - \varphi - \lambda \theta^T G x}\, E \exp\left\{ -\lambda \theta^T K \varepsilon_{t+1} - \tfrac{1}{2} (Ax + H\varepsilon_{t+1})^T Q (Ax + H\varepsilon_{t+1}) \right\} \right]$$

$$= \min_{c,\theta} \left[e^{-\gamma c} + \exp\left(-\rho - \varphi - \lambda r(w-c) - \lambda \theta^T G x - \tfrac{1}{2} x^T A^T Q A x\right) E \exp\left(-\tfrac{1}{2} \varepsilon^T H^T Q H \varepsilon - \varepsilon^T(\lambda K^T \theta + H^T Q A x) \right) \right]$$

$$= \min_{c,\theta} \left[e^{-\gamma c} + \exp\left(\lambda r c - \rho - \varphi - \lambda r w - \tfrac{1}{2} x^T A^T Q A x - \lambda x^T G^T \theta\right) . \frac{\exp\left\{ \tfrac{1}{2} (\lambda K^T \theta + H^T Q A x)^T \Sigma (\lambda K^T \theta + H^T Q A x) \right\}}{\det \Sigma^{1/2}} \right]$$

where $\Sigma^{-1} = I + H^T Q H$. The minimising θ is easily shown to be

$$\theta = \lambda^{-1} (K \Sigma K^T)^{-1} (G - K \Sigma H^T Q A) x \equiv Mx, \text{ say,}$$

which gives as a minimised value for the second term

$$\det \Sigma^{\frac{1}{2}} . \exp\Big[\lambda r c - \rho - \varphi - \lambda r w - \tfrac{1}{2} x^T A^T Q A x + \tfrac{1}{2} x^T A^T Q H \Sigma H^T Q A x$$
$$- \tfrac{1}{2} x^T M^T (\lambda G - \lambda K \Sigma H^T Q A) x \Big]$$

$$\equiv \det \Sigma^{\frac{1}{2}} \exp\left[\lambda r c - \rho - \varphi - \lambda r w - \tfrac{1}{2} x^T \tilde{Q} x \right]$$

$$\left[\tilde{Q} \equiv A^T Q A - A^T Q H \Sigma H^T Q A + M^T (K \Sigma K^T)^{-1} M \right]$$

$$\equiv \Lambda e^{\lambda r c}, \text{ say,}$$

with $\Lambda = \sqrt{\det \Sigma}\, \exp(-\rho - \varphi - \lambda r w - \tfrac{1}{2} x^T \tilde{Q} x)$. Minimising the overall expression over c is now easy: we pick

$$c = \frac{1}{\gamma + \lambda r} \log\left(\frac{\gamma}{\lambda r \Lambda} \right)$$

and get a value of

$$\left(1 + \frac{\gamma}{\lambda r} \right) . \left(\frac{\gamma}{\lambda r} \right)^{-\gamma/(\gamma + \lambda r)} \Lambda^{\gamma/(\gamma+\lambda r)}$$

Hence $\lambda = \gamma(r-1)/r$, and we get a Riccati-like equation for Q.

<u>Liquidity modelling again</u> 12/3/03

1) Let's return to the model on p 36–37, and write ψ for the inverse function of φ. Certain properties are required of the model, to do with what happens if the agent suddenly changes holdings by an $O(1)$ amount; if this is done, the random components get swamped, and we are in effect looking at a deterministic model, where increasing ones holding by x will raise the current price by a factor $e^{\psi(x)}$, and will cost

$$\int_0^x e^{\psi(v)}\,dv.$$

To prevent absurdities, we shall have to have

<u>Round trips are expensive</u>

(that is, suddenly buying x and then immediately selling x must leave you worse off). If this is to happen, we must have

$$\boxed{\psi(-y) = -\psi(y) \qquad \forall y \in \mathbb{R}.}$$

at least <u>if the liquidation value of positions are unbounded above and below</u>. The argument is simple. If we start with x, rapidly buy y (in one trade) then immediately sell it, we shall have at the end that our liquidation value is

$$-\int_0^y e^{\psi(v)}\,dv + e^{\psi(y)}\int_{-y}^0 e^{\psi(v)}\,dv + L(x).\ e^{\psi(y)+\psi(-y)}$$

and this must be at most the original liquidation value $L(x)$: thus

$$L(x)\left\{1 - e^{\psi(y)+\psi(-y)}\right\} \geq \int_{-y}^0 e^{\psi(y)+\psi(v)}\,dv - \int_0^y e^{\psi(v)}\,dv$$

and if $\sup_x L(x) = +\infty$, $\inf_x L(x) = -\infty$, then the only possibility is $\psi(y)+\psi(-y) = 0$.

2) <u>How should the liquidation value be defined?</u>

The obvious definition is

$$L(x) \equiv \sup\left\{ -\sum_{j=1}^N \int_0^{\xi_j} e^{\psi(v)}\,dv\ e^{\sum_{r=1}^{j-1}\psi(\xi_r)} \ :\ \sum_1^N \xi_r = -x \right\}$$

which is the greatest amount of cash you could make by liquidating the position in a sequence of trades of the sizes you choose. It is evident that L is increasing. What more can we say? It seems like we need to generalize the idea that round trips are expensive to say

[This page is a dead end]

Overshooting deals are costly.

What this means is that if we start with x, sell $a>0$, then buy back immediately $b\in(0,a)$, it would have been better simply to sell $a-b$ in one go. Before we go any further, let's make certain natural simplifying assumptions:

$$\boxed{\begin{array}{l}\psi' \text{ decreases in } \mathbb{R}^+, \quad \text{increases in } \mathbb{R}^- \\[2ex] \psi(y)+\psi(-y)\geq 0 \qquad \forall y\end{array}}$$

Now let's check whether an overshooting deal is costly. If we overshoot, the value after doing this will be

$$\int_{-a}^{0} e^{\psi(v)}dv - e^{\psi(-a)}\int_0^b e^{\psi(v)}dv + e^{\psi(-a)+\psi(b)}L(x-a+b)$$

compared with

$$\int_{-(a-b)}^{0} e^{\psi(v)}dv + e^{\psi(b-a)}L(x-a+b)$$

So we need

$$\left(e^{\psi(b-a)}-e^{\psi(-a)+\psi(b)}\right)L(x-a+b) \geq \int_{-a}^{b-a} e^{\psi(v)}dv - e^{\psi(-a)}\int_0^b e^{\psi(v)}dv$$

Now we shall have $\psi(-a)+\psi(b)\geq\psi(b-a)$, because if we consider the difference as a function of b, the inequality is satisfied at $b=0$, $b=a$, and differentiating wrto b gives $\psi'(b)-\psi'(b-a)$, which is decreasing in b. Hence the inequality is good throughout the interval. We will need to assume

$$\boxed{\bar{L}\equiv\sup_x L(x) <\infty}$$

otherwise the only possibility would be $\psi(x)=\lambda x$ for some λ. We shall then find that to make overshoot deals costly we have to have for all $0<b<a$

$$-\bar{L}\left(e^{\psi(-a)+\psi(b)}-e^{\psi(b-a)}\right)\geq\int_{-a}^{b-a}e^{\psi(v)}dv-\int_0^b e^{\psi(v)+\psi(-a)}dv$$

$$=\int_0^b\left\{e^{\psi(v-a)}-e^{\psi(v)+\psi(-a)}\right\}dv$$

The derivative wrto b at $b=0$ of LHS is <0, and of RHS $=0$; so this cannot be expected to hold.

Maybe the whole idea of round trips is flawed, cos the idea of instantaneously shifting large quantities of the asset is not what liquidity is about

3) If we accept the notion that the liquidation value should be the sup over all liquidation routes of the cash achieved, then we would have to have $\forall x, \forall a$

(*) $\qquad L(x) \geq e^{\psi(a)} L(x+a) - \int_0^a e^{\psi(v)}\, dv$

Differentiating the RHS wrt a at $a=0$ (assuming L is differentiable) leads to

$$0 = e^{\psi(0)} \left[\psi'(0) L(x) + L'(x) - 1 \right]$$

so this gives the solution

$$\boxed{L(x) = \frac{1 - e^{-x\psi'(0)}}{\psi'(0)}}$$

Abbreviating $\psi'(0) = \varepsilon$, and returning to (*), we get

$$1 - e^{-\varepsilon x} \geq -e^{\psi(a) - \varepsilon(x+a)} - \varepsilon \int_0^a e^{\psi(v)}\, dv + e^{\psi(a)}$$

$$\therefore \quad e^{-\varepsilon x} \left[e^{\psi(a) - \varepsilon a} - 1 \right] \geq -1 + e^{\psi(a)} - \varepsilon \int_0^a e^{\psi(v)}\, dv$$

Letting $x \to -\infty$ shows that we must have $\boxed{\psi(a) \geq \varepsilon a}$ for all a. This being the case, letting $x \to \infty$ shows that we require

$$\boxed{\varepsilon \int_0^a e^{\psi(v)}\, dv \geq e^{\psi(a)} - 1}$$

for all a. Notice, incidentally, that $\psi(a) \geq \varepsilon a \;\; \forall a$ implies the costly-round-trip condition $\psi(y) + \psi(-y) \geq 0$.

Now suppose we have $x > 0$, and we think to liquidate in one go; this would generate cash

$$\int_{-x}^0 e^{\psi(v)}\, dv \geq \int_{-x}^0 e^{\varepsilon v}\, dv = \frac{1 - e^{-\varepsilon x}}{\varepsilon} = L(x)$$

so it must be that $\psi(a) = \varepsilon a$ for all $a \leq 0$. But in $\mathbb{R}^+$ we can write $f(a) = e^{\psi(a)}$, so that the last boxed inequality says

$$f(x) \leq 1 + \varepsilon \int_0^x f(t)\, dt \qquad \forall x \geq 0$$

Gronwall gives $\quad f(x) \equiv e^{\psi(x)} \leq e^{\varepsilon x}$, and hence $\boxed{\psi(x) = \varepsilon x \text{ for all } x}$

<u>In particular, round trips cost nothing</u> ?!

Limited liability in a simple exchange economy again (18/3/03)

Let's return to the situation on pp 38-39, where the aggregate supply of the assets is now denoted by x, where $0 \le x \le 1$. To some extent, we may choose x. Suppose we write

$$S_0 = \xi - \eta$$

where $\xi, \eta \geq 0$. We require the complementary slackness conditions

$$\left\{ \begin{array}{lll} \eta \cdot x = 0 & & \text{(if price is negative, we hold none of the asset)} \\ \zeta \cdot \xi = 0 \;, & \zeta = x - w & \text{(if price is positive, we throw none away, or all)} \\ w \cdot \tilde{w} = 0 & \zeta = 1 - x - \tilde{w} & w, \tilde{w} \geq 0 \end{array} \right.$$

$\zeta = x \wedge (1-x)$

The process of discarding assets we don't like should end up at an equilibrium, but is it a good one? We have that agent j gets expected utility

$$-\exp\{ -\gamma_j \, \alpha_j (\mu - \Gamma V x) - \tfrac{1}{2}\Gamma^2 x \cdot Vx \}$$

So if we take logs of the negative of this, multiply by γ_j^{-1} and add, we find that the quantity

$$\tfrac{1}{2}\Gamma \, x \cdot Vx - \mu \cdot x$$

is what we would be trying to minimize (this is a crude measure, but gives us a way to distinguish between possible equilibria.) So we have

$$S_0 = \xi - \eta = (\mu - \Gamma V x)/(1+r)$$

and we now want to

$$\min \ \tfrac{1}{2}\Gamma \, x \cdot Vx - \mu x \qquad \text{subject to} \qquad \eta \cdot x = 0, \ \zeta \cdot \xi = 0, \ w \cdot \tilde{w} = 0,$$
$$\zeta = x - w = 1 - x - \tilde{w},$$
$$\xi - \eta = \mu - \Gamma V x$$

One way to approach this would be to take very large M and

$$\boxed{\begin{array}{l} \min \ \tfrac{1}{2}\Gamma x \cdot Vx - \mu x + M(\eta \cdot x + \zeta \cdot \xi + w \cdot \tilde{w}) \\ \text{subject to} \quad \zeta = x - w \ , \ \zeta = 1 - x - \tilde{w} \\ \qquad\qquad \xi - \eta = \mu - \Gamma V x \end{array}}$$

This can in principle be done, as a quadratic program.

Rational expectations equilibria again (24/3/03)

1) Let's return to the problem of a linear-Gaussian system

$$dx = a\,dW + bx\,dt$$

underlying some vector of assets, which have dividend process δ; assume wlog that $\delta = Dx$ for some matrix D. Agents see some public signal process S,

$$dS = \sigma_S\,dW + M_S\,x\,dt$$

which includes the dividend process δ and the price process p, and additionally some individual signal

$$dy_i = \sigma_i\,dW + M_i\,x\,dt$$

There is therefore a filtering problem for each agent; then there is an optimal investment/consumption problem for each agent (note that optimal investments must be functions only of what the agents see); then there is the market-clearing constraint for an equilibrium (clearing both ownership of shares and consumption of good); and finally we will have to check that for each agent the current prices of shares are the NPV of all future dividends.

2) <u>The filtering problem</u>: Agent i sees signal process $Y_i = \binom{S}{y_i}$, solving

$$dY_i = \eta_i\,dW + \beta_i\,x\,dt \qquad\qquad \eta_i \equiv \begin{pmatrix}\sigma_S\\ \sigma_i\end{pmatrix}, \quad \beta_i = \begin{pmatrix}M_S\\ M_i\end{pmatrix}$$

giving filtering problem

$$\begin{cases} dx = a\,dW + bx\,dt \\ dY = \eta\,dW + \beta x\,dt \end{cases} \qquad \text{(index } i \text{ omitted for now)}$$

If we set $d\nu = dY - \beta\hat{x}\,dt$ as the innovations process, we shall get the estimate $\hat{x}$ solving

$$\boxed{d\hat{x} = H\,d\nu + b\hat{x}\,dt}$$

where

$$\boxed{H\,\eta\eta^T = V\beta^T + a\eta^T}$$

and

$$\boxed{0 = a\left(I - \eta^T(\eta\eta^T)^{-1}\eta\right)a^T + \left(b - a\eta^T(\eta\eta^T)^{-1}\beta\right)V + V\left(b - a\eta^T(\eta\eta^T)^{-1}\beta\right)^T - V\beta^T(\eta\eta^T)^{-1}\beta V}$$

Note: $\alpha \Phi \alpha^T = dp\,dp^T/dt$, which will be same for all agents.

Notice also that since $S = (I \ 0) Y$ we find

$$dS = (I \ 0)\,(d\nu + \beta \hat{x}\,dt) = (I \ 0)\,d\nu + M_S \hat{x}\,dt$$

3) <u>The optimal investment/consumption problem.</u>

It turns out that if we are to get market clearing, we have to have the price vector represented in the form

$$(?) \qquad p = \mu \hat{X}, \qquad \text{where} \quad \hat{X} = \begin{pmatrix} \hat{x}^1 \\ \vdots \\ \hat{x}^\sigma \end{pmatrix} \quad \text{is the vector of agents' estimates.}$$

Since p is a subvector of the public signal S, we have for every j

$$dp = \alpha_j\, d\nu_j + m_j\, \hat{x}_j\, dt$$

for matrices α_j, m_j which are known once the filtering problem for agent j is solved. The agent's wealth equation (dropping subscripts) is

$$dw = (rw - c)\,dt + \theta\left[\,dp - rp\,dt + \delta\,dt\,\right]$$

and we suspect a value function

$$V(x,w) \equiv \sup E\left[\int_0^\infty -\exp(-\rho s - \gamma c_s)\,ds \;\middle|\; \hat{x}_0 = x,\ w_0 = w\right]$$

$$= -\exp\left[-r\gamma w - \tfrac{1}{2} x^T Q x - \varphi\right],$$

so that if $\xi_t = r\gamma w_t + \frac{1}{2}\hat{x}_t \cdot Q\hat{x}_t + \varphi$, we get

$$d\xi = r\gamma\,dw + Q\hat{x}\cdot d\hat{x} + \tfrac{1}{2}\,\mathrm{tr}\,(Q H \Sigma H^T)\,dt \qquad \left(\Sigma = d\nu d\nu^T/dt = (\eta\eta^T)dt\right)$$

$$\equiv r\gamma\,dw + Q\hat{x}\cdot d\hat{x} + k\,dt, \quad \text{say.}$$

Doing the optimisation, we obtain the conditions:

$$\begin{cases} c^* = \frac{1}{\gamma}(\xi - \log r) \\[2ex] \theta^* = (r\gamma)^{-1}\,(\alpha\Sigma\alpha^T)^{-1}\left\{ (m + D - \alpha\Sigma H^T Q)\hat{x} - rp \right\} \end{cases} \qquad (\Rightarrow p = \mu\hat{X}, \text{ as claimed})$$

and for $V(\hat{x}_t, w_t)$ to be a martingale under this control we need

$$\boxed{\begin{aligned} 0 \equiv\ & (r\log r - r + \rho + k - r\varphi) - \tfrac{1}{2} r\, \hat{x}^T Q \hat{x} + \tfrac{1}{2}\hat{x}\,(Q b + b^T Q)\,\hat{x} \\ & - \tfrac{1}{2}\hat{x}^T Q H \Sigma H^T Q \hat{x} + \tfrac{1}{2}\left((m+D-\alpha\Sigma H^T Q)\hat{x} - rp\right)^T (\alpha\Sigma\alpha^T)^{-1}\left((m+D-\alpha\Sigma H^T Q)\hat{x} - rp\right) \end{aligned}}$$

$h = h_j$, $k = k_j$, $K = K_j$, $\Sigma = \Sigma_j$, $\alpha = \alpha_j$

$\alpha_j \Sigma_j \alpha_j^T$ same for all j

$$d\hat{x}_j = H_j \, dV_j + b \hat{x}_j \, dt$$

$$dp = \alpha_j \, dV_j + m_j \hat{x}_j \, dt$$

Notice the implications of this: if the estimates $\hat{x}^i$ were not the same, we could not have an equilibrium of this form!!

We seem to be coming back to the results of Futia (Econometrica $\underset{\sim}{49}$, 171–192)?

Perhaps: but then again perhaps we need to seek a value function of the form

$$-\exp\left[-r\gamma w - \tfrac{1}{2}\begin{pmatrix}\hat{x}\\ p\end{pmatrix}^T Q \begin{pmatrix}\hat{x}\\ p\end{pmatrix} - \varphi\right]$$

Let's stack $z \equiv \begin{pmatrix}\hat{x}\\ p\end{pmatrix}$, so that $dz = h\,d\nu + kz\,dt$, $h = \begin{pmatrix}H\\ d_j\end{pmatrix}$, $k = \begin{pmatrix}b & 0\\ m_j & 0\end{pmatrix}$, and write

$$dw = (rw-c)dt + \theta^T\left(dp - rp\,dt + D\hat{x}\,dt\right)$$

$$= (rw-c)dt + \theta^T\left(\alpha d\nu + Kz\,dt\right) \qquad K = (D+m, -r)$$

$$= (D+m_j, -r)$$

Optimising leads again to

$$\begin{cases} c^* = \frac{1}{\gamma}(\xi - \log r) \\ r\gamma\theta^* = (\alpha\Xi\alpha^T)^{-1}(K - \alpha^T\Xi hQ)z \end{cases}$$

and

$$\boxed{\begin{aligned} 0 = \{\rho - r + r\log r + \kappa - r\varphi\} + z^T Qkz - \tfrac{1}{2}r z^T Q z - \tfrac{1}{2}z^T Qh\Xi h^T Q z \\ + \tfrac{1}{2}z^T(K - \alpha^T\Xi hQ)^T(\alpha\Xi\alpha^T)^{-1}(K - \alpha^T\Xi hQ)z \end{aligned}}$$

<u>Observe</u> that there is no loss of generality in assuming x is a zero-mean process. The information conveyed is not altered by changing the mean, and if we want non-zero mean to give an interesting dividend process, we could in fact write $x = \bar{x} + \tilde{x}$ ($\bar{x}$ is the mean of x), and $\tilde{p} = p - r^{-1}D\bar{x}$, and this now makes $\tilde{x}$ zero mean, $\tilde{p}$ conveys exactly the same info as p, and the wealth equation is good using $\tilde{p}$, $\hat{x} - \bar{x}$.

4) Do we have the arbitrage-pricing relation

$$\zeta_j(t)\,p_t + \int_0^t \zeta_j(s)\,\delta(s)\,ds \qquad \underline{\text{is a martingale for each } j}\ ?$$

<u>Yes</u>: this follows by general arguments (as, for example, in Breeden).

Convertible bonds again (30/1/03)

It appears that if we define functions

$$\varphi_0(t) \equiv \int_0^t \alpha\beta\,(e^{\beta u} - e^{-\alpha u})\,du = \alpha e^{\beta t} + \beta e^{-\alpha t} - \alpha - \beta$$

$$\varphi_1(t) = \int_0^t (\alpha+1)(\beta-1)\,(e^{\beta u} - e^{-\alpha u})\,e^{-u}\,du = (\alpha+1)\,e^{(\beta-1)t} + (\beta-1)\,e^{-(\alpha+1)t} - \alpha - \beta$$

then these may play a rôle in the study. Notice firstly that both are convex and non-negative, zero at zero. In the no-calling case, we shall have

(1) $$\boxed{S(m,V) = \frac{mp'\,\varphi_0(\log V/\xi) - rV\varphi_1(\log V/\xi)}{r(\alpha+\beta)(n-m)}}$$

and

(2) $$\boxed{B(m,V) - S(m,V) = \frac{rV\,\varphi_1(\log V/\eta) - (mp' + (n-m)p)\,\varphi_0(\log V/\eta)}{r(n-m)(\alpha+\beta)}}$$

and

(3) $$\boxed{\xi(m) = \frac{(n-m\tau)\,\varphi_0(\log\theta)\,p/r}{\varphi_1(\log\theta) - (\alpha+\beta)\,p\,(n-m)/m}}$$

$\theta \equiv \theta(m) \equiv \xi(m)/\eta(m)$

Observe that for $t>0$, $\varphi_1(-t)/\varphi_0(-t)$ increases, $\varphi_0(-t)$ increases, so

$$t \mapsto \frac{(n-m\tau)\,\varphi_0(-t)\,p/r}{\varphi_1(-t) - (\alpha+\beta)\,p(n-m)/m}$$

decreases (while >0)

Differentiating the expression for S and using $\frac{\partial S}{\partial m} = 0$ at $V=\eta$, we derive

(4) $$\boxed{\frac{\partial\xi}{\partial m} = \frac{\theta\xi\,\{np'\,\psi_0(1/\theta) - r\eta\,\psi_1(1/\theta)\}}{\psi_0'(1/\theta)\,(n-m)\,(mp' - \delta\xi)}}$$

and differentiating the expression for ξ in terms of θ gives

(5) $$\boxed{\frac{d}{dm}\log\xi = -\frac{\tau}{n-m\tau} - \frac{(\alpha+\beta)\,p\,n/m}{m\psi_1(\theta) - p(\alpha+\beta)(n-m)} + \frac{d\theta}{dm}\left[\frac{\psi_0'(\theta)}{\psi_0(\theta)} - \frac{\psi_1'(\theta)}{\psi_1(\theta) - p(\alpha+\beta)(n-m)/m}\right]}$$

(6) $$= \frac{np'\,\theta\,\psi_0(1/\theta) - r\xi\,\psi_1(1/\theta)}{(n-m)\,\psi_0'(1/\theta)\,(mp' - \delta\xi)}$$

from (4)

(PTO)

More on convertible bonds (28/3/03)

1) The differential equation for θ seems to be very hard to work with near 0. Suppose we consider instead

$$\lambda(m) \equiv \theta(m)/m$$

which we shall require converges to a positive finite value λ_0 as $m \downarrow 0$. Inspection of (3) on p.35 reveals that

$$\xi(m)/m \rightarrow \frac{n\rho\beta}{r(\beta-1)}\cdot\lambda_0 \qquad (m\downarrow 0)$$

Consider now the expression (6) on page 35; as $m\downarrow 0$, this goes like

$$\frac{\rho'}{\beta(\rho' - n\rho\lambda_0(\alpha+1)/\alpha)}\cdot\frac{1}{m}$$

and the constant is 1 if and only if

$$\lambda_0 = \frac{n\rho'\alpha}{r(1+\alpha)} \Big/ \frac{n\beta\rho}{r(\beta-1)} = a_0/\eta_0 \quad \text{(say)}.$$

Henceforth, we suppose that we are looking only at solutions for which λ_0 is this value.
We also know that

$$(7)\quad \eta(m) = \eta_0 - \frac{\beta\rho\pi}{r(\beta-1)}m - \frac{\alpha+\beta}{\beta-1}\frac{n\rho}{r}(\lambda_0 m)^\alpha\left[1-\frac{\rho\alpha(1-\pi)}{1+\alpha}\right] + o(m^\alpha),\quad \xi(m) = a_0 m + o(m^\beta),$$

for the solution that we're concerned with.

2) Using Maple, we find that

$$(8)\quad \theta\left[\frac{\psi_0'(\theta)}{\psi_0(\theta)} - \frac{\psi_1'(\theta)}{\psi_1(\theta) - p(\alpha+\beta)(n-m)/m}\right] = 1 - \frac{(\alpha+\beta)(\alpha(\beta-1)-(\alpha+1)\beta\rho n\lambda)(\lambda m)^\alpha}{\beta(\beta-1)} + o(m^\alpha)$$

as $m\downarrow 0$. Using (5), we get an expression for $\left(\frac{1}{\theta}\frac{d\theta}{dm} = \frac{1}{\lambda}\frac{d\lambda}{dm} + \frac{1}{m}\right)$

$$\left\{\frac{1}{\lambda}\frac{d\lambda}{dm} + \frac{1}{m}\right\}\theta\left[\frac{\psi_0'(\theta)}{\psi_0(\theta)} - \frac{\psi_1'(\theta)}{\psi_1(\theta) - p(\alpha+\beta)(n-m)/m}\right]$$

$$(9)\quad = \frac{n\rho'\theta\psi_0(1/\theta)}{(n-m)\psi_0'(1/\theta)(m\rho'-\delta\xi)} - \frac{r\xi\psi_1(1/\theta)}{(n-m)\psi_0'(1/\theta)(m\rho'-\delta\xi)} + \frac{\pi}{n-m\pi} + \frac{(\alpha+\beta)p\,n/m}{m\psi_1(\theta) - p(\alpha+\beta)(n-m)}.$$

$$\psi_0(\theta) = \alpha s \lambda^{\beta} + \beta \lambda^{-\alpha}/t - \alpha - \beta \quad , \quad \psi_1(\theta) = (\alpha+1)\frac{s\lambda^{\beta-1}}{m} + (\beta-1)\frac{\lambda^{-\alpha-1}}{tm} - \alpha - \beta$$

$$\psi_0'(\theta) = \frac{\alpha\beta}{m}\left[s\lambda^{\beta-1} - \frac{\lambda^{-\alpha-1}}{t}\right], \qquad \psi_1'(\theta) = \frac{(\alpha+1)(\beta-1)}{m^2}\left[s\lambda^{\beta-2} - \frac{\lambda^{-\alpha-2}}{t}\right]$$

$$\psi_0(1/\theta) = \frac{\alpha\lambda^{-\beta}}{s} + \beta t \lambda^{\alpha} - \alpha - \beta \quad , \quad \psi_1(1/\theta) = (\alpha+1)\frac{m\lambda^{1-\beta}}{s} + (\beta-1)mt\lambda^{1+\alpha} - \alpha - \beta$$

$$\psi_0'(1/\theta) = \alpha\beta m\left[\frac{\lambda^{1-\beta}}{s} - t\lambda^{1+\alpha}\right]$$

<u>The case $0<\alpha<1$.</u> Here we have to leading order

$$\Theta(m) \equiv m\lambda(m) = \frac{a_0 m}{\eta_0 - \frac{\alpha+\beta}{\beta-1}\frac{n\rho}{r}\lambda_0^\alpha m^\alpha\left(1-\frac{p\alpha(1-r\tau)}{1+\alpha}\right)}$$

$$= m\lambda_0\left[1+\frac{\alpha+\beta}{\beta}q\lambda_0^\alpha m^\alpha\right] + \text{higher order} \qquad q = 1-\frac{\alpha p(1-r\tau)}{1+\alpha}$$

This suggests that we write

$$\lambda = \lambda_0(1+L)$$

and think of $L = L(t)$ where $\boxed{t \equiv m^\alpha}$ Similarly, we introduce $\boxed{s = m^\beta}$ as another variable, and now all the functions $\psi_{0/1}(\Theta, 1/\Theta)$ etc can be expressed in terms of <u>integer</u> powers of s, t, m, with the only non-integer powers appearing as $(1+L)^\alpha$, which is of course analytic

Introducing the notations

$$F_0 = \Theta\left[\frac{\psi_0'(\Theta)}{\psi_0(\Theta)} - \frac{\psi_1'(\Theta)}{\psi_1(\Theta)-p(\alpha+\beta)(n-m)/m}\right]$$

$$F_1 = \frac{n\rho'\,\Theta\,\psi_0(1/\Theta)}{(n-m)\psi_0'(1/\Theta)(m\rho'-\delta\xi)}, \qquad F_2 = \frac{-r\xi\,\psi_1(1/\Theta)}{(n-m)\psi_0'(1/\Theta)(m\rho'-\delta\xi)}$$

$$F_3 = \frac{(\alpha+\beta)pn/m}{m\psi_1(\Theta) - p(\alpha+\beta)(n-m)}, \qquad F_4 = \frac{\tau}{n-m\tau}$$

we have the differential equation to be solved is

$$\left(\frac{1}{\lambda}\frac{d\lambda}{dm} + \frac{1}{m}\right)F_0 = F_1 + F_2 + F_3 + F_4$$

or again

$$\left(\frac{\alpha t L'}{1+L} + 1\right) = \frac{m(F_1+F_2+F_3+F_4)}{F_0}$$

so

$$\alpha t L' = (1+L)\left\{\frac{m(F_1+F_2+F_3+F_4)}{F_0} - 1\right\}$$

(11/7/03)

... this proved to be too onerous, so we passed things over to Maple, where the strategy was to represent m as a function of Θ, setting m up as a power series in $(\Theta, s, u) \equiv (\Theta, \Theta^\beta, \Theta^\alpha)$. We also think now of ξ as a function of Θ. Differentiating (3) with respect to Θ gives a linear relation between $\frac{\partial\xi}{\partial\Theta}$ and $\frac{\partial m}{\partial\Theta}$, and comparing with (4) we are able to deduce an expression for $\frac{\partial m}{\partial\Theta}$ in terms of m, Θ (and ξ, which is given by (3) in terms of m, Θ). Now if we have

The power series

$$m = \sum_{i,j,k \geq 0} a_{i,j,k}\, \theta^i s^j u^k$$

we have

$$\frac{\partial m}{\partial \theta} = \left(\sum_{i,j,k} (i + j\beta + k\alpha)\, a_{i,j,k}\, \theta^i s^j u^k \right) / \theta$$

Popping this into Maple + doing some calculations, we get to

$$m = \theta \Bigg\{ \frac{n\beta(1+\alpha)}{\alpha(\beta-1)(1-\tau)} - \frac{n\beta^2(1+\alpha)^2\tau}{\alpha^2(\beta-1)^2(1-\tau)^2}\,\theta - \frac{n\beta(\alpha+\beta)(1+\alpha)(\beta\tau+1-\tau)}{\alpha^2(\beta-1)^2(1-\tau)^2}\,s$$

$$- \frac{n(\alpha+\beta)(1+\alpha-\alpha\beta(1-\tau))}{\alpha(\beta-1)(1-\tau)}\,u$$

$$+ \frac{n\theta^2}{\tau}\left(\frac{\beta\tau(1+\alpha)}{(1-\tau)\alpha(\beta-1)}\right)^3 + \frac{n\beta(1+\alpha)(\beta\tau+1-\tau)^2(\alpha+\beta)^2}{\alpha^3(\beta-1)^3(1-\tau)^3}\,s^2$$

$$+ \frac{n\beta^2\tau(1+\alpha)^2(\alpha+\beta)(\tau(\beta-1)(\beta+4)+2(\beta+2))}{2\alpha^3(\beta-1)^3(1-\tau)^3}\,\theta s$$

$$+ \frac{n\beta(\alpha+\beta)(1+\alpha)(2\tau+(1+\tau)\alpha-\alpha\beta(1-\tau^2))}{(1-\tau)^2(\beta-1)^2\alpha^2}\,\theta u$$

$$+ \left\{ - \frac{n\beta\beta(\alpha+\beta)^2(\alpha+1+\alpha\tau(\beta-1))}{\alpha(\beta-1)^2(1-\tau)(\alpha+1)} + \frac{n\beta(\alpha+\beta)(\tau((\alpha+\beta)\alpha\beta+2(\beta-1))+(\alpha+1)^2+\alpha\beta+1)}{(\beta-1)^2(1-\tau)^2\alpha^2} \right\} su \Bigg\}.$$

I've got higher-order terms also (next level); we have for example

$$- \frac{n}{\tau}\left(\frac{\beta\tau(1+\alpha)}{\alpha(\beta-1)(1-\tau)}\right)^4 \theta^3$$

4) Alessandro (a) An easy thing would be to try to fit directly the model to historical vols + see what estimates arise, and how these compare

(b) However the model is parametrised, we really should set it up in such a way that the BS model corresponds to (say) θ, ∞.

(c) Are we getting parameter stability because of a ridge regression effect (i.e., there is a direction whose value changes so little that we never get information to move us in that direction)? Finding the e-values of the Hessian at optimal points would tell us something about that.

(d) Try making predictions of one-step-ahead vol using the model

(e) See whether this model hedges better than BS

(f) What about transaction costs?

Gunther Shouldn't we also allow the possibility that the firm might buy back bonds to prevent the losses on bankruptcy?

Phil Dybvig's visit, March 03. Some very helpful comments/references on various projects including

1) John Aquilina: The model is the standard CAPM, where time-zero prices of the asset-vector $S_1 \sim N(\mu, V)$ would be $\frac{1}{1+r}(\mu - \Gamma V A)$, where A is the initial aggregate supply of the assets, $\Gamma^{-1} = \sum_{j=1}^{J} \gamma_j^{-1}$. Setting $\hat{S}_0 \equiv (\mu - \Gamma V A)/(1+r)$, we ask what would happen if agents could costlessly discard positive quantities of the assets at time 0. Then it would be impossible for $\hat{S}_0$ to be the asset prices in a competitive equilibrium, if $\hat{S}_0^i < 0$ for some i, because agents would never want to hold positive quantities of asset i. So we allow A to be reduced to $0 \leq A - z \leq A$, and try to get non-negative prices. Phil makes a number of remarks.

(a) Why should we be able to walk away from an asset of negative value at time 0 if we couldn't also walk away from it at time 1?

(b) Is there inefficiency here (i.e. by allowing this extra freedom, agents end up worse off)?

(c) Can one prove ∃ competitive equilibrium?

(d) Why not allow assets to be reintroduced once others have been dumped?

(e) This process of dumping assets sequentially is a form of tâtonnement

(f) Giving agents private endowments (not publicly known) could create inefficiency.

(g) What if agents have heterogeneous beliefs?

(h) There is some classical work of Hirshleifer which shows that giving people more info can make them worse off. Also a paper of Oliver Hart ~ 1978 which designs a two-period model with a similar 'spanning dimension collapse' which does strange things...

2) Sevo Singh In the study of liquidity effects, refer to existing economics literature for how this topic gets treated there:

Glosten-Milgrom, Dybvig-Diamond, Amihud-Mendelson, Kerry Back papers, Kyle model

3) Armand Didn't like the form we have it in, on the grounds that the equilibrium isn't well based - equilibrium prices for what? For example, need to rule out the possibility that nobody invests in any asset in equilibrium. Suggests

(a) introduce dividend processes to base things on (will there prices be NPV of future dividends under constant discounting...?))

(b) Agent endowments?

(c) Maybe allow true values of assets to be revealed at $T \gg 0$?

(d) Various references.

(e) Perhaps introduce exponential killing as way of attaching prices to something

Interesting questions

1) Christer Borell (Gothenburg) asks the following: if $AP(t,S)$ is the American put price then it seems that

$$x \longmapsto \log AP(t, e^x) \quad \text{is concave.}$$

Proof or disproof?

2) Per Hörfelt asks: suppose we want to construct a distribution on the integers such that the first $2m$ moments match those of a $N(0,\sigma^2)$; can this be done? When?

3) LEF asks the question: Let K be a convex body (compact) in $\mathbb{R}^d$, and let μ be surface measure on ∂K. Suppose we wish to maximise $\mu(\partial K)$ subject to

$$\int x\,\mu(dx) = 0, \qquad \int |x|^2 \mu(dx) = 1$$

- What K does the job? (in $d=2$, it's an equilateral triangle)

4) According to Jeff Steif, it is a known but difficult result that if Y is a stationary process, then $\bigcap_n \sigma(Y_k : k \geq n)$ is trivial iff $\bigcap_n \sigma(Y_k : k \leq -n)$ is trivial..

5) A bunch of problems from Sasha Cherny. Let $(M_t)_{t\geq 0}$, $(N_t)_{t\geq 0}$ be independent processes, with product $Z_t \equiv M_t N_t$.

(a) If M and N are $\left\{\begin{matrix}\text{general}\\ \text{continuous}\end{matrix}\right\}$ $\left\{\begin{matrix}\text{martingales}\\ \text{local martingales}\end{matrix}\right\}$ in their own filtrations, is Z a process of the same type relative to its own filtration?

(b) Suppose that M, N are both adapted to some filtration $(\mathcal{F}_t)_{t\geq 0}$ w.r.t. which they are martingales; same questions.

Apparently 5 of the answers are 'Yes', 3 are 'No'.

6) Do we really believe in $\int_0^\infty U(t, c_t)\,dt$ as a criterion? Hans F. points out a paper of Kreps, Huang, ... which suggests that the answer should be 'No', & suggests some alternatives

7) Bernt Øksendal has some nice work on informed vs uninformed traders