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Pooling of beliefs (8/10/07)

1) If one considers the issuance of insurance cover for a rare loss, whose frequency and magnitude is unknown, there is the tendency for premia to drop during a period where there is little going on, for two reasons. One is that there is learning about the parameters of the loss process; the other is that competition drives premia down to the level charged by the most sanguine insurer. However, there is another possible effect, and this is that as some insurers quote premia, the other insurers learn from this what others think of the risk. Let's look at a very simple first story for this.

2) Each of N agents starts with his own prior dist^n for some RV. One agent i is then selected at random and his belief (prior) is made known. Every other agent j then modifies his law to $\alpha_j \mu_0^j + (1-\alpha_j)\mu_0^i$ (so he shifts a little towards the views of agent i). This gets repeated. The state at time t is an $N \times N$ matrix Π_t where $\pi_t(\cdot, j)$ is the vector of weights agent j currently assigns to the original beliefs of the agents. We have $A = \mathrm{diag}(\alpha_j)$

$$\Pi_{t+1} = \Pi_t A + \Pi_t Z_{t+1} (1-\alpha) \qquad \left(\begin{array}{c}1-\alpha \text{ is a} \\ \text{row vector}\end{array}\right)$$

where (Z_t) are IID, uniform on $\{e_1, \dots, e_N\}$. Thus we have the random matrix evolution

$$\Pi_{t+1} = \Pi_t (A + Z_{t+1} (1-\alpha))$$

3) A special case is when $\alpha_i = a \ \forall i$, and then we have by induction

$$\Pi_t = a^t I + \mathfrak{z}_t 1^T$$

where we check

$$\Pi_{t+1} = \Pi_t (aI + Z_{t+1}(1-a) 1^T) = a^{t+1} I + a^t Z_{t+1} (1-a) 1^T + a\mathfrak{z}_t + \mathfrak{z}_t (1-a) 1^T$$

$$= a^{t+1} I + \mathfrak{z}_{t+1} 1^T$$

with

$$\boxed{\mathfrak{z}_{t+1} = \mathfrak{z}_t + a^t (1-a) Z_{t+1}.}$$

This could lead to the law of Π in the limit, but it doesn't look so simple. Nonetheless, it shows quite clearly the importance of the early selection of an agent's views.

With more general α, the thing that matters is $A^n v$; eventually, enough of these green, so we can work in terms of these, but it's not very beautiful

Investing with insurance business (8/10/07)

1) We have wealth dynamics for the firm

$$dw_t = rw_t\,dt + \theta_t(\sigma dW_t + (\mu - r)dt) + (pq - \delta_t)dt - dY_t$$

where p is the premium charged, $q(p)$ the resulting volume of business. HJB here for objective $E\left[\int_0^\tau e^{-\rho t} U(\delta_t)\,dt - K e^{-\rho\tau}\right]$ is

$$0 = \sup\left[U(\delta) - \rho V + \tfrac{1}{2}\sigma^2\theta^2 V'' + ((\mu - r)\theta + rw + pq - \delta)V' + \varepsilon q\int (V(w-y) - V(w))F(dy)\right]$$

where F is dist^n of jumps, ε is intensity if $q = 1$. Assuming $U(x) = x$, we get $V' \geq 1$,

$$0 = -\rho V - \tfrac{1}{2}\kappa^2 \frac{V'^2}{V''} + (rw + pq)V' + \varepsilon q\left\{-V(w) + \int V(w-y)F(dy)\right\}$$

Perhaps here if we took F to be a point mass (or more ambitiously, finitely many point masses) we could look for a solution where dividend gets paid out at $w = w^*$ so as to keep $w \leq w^*$, and the value is quadratic in w below w^* (or piecewise quadratic?) However, this is not how it works: the true story is far more complicated.

2) Let's try to work out the value if we use $U(x) = x$ and use a strategy $\delta_t = a + bw_t$. The wealth dynamics are therefore (assuming all wealth is invested in stock index)

$$dw_t = w_t(\sigma dW_t + \mu dt) + (pq - a - bw)dt - dY_t$$

so that the value function satisfies

$$0 = -\rho V + \tfrac{1}{2}\sigma^2 w^2 V'' + (\mu w + pq - a - bw)V' + q\varepsilon\int_0^\infty F(dy)(V(w-y) - V(w)) + U(a+bw)$$

$V(x) = -K$ for $x < 0$. Even if $\varepsilon = 0$ this is only solved in terms of Kummer functions. But maybe it's worth taking Laplace transforms: $\tilde{V}(s) \equiv \int_0^\infty e^{-sx}V(x)\,dx$, $\tilde{F}(s) = \int_0^\infty e^{-sy}F(dy)$, to give us

$$\boxed{\begin{aligned}&-(\rho + q\varepsilon)\tilde{V}(s) + \tfrac{1}{2}\sigma^2\left(2\tilde{V} + 4s\tilde{V}'(s) + s^2\tilde{V}''(s)\right) + (\mu - b)\left(-\tilde{V} - s\tilde{V}'\right) + (pq - a)\left(-V(0+) + \underline{s\tilde{V}(s)}\right)\\ &+ \frac{a}{s} + \frac{b}{s^2} \quad + \left(\tilde{F}(s)\tilde{V}(s) - K(1 - \tilde{F}(s))/s\right)q\varepsilon = 0\end{aligned}}$$

The underlined term is what makes life difficult.

3) Can we even do a hugely simplified problem? Take $\sigma = \mu = b = 0$, so in effect it's just a ruin probability story. Then

$$\boxed{\tilde{V} = \frac{q\varepsilon(1-\tilde{F})K - a + (pq - a)sV_{0+}}{s\{q\varepsilon\tilde{F} + (pq-a)s - \rho - q\varepsilon\}}}$$

How can we identify V_{0+}? Certainly we should have $V_{0+} = \lim_{s\to\infty} s\int_0^\infty e^{-sx} V(x)dx$, and this is indeed satisfied whatever V_{0+} is. Looking at the original equation at $0+$ gives us

$$V'(0+) = \frac{(\rho - q\varepsilon)V_{0+} + q\varepsilon K - a}{pq - a}$$

and this too is consistent with behaviour of $s\tilde{V}(s) - V(0+)$ at $s=\infty$, whatever V_{0+}.

4) Back to main proof formulation. If we have SN Lévy process $X_t = \gamma t - Y_t$, where jumps of Y have distn F, intensity λ, then

$$\psi(z) = \gamma z + \lambda\int_0^\infty F(dy)(e^{-zy}-1)$$

As we have the WH factorisation

$$\frac{\rho}{\rho-\psi(z)} = E\left[e^{z\bar{X}(T_\rho)}\right]E\left[e^{z\underline{X}(T_\rho)}\right] = \frac{\eta_+}{\eta_+ - z}\cdot E\left[e^{z\underline{X}(T_\rho)}\right]$$

$$\Rightarrow \quad E\left[e^{z\underline{X}(T_\rho)}\right] = \frac{(\eta_+ - z)\rho}{\eta_+(\rho-\psi(z))}$$

which is analytic in $\mathrm{Re}\, z > 0$; thus $\boxed{\psi(\eta_+) = \rho}$ and we deduce

$$\boxed{P\left[\underline{X}(T_\rho) = 0\right] = 1 - E\left[e^{-\rho\tau}\right] = \lim_{z\uparrow\infty} E\, e^{z\underline{X}(T_\rho)} = \rho/\eta_+\gamma}$$

From this,

$$V(w) = E^w\left[\int_0^\tau e^{-\rho t} a\, dt - e^{-\rho\tau}K\right]$$

$$= \frac{a}{\rho} - \left(K + \frac{a}{\rho}\right)E^w\left[e^{-\rho\tau}\right]$$

$$\rightarrow \quad \frac{a}{\rho} - \left(K + \frac{a}{\rho}\right)\left(1 - \frac{\rho}{\eta_+\gamma}\right) \equiv V(0+) \quad \text{as } w \downarrow 0,$$

where $\gamma \equiv pq - a$, $\lambda \equiv q\varepsilon$ in earlier notation. Thus $V_0 = \frac{a}{\eta_+\gamma} - K\left(1 - \rho/\eta_+\gamma\right)$, and

$$\boxed{\tilde{V}(s) = \frac{a - s\gamma V_0 + K(\psi(s) - \gamma s)}{s(\rho - \psi(s))} = \frac{a(\eta_+ - s) + K(\eta_+\psi(s) - \rho s)}{\eta_+ s(\rho - \psi(s))}}$$

[We might try to optimise $V(0+)$ over a, but this seems too artificial.]

Competition in pricing (12/10/07)

1) Suppose that firms $1, \dots, N$ provide a product at price p_i. Firm i's cost for producing it is a_i per unit, and the quantity sold is $F_i(p)$, depending on all the prices. We expect $F_i(\cdot)$ is decreasing in p_i, and increasing in p_j. It may be that we want

$$F(\lambda p) = \varphi(\lambda) F(p) \qquad \text{for } \lambda > 0,$$

a proportional scaling property.

Firm j's profit is therefore

$$(p_j - a_j) F_j(p);$$

Can we find any nice functional forms for F which would allow us to find Pareto-efficient price vectors for given a ??

2) Suppose we were to take the equations (FOCs) for Pareto efficiency:

$$0 = F_j + (p_j - a_j) D_j F_j \quad \text{, equivalently} \quad D_j(\log F_j) = -\frac{1}{p_j - a_j}$$

this would suggest one natural candidate could be

$$\boxed{D_j(\log F_j) = -\alpha / \Sigma p_k}$$

for some $\alpha > 0$. To find optimal p_j, we solve

$$\alpha(p_j - a_j) = \sum_k p_k \iff (\alpha - 1) p_j = \sum_{k \neq j} p_k + \alpha a_j$$

which implies $(\alpha - 1)\Sigma p_j = (N-1)\Sigma p_j + \alpha \Sigma a_j \Rightarrow \Sigma p_j = \frac{\alpha}{\alpha - N} \Sigma a_j$

and therefore we must have

$$\alpha > N.$$

This leads to $\boxed{p_j = a_j + \frac{1}{\alpha - N} \Sigma a_k.}$

We could achieve this with $F_j(p) = \left(\sum_{k \neq j} p_k / \Sigma p_k \right)^{\alpha}$, for example.

3) This could be generalised somewhat if we took

$$\boxed{D_j(\log F_j) = -\frac{\alpha_j}{p_j + \lambda_j \sum_{k \neq j} p_k}}$$

for then we have

$$(\alpha_j - 1)\, p_j = \lambda_j \sum_{k \neq j} p_k + \alpha_j q_j$$

$$\therefore \quad (\alpha_j - 1 + \lambda_j)\, p_j = \lambda_j \sum_k p_k + \alpha_j q_j$$

and so

$$\boxed{p_j = \frac{\alpha_j q_j + \lambda_j \sum p_k}{\alpha_j - 1 + \lambda_j}}$$

Summing on j,

$$\sum p_j = \left(\sum_j \frac{\lambda_j}{\alpha_j - 1 + \lambda_j} \right) \sum p_k + \sum \frac{\alpha_j q_j}{\alpha_j - 1 + \lambda_j}$$

which shows we need

$$\boxed{\sum_j \frac{\lambda_j}{\alpha_j - 1 + \lambda_j} < 1}$$

4) We might therefore consider

$$F_j(p) = \frac{\left(\sum_{k \neq j} p_k \right)^{\alpha_j - \varepsilon}}{\left(p_j + \lambda_j \sum_{k \neq j} p_k \right)^{\alpha_j}}$$

whence we have

$$F(tp) = t^{-\varepsilon} F(p)$$

for $t > 0$, capturing the idea that total demand falls as prices rise.

For (i), $E[Y|\theta] = k/\theta$, and $E(\frac{1}{\theta}|y_t) = \dfrac{b_0 + \sum_1^{N_t} y_j}{a_0 - 1 + N_t}$

so $E\left[Y \middle| y_t\right] = \dfrac{k\,(b_0 + \sum_1^{N_t} y_j)}{a_0 - 1 + k N_t}$

Insurance and Bayes (15/10/07)

1) Suppose that the claims experienced in a line of insurance business are IID with some common density $f(y|\theta)$ and come at the times of a Poisson process of rate λ.

An insurance company has a prior density $\pi_0(\theta)$ for the parameter θ of the claims distribution, and a $\Gamma(\alpha_0, \beta_0)$ prior for λ. As the firm observes the claims process, how does its prior update?

2) The updating of the density of λ is quite simple. If we see n claims by time t, then the prior has become posterior

$$\propto \quad (\beta_0\lambda)^{\alpha_0-1}\beta_0 \exp(-\beta_0\lambda) \cdot (\lambda t)^n e^{-\lambda t}$$

$$\propto \quad \lambda^{\alpha_0+n-1} \exp(-(\beta_0+t)\lambda),$$

so the posterior is $\Gamma(\alpha_0+N_t, \beta_0+t)$.

3) If we see claims $y_1, y_2, \dots, y_{N_t}$, the posterior for θ will be proportional to

$$\pi_0(\theta)\prod_{r=1}^{N_t} f(y_r|\theta)$$

and it remains to do some nice examples.

(i) $f(y|\theta) = \theta(\theta y)^{K-1}\exp(-\theta y)/\Gamma(K)$, $\pi_0(\theta) = \theta^{a_0-1}e^{-b_0\theta}b_0^{a_0}/\Gamma(a_0)$,

where K is a known constant, will keep a Gamma posterior for θ,

$$\pi_t(\theta) = \Gamma\left(a_0 + KN_t, \; b_0 + \sum_1^{N_t} Y_j\right),$$

which has mean $(a_0+KN_t)/(b_0+\sum_1^{N_t} Y_j)$.

(ii) If we were to assume that $Z_j = 1/Y_j$ were $\Gamma(K,\theta)$ distributed, with a $\Gamma(a_1, b_1)$ prior for θ, and $K>1$ known and fixed, the posterior for θ will be

$$\pi_t(\theta) = \Gamma\left(a_1 + KN_t, \; b_1 + \sum_1^{N_t} Y_j^{-1}\right)$$

We shall have

$$E(Y|\theta) = \frac{\theta}{K-1}$$

so

$$E[Y|\mathcal{Y}_t] = \frac{a_1 + a N_t}{b_1 + \sum_1^{N_t} Y_j^{-1}} \cdot \frac{1}{K-1}$$

But from a modelling point of view this is dumb – large values of Y have little impact on our beliefs!!

(iii) Suppose we have for $K>0$ fixed that the law of the Y's is $P(Y>y) = \left(\frac{K}{K+y}\right)^\theta$

for some $\theta > 1$. Then

$$E[Y|\theta] = \kappa/(\theta-1).$$

The density of Y is

$$f(y|\theta) = \frac{\theta}{\kappa}\left(\frac{\kappa}{\kappa+y}\right)^{\theta+1}$$

Prior $\pi_0(\theta)$ transforms to posterior

$$\propto \pi_0(\theta)\ \theta^{N_t}\left(\prod \frac{\kappa}{\kappa+Y_i}\right)^{\theta+1}$$

To keep expectations finite, let's try

$$\pi_0(\theta) \propto (\theta-1)^+\ e^{-b\theta}$$

so that the posterior is

$$\propto (\theta-1)^+\ \theta^{N_t} \exp\left[-\left(b + \sum_1^{N_t} \log\frac{(\kappa+Y_i)}{\kappa}\right)\theta\right].$$

To do the calculations here, we need to find for positive integer m

$$J_m \equiv \int_1^\infty \theta^m e^{-\mu\theta}\, d\theta = \frac{e^{-\mu}}{\mu} + \frac{m}{\mu} J_{m-1} \qquad (*)$$

Know $J_0 = e^{-\mu}/\mu$, and if $Q_m \equiv \mu^m J_m/m!$ we get

$$\frac{m!\,Q_m}{\mu^m} = \frac{e^{-\mu}}{\mu} + \frac{m!}{\mu^m} Q_{m-1} \Rightarrow Q_m = Q_{m-1} + \frac{\mu^{m-1}e^{-\mu}}{m!} = Q_{m-1} + \frac{e^{-\mu}}{\mu}\cdot\frac{\mu^m}{m!}$$

$$\Rightarrow \boxed{Q_m = \left(\sum_{j=0}^{m} \frac{\mu^j}{j!}\right)\frac{e^{-\mu}}{\mu}}$$

(numerically, it may be best to generate J_m from the recursion (*)

Numerics suggest $a_B(m) < 0$ always, and $a_B(\cdot)$ decreasing $a_B(0) = 0$

~~$b_B(m) > 0$ always, and $b_B(\cdot)$ increasing~~

$a_S(m) > 0$ always, and $a_S(\cdot)$ increasing, $a_S(0) = 0$

$b_S(m) < 0$ always, and $b_S(\cdot)$ decreasing. , $b_S(0) = 0$

(hence: $a_Y(m) < 0$ always, decreasing, $a_Y(0) = 0$

$b_Y(m) > 0$ always, increasing)

PDCCB: a slightly different perspective (23/10/07)

1) Let's look at the no-calling situation only. We have expressions

$$\begin{cases} B(m,V) = \frac{\rho}{r} + a_B(m)\,V^{-\alpha} + b_B(m)\,V^{\beta} \\ S(m,V) = \frac{rV - m\rho'}{r(n-m)} + a_S(m)\,V^{-\alpha} + b_S(m)\,V^{\beta} \end{cases}$$

and if we multiply throughout by V^α, write $z \equiv V^{\alpha+\beta}$, $\nu \equiv \frac{\alpha}{\alpha+\beta}$, $\lambda \equiv \frac{\alpha+1}{\alpha+\beta}$, we can express $V^\alpha S(m,V) \equiv \tilde{S}(m,z)$, $V^\alpha B(m,V) \equiv \tilde{B}(m,z)$ as

$$\begin{cases} \tilde{B}(m,z) = \frac{\rho}{r} z^{\nu} + a_B(m) + b_B(m)\,z \\ \tilde{S}(m,z) = \frac{r z^{\lambda} - m\rho' z^{\nu}}{r(n-m)} + a_S(m) + b_S(m)\,z \end{cases}$$

2) If we write $x(m) = \xi(m)^{\alpha+\beta}$, $y(m) = \eta(m)^{\alpha+\beta}$, $\tilde{\tilde{S}}(z) = \tilde{S}(m(\eta),\eta)$ $\quad (z = \eta^{\alpha+\beta})$ we are looking to have $\tilde{S} \geq 0$ making smooth contact to 0 at $z = x(m)$, which leads to

$$a_S(m) + b_S(m)\,z \geq \varphi(m,z) = \frac{m\rho' z^{\nu} - r z^{\lambda}}{r(n-m)}.$$

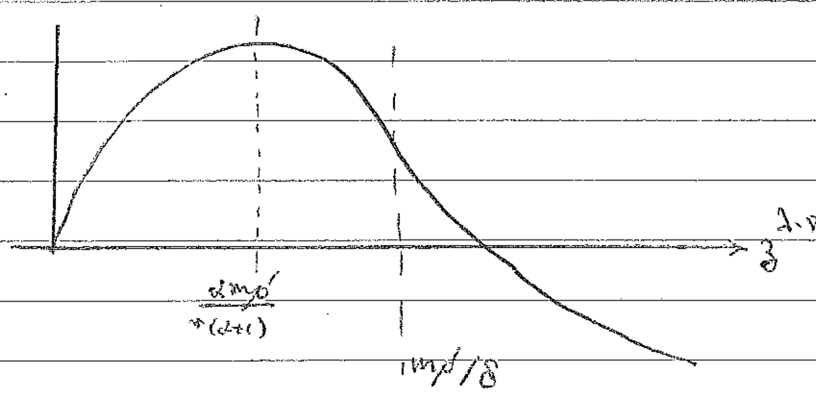
Notice

$$\varphi'(m,z) = \frac{z^{\nu-1}}{r(n-m)}\left(m\rho'\nu - r\lambda z^{\lambda-\nu}\right)$$

which changes sign at $z^{\lambda-\nu} = \frac{\alpha m\rho'}{r(\alpha+1)}$, being positive to the left of this. Likewise

$$\varphi''(m,z) = \frac{z^{\nu-2}}{r(n-m)}\left(m\rho'\nu(\nu-1) - \lambda(\lambda-1)\,r\,z^{\lambda-\nu}\right)$$ changes sign at $z^{\lambda-\nu} = \frac{m\rho'}{\delta}$

$= \frac{\beta}{\beta-1}\cdot\frac{\alpha m\rho'}{r(\alpha+1)}$, which is bigger than $\alpha m\rho'/r(\alpha+1)$. So we have that the graph of φ looks like



and we are trying to put a tangent over this function. Of course, this will depend on m, so maybe the picture doesn't help but it does make clear that contact point is to the left of $m\rho'/\delta$. However, this also makes clear that the intercept $a_S(m) > 0$. Is $b_S(m) < 0$??

[Numerics always result in $b_S(m) < 0$, and $b_S(\cdot)\downarrow$]

$$b_B(m) = b_S(m) + \frac{r\lambda y^{\lambda-1} - m\rho' \nu y^{\nu-1}}{r(n-m)}$$

$$= \frac{m\rho' \nu x^{\nu-1} - \lambda r x^{\lambda-1} + \lambda r y^{\lambda-1} - \nu \rho' y^{\nu}(n-mr)}{r(n-m)}$$

$$a_B(m) = a_S(m) + \frac{r y^{\lambda} - \rho(n-mr)y^{\nu}}{r(n-m)} + \frac{\nu \rho(n-mr)y^{\nu} - \lambda r y^{\lambda}}{r(n-m)}.$$

3) Collect the conditions:

(1) $a_S(m) + b_S(m)\,x = \dfrac{m\rho' x^{\nu} - r x^{\lambda}}{r(n-m)}$ $\qquad (S=0 \text{ at } \xi)$

(2) $b_S(m) = \dfrac{m\rho'\nu x^{\nu-1} - r\lambda x^{\lambda-1}}{r(n-m)}$ $\qquad (S'=0 \text{ at } \xi)$

(3) $a_B(m) + b_B(m)\,x = -\dfrac{\rho}{r} x^{\nu} + p x^{\lambda}/m$ $\qquad (B = pV/m \text{ at } \xi)$

(4) $\tilde{s} = \dfrac{\rho}{r} y^{\nu} + a_B(m) + b_B(m)\,y$

(5) $\phantom{\tilde{s}} = \dfrac{r y^{\lambda} - m\rho' y^{\nu}}{r(n-m)} + a_S(m) + b_S(m)\,y$ $\qquad (B=S=\tilde{s} \text{ at } \eta)$

(6) $\tilde{s}' = b_B(m) + \dfrac{\nu\rho}{r} y^{\nu-1}$ $\qquad$ (gradient of s at η)

(7) $\phantom{\tilde{s}'} = b_S(m) + \dfrac{\lambda r y^{\lambda-1} - m\rho'\nu y^{\nu-1}}{r(n-m)}$

From (1) and (2) we can express $b_S(m)$ and $a_S(m)$ in terms of m and x:

$$a_S(m) = \frac{m\rho'(1-\nu)x^{\nu} - r(1-\lambda)x^{\lambda}}{r(n-m)}$$

The equality (6)-(7) gives us $b_B(m)$ in terms of x, m, and y, and then (4)-(5) allows us to get $a_B(m)$ in terms of m, x, y. Next, $\tilde{s}$ = (5) allows us to express m in terms of $\tilde{s}$, x and y. Then the ODE $\tilde{s}' = (6)$ can be seen as

$$\tilde{s}'(y) \equiv \tilde{s}' = F(\tilde{s}, x, y)$$

where we also have the relation (3), which is a constraint of the form $H(\tilde{s}, x, y) = 0$ which determines x implicitly as a function of $(\tilde{s}, y)$. However, we don't need to solve the implicit equation for x!! Instead, we consider the two-dimensional ODE

$$\frac{d}{dy}\begin{pmatrix} \tilde{s}(y) \\ x(y) \end{pmatrix} = \begin{pmatrix} F(\tilde{s}, x, y) \\ -\left(H_y + H_s\, F(\tilde{s}, x, y)\right)/H_x \end{pmatrix}$$

which could certainly help with the numerics if nothing else.

4) A useful bound?

$$B(m, V) \le \frac{V}{n} + \frac{\rho}{r}$$

because if you were offered a perpetual coupon of $\rho\,dt$ + a zero-coupon convertible bond, you would prefer that to the original bond, and would convert immediately; the value of this preferred alternative is evidently $V/n + \rho/r$.

5) We may similarly study behaviour of $B(m,V) = \rho/r + a_B(m)V^{-\alpha} + b_B(m)V^{\beta}$ as $m \downarrow 0$, and we find

$$a_B(m) \to 0, \qquad b_B(m) \to \left(\frac{\eta}{n} - \frac{\rho}{r}\right)\eta^{-\beta}$$

so

$$\lim_{m\to 0} B(m,V) = \frac{\rho}{r} + \left(\frac{\eta}{n} - \frac{\rho}{r}\right)\left(\frac{V}{\eta}\right)^{\beta}.$$

6) Smooth pick-up after the NC solution finishes? In the case $K > K_c$, the no-calling solution works for a while, then S clips the corner. Can there be a resumption of case (2,4) later? This is a root-finding question. If we take $\eta \in (\eta(m^*), \eta(0))$ and $m > m^*$, then smooth-pasting S to s at η gives us one expression for S:

$$S = \frac{m\rho'\psi_0(V/\eta) - rV\psi_1(V/\eta)}{r(n-m)(\alpha+\beta)} + s(\eta)\frac{\psi_0(V/\eta)+\alpha\beta}{\alpha+\beta} + s'(\eta)\frac{V\psi_0'(V/\eta)}{\alpha\beta(\alpha+\beta)}$$

which has to equal the other:

$$S = \frac{m(\rho'-rK)}{r(n-m)(\alpha+\beta)}\psi_0\left(V/\xi\right) + \frac{V-mK}{n-m}$$

Matching coeffs of $V^{-\alpha}$, V^{β} gives us

$$\begin{cases} (1) & \alpha m(\rho'-rK)\xi^{-\beta} = m\rho'\alpha\eta^{-\beta} - r(\alpha+1)\eta^{1-\beta} + r(n-m)s(\eta)\alpha\eta^{-\beta} + s'(\eta)r(n-m)\eta^{-\beta} \\ (2) & \beta m(\rho'-rK)\xi^{\alpha} = m\rho'\beta\eta^{\alpha} - r(\beta-1)\eta^{\alpha+1} + r(n-m)s(\eta)\beta\eta^{\alpha} + s'(\eta)r(n-m)(-\eta^{\alpha}) \end{cases}$$

In addition we require $B(m,\xi) = K$, which comes out as

$$(3) \qquad K = s(\eta) + \frac{s-\rho/r}{\alpha+\beta}\psi_0(\xi/\eta) + \frac{\xi s'(\eta)}{\alpha\beta(\alpha+\beta)}\psi_0'(\xi/\eta)$$

We can use (1) to find a value for ξ, and then (2)+(3) have to hold.

BUT THERE CAN BE NO SUCH PICK-UP: WE'D HAVE TO HAVE $K_c > \rho'/r > K$!!

7) In view of the fact that $\lim_{m\downarrow 0} B(m,V)$ is not degenerate, it may be preferable to try to work with $b(\eta) = B(m, \eta(m))$. As always, we expect it may be round $m=0$ that things are trickiest, so introducing the variable $z \equiv \frac{\xi}{m} \to \frac{\alpha\rho'}{(1+\alpha)r}$ may help.

We have $B(m,\xi) = p\xi/m \equiv pz$, $B = \rho_r + a_B(m)V^{-\alpha} + b_B(m)V^{\beta}$, so we get

$$\begin{pmatrix} a_B \\ b_B \end{pmatrix} = \frac{1}{\eta^{\alpha+\beta} - \xi^{\alpha+\beta}} \begin{pmatrix} \xi^{\alpha}\eta^{\alpha+\beta}(pz - \rho/r) - \xi^{\alpha+\beta}\eta^{\alpha}(b(\eta) - \rho/r) \\ \xi^{\alpha}(\rho_r - pz) + \eta^{\alpha}(b(\eta) - \rho/r) \end{pmatrix} \quad (1)$$

and we have

$$\begin{cases} b(\eta) = H(m,z,\eta) \equiv S(m,\eta) \equiv \dfrac{m\rho'\psi_0(\eta/mz) - r\eta\psi_1(\eta/mz)}{r(\alpha+\beta)(n-m)} & (2) \\ b'(\eta) = H_\eta \Rightarrow H_m \dfrac{dm}{d\eta} + H_z \dfrac{dz}{d\eta} = 0 & (3) \end{cases}$$

and $b'(\eta) = \beta\, b_B(m)\,\eta^{\beta-1} - \alpha\, a_B(m)\,\eta^{-\alpha-1} = H_\eta(m,z,\eta)$ (4)

So the strategy would be to solve the system

$$\begin{cases} b'(\eta) = G(m,z,\eta) \\ H_m m' + H_z z' = 0 \\ 0 = (G_\eta - H_{\eta\eta}) + (G_m - H_{\eta m})m' + (G_z - H_{\eta z})z' \end{cases}$$

with initial condition $\eta = \eta_0$, $b(\eta) = \eta_0/n$, $m_0 = 0$, $z_0 = \alpha\rho'/(1+\alpha)r$. Here, we define $G(m,z,\eta)$ via (4) using (1).

PDCCB base case near 0 (6/11/07)

(i) Let's go back to the no-ceiling case, and try to understand the ODE for $s(\eta) = S(m(\eta), \eta)$ near to zero. Using $\partial S/\partial m = 0$ at conversion, and the equations for S, Y, we get

$$\boxed{\begin{aligned}
&(1) \quad r(\alpha+\beta)(n-m)\, s'(\eta) = \frac{1}{\xi} \left\{ m\rho' \psi_0'(\eta/\xi) - r\xi \left\{ \frac{\eta}{\xi} \psi_1'(\eta/\xi) + \psi_1(\eta/\xi) \right\} \right\} \\
&(2) \quad r(\alpha+\beta)(n-m)\, p\xi/m = r\xi\, \psi_1(\xi/\eta) - \rho(n-m r c)\, \psi_0(\xi/\eta) \\
&(3) \quad r(\alpha+\beta)(n-m)\, s(\eta) = m\rho' \psi_0(\eta/\xi) - r\eta\, \psi_1(\eta/\xi).
\end{aligned}}$$

(ii) Look at (3) when m, ξ are small: RHS is

$$\left(\frac{\eta}{\xi}\right)^{\beta} \left[\alpha m\rho' - r(\alpha+1)\xi \right] + \left(\frac{\eta}{\xi}\right)^{-\alpha} \left[\beta m\rho' - r(\beta-1)\xi \right] - m\rho'(\alpha+\beta) + r\eta(\alpha+\beta)$$

$$\longrightarrow r(\alpha+\beta)\, n\, (\eta_0/n) \qquad \text{as } m \to 0$$

This gives us the conclusion

$$\boxed{\xi^{-\beta} \left[\alpha m\rho' - r(\alpha+1)\xi \right] \longrightarrow 0} \qquad (m \to 0)$$

so in particular

$$\boxed{\frac{\xi}{m} \longrightarrow \frac{\alpha\rho'}{r(\alpha+1)}.}$$

(iii) Look at (2) when m, ξ are small: RHS is

$$\left(\frac{\xi}{\eta}\right)^{-\alpha} \left[(\beta-1) r\eta - \rho\beta(n-m r c) \right] + \left(\frac{\xi}{\eta}\right)^{\beta} \left[(\alpha+1) r\eta - \rho\alpha(n-m r c) \right] - r\xi(\alpha+\beta) - \rho(n-m r c)(\alpha+\beta)$$

$$\longrightarrow r(\alpha+\beta) n\, p \frac{\alpha\rho'}{r(\alpha+1)} = \frac{\alpha(\alpha+\beta) n\rho(1-c) p}{\alpha+1}$$

Thus

$$\boxed{\left(\frac{\xi}{\eta}\right)^{-\alpha} \left[(\beta-1) r\eta - \rho\beta(n-m r c) \right] \longrightarrow n\rho(\alpha+\beta) \left\{ \frac{(1-c)\alpha p}{1+\alpha} - 1 \right\} < 0}$$

(iv) We can rearrange (3) to express m in terms of s, η and $\theta \equiv \xi/\eta$

$$m = \frac{r n(\alpha+\beta) s + r\eta\, \psi_1(1/\theta)}{\rho' \psi_0(1/\theta) + r(\alpha+\beta) s}$$

While it would be possible to eliminate m from (1) and (2) using this, the resulting expression for (2)

Note: for very small θ, this is saying

$$(ns-\eta)' \approx \eta' \left\{ a_1 (ns-\eta) + a_2 \theta \right\}$$

$$a_1 \equiv \beta / \eta(0)$$

$$a_2 \equiv \frac{\beta(1+\alpha)(\rho' - rs)}{\alpha \rho'} + \frac{W(0)}{n}$$

$$\Rightarrow \frac{d}{d\theta} e^{a_1 \eta} (ns-\eta) \doteq e^{a_1 \eta} a_2 \theta \eta'$$

Thus if we have $\eta(\theta) - \eta(0) = b\,\theta^{\gamma}$ + smaller order terms, we'll expect

$$\boxed{(ns-\eta)(\theta) \approx \frac{a_2 b \gamma}{\gamma+1} \theta^{\gamma+1}}$$

is quite clumsy. However, doing the substitution in (1) pays off, because we get on the RHS

$$\frac{1}{\xi}\left[m\rho' \psi_0'(\tfrac{1}{\theta}) - r\xi\left\{\tfrac{1}{\theta}\psi_1'(\tfrac{1}{\theta}) + \psi_1(\tfrac{1}{\theta})\right\}\right] = Q(\theta, \eta, w) \qquad (w = m/\theta)$$

$$= \frac{1}{\xi}\left[(m\rho'\alpha\beta - r\xi(\alpha+1)\beta)\,\theta^{1-\beta} + (r\xi\alpha(\beta-1) - \alpha\beta m\rho')\,\theta^{\alpha+1} + r\xi(\alpha+\beta)\right]$$

$$= \frac{1}{\xi}\left[(\alpha m\rho' - r\xi(1+\alpha))\beta\theta^{1-\beta} + (r\xi(\beta-1) - \beta m\rho')\alpha\theta^{\alpha+1} + r\xi(\alpha+\beta)\right] \qquad (*)$$

and now

$$\alpha m\rho' - r\xi(1+\alpha) = \left[\alpha\rho'(rn(\alpha+\beta)s + r\eta\psi_1(\tfrac{1}{\theta})) - r\xi(1+\alpha)(\rho'\psi_0(\tfrac{1}{\theta}) + r(\alpha+\beta)s)\right] / (\rho'\psi_0(\tfrac{1}{\theta}) + r(\alpha+\beta)s)$$

$$= \left[r\eta\rho'(\alpha\psi_1(\tfrac{1}{\theta}) - (1+\alpha)\theta\psi_0(\tfrac{1}{\theta})) + \alpha\rho' rn(\alpha+\beta)s - r^2\eta\theta(1+\alpha)(\alpha+\beta)s\right] / (\rho'\psi_0(\tfrac{1}{\theta}) + r(\alpha+\beta)s)$$

$$= \Big[r\eta\rho'\left(\alpha(\beta-1)\theta^{1+\alpha} - (1+\alpha)\beta\theta^{1+\alpha} - \alpha(\alpha+\beta) + \theta(1+\alpha)(\alpha+\beta)\right)$$

$$+ \alpha\rho' rn(\alpha+\beta)s - r^2\eta\theta(1+\alpha)(\alpha+\beta)s\Big] / (\rho'\psi_0(\tfrac{1}{\theta}) + r(\alpha+\beta)s)$$

The numerator is $o(1)$ as $\theta \to 0$, the denominator is $O(\theta^{-\beta})$ so all in all the expression $(*) \longrightarrow r(\alpha+\beta)$ as $\theta \to 0$, with no explosions!

(v) Developing a little further,

$$\alpha m\rho' - r\xi(1+\alpha) = \frac{r(\alpha+\beta)\left[-\eta\rho'\theta^{1+\alpha} + \alpha\rho'(ns-\eta) + \theta\eta(1+\alpha)(\rho'-rs)\right]}{\rho'\psi_0(\tfrac{1}{\theta}) + r(\alpha+\beta)s}$$

Hence

$$\boxed{\frac{Q - r(\alpha+\beta)}{r(\alpha+\beta)} = \frac{\beta}{\eta}\,\frac{\alpha\rho'(ns-\eta) + \theta\eta(1+\alpha)(\rho'-rs) - \eta\rho'\theta^{1+\alpha}}{\theta^{\beta}(\rho'\psi_0(\tfrac{1}{\theta}) + r(\alpha+\beta)s)} + \frac{r\xi(\beta-1) - \beta m\rho'}{r\xi(\alpha+\beta)}\,\alpha\theta^{\alpha+1}}$$

Rearranging (1) leads to

$$ns' - \eta' = \frac{n}{n-m}\eta'\left[\frac{Q}{r(\alpha+\beta)} - \frac{n-m}{n}\right]$$

$$= \frac{n}{n-m}\eta'\left\{\frac{m}{n} + \frac{\beta}{\eta}\,\frac{\alpha\rho'(ns-\eta) + \theta\eta(1+\alpha)(\rho'-rs) - \eta\rho'\theta^{1+\alpha}}{\theta^{\beta}(\rho'\psi_0(\tfrac{1}{\theta}) + r(\alpha+\beta)s)} + \frac{r\xi(\beta-1) - \beta m\rho'}{r\xi(\alpha+\beta)}\,\alpha\theta^{\alpha+1}\right\}$$

$X_1, \ldots, X_n$ are IID $N(0, V)$ (p-variate observations) $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$

Then

$S = X^T X$ is $p \times p$, with Wishart distr^n

~~$S \sim W_p(V, n)$, density~~

$$|W|^{(n-p-1)/2} \exp\left\{-\tfrac{1}{2} \operatorname{tr}(V^{-1}W)\right\} \Big/ 2^{np/2} |V|^{n/2} \Gamma_p(n/2)$$

where $\Gamma_p(n/2) \equiv \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\left[(n+1-j)/2\right]$

(ii) NB we could handle this if we were to do the old trick of writing $\tau_0 = \kappa K_0$ ...

Inference on data? (8/11/07)

(i) Let's suppose we have a univariate data process

$$X_{t+1} = \beta X_t + \mu + \varepsilon_{t+1} \qquad \text{where } \varepsilon_t \sim N(0, \tau^{-1})$$

and precision τ is known. If we have at time t a $N(z_0, T_0^{-1})$ prior for $z = (\mu, \beta)^T$, then after observing X_{t+1} we have posterior

$$\propto \exp\left[-\tfrac{1}{2}(z-z_0)\cdot T_0 (z-z_0) - \tfrac{1}{2}\tau(x_{t+1} - \mu - \beta x_t)^2 \right]$$

$$= \exp\left[-\tfrac{1}{2}(z-z_0)\cdot T_0(z-z_0) - \tfrac{1}{2}(z-z_1)\cdot T_1(z-z_1) \right] , \quad T_1 \equiv \begin{pmatrix} 1 & x_t \\ x_t & x_t^2 \end{pmatrix}\tau$$

$z_1 = (x_{t+1}, 0)^T$. Thus the posterior for z will be

$$N(\tilde{z}, \tilde{T}) \qquad \tilde{z} \equiv (T_0 + T_1)^{-1}(T_0 z_0 + T_1 z_1), \quad \tilde{T} = T_0 + T_1 .$$

What we might do to keep things nice and recursive is to take θT_0 in place of T_0, where $0 < \theta < 1$ to allow for some shrinkage of the influence of history.

(ii) Could we allow a Γ prior on τ? The recursions become a mess ...

(iii) Predicting ahead is not too difficult; If we have a posterior $N(z_0, T_0^{-1})$ at time t, then the predicted mean at time $(t+1)$ is

$$z_0 \cdot (1, X_t)$$

with variance $(1, X_t)\cdot T_0^{-1}(1, X_t)^T + \tau^{-1}$.

(iv) Let's now look at a MV version of this, where $X_{t+1} = BX_t + \mu + \varepsilon_{t+1}$, with the ε_t IID $N(0, \tau^{-1})$. If we stack (μ, B) into a single vector z, which is assumed to have a $N(z_0, T_0^{-1})$ prior, then the posterior is

$$\propto \exp\left[-\tfrac{1}{2}(z-z_0)\cdot T_0 (z-z_0) - \tfrac{1}{2}(\mu - X_{t+1} + BX_t)\cdot \tau (\mu - X_{t+1} + BX_t) \right]$$

$$\propto \exp\left[-\tfrac{1}{2}(z-z_0)\cdot T_0 (z-z_0) - \tfrac{1}{2}(z-z_1)\cdot T_1 (z-z_1) \right]$$

when we set up the quadratic form T_1 correctly. Updating to new distn is similar to before.

(v) Could we allow a prior on the precision? We could consider $\tau = \tau_0 K$, where τ_0 is a scalar, K is known, and a gamma-gaussian story could be told, but part of the difficulty would be with confidence intervals.

PDCCB: learning the behaviour at 0 (13/11/07)

(i) It seems that the good choice is to take $\theta \equiv \xi/\eta$ as the independent variable, and use the equations ($t \equiv 1/\theta$):

$$(1) \quad r(n-m)(\alpha+\beta)A'(\theta) = \frac{1}{\xi}\frac{d\eta}{d\theta}\left[m\rho'\psi_0'(t) - r\xi\left(\psi_1(t) + t\psi_1'(t)\right)\right]$$

$$(2) \quad r(n-m)(\alpha+\beta)p\frac{\xi}{m} = r\xi\psi_1(\theta) - \rho(n-m\tau)\psi_0(\theta)$$

$$(3) \quad r(n-m)(\alpha+\beta)A = m\rho'\psi_0(t) - r\eta\psi_1(t)$$

(ii) Now from (3) we learn, introducing $\boxed{w \equiv m/\theta}$,

$$\boxed{\begin{aligned} r\psi_1(t)\eta &= m\rho'\psi_0(t) - r(n-m)(\alpha+\beta)A \\ &= w\rho'\theta\psi_0(t) - r(n-\theta w)(\alpha+\beta)A \end{aligned}}$$

Multiplying all through by $\theta^{\beta-1}$ gives us

$$\boxed{r\, t^{1-\beta}\psi_1(t)\eta = w\rho'\, t^{-\beta}\psi_0(t) - r\theta^{\beta-1}(n-\theta w)(\alpha+\beta)A}$$

an equation which involves only quantities which are $O(1)$ as $\theta \to 0$.

(iii) Let's next take (2), eliminate $\eta = \xi/\theta$ with the aid of (3), to get a quadratic form.

$$0 = r(n-m)(\alpha+\beta)p\eta\theta - rm\eta\theta\psi_1(\theta) + \rho m(n-m\tau)\psi_0(\theta)$$

$$= r\eta\theta\left[p(n-m)(\alpha+\beta) - m\psi_1(\theta)\right] + \rho m(n-m\tau)\psi_0(\theta)$$

$$= \theta\left[m\rho'\psi_0(t) - r(n-m)(\alpha+\beta)A\right].\left[p(n-m)(\alpha+\beta) - m\psi_1(\theta)\right] + \rho m(n-m\tau)\psi_0(\theta)\psi_1(t)$$

$$= \theta\left[m\left(\rho'\psi_0(t) + r(\alpha+\beta)A\right) - rn(\alpha+\beta)A\right].\left[pn(\alpha+\beta) - m\left(p(\alpha+\beta) + \psi_1(\theta)\right)\right] + \rho m(n-m\tau)\psi_0(\theta)\psi_1(t)$$

$$= -m^2\left\{\rho\tau\psi_0(\theta)\psi_1(t) + \left(\rho'\psi_0(t) + r(\alpha+\beta)A\right)\left(p(\alpha+\beta) + \psi_1(\theta)\right)\theta\right\}$$

$$+ m\left\{n\rho\psi_0(\theta)\psi_1(t) + rn(\alpha+\beta)A\left(p(\alpha+\beta) + \psi_1(\theta)\right)\theta + pn(\alpha+\beta)\left(\rho'\psi_0(t) + r(\alpha+\beta)A\right)\theta\right\}$$

$$- rn^2(\alpha+\beta)^2 p\theta A$$

Now some asymptotics on the coefficients as $\theta \downarrow 0$:

$m^2] \sim \rho'\alpha(\beta-1)\theta^{-\alpha-\beta}$ $\qquad$ $m^0] \sim rn^2(\alpha+\beta)^2 pA\theta$

$m] \sim n\rho\beta(\alpha+1)\theta^{1-\alpha-\beta}$

PDCCB: taming the behaviour at 0 (13/11/07)

(i) It seems that the good choice is to take $\theta \equiv \xi/\eta$ as the independent variable, and use the equations ($t = 1/\theta$):

$$(1)\quad r(n-m)(\alpha+\beta)A'(\theta) = \frac{1}{\xi}\frac{d\eta}{d\theta}\left[m\rho'\psi_0'(t) - r\xi\left(\psi_1(t) + t\psi_1'(t)\right)\right]$$

$$(2)\quad r(n-m)(\alpha+\beta)\,p\,\frac{\xi}{m} = r\xi\psi_1(\theta) - \rho(n-m\tau)\psi_0(\theta)$$

$$(3)\quad r(n-m)(\alpha+\beta)A = m\rho'\psi_0(t) - r\eta\psi_1(t)$$

(ii) Now from (3) we learn, introducing $\boxed{w \equiv m/\theta}$,

$$\boxed{\begin{aligned} r\psi_1(t)\eta &= m\rho'\psi_0(t) - r(n-m)(\alpha+\beta)A \\ &= w\rho'\,\theta\psi_0(t) - r(n-\theta w)(\alpha+\beta)A \end{aligned}}$$

Multiplying all through by $\theta^{\beta-1}$ gives us

$$\boxed{r\,t^{1-\beta}\psi_1(t)\eta = w\rho'\,t^{-\beta}\psi_0(t) - r\theta^{\beta-1}(n-\theta w)(\alpha+\beta)A}$$

an equation which involves only quantities which are $O(1)$ as $\theta \to 0$.

(iii) Let's next take (2), eliminate $\eta = \xi/\theta$ with the aid of (3), to get a quadratic form.

$$0 = r(n-m)(\alpha+\beta)p\,\eta\theta - rm\eta\theta\psi_1(\theta) + \rho m(n-m\tau)\psi_0(\theta)$$

$$= r\eta\theta\left[p(n-m)(\alpha+\beta) - m\psi_1(\theta)\right] + \rho m(n-m\tau)\psi_0(\theta)$$

$$= \theta\left[m\rho'\psi_0(t) - r(n-m)(\alpha+\beta)A\right]\cdot\left[p(n-m)(\alpha+\beta) - m\psi_1(\theta)\right] + \rho m(n-m\tau)\psi_0(\theta)\psi_1(t)$$

$$= \theta\left[m\left(\rho'\psi_0(t) + r(\alpha+\beta)A\right) - rn(\alpha+\beta)A\right]\cdot\left[pn(\alpha+\beta) - m\left(p(\alpha+\beta) + \psi_1(\theta)\right)\right] + \rho m(n-m\tau)\psi_0(\theta)\psi_1(t)$$

$$= -m^2\left\{\rho\tau\psi_0(\theta)\psi_1(t) + \left(\rho'\psi_0(t) + r(\alpha+\beta)A\right)\left(p(\alpha+\beta) + \psi_1(\theta)\right)\theta\right\}$$

$$+ m\left\{n\rho\psi_0(\theta)\psi_1(t) + rn(\alpha+\beta)A\left(p(\alpha+\beta) + \psi_1(\theta)\right)\theta + pn(\alpha+\beta)\left(\rho'\psi_0(t) + r(\alpha+\beta)A\right)\theta\right\}$$

$$- rn^2(\alpha+\beta)^2 p\theta A$$

Now some asymptotics on the coefficients as $\theta \downarrow 0$.

$m^2]\ \sim \rho'\alpha(\beta-1)\,\theta^{-\alpha-\beta}$ $\qquad$ $m^0]\ \sim rn^2(\alpha+\beta)^2 pA\,\theta$

$m]\ \sim n\rho\beta(\alpha+1)\,\theta^{1-\alpha-\beta}$

What this suggests therefore is to multiply throughout by $\theta^{\alpha+\beta-2}$, giving the quadratic for $w \equiv m/\theta$

$$\boxed{0 = -w^2 C_2(\theta,s) + w C_1(\theta,s) - C_0(\theta,s) \equiv G(\theta,w,s)}$$

where:

$$C_2(\theta,s) \equiv \theta\rho\pi.\ \theta^\alpha \psi_0(\theta).\ t^{1-\beta}\psi_1(t) + \left(\rho' t^{-\beta}\psi_0(t) + r(\alpha+\beta)\theta^\beta s\right)\left(\theta^{\alpha+1}\psi_1(\theta) + \theta^{\alpha+1} p(\alpha+\beta)\right)$$

$$\to \rho' \alpha(\beta-1) \quad \text{as } \theta \downarrow 0$$

$$C_1(\theta,s) \equiv n\rho\ \theta^\alpha \psi_0(\theta)\ t^{1-\beta}\psi_1(t) + rn(\alpha+\beta)s\left(\theta^{\alpha+1}\psi_1(\theta) + \theta^{\alpha+1}p(\alpha+\beta)\right)\theta^{\beta-1}$$

$$+ pn(\alpha+\beta)\left(\rho' t^{-\beta}\psi_0(t) + r(\alpha+\beta)s\theta^\beta\right)\theta^\alpha$$

$$\to n\rho\ \beta(\alpha+1) \quad \text{as } \theta \downarrow 0$$

$$C_0(\theta,s) = rn^2(\alpha+\beta)^2 p s\ \theta^{\alpha+\beta-1} \longrightarrow 0 \qquad (\theta \downarrow 0).$$

As $\theta \downarrow 0$, we get $w \to n\rho\,\beta(\alpha+1)/\rho'\alpha(\beta-1)$, as we should.

(iv) We have ODE (1) evolves subject to the constraints

$$\left.\begin{array}{l} H(\theta,\eta,s,w) \equiv r\eta\ t^{1-\beta}\psi_1(t) - w\rho' t^{-\beta}\psi_0(t) + r\theta^{\beta-1}(n-\theta w)(\alpha+\beta)s = 0 \\ G(\theta,w,s) \equiv -w^2 C_2(\theta,s) + w C_1(\theta,s) - C_0(\theta,s) = 0 \end{array}\right\}$$

We find

$$H_\theta = r(\alpha+\beta)(\beta-1)(ns-\eta)\,\theta^{\beta-2} + (\alpha+\beta)\,\theta^{\beta-1}\left[(\beta-1)r\eta\,\theta^\alpha - w\rho'\beta(\theta^\alpha-1) - \beta r s w\right]$$

$$\frac{\partial C_2}{\partial\theta} \to \rho\pi\,\beta(\alpha+1)$$

$$\frac{\partial C_1}{\partial\theta} \sim n(\alpha+\beta)(\beta-1)\,\theta^{\beta-2}\left(r(\beta-1)s - \beta p\right) + n\rho\,\alpha(\alpha+\beta)\left(\alpha p(1-\pi) - (\alpha+1)\right)\theta^{\alpha-1}$$

$$\frac{\partial C_0}{\partial\theta} \sim n^2 r\,(\alpha+\beta)^2 p s\,(\alpha+\beta-1)\,\theta^{\alpha+\beta-2} \qquad = o\left(\frac{\partial C_1}{\partial\theta}\right)$$

We have

$$\left.\begin{array}{l} H_\eta \to r(\alpha+1) \\ H_w \to -\rho'\alpha \\ H_s \to 0 \end{array}\right\} \qquad \left.\begin{array}{l} G_w \to -n\rho\beta(\alpha+1) \\ G_s \to 0 \end{array}\right\}$$

Not so !!

. This is better; there is only one price in the market for the product

? Should we bound jumps of Z for this latter [illegible]?

Modifying the model with Angus (15/11/07).

(i) Bill Janeway says that the volume of business an insurance firm takes on is determined by leverage - by arguing that a line of business is less risky, they can do more of it. He also emphasises that the investment of capital is an important feature; insurance companies frequently subsidise the insurance business from investment returns.

(ii) To try to understand this, we could propose that the volume of business done by firm i at time t, $V_i(t)$, must be constrained by

$$(1) \qquad V_i(t) \leq K_i(t)\, w_i(t)$$

where $w_i(t)$ is firm i's wealth at time t, and $K_i(t)$ is some number obtained from risk-management considerations. What I'd propose is that if Z_t is the accumulated claims by time t, then we'll demand

$$(2) \qquad E\,(Z_T - w)^+ \leq \varepsilon w$$

for some $\varepsilon > 0$, $T > 0$ (this is a fairly common approach - if you do VaR, you say prob^y of a loss in excess of ... should be no more than ... in the next 2 weeks) If we use this expected shortfall approach to risk management, then we are led immediately (?) to a proportional constraint of the form (1), though it may in general be difficult to find K analytically.

(iii) The price at which the product trades should be determined by market clearing:

$$p_t = \varphi\Big(\sum_{i=1}^{d} V_i(t) \Big)$$

where φ is decreasing (could do $\varphi(v) = a + v^{-\alpha}$, so that there's a floor for the price)

(iv) The wealth dynamics of firm i would therefore be

$$dw_i(t) = -\,dZ_i(t) + p_t V_i(t)\,dt + w_i(t)(\sigma\, dW_t + b\,dt)$$

$$= -\,dZ_i(t) + w_i(t)\big(\sigma\, dW_t + (b + K_i(t)\,p_t)\,dt\big)$$

In an insurance interpretation of the model, we would suppose that the loss processes Z_i are independent, but for some story about credit derivatives it would be more natural to think of some overall loss process shared pro rata by the firms:

$$dw_i(t) = w_i(t)\left[\frac{-K_i(t)\,dZ_t}{\sum_j K_j(t) w_j(t)} + \sigma\, dW_t + (b + K_i(t)\,p_t)\,dt \right]$$

More on the asymptotics of PDCCB (21/11/07)

(i) We can write equations (1), (2), (3) on p15 in the form

$$\begin{cases} r(n-m)(\alpha+\beta)s'(\theta) = \eta'(\theta).\ Q(\theta,\eta,w) \\ H(\theta,\eta,s,w) = 0 \\ G(\theta,s,w) = 0 \end{cases}$$

where $Q(\theta,\eta,w) \to r(\alpha+\beta)$ as $\theta \downarrow 0$ (see p13). Using the differentiated form of the last two we see that

$$(*)\quad \begin{pmatrix} 0 \\ -H_\theta \\ -G_\theta \end{pmatrix} = \begin{pmatrix} r(n-m)\alpha+\beta & -Q & 0 \\ H_s & H_\eta & H_w \\ G_s & 0 & G_w \end{pmatrix} \begin{pmatrix} s' \\ \eta' \\ w' \end{pmatrix} \equiv A(\theta) \begin{pmatrix} s' \\ \eta' \\ w' \end{pmatrix}$$

and by the asymptotics at the foot of p16, we get

$$A(\theta) \to \begin{pmatrix} rn(\alpha+\beta) & -r(\alpha+\beta) & 0 \\ 0 & r(\alpha+1) & -\rho'\alpha \\ 0 & 0 & -n\rho\beta(1+\alpha) \end{pmatrix}$$

which is invertible. So to understand how the solution (s,η,w) moves out from $\theta=0$, we must understand the LHS of $(*)$, namely, H_θ and G_θ near to 0.

(ii) We have

$$H_\theta \sim r(\alpha+\beta)(\beta-1)(ns-\eta)\theta^{\beta-2} + (\alpha+\beta)\theta^{\beta-1}\beta w(\rho'-rs)$$

$$\begin{cases} \dfrac{\partial c_2}{\partial\theta} \to \rho\tau\beta(\alpha+1) \\ \dfrac{\partial c_1}{\partial\theta} \sim n(\alpha+\beta)\alpha\rho\left(\alpha\rho(1-\tau)-(\alpha+1)\right)\theta^{\alpha-1} + n(\alpha+\beta)(\beta-1)\left(r(\beta-1)s-\beta\rho\right)\theta^{\beta-2} \\ \dfrac{\partial c_0}{\partial\theta} \sim n^2 r(\alpha+\beta)^2 \rho s(\alpha+\beta-1)\theta^{\alpha+\beta-2} \end{cases}$$

From (1) we have that

$$ns(\theta)-\eta(\theta) = o(\eta(\theta)-\eta(0))$$

and

$$ns(\theta)-ns(0) \sim \eta(\theta)-\eta(0)$$

In fact, we show on the reverse of p12 that

$$ns(\theta)-\eta(\theta) \sim \text{const.}\ \theta(\eta(\theta)-\eta(0))$$

so

$$\boxed{H_\theta \sim (\alpha+\beta)\theta^{\beta-1}\beta w(\rho'-rs)} \to 0 \quad (\theta\to 0).$$

Using the middle equation of (*), we derive the approximate relation

$$0 \approx r(\alpha+1)\eta' - \frac{\rho'\alpha}{n\rho\beta(1+\alpha)} G_\theta$$

$$\approx r(\alpha+1)\eta' - \frac{\alpha\rho'}{n\rho\beta(1+\alpha)}\left\{ -W_0^2 \rho r c\beta(1+\alpha) + w_0 \frac{\partial G}{\partial\theta} \right\}$$

$$\approx r(\alpha+1)\eta' + \frac{\alpha\rho'}{n\rho\beta(1+\alpha)}\Big\{ W_0^2 \rho r c\beta(1+\alpha) + w_0\, n(\alpha+\beta)\alpha\rho(\alpha+1-\alpha\beta(1-c))\,\theta^{\alpha-1}$$
$$- w_0\, n(\alpha+\beta)(\beta-1)(r(\beta-1)s - \beta\rho)\,\theta^{\beta-2} \Big\}$$

$$\approx r(\alpha+1)\eta' + \frac{\alpha\rho'}{n\rho\beta(1+\alpha)}\Big[W_0^2 \rho c\beta(1+\alpha) + w_0\, n(\alpha+\beta)\alpha\rho(\alpha+1-\alpha(1-c)\beta)\,\theta^{\alpha-1}$$
$$- w_0\, r(\alpha+\beta)(\beta-1)^2(\eta-\eta_0)\,\theta^{\beta-2} \Big]$$

Could the dominant term in $[\dots]$ be $\theta^{\beta-2}$? This would give the approximate ODE

$$\eta' \doteq -c(\eta_0-\eta)\,\theta^{\beta-2} \qquad (c>0 \text{ some constant})$$

so that $\log(\eta_0-\eta(\theta)) \doteq A + \frac{c}{\beta-1}\theta^{\beta-1}$

and the LHS $\to -\infty$ as $\theta\to 0$, the RHS does not. So our conclusion is

(a) If $\alpha>1$, then

$$\eta(0)-\eta(\theta) \sim \theta\,\frac{\alpha\rho' c}{n r(1+\alpha)}\,W_0^2$$

(b) If $\alpha<1$, then

$$\eta(0)-\eta(\theta) \sim \frac{n\rho(\alpha+\beta)(\alpha+1-\alpha\beta(1-c))}{r(\beta-1)(1+\alpha)}\,\theta^\alpha$$

$$\left[W_0 = \frac{n\rho\,\beta(1+\alpha)}{\rho'\alpha(\beta-1)} \right]$$

How would we generate an increase in population variance?

Developing a suggestion of Angus for a meeting model (3/12/07)

Angus suggests we could have some story where agents with diverse beliefs meet and update their beliefs about the population distribution of the value of some asset, perhaps trading as they do so. Let's see how we could tell such a story firstly for the evolution of beliefs (only) in discrete time.

(i) At time t, the population holds beliefs about the value of the asset which follow a $N(A, T_t^{-1})$ distribution. Individual i believes the value of the asset is m_t^i, and has a Gaussian prior with precision τ_t, same precision for all agents. He meets a randomly-chosen individual whose mean value is y, after which his posterior mean has changed to

$$m_{t+1}^i = \frac{\tau_t m_t^i + T_t y}{\tau_t + T_t}$$

with posterior precision

$$\boxed{\tau_{t+1} = \tau_t + T_t .}$$

After all this, the population still has a normal law, $N(A, T_{t+1}^{-1})$, where $v_t \equiv T_t^{-1}$ satisfies

$$v_{t+1} = v_t \frac{\tau_t^2 + T_t^2}{(\tau_t + T_t)^2}$$

since m_t^i and y are independent $N(A, v_t)$. Thus we have

$$\boxed{T_{t+1} = T_t \frac{(\tau_t + T_t)^2}{\tau_t^2 + T_t^2}}$$

Setting $a_t \equiv \tau_t / T_t$, we get

$$\frac{\tau_{t+1}}{\tau_t} = 1 + a_t^{-1}, \qquad \frac{T_{t+1}}{T_t} = \frac{(1+a_t)^2}{1+a_t^2}$$

and then we have

$$\frac{\tau_{t+1}}{T_{t+1}} \cdot \frac{T_t}{\tau_t} = \frac{a_{t+1}}{a_t} = \frac{(1+a_t^{-1})(1+a_t^2)}{(1+a_t)^2} = \frac{1+a_t^2}{a_t(1+a_t)}$$

whence

$$\boxed{a_{t+1} = \frac{1+a_t^2}{1+a_t}}$$

This simple recursion converges monotonically to $a_\infty = 1$, but there does not appear to be a useable expression for the general term.

on central planner problem?

Heterogeneous beliefs done properly (8/12/07)

(i) There's a paper by M. Kurz which eloquently explains good reasons why we should treat heterogeneous agents as seeing the same things, but having different beliefs about those things. So if there is a reference measure P on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ and agent i's beliefs Q^i satisfy

$$\left.\frac{dQ^i}{dP}\right|_{\mathcal{F}_t} = \Lambda^i_t$$

then we suppose that the agents see the smaller observation filtration $\mathcal{Y}_t$, and form their conditional expectations $\tilde{\Lambda}^i_t \equiv E[\Lambda^i_t \mid \mathcal{Y}_t]$. Agent i's marginal price at time t for claim ξ at time $T > t$ ($\xi \in b\mathcal{Y}_T$) will be

$$E_t\left[U_i'(c^*_{T,i}) \Lambda^i_T \xi\right] / \tilde{\Lambda}^i_t U_i'(c^*_{t,i}) = E_t\left[U_i'(c^*_{T,i}) \tilde{\Lambda}^i_T \xi\right] \Big/ U_i'(c^*_{t,i})\tilde{\Lambda}^i_t$$

$\Rightarrow$ If we had complete market setting, there have to be positive scalars λ_i s.t.

$$U_i'(c^*_{t,i}) \tilde{\Lambda}^i_t = \lambda_i \zeta_t$$

So the usual story tells us that

$$\Delta_t = \sum_j \delta^j_t = \sum_i c^*_{t,i} = \sum_j I_j(\lambda_j \zeta_t / \tilde{\Lambda}^j_t)$$

or more generally if the utility depends on time also

$$\boxed{\Delta_t = \sum_j I_j(t, \lambda_j \zeta_t / \tilde{\Lambda}^j_t).}$$

(ii) Previously, Angus + I were supposing that $\mathcal{Y}_t = \mathcal{F}_t$, and the agents are filtering values of constant parameters from observations. However, this leads to eventual learning of true values, + everything converges, so to prevent that we were telling a story with random lifetimes to keep the randomness stationary. But we don't need to do this if we are actually filtering a random process as well!

(iii) Let's take this in stages for a simplest example. Suppose dividend process δ_t of single asset is

$$\delta_t = a + v \cdot X_t$$

where

$$dX_t = dW_t - BX_t\,dt$$

and wlog we assume that $v = \beta e_1$, with β, B known, and a Gaussian prior for (X_0, a). Since the agent will eventually learn a from the data, we may assume that a is known, and so wlog $a = 0$ for now. So how do we deal with $\tilde{\Lambda}_t$ (for now, omit agent superscript)? Using Wiener measure as the reference measure, we have

KF story: If $dX = \Sigma\, dW + KX\, dt$, where W, Z are BM, $E[W_t Z_t^T] = t\rho$

$dY = CX\, dt + \sigma\, dZ$

we get filtering equations

$$d\hat{X}_t = \theta_t\, d\hat{Z}_t + K\hat{X}_t\, dt$$

where $\sigma\, d\hat{Z}_t = dY - C\hat{X}_t\, dt$, and

$$\theta_t \sigma^T = V_t C^T + \Sigma\rho\sigma^T \qquad (1)$$

where $$\dot{V}_t = V_t K^T + K V_t + \Sigma\Sigma^T - \theta_t\theta_t^T \qquad (2).$$

For our application, let's seek the steady state solution, taking $C = e_1^T$, $\sigma = \varepsilon$, $\Sigma = I$, $K = -B = -\begin{pmatrix} b_{11} & b^T \\ b & B_0 \end{pmatrix}$. We then have from (1) that $\theta = \varepsilon^{-1} V e_1$, so we write $V = \begin{pmatrix} \varepsilon a & \varepsilon\theta^T \\ \varepsilon\theta & v \end{pmatrix}$. Substituting in, matching terms, and letting $\varepsilon \downarrow 0$, we get finally

$$\begin{cases} \theta = -vb \\ 0 = I - vB_0^T - B_0 v - v b b^T v \end{cases}$$

$$\Lambda_t = \exp\left\{ -\int_0^t (BX_s, dX_s) - \frac{1}{2}\int_0^t |BX_s|^2 ds \right\},$$

$$d\Lambda_t = \Lambda_t \; (-BX_s, dX_s)$$

and $\mathcal{Y}_t = \sigma(\{X^1_s : 0 \le s \le t\})$. The $(\mathcal{Y}_t)$-optional projection $\tilde{\Lambda}_t$ will be representable as

$$\tilde{\Lambda}_t = 1 + \int_0^t H_s \, dX^1_s$$

for some $(\mathcal{Y}_t)$-previsible process H, and what we must have is that for any $L^\infty(\mathcal{Y}_t)$ random variable $Z = \int_0^t \theta_s \, dX^1_s$

$$E[\tilde{\Lambda}_t Z] = E[\Lambda_t Z] = E\left[\int_0^t \Lambda_s (-BX_s, e_1)\theta_s \, ds\right]$$

$$= E\left[\int_0^t H_s \cdot \theta_s \, ds\right]$$

Hence

$$H_s = E[-e_1 \cdot BX_s \Lambda_s | \mathcal{Y}_s] = \frac{E[-e_1 \cdot BX_s \Lambda_s | \mathcal{Y}_s]}{E[\Lambda_s | \mathcal{Y}_s]} \cdot \tilde{\Lambda}_s$$

The interpretation is now very intuitive, since the first factor here is just the conditional expectation of the first component of the drift, given $\mathcal{Y}_s$. Thus what we get is

$$\boxed{d\tilde{\Lambda}_t / \tilde{\Lambda}_t = -e_1 B\hat{X}_t \, dX^1_t}$$

and now we've reduced the problem to a straightforward Kalman filtering!

(iv) When you go through the calculations, you find that the steady-state covariance v of $x_t \equiv (X^2_t, X^3_t, \dots, X^n_t)$ given $\mathcal{Y}_t$ will be the pds solution to

$$\boxed{0 = I - vB_0^T - B_0 v - v bb^T v} \qquad \left[B = \begin{pmatrix} b_{11} & b^T \\ b & B_0 \end{pmatrix} \right]$$

why symmetric?

and then $\boxed{\theta = -vb}$ to get the filtering equation

$$d\hat{x}_t = \theta \, d\hat{W}^1 - (bx^1 + B_0 \hat{x}_t)dt.$$

where

$$d\hat{W}^1_t = d\hat{x}^1_t + (b_{11} x^1_t + b^T \hat{x}_t)\, dt$$

$$= dx^1_t + (b_{11} x^1_t + b^T \hat{x}_t)\, dt$$

from which finally

$$\boxed{d\hat{x}_t = \theta \, dx^1_t + \{\theta(b_{11} x^1_t + b^T \hat{x}_t) - bx^1_t - B_0 \hat{x}_t\} dt}$$

Contracting to manage risk (11/12/07)

(i) Let's consider the situation of a principal with utility U_p who contracts with an agent (such as a trader) whose utility is U_A to invest on his behalf in a market. For simplicity to begin with, we may assume the wealth equation is

$$dw_t = rw_t\,dt + \theta_t\,(\sigma\,dW_t + (\mu - r)\,dt)$$

and the horizon is T. If the agent has amassed a wealth x at time T, the deal is that he will receive $\varphi(x)$. How would the principal set this up to his own best advantage?

One simple answer to the question is that he would take some $k>0$ and $a \in \mathbb{R}$, and define φ by

$$u(x) \equiv \boxed{U_p(x-\varphi(x)) = k\,U_A(\varphi(x)) - a} \equiv u_A(\varphi(x))$$

It should be possible to work this out numerically (and only numerically!) by stick some points y in the region of common values of $U_p(\cdot)$ and $k\,U_A(\cdot) - a$, and inverting the two utilities.

Strict monotonicity of U_p and U_A would imply there exists a unique solution $\varphi(x)$ for any x for which

$$U_p(x) > k\,U_A(0) - a, \qquad U_p(0) < k\,U_A(x) - a.$$

This defines an interval in which φ can be defined, $(\underline{x}, \infty)$. Probably this will require us to set $\varphi(x) = 0$ for $x \leq \underline{x}$, and to stop the wealth process if it falls to $\underline{x}$.

Notice that $\varphi(\cdot)$ is clearly increasing, and it can be shown to be concave in $(\underline{x}, \infty)$. Indeed, if $x_1, x_2 > \underline{x}$, $p = 1-q \in (0,1)$ and concavity of u fails, then $(\bar{x} = px_1 + qx_2)$

$$u(\bar{x}) = u_A(\varphi(\bar{x})) < p\,u(x_1) + q\,u(x_2) = p\,u_A(\varphi(x_1)) + q\,u_A(\varphi(x_2)) \leq u_A(p\varphi(x_1) + q\varphi(x_2))$$

and so $\varphi(\bar{x}) < p\,\varphi(x_1) + q\,\varphi(x_2)$. Hence

$$\begin{aligned}
u_A(\varphi(\bar{x})) = U_p(\bar{x} - \varphi(\bar{x})) &> U_p(\bar{x} - p\varphi(x_1) - q\varphi(x_2)) \\
&= U_p\big(p(x_1 - \varphi(x_1)) + q(x_2 - \varphi(x_2))\big) \\
&\geq p\,U_p(x_1 - \varphi(x_1)) + q\,U_p(x_2 - \varphi(x_2)) \\
&= p\,u_A(\varphi(x_1)) + q\,u_A(\varphi(x_2))
\end{aligned}$$

Contradicting the first inequality.

(ii) How would we try to maximise expected utility of terminal wealth subject to a risk-measure constraint? Suppose that the constraint we aim to impose is a law-invariant coherent risk measure; such things are characterised by Kusuoka (and Wang ... + Panjer) to be of the form

$$\rho(X) = \sup\{\rho^\mu(X) : \mu \in \mathcal{M}_0\}$$

where $\mathcal{M}_0$ is a set of prob^{ys} on $[0,1]$, and

$$\rho^\mu(X) \equiv \int \rho_\lambda(X)\,\mu(d\lambda), \qquad \rho_\lambda(X) \equiv -\lambda^{-1} E[X : X \leq F_X^{-1}(\lambda)] = -\lambda^{-1}\int_0^\lambda F_X^{-1}(x)\,dx.$$

What we shall have is that the optimal X will be $X = \psi(\zeta)$ for some decreasing function ψ so that

$$F_X^{-1}(\lambda) = \psi\left(F_\zeta^{-1}(1-\lambda)\right)$$

Hence

$$\begin{aligned}\rho_\lambda(X) &= -\lambda^{-1}\int_0^\lambda F_X^{-1}(x)\,dx \\ &= -\lambda^{-1}\int_0^\lambda \psi(F_\zeta^{-1}(1-t))\,dt \\ &= -\lambda^{-1}\int_{1-\lambda}^1 \psi(F_\zeta^{-1}(y))\,dy \\ &= -\lambda^{-1}\int_{F_\zeta^{-1}(1-\lambda)}^\infty \psi(z)\,F_\zeta(dz)\end{aligned}$$

and so

$$\begin{aligned}\rho^\mu(X) &= -\int_0^1 \mu(d\lambda)\,\lambda^{-1}\int_{F_\zeta^{-1}(1-\lambda)}^\infty \psi(z)\,F_\zeta(dz) \\ &= -\int_{-\infty}^\infty \psi(z)\left(\int_{1-F_\zeta(z)}^1 \lambda^{-1}\mu(d\lambda)\right)F_\zeta(dz) \\ &= -E[\psi(\zeta)\,g(\zeta)]\end{aligned}$$

where g is non-negative increasing. However, we don't necessarily have g continuous, so no hope of convex/concave in general. But note that all the business to do with quantiles of X is gone!!

(iii) Taking on the general coherent LI risk measure may not be so easy but we could certainly try the situation where $\mathcal{M}_0$ is finite. This would give rise to a finite collection $g_1, \dots, g_n$ of non-negative increasing functions, and the risk-measure constraint

$$\max\left(\rho^{\mu_i}(X)\right) \leq -b \qquad \text{(notice } \rho_\lambda(X) \leq 0 \text{ since } X \geq 0\text{)}$$

becomes the condition

$$E[\psi(\zeta)\,g_i(\zeta)] \geq +b \quad \text{for each } i.$$

The Lagrangian is

$$\max\ E\left[U(\psi(\zeta)) - \lambda\zeta\psi(\zeta) + \sum_{i=1}^n \alpha_i\{\psi(\zeta)g_i(\zeta) - b - z_i\} + \lambda w_0\right]$$

for slack variables $z_i \geq 0$. Complementary slackness $\Rightarrow \alpha_i \geq 0$, and optimising over $X = \psi(\zeta)$ gives us $\boxed{U'(X) = \lambda\zeta - \sum_i^n \alpha_i g_i(\zeta)}$. The maximised Lagrangian is thus

Rubbish! φ is determined <u>only</u> by the utilities, not at all by the strategies chosen

$$\lambda w_0 - \sum_{i=1}^n \alpha_i b + E\left[\tilde{U}\left(\lambda\zeta - \sum_1^n \alpha_i g_i(\zeta)\right)\right].$$

There might be some hope that this could be minimised over the multipliers.

(iv) But there's a bit of a snag here. Suppose there was just one ρ^μ, and that μ were a point mass at $a \in (0,1)$. Then $g_1(\cdot)$ stays at 0 for a bit, then jumps (at $F_\zeta^{-1}(1-a)$) to a^{-1}.

However, what this means is that $X = \psi(\zeta)$ for a ψ which is not monotone...

Here's how to handle that. We know that whatever optimal X^* we come up with, $E[X^*|\zeta]$ will improve $E\,U(X)$ (Jensen) and will still respect the risk constraints (Cherny-Grigoriev show that for atomless dist^n lawinvariance $\Leftrightarrow$ dilatation monotone), so we may wlog suppose X^* is a function of ζ. If X^* were not a monotone decreasing function of ζ, then we could obtain the same law of X by using a different (now decreasing) function of ζ, with lower cost (Dybvig's million-dollar idea). Thus we can insist on ψ being monotone decreasing.

So the trick will be to solve

$$U'(X) = h(\zeta)$$

where $h(y) = \sup\{\lambda x - \sum_1^n \alpha_i g(x) : x \leq y\}$. This will typically create atoms in the law of X^*, as ζ moves through flat regions for h....!

Numerical solution maybe possible, though quite tough - minimising the dual function would involve repeated evaluations, each of which is a 1-D integral.

(v) What contract would be offered to the agent? This could be a bit tricky if we are after optimising $E\,U_p(X_T - \varphi(X_T))$ subject to risk measure constraints, as we can't know φ til we've solved, but we need φ to solve. Working on the assumption that the compensation to the agent is not so large, we may instead just max $E\,U_p(X_T)$, as we were doing before. This then gives us some optimal $X_T^* = \psi(\zeta_T)$ and for this to be optimal for some agent with utility $u \equiv U_A \circ \varphi$ who is not constrained by RM worries, we would need

$$u'(X_T^*) = \lambda\zeta_T$$

so that $u'(x) = \lambda\psi^{-1}(x)$, and we could find u by integrating ψ^{-1}, and then we can deduce φ.

Cleaning up the Bayesian agents story with Angus (8/1/08)

The first Bayesian agent story with Angus (where agents learned about parameters, but got replaced from time to time by ignorant descendants) is in a rather untidy form at the moment, so let's clean up the presentation a bit.

1) Take $\Omega = C(\mathbb{R},\mathbb{R})$, $X_t(\omega) = \omega(t)$, $\mathcal{F}_t = \sigma(\{X_s \mid s \le t\})$ and $\mathbb{P}_0$ the reference probability being such that X_t is a stationary OU process, $dX_t = dW_t - \lambda X_t dt$. Define

$$W_t = X_t - X_0 + \int_0^t \lambda X_s\, ds \qquad (t \in \mathbb{R}).$$

The aim is to tell a stationary story in the end.

2) Think of agent i as a dynasty; at the times τ_k^i $(k \in \mathbb{Z})$ of a stationary renewal process, the old agent is succeeded by his ignorant son. If $(c_t^i)_{t\in\mathbb{R}}$ is the consumption process of dynasty i, the objective is to

$$\max \; E_0\left[\int_{t_0}^{\infty} e^{-\rho_i t}\, U_i(c_t^i)\, \Lambda_t^i\, dt\right]$$

where t_0 is some large negative start value (to be allowed to go to $-\infty$ in a while), $U_i(x) = -\frac{1}{\gamma_i} e^{-\gamma_i x}$ and Λ_t^i is the relevant likelihood ratio; this will experience jumps at the reset times τ_k^i. We'll allow the γ_i to be different, but for a stationary solution we shall need all ρ_i equal to ρ (else one dynasty dominates the others in the end) [In fact, even this we could modify by considering $\tilde{c}_t^i = c_t^i - \rho_i t/\gamma_i$, but it's getting a bit far-fetched].

3) If we assume complete market, then for $t_0 < t < s$ if we consider agent i's time-t price $\pi_t(Y)$ for (marginal) Y at time s, $Y \in b\mathcal{F}_s$, we shall have

$$\pi_t(Y)\; e^{-\rho t} U_i'(c_t^i)\Lambda_t^i = E_0\left[\,Y e^{-\rho s} U_i'(c_s^i)\,\Lambda_s^i \,\middle|\, \mathcal{F}_t\right]$$

and since agents agree on the price of Y, and $Y \in b\mathcal{F}_s$ is arbitrary, we must have

$$\zeta_{ts}^i \equiv \frac{e^{-\rho s} U_i'(c_s^i)\Lambda_s^i}{e^{-\rho t} U_i'(c_t^i)\Lambda_t^i}$$

is the same for all agents. Taking $t = t_0$, we deduce that there is an adapted process ζ_t such that $\forall i$

$$\boxed{e^{-\rho t}\, U_i'(c_s^i)\,\Lambda_s^i = \nu_i\, \zeta_s}$$

for some $\mathcal{F}_{t_0}$-measurable ν_i. Taking logs, and dividing by γ_i,

$$-\frac{\rho t}{\gamma_i} - c_t^i + \frac{1}{\gamma_i}\log\Lambda_t^i = \frac{1}{\gamma_i}\left(\log \nu_i + \log \zeta_t\right)$$

Now suppose that there are N dynasties, $\bar{\gamma} \equiv \left(\frac{1}{N}\sum_1^N 1/\gamma_i\right)^{-1}$ for the harmonic mean of the γ_i. By market clearing,

$$-\frac{\rho t}{\bar{\gamma}} - \frac{1}{N}\delta_t + \frac{1}{N}\sum \frac{1}{\gamma_i}\log\Lambda_t^i = \frac{1}{N}\sum \frac{1}{\gamma_i}\log \nu_i + \frac{1}{\bar{\gamma}}\log \zeta_t$$

How does this behave as $N \to \infty$? Let's suppose that $\bar{\gamma}$ has a limit (again denoted $\bar{\gamma}$, with slight notational abuse.) Then we get

$$\log \zeta_t = \text{const} - \rho t + \bar{\gamma}\lim \sum_i \frac{1}{N\gamma_i}\log\Lambda_t^i$$

so rather remarkably the dividend process δ_t goes away in the limit ... perhaps a better idea is to set $\delta_t = N\sigma X_t$, so that the total dividend scales with the population size. If we do this, we shall find

$$\boxed{\log \zeta_t = \text{const} - \rho t - \sigma X_t + \bar{\gamma}\lim \sum_i \frac{1}{N\gamma_i}\log\Lambda_t^i}$$

Of course, what I've written as a constant is actually $\mathcal{F}_{t_0}$-measurable but now by letting $t_0 \to -\infty$ we get an $\mathcal{F}_{-\infty}$-measurable random variable – but $\mathcal{F}_{-\infty}$ is trivial. Really all that matters is $d(\log \zeta_t)$.

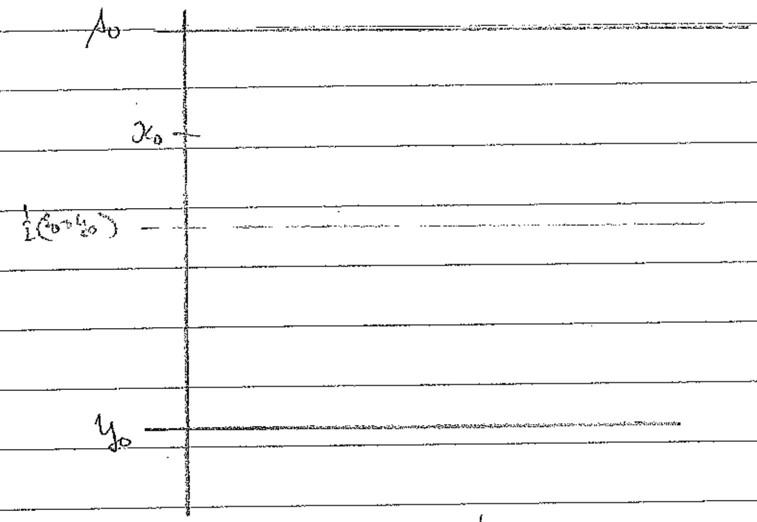
4) Now we can run the story as before to analyse $\lim\left\{\sum_i \frac{1}{N\gamma_i}\log\Lambda_t^i\right\}$, but one thing needs to be noticed. The density $\varphi(\cdot)$ of the time since the last change of generation is naturally taken to be

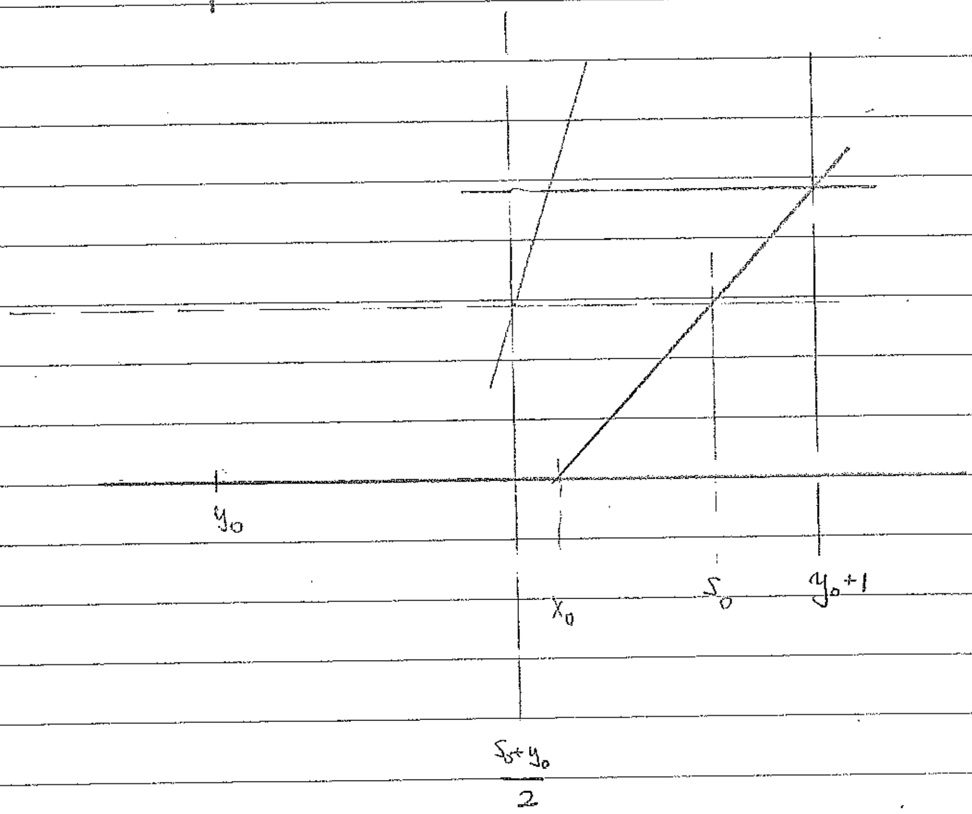
$$\varphi(u) = \frac{\lambda}{1+\varepsilon\lambda}(\varepsilon+u)\lambda e^{-\lambda u}$$

but for this to be the stationary dist^n of the age in a renewal process, we must have that $\varphi(\cdot)$ is decreasing, so we find the necessary condition

$$\lambda\varepsilon \geq 1$$

This isn't all that restrictive.





Interesting harmonic functions of (B_t, S_t, Y_t) (31/1/08)

(i) Jim Obloj has a conjecture about harmonic functions of (B_t, S_t, Y_t), where $S_t = \sup_{s\le t} B_s$, $Y_t = \sup_{s\le t}\{-B_s\}$, that the only possibilities are that

$$h(x, s, y) = h_1(x,s) + h_2(x,y)$$

where h_1, h_2 are two harmonic f^{ns} – so, in short, there are no 'interesting' examples of the three variables all together. But that's not the situation.

(ii) Clearly $h(\cdot, s, y)$ must be linear in $[-y, s]$, so we just need to understand

$$h_+(s,y) \equiv h(s,s,y), \qquad h_-(s,y) \equiv h(-y,s,y).$$

Let's construct interesting martingales by firstly time changing by

$$\tau_t \equiv \inf\{u : S_u + Y_u > t\}$$

to obtain $M_t \equiv B(\tau_t)$, and then projecting down onto the filtration of M. If we use the standard Brownian exponential martingales $\exp\{\theta B_t - \theta^2 t/2\}$, what does this become under these transformations?

(iii) If $M_t = s > 0$, and $Y(\tau_t) = y$, the sample path of M rises at unit speed until it makes a downward jump, which comes at intensity $(M_t + y)^{-1}$. What happens to the exponential martingale as this happens?

~~When an excursion crosses over~~ from s to $-y$, the prob^y it contains no $\frac12\theta^2$ mark is

$$\frac{\theta(s+y)}{\sinh\theta(s+y)}$$

Excursions which don't get more than a away and contain an $(\frac12\theta^2)$-mark come at rate $\theta\coth\theta a - 1/a$, so if M starts at s, with $Y = y$, and goes all the way up to $t > s$ without crossing over to y, then the LT of the time it took for BM to do that is

$$\tfrac12\theta^2 \mapsto \exp\Big[-\int_s^t \Big\{\theta\coth\theta(v+y) - \frac{1}{v+y}\Big\}dv\Big]$$

$$= \exp\Big[-\Big[\log\sinh\theta(v+y) - \log(v+y)\Big]_s^t\Big]$$

$$= \frac{t+y}{s+y}\,\frac{\sinh\theta(s+y)}{\sinh\theta(t+y)}$$

This suggests we use $\boxed{h_+(s,y) = \dfrac{e^{\theta s}(s+y)}{\sinh\theta(s+y)}}$.

(iv) What happens when the process crosses over? It jumps from s to $-y$ at rate $(s+y)^{-1}$, so if $h_-(s,y)$ is to be correct to give a martingale, we shall need

$$0 = \frac{\partial h_+}{\partial s} + \frac{1}{s+y}\left(h_- \frac{\theta(s+y)}{\sinh\theta(s+y)} - h_+\right)$$

and solving this for h_- is quite straightforward calculation:

$$\boxed{h_-(s,y) = \frac{(s+y)\,e^{-\theta y}}{\sinh\theta(s+y)}}$$

But notice that this example admits killing when the gap from top to bottom (or vice versa) is jumped.

Portfolio-constrained Bayesian Merton wealth investor (1/2/08)

(i) Suppose that we have an agent who is allowed to invest in a vector of assets

$$\cancel{dS = S\ \sigma(dW + \lambda dt)}$$

where $\lambda \sim N(\hat{\lambda}_0, \tau_0^{-1} I)$ ~~is~~ not known to the agent, and he has to filter it from observation of $X_t = W_t + \lambda t$. After observing from time 0 to time t, his posterior distribution for λ_t is $\lambda \sim N(\hat{\lambda}_t, (\tau_0 + t)^{-1} I)$, where

$$\hat{\lambda}_t = \frac{X_t + \tau_0 \hat{\lambda}_0}{\tau_0 + t}$$

~~and~~

$$dX_t = d\hat{W}_t + \hat{\lambda}_t\, dt, \qquad d\hat{\lambda}_t = \frac{d\hat{W}_t}{\tau_0 + t}.$$

(ii) Suppose we have an investor who is trying to max $E\, U(w_T)$ subject to some constraint on the portfolio proportions. So if θ_t is the vector of cash-values of the holdings in the risky assets at time t, we shall insist $\theta_t / w_t \in K \quad \forall t$. The value function $V(t, \lambda, \tau, w)$ has the scaling form $V(t,\lambda,\tau,w) = U(w)\, f(t,\lambda,\tau)$, and the wealth dynamics are

$$\begin{cases} dw_t = r w_t\, dt + \theta_t \cdot \{\sigma(d\hat{W}_t + \hat{\lambda}_t dt) - r 1\, dt\} \\ d\hat{\lambda}_t = d\hat{W} / (\tau_0 + t) \end{cases}$$

Working through the HJB equations gets us to

$$0 = \sup_x \left[-\frac{R}{2} x \cdot Vx\, f + r + x \cdot (\sigma \lambda - r) + \frac{x \cdot \sigma f_\lambda}{\tau} + \frac{1}{1-R}\left(\dot{f} + f_\tau + \tfrac{1}{2}\tau^{-2} f_{\lambda\lambda}\right)\right],$$

where x is constrained to lie in K, $x \equiv w^{-1}\theta$.

(iii) For a general constraint set, this seems hard to handle, and even if K were a cone it's likely to be difficult. With no constraints, we know that x maximises

$$-\tfrac{1}{2} R\, x \cdot Vx + x \cdot (\sigma\lambda - r) \frac{\tau}{\tau + (T-t)(1 - 1/R)}$$

so as a heuristic guess maybe it would be to maximise the same quadratic under the constraint – a bit physicsy, but maybe not too bad..

One issue in practice would be what precision to use....

with $c \equiv t w$, $\theta = y w$, the thing we have to optimize is

$$\frac{1}{2} f^2 \psi'' + \left(\frac{1}{2} f^2 - \lambda y - (1-R) f^2 y \right) \psi'$$

$$- \left\{ \rho + (R-1)\left(r + y(\mu - r) - t \right) - \frac{1}{2} R(R-1) f^2 y^2 \right\} \psi \quad + t^{1-R}$$

Optimal investment/consumption with H-R dynamics (2/2/08)

(i) Suppose we can invest in a single risky asset

$$dS_t = S_t(\sigma_t\, dW_t + \alpha dt)$$

where σ_t is a function of the effect $Z_t \equiv \int_{-\infty}^t \lambda e^{\lambda(s-t)}(X_s - X_t)\, ds$, where $X_t = \log S_t$.

Thus

$$dX_t = \sigma_t\, dW_t - \tfrac{1}{2}\sigma_t^2 dt$$

and $dZ_t = d\left(e^{-\lambda t}\int_{-\infty}^t \lambda e^{+\lambda s} X_s\, ds - X_t \right) = -\lambda Z_t dt - dX_t$, so we have

$$dZ_t + \lambda Z_t dt = -f(Z_t)\, dW_t + \tfrac{1}{2} f(Z_t)^2 dt \qquad (\sigma_t \equiv f(Z_t))$$

therefore Z is an autonomous diffusion.

(ii) An agent wishes to obtain

$$\max\ E\left[\int_0^\infty e^{-\rho t} U(c_t) dt \,\middle|\, w_0 = w,\ Z_0 = z\right] \equiv V(w,z)$$

where wealth satisfies

$$dw_t = r w_t\, dt + \theta_t\left(\sigma_t\, dW_t + (\alpha - r)dt\right) - c_t\, dt,$$

and U is CRRA. The HJB equation here is

$$0 = \sup_{\theta, c}\Big[U(c) - \rho V + (rw + \theta(\alpha - r) - c)V_w + (\tfrac{1}{2}f(z)^2 - \lambda z)V_z$$
$$+ \tfrac{1}{2}\left(\sigma^2\theta^2 V_{ww} - 2\sigma\theta f V_{wz} + f(z)^2 V_{zz}\right)\Big]$$

and by scaling we have

$$V(w,z) = \psi(z)\, U(w).$$

The HJB equation works out as

$$\boxed{0 = \sup_y \Big[-\frac{R}{2} f^2 y^2 \psi + y\{(\alpha - r)\psi - f^2\psi'\} + \frac{R\psi^{1-1/R}}{1-R} - \frac{\rho\psi}{1-R} + r\psi + (\tfrac{1}{2}f^2 - \lambda z)\frac{\psi'}{1-R} + \tfrac{1}{2} f^2 \frac{\psi''}{1-R} \Big]}$$

and this gives us $y = \{(\alpha - r)\psi - f^2\psi''\}/Rf^2\psi$, and the HJB equation for ψ.

At this stage, it looks like only numerics will work.... but we could fake things, and assume the form of ψ, then derive f^2 (we get a quadratic for f^2)... though we would have to watch that the solution to the quadratic did not go negative.

Controlling drawdown for wealth (6/2/08)

(i) Suppose we have the usual dynamics

$$dw_t = rw_t\,dt + \theta\cdot(\sigma dW_t + (\mu - r)dt)$$

with objective

$$\max E\left[\int_0^\infty e^{-\rho t} U(w_t)\,dt \;\Big|\; w_0 = w,\ \bar{w}_0 = \bar{w}\right] \equiv V(w, \bar{w})$$

where $\bar{w}_t = \sup_{s\le t} w_s$, and we require $w_t \geq b\,\bar{w}_t \ \forall t$, where $b \in (0,1)$ is fixed.
Scaling gives

$$V(w,\bar{w}) = \bar{w}^{1-R}\, V(w/\bar{w}, 1) = \bar{w}^{1-R}\, v(x) \qquad (x = w/\bar{w} \in [b,1])$$

HJB gives

$$0 = \sup_\theta \left[U(w) - \rho V + \tfrac{1}{2}\, \theta^T \sigma\sigma^T \theta\, V_{ww} + (rw + \theta\cdot(\mu - r))\, V_w \right]$$

So optimal θ is given by $\boxed{\theta^* = (\sigma\sigma^T)^{-1}(\mu - r)\; V_w / V_{ww}}$ and we get

$$0 = U(w) - \rho V + rw\,V_w - \tfrac{1}{2}\kappa^2 \frac{V_w^2}{V_{w}} \qquad \left[\kappa^2 = (\mu-r)^T(\sigma\sigma^T)^{-1}(\mu - r)\right]$$

(ii) Now $V_w = \bar{w}^{-R}\, v'(x)$, $V_{ww} = \bar{w}^{-1-R}\, v''(x)$ and thus the HJB becomes

$$\boxed{0 = U(x) - \rho v(x) + rx\, v'(x) - \tfrac{1}{2}\kappa^2 \frac{v'(x)^2}{v''(x)}}$$

We have that the term in $d\bar{w}$ must vanish, giving the BC

$$\boxed{(1-R)\, v(1) - v'(1) = 0}$$

and the other condition is that the optimal θ must be zero at $x = b$.

(iii) In terms of the dual variables $z = v'(x)$, $J(z) = v(x) - zx$, we obtain

$$\boxed{0 = U(-J_z) + \tfrac{1}{2} z^2 \kappa^2 J_{zz} + (\rho - r) z J_z - \rho J}$$

which is (unfortunately!) non-linear. The form of the solution will be that $\exists\ 0 < z_1 < z_b$ s.t.

$$J(z) = A\tilde{U}(z) \qquad \text{for } z \le z_1$$
$$\quad = v(b) - bz \qquad \text{for } z \ge z_b$$

J solves the ODE inbetween, with C^1 at z_1, C^2 at z_b. Should be OK numerically.

Optimal investment with price impact effect (7/2/08)

(i) One of Ralf's students, Michael Busch, talked about modelling hedge fund returns, and this suggested that it could be fun to try to incorporate a price impact effect. Let's go back in history to WN X, pp 16–20, where I describe a simple model for price impact. If there is some underlying driving semimartingale (X_t) such that the price S_t at time t is represented as

$$S_t = f(\eta_t, X_t)$$

where f is some suitably smooth function, and η_t is the number of units of the stock you hold at time t, then the liquidation value of the holding of stock is

$$F(\eta_t, X_t) = \int_0^{\eta_t} f(y, X_t)\, dy$$

If the agent holds ξ_t in cash at time t, and η_t shares, the liquidation value of his portfolio is

$$v_t = \xi_t + F(\eta_t, X_t)$$

Assuming zero interest rate, the changes in liquidation value of the portfolio must come entirely from changes in X:

$$dv_t = F_X\, dX_t + \tfrac{1}{2} F_{XX}\, d\langle X\rangle_t$$

If we had an interest rate different from 0, we would see

$$dv_t = r_t \xi_t\, dt + F_X\, dX_t + \tfrac{1}{2} F_{XX}\, d\langle X\rangle_t$$

$$= r_t v_t\, dt + F_X\, dX_t + \tfrac{1}{2} F_{XX}\, d\langle X\rangle_t - r_t F\, dt .$$

The mark-to-market value of his portfolio is

$$w_t = \xi_t + \eta_t f(\eta_t, X_t).$$

Suppose his objective is

$$\max E \int_0^\infty e^{-\rho t} U(w_t)\, dt$$

– how should he behave?

(ii) This is hard to do at this level of generality, but under some more specific assumptions we can go a bit further. Letting $f(\eta, x) \equiv \psi(\eta) . x$, and $dX = X(\sigma dW + \mu dt)$, we have a value function

$$V(v, \eta, X) = \sup E\left[\int_0^\infty e^{-\rho t} U(w_t)\, dt \;\middle|\; v_0 = v, \eta_0 = \eta, X_0 = X \right]$$

which scales: for $\lambda > 0$, $V(\lambda v, \eta, \lambda X) = \lambda^{1-R} V(v, \eta, X)$.

However, there's a snag: if $f' > 0$, then you borrow unboundedly, putting the money into the risky asset, and the value is unbounded... needs another element to make it well posed.

If $z \equiv v/X$, then a little calculation gets us to

$$dz = (\psi - z)(\sigma dW + (\mu - r - \sigma^2) dt)$$

(iii) How if we were to introduce a proportional TC, so that

$$dv = rv\,dt + \psi(\eta)(dX - rX\,dt) - \epsilon X \psi(\eta)\, |d\eta| \quad ?$$

Let's try representing the value function as

$$V(v, \eta, X) = X^{1-R} g(z, \eta)$$

— we need to keep track of cash account and share account separately, and we've lost the scaling, so it's genuinely 2-dimensional; also, need to put some imperfections in to stop the problem being ill-posed, so all in all this doesn't look v. promising.

Care – if $0 < R < 1$, we do indeed want $\sup \{ g(p, \xi) : 0 \leq p < 1 \}$, but for $R > 1$ we want inf. (!) $\longrightarrow$

Optimal investment/consumption with variable liquidity (7/2/08)

(i) Another spin-off from the visit of Ralf's group was a problem presented by Frank Seifried on work by him, Peter Diesinger + Gerald Kraft, which envisaged a standard terminal wealth optimisation, but with the additional feature that there's a 2-state Markov chain such that in state 0 you can't trade the stock, in state 1 you can trade as usual – representing a liquidity lock-down. They were able to compute through the case of log utility, but CRRA is apparently harder.

(ii) A variant of this story would be to suppose that for each state ξ of the chain the opportunities to trade come with intensity $\lambda(\xi)$ – like the old RZ liquidity study! The case $\lambda(0)=0$, $\lambda(1)=\infty$ is what the ITWM guys have done. Let's also go to a consumption problem, to make it all time homogeneous, though perhaps discounted utility of wealth might be a good one too.

If x is wealth in cash, y the wealth in stock, then between rebalancings

$$\left.\begin{aligned} dx &= (rx-c)\,dt \\ dy &= y(\sigma dW+\mu dt) \end{aligned}\right\}$$

and at the times of rebalancing we select x, y to max $V(x,y;\xi')$ subj to $x+y=w$, if we jump to state ξ', or to max $V(x,y;\xi)$ if we don't jump. Usual scaling story tells us that $V(\lambda x, \lambda y;\xi) = \lambda^{1-R}V(x,y;\xi)$. We have if $\bar V(w;\xi) \equiv \sup\{V(x,y;\xi) : x+y=w\}$ that

$$0 = \sup\Big[U(c)-\rho V+(rx-c)V_x+\tfrac12\sigma^2y^2V_{yy}+\mu yV_y+\lambda(\xi)(\bar V-V)+Q(\bar V-V)\Big]$$

If we set $p \equiv \frac{y}{x+y}$, $V(x,y;\xi) = U(x+y)\,g(p;\xi) \equiv U(w)\,g(p;\xi)$, and $\bar g(\xi) \equiv \sup\{g(p;\xi) : 0\le p\le 1\}$, then we shall have ($c=wz$)

$$0 = \sup_z \frac{1}{1-R}\Big[z^{1-R}+\tfrac12\sigma^2p^2(1-p)^2g''+\{(\mu-r)(1-p)-\sigma^2pR(1-p)+z\}g'p+\lambda(\bar g-g)+Qg$$
$$+\left(-\rho+(\mu-r)p(1-R)+r(1-R)-(1-R)z-\tfrac12R\sigma^2(1-R)p^2\right)g\Big]$$

as the HJB equations. Optimal z comes out to be

$$z^* = \left(\frac{(1-R)g-pg'}{1-R}\right)^{-1/R}.$$

Should be OK numerically. Take initial policy to be never alter holding of stock, consume at rate $c=rx$ from cash, so $V(x,y;\xi) = \rho^{-1}U(rx) = \rho^{-1}U(w)\left(\frac{rx}{w}\right)^{1-R} = \rho^{-1}U(w)(r(1-p))^{1-R}$, so that initially $g(p,\xi) = \rho^{-1}(r(1-p))^{1-R}$.

(iii) BC at the max value of p? Assume we just consume the interest until the first jump. If $\bar p$ is biggest value of p, we get value

$$U(1)\,g(\bar p,\xi) = \frac{U(r(1-\bar p))}{\rho+\lambda(\xi)} + \frac{\bar g(\xi)\,U(\bar p)\,\lambda(\xi)}{\lambda(\xi)+\rho+(R-1)(\mu-\frac12\sigma^2R)}$$

Joint law of (M_t, S_t, Y_t) for a cts martingale (13/2/08)

(i) Suppose M is a continuous martingale, $M_0 = 0$, and $S_t = \sup_{s\le t} M_s$, $Y_t = \sup_{s\le t}(-M_s)$. Can we characterise the possible joint laws of the triple (M_t, S_t, Y_t)? Earlier we found N+S conditions for the joint law of (M_t, S_t); the work of Cox + Obłój suggests there may be something similar for the trivariate law.

(ii) Some necessary conditions.

$$(0)\quad \left.\begin{aligned} E[M_t : S_t \ge a] &= a\,P[S_t \ge a] & \forall a>0 \\ E[M_t : Y_t \ge b] &= -b\,P[Y_t \ge b] & \forall b>0 \end{aligned}\right\}$$

are obvious necessary conditions, from OST. Likewise, using OST again gives

$$(1)\quad \begin{aligned} 0 = E[M(H_a \wedge H_{-b})] &= E[M_t : S_t < a, Y_t < b] + a\,P[H_a < H_{-b}, H_a < t] \\ &\quad - b\,P[H_{-b} < H_a, H_{-b} < t] \end{aligned}$$

The first term, $E[M_t : S_t < a, Y_t < b]$ can be evaluated directly from the joint law, but what of the other two? Notice

$$\left.\begin{aligned} P[H_a < H_{-b}, H_a < t] &= P[H_a < t] - P[H_{-b} < H_a < t] \\ P[H_{-b} < H_a, H_{-b} < t] &= P[H_{-b} < t] - P[H_a < H_{-b} \le t] \end{aligned}\right\}$$

so adding these gives

$$(2)\quad P[H_a < H_{-b}, H_a < t] + P[H_{-b} < H_a, H_{-b} < t] = P[S_t > a] + P[Y_t > b] - P[S_t > a, Y_t > b]$$

Thus (1)+(2) is a pair of simultaneous linear equations for the two unknowns $P[H_a < H_{-b}, H_a < t]$ and $P[H_{-b} < H_a, H_{-b} < t]$ in terms of the joint dist^n of (M_t, S_t, Y_t). Solving gives

$$(3)\quad P[H_a < H_{-b} \wedge t] = \frac{b - \int_{\{s<a,\, y<b\}} (b+m)\,dF}{a+b} \le \frac{b}{a+b}$$

$$(4)\quad P[H_{-b} < H_a \wedge t] = \frac{a + \int_{\{s\le a,\, y\le b\}} (m-a)\,dF}{a+b} \le \frac{a}{a+b}$$

Check that (4) ≥ 0 (similarly for (3))

$$\begin{aligned} a + E[M-a\,;\, S\le a, Y\le b] &= a + E[M-a\,;\, S\ge a \text{ or } Y\le b] \qquad \text{(by (0))} \\ &= a\,P[S<a, Y>b] + E[M : S\ge a \text{ or } Y\le b] \end{aligned}$$

$$= a\, P[S<a,\, Y>b] - E[M : S<a,\, Y>b] \qquad (EM=0)$$

$$= E[a-M : S<a,\, Y>b]$$

$$\geq 0 .$$

Notice that (S) allows us to express

$$P[H_{-b} < H_a < t] = P[H_a < t] - P[H_a < H_{-b} \wedge t]$$
$$= P[S>a] - P[H_a < H_{-b} \wedge t]$$

in terms of the joint law of (M, S, Y).

Optimization under soft drawdown constraints (14/2/08)

(i) Drawdown constraints of the form $w_t \geq b\bar{w}_t = b\sup_{s\leq t} w_s$ don't seem to do too well in practice... so maybe we take an objective of the form

$$\max E\int_0^\infty e^{-\rho t}\{U(c_t) + \gamma w_t^\alpha \bar{w}_t^\beta\}\,dt$$

where we'll want $\alpha+\beta = 1-R$ to preserve scaling, and $\alpha\gamma > 0 > \beta\gamma$ (so for $R>1$ we would presumably do $\alpha < 1-R$, $\beta > 0$ and for $R<1$ we'd do $\alpha > 1-R$, $\beta < 0$, $\gamma > 0$).

(ii) The value f$^{\text{n}}$ $V(w,\bar{w}) = \bar{w}^{1-R} f(z)$, where $z \equiv w/\bar{w}$, and writing $c = qw$, the HJB becomes

$$0 = \sup\Big[\bar{w}^{1-R}U(q)z^{1-R} + \gamma\bar{w}^{1-R}z^\alpha + \bar{w}^{1-R}zf'\{r+\theta(\mu-r)-q\} + \tfrac{1}{2}\sigma^2\theta^2z^2f''\,\bar{w}^{1-R} - \rho\bar{w}^{1-R}f\Big]$$

so

$$0 = \sup\Big[U(q)z^{1-R} + \gamma z^\alpha - \rho f + zf'\{r+\theta(\mu-r)-q\} + \tfrac{1}{2}\sigma^2\theta^2z^2f''\Big]$$

$$\Rightarrow\quad zq = \frac{c}{\bar{w}} = (f')^{-1/R}, \qquad \theta = -\frac{(\mu-r)f'}{\sigma^2zf''}$$

and HJB is

$$0 = \tilde{U}(f') + \gamma z^\alpha - \rho f + zrf' - \tfrac{1}{2}\kappa^2\frac{f'^2}{f''}$$

Note the condition $V_{\bar{w}} = 0$ when $w = \bar{w}$ comes out as

$$(1-R)f(1) - f'(1) = 0$$

What about behaviour at the bdy $z=0$?? Let's approximate the story by saying that at the bottom grid point x_0, we come out of the stock and consume at rate $\lambda w \equiv (r-\varepsilon)w$ until wealth rises to $x_1 > x_0$. The time t_1 when this happens is $t_1 = \varepsilon^{-1}\log(x_1/x_0)$, and the value we get at x_0 will be

$$f(x_0) = U(\lambda x_0)\,\frac{1-e^{-(\rho+\varepsilon(R-1))t_1}}{\rho+\varepsilon(R-1)} + e^{-\rho t_1}f(x_1)$$

This fixes a boundary condition at x_0.

Corporate finance + the monetary transmission mechanism (18/2/08)

(i) There's a paper so titled in RFS 19 by Patrick Bolton + Xavier Freixas, which tells an interesting story involving banks, firms, households and the government. Generalising slightly in some respects, there's a unit mass of firms, each of which has success prob'y $p \in [0,1]$, $\pi(dp)$ being the mass of firms with success prob'y p. Everyone knows for each firm what p is. The owner of each firm has wealth $W < I$, where I is the amount required to launch a venture, which delivers $V > I$ at time 1 if successful, else it delivers 0.

The required $I - W = \varepsilon$ needed to launch the venture can either be borrowed directly, or via a bank. The bank's manager has wealth w initially, and if he raises x through deposits or equity issuance, may make loans up to $(w+x)/\kappa$ in total. There is a per-unit cost c for making loans. The good thing about a bank loan is that if the venture fails, then the bank recovers v if it's good (proportion $1-\mu$) or βv if it's bad (proportion μ), so the mean recovered value is $(1-\mu)v + \mu\beta v \equiv v_0$, say.

The government issues bonds with face value G and interest rate R_G. The households want to

$$\max \quad U(c_0) + U(c_1)$$

where c_i is period-i consumption, and we'll suppose that the banks want to maximize EU_b (profit), firms want to max EU_f (profit). Households start with wealth 1.

(ii) It looks to me as if it will be helpful to consider a central planner problem. At time 0, the planner will directly allocate $\varepsilon\varphi_1(p)\pi(dp)$ to firms with prob'y p of success and give x_2 to the banks, invest x_3 in govt. bonds. The banks now allocate $\varepsilon\varphi_2(p)\pi(dp)$ to firms with prob'y p of success, where

$$\int \varepsilon\varphi_2(p)\pi(dp) = (x_2+w)/\kappa \quad , \quad 0 \le \varphi_1 + \varphi_2 \le 1 \quad , \quad \varphi_1, \varphi_2 \in \{0,1\}.$$

The aggregate value available at time 1 is therefore

$$V\int p\,\varphi_1\,d\pi + \int (pV + (1-p)v_0 - c)\,\varphi_2\,d\pi + x_3 R_G \equiv X_1 \text{ , say.}$$

At time 0, we started with total wealth $1 + w + W \equiv X_0$, say. At time 1 we will pay $\eta_+(p)$ to each successful p-firm, $\eta_0(p)$ to each unsuccessful p-firm, and y_{bG} to each good bank, y_{bB} to each bad bank. Total payments to banks + firms come to $Y_1 = \mu y_{bB} + (1-\mu)y_{bG} + \int\{\eta_+(p)p + \eta_0(p)(1-p)\}\pi(dp)$, so the central planner's objective is

$$\boxed{\begin{aligned} &U(X_0 - \underbrace{x_1}_{-W} - x_2 - x_3) + U(X_1 - Y_1) + \alpha_b\{\mu U_b(y_{bB} - w) + (1-\mu)U_b(y_{bG} - w)\} \\ &+ \alpha_f \int\{p U_f(\eta_+(p) - W) + (1-p)U_f(\eta_0(p) - W)\}\pi(dp) \end{aligned}}$$

$$\left.\begin{aligned} x_1 &= \varepsilon \int \varphi_1(p)\, \pi(dp) \\ \frac{x_2 + w}{\kappa} &= \varepsilon \int \varphi_2(p)\, \pi(dp) \end{aligned}\right\} \quad \text{so} \quad Y_0 = \varepsilon \int \varphi_1(p)\, d\pi + \left(\kappa\varepsilon \int \varphi_2(p)\, d\pi\right) + x_3$$

($\geq$ in general)

(iii) It could be that we should weight the utilities of good and bad banks differently; or that the weights for different p-firms should be different. In B+F, the parameter W seems superfluous – it never enters into any of the conditions or results. So let's for now make the simplifying assumption that $\boxed{W=0}$ and interpret a p-firm as an entrepreneurial opportunity which requires initial investment ε, and returns V with prob p. This way, there is no "agent" associated with a firm, and no need to consider preferences for any such agent. If we write $Y_0 = x_1 + x_2 + x_3 + w$, the total amount invested at time 0, we then get the objective

$$U(X_0 - Y_0) + U(X_1 - Y_1) + \alpha_G (1-\mu) U_b(y_G - w) + \alpha_B \mu U_b(y_B - w)$$

for non-negative α_G, α_B, with y_{bG}, y_{bB} shortened to y_G, y_B resp.

Optimising over x_3 gives us

$$\boxed{U'(X_0 - Y_0) = R_G U'(X_1 - Y_1),}$$ ~~assuming~~ an interior solution

Optimising over y_G, y_B leads to the FOC

$$\boxed{U'(X_1 - Y_1) = \alpha_G U_b'(y_G - w) = \alpha_B U_b'(y_B - w)}$$

Considering now the dependence on $\varphi_1(p), \varphi_2(p)$, it will be worth raising $\varphi_1(p)$ from zero if

$$\boxed{pV - \varepsilon R_G \geq 0,}$$

it will be worth raising φ_2 from zero if

$$\boxed{(pV + (1-p)v_0 - c) - R_G \varepsilon \kappa \geq 0}$$

and if both of these apply, we choose to raise φ_1 or φ_2 depending on which is the larger. Notice that if R_G is too small, the total $x_1 + x_2 + W$ allocated to firms may exceed X_0. Notice also that the optimal choices of x_1, x_2 depend only on R_G, V, v_0, c, and not on α_B, α_G, y_G, y_B, … nor on G. Thus open-market operations (varying G) will not affect the optimal loan policy.

Optimal Contracting in a dynamic model (22/2/08)

(i) There's an interesting paper by Peter DeMarzo + Yuliy Sannikov (JF 61, 2681–2724, 2006) "Optimal security design and dynamic capital structure in a continuous-time agency model", which tells the following story. There's an enterprise which produces random returns, its value at time t, Y_t, given by

$$Y_t = \sigma W_t + \mu t$$

(wlog, let's simplify to $\sigma = 1$). A principal intends to hire an agent to manage this project, with a wage schedule (I_t) increasing adapted, and a termination time τ. At τ, the principal gets L, the agent gets R. The agent may misreport Y_t, telling the principal that in fact the results were

$$\hat{Y}_t = Y_t - \int_0^t a_s\,ds.$$

The agent can secretly save, as well as misreporting. His savings S_t at time t being given by

$$dS_t = \rho S_t\,dt + dI_t - dC_t + (\lambda a_t^+ - a_t^-)\,dt$$

where $\lambda \in (0,1]$, and C is his consumption. The agent's objective is

$$E\left[\int_0^\tau e^{-\gamma s}dC_s + e^{-\gamma\tau}R\right] = R + E\left[\int_0^\tau e^{-\gamma s}(dC_s - \gamma R\,ds)\right]$$

which must exceed some reservation value v_0, and the principal's objective is

$$E\left[\int_0^\tau e^{-rs}(d\hat{Y}_s - dI_s) + e^{-r\tau}L\right] = L + E\left[\int_0^\tau e^{-rs}(d\hat{Y}_s - dI_s - rL\,ds)\right]$$

What is the principal's optimal choice of I, τ, and what is the agent's optimal response to it?

(ii) DMS assume that

$$\gamma > r \geq \rho$$

so under this hypothesis, we have

Proposition 1 The agent will never save, if constrained to keep $S \geq 0$.

Proof Set $\varepsilon \equiv \gamma - \rho > 0$, and $dK_t = dI_t + (\lambda a_t^+ - a_t^-)\,dt$, supposed fixed. The dynamics of the savings account are expressed as $(\tilde{S}_t = e^{-\rho t}S_t)$

$$d\tilde{S}_t = e^{-\rho t}(dK_t - dC_t)$$

and so if we develop

$$\int_0^\tau e^{-\gamma s}dC_s = \int_0^\tau e^{-\varepsilon t}(e^{-\rho t}dC_t)$$

$$= \int_0^\tau e^{-\varepsilon t}(e^{-\rho t}dK_t - d\tilde{S}_t)$$

$$= \int_0^\tau e^{-\varepsilon t-\rho t}dK_t - \left[e^{-\varepsilon t}\tilde{S}_t\right]_0^\tau - \int_0^\tau \varepsilon e^{-\varepsilon t}\tilde{S}_t\,dt$$

$$= \int_0^\tau e^{-\gamma t} dK_t + \tilde{S}_\tau - e^{-(\varepsilon+\gamma)\tau}\tilde{S}_0 - \varepsilon \int_0^\tau e^{-\varepsilon t}\tilde{S}_t\, dt$$

$$\le \int_0^\tau e^{-\gamma t} dK_t + \tilde{S}_0$$

which bound is attained if we immediately consume $\tilde{S}_0$ and then consume dK.

We may therefore wlog assume

$$dC_t = dI_t + (\lambda a_t^+ - a_t^-)\, dt.$$

(iii) Let's now restrict attention to the situation where $\lambda = 1$; what we hope to prove is that in this situation the agent will never benefit by mis-reporting, from which it is obvious that for $\lambda \in (0,1)$ there can be no benefit to mis-reporting.

The wage schedule and the stopping time are going to respond to the agent's reports, we expect, but a neat way to handle this is to write the principal's objective as

$$E\left[Z^a_\tau \int_0^\tau e^{-rs} (dY_s - dI_s - rL\, ds) \right]$$

and the agent's as

$$E\left[Z^a_\tau \int_0^\tau e^{-\gamma s} (dC_s - \gamma R\, ds) \right]$$

where

$$dZ^a_t = -Z^a_t\, a_t\, dW_t$$

is the Cameron-Martin change-of-measure martingale. In this formulation, the processes Y, I and the stopping time τ are <u>not</u> altered by the choice a made by the agent, and the agent's objective is expressed as

$$E\left[Z^a_\tau \int_0^\tau e^{-\gamma s} (dI_s + a_s\, ds - \gamma R\, ds) \right]$$

$$= E\left[Z^a_\tau \int_0^\tau e^{-\gamma s} (dI_s - \gamma R\, ds) + \int_0^\tau Z^a_t\, a_t\, e^{-\gamma t} dt \right]$$

$$= E\left[Z^a_\tau \int_0^\tau e^{-\gamma s} (dI_s - \gamma R\, ds) - Z^a_\tau \int_0^\tau e^{-\gamma t} dW_t \right]$$

$$= E\left[Z^a_\tau \left\{ \int_0^\tau e^{-\gamma s} (dI_s - \gamma R\, ds) - \int_0^\tau e^{-\gamma t} dW_t \right\} \right]$$

The point of this is that the agent's influence is only in Z^a_τ, the other part $\xi \equiv \int_0^\tau e^{-\gamma s} (dI_s - \gamma R\, ds) - \int_0^\tau e^{-\gamma t} dW_t$ being set by the principal. Thus what the agent wants to do is to pick a positive $\mathcal{F}_\tau$-measurable random variable

Z_τ with mean 1 so as to maximise $E[Z_\tau \xi]$, and this will require that ξ is bounded, with Z concentrated on the set where ξ attains its maximum.

~~If the principal chooses a policy so that~~

$$\xi = A \quad \text{a.s.}$$

where A is the reservation level of the agent's utility, then it is immaterial what action the agent chooses!

Thus we can attack instead this version of the principal's problem:

$$\boxed{\begin{aligned} &\max \quad E \int_0^\tau e^{-rt}(dY_t - dI_t - rL\,dt) \\ &\text{subj to} \quad \xi \equiv \int_0^\tau e^{-\gamma s}(dI_s - \gamma R\,ds) - \int_0^\tau e^{-\gamma s} dW_s = A \end{aligned}}$$

(iv) The principal will select an increasing process I, and stop when

$$\xi_t \equiv \int_0^t e^{-\gamma s}(dI_s - \gamma R\,ds - dW_s)$$

gets up to A. If we set

$$V(t,\xi) = \sup E\left[\int_t^\tau e^{-rs}((\mu - rL)ds - dI_s) \;\middle|\; \xi_t = \xi\right]$$

then $V(t,\xi_t) + \int_0^t e^{-rs}((\mu - rL)ds - dI_s)$ is a supermartingale and a martingale under optimal control. Now

$$dV \doteq \left\{V_t - \gamma R e^{-\gamma t} V_\xi + \tfrac{1}{2} e^{-2\gamma t} V_{\xi\xi}\right\}dt + \left(e^{-\gamma t}V_\xi - e^{-rt}\right)dI_t + (\mu - rL)e^{-rt}dt$$

so we get the HJB equation looking like

$$\begin{cases} 0 = V_t - \gamma R e^{-\gamma t} V_\xi + \tfrac{1}{2} e^{-2\gamma t} V_{\xi\xi} + e^{-rt}(\mu - rL) \\ e^{-\gamma t}V_\xi - e^{-rt} \le 0 \end{cases},$$

with $V(t,A) = 0$. Write

$$V(t,\xi) = e^{-rt}\varphi(t, e^{\gamma t}(A-\xi)) \equiv e^{-rt}\varphi(t,z)$$

to find

$$\boxed{\begin{cases} 0 = -r\varphi + \varphi_t + \gamma(R+z)\varphi_z + \tfrac{1}{2}\varphi_{zz} + \mu - rL \\ \varphi_z \ge -1, \quad \varphi(t,0) = 0 \end{cases}}$$

in $z \ge 0$. Should be OK numerically if we enforce stopping by time N; this is probably all we can do.

A simple example with transaction costs (25/2/08)

There was a story Phil Dybvig once told me about transaction costs, to do with exponential utility and Brownian assets, where the answer came out to be quite explicit ... I'm not sure I remember it, but I think it might have gone something like this.

(i) If we have assets

$$dS_t = \sigma\, dW_t + \mu\, dt$$

and an agent whose wealth equation is

$$dw_t = r w_t\, dt + \theta_t \cdot (\sigma\, dW_t + (\mu - r)\, dt) - c_t\, dt,$$

aiming to

$$\max E\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt\right] \qquad U'(x) = e^{-\gamma x}$$

then the optimal solution is to keep θ constant, and to consume at rate $c_t = r w_t + b$ for some b. Let's work out the objective if we follow this policy. We have

$$w_t \sim N(w_0 + kt,\ vt) \qquad \left.\begin{aligned} k &= \theta(\mu - r) - b \\ v &= \theta^T \sigma \sigma^T \theta \end{aligned}\right\}$$

so we have

$$-\frac{1}{\gamma} E \int_0^\infty \exp\{-\rho t - \gamma r w_t - \gamma b\}\, dt$$

$$= -\frac{1}{\gamma} \int_0^\infty \exp\{-\rho t - \gamma b - \gamma r (w_0 + kt) + \tfrac{1}{2}\gamma^2 r^2 v t\}\, dt$$

$$= -\frac{1}{\gamma} \cdot \frac{e^{-\gamma b - \gamma r w_0}}{\rho + \gamma r k - \frac{1}{2}\gamma^2 r^2 v}$$

Optimising over b gives

$$\rho + \gamma r k - \tfrac{1}{2}\gamma^2 r^2 v = r$$

so that

$$\boxed{\gamma r b = \rho - r + \gamma r\, \theta(\mu - r) - \tfrac{1}{2}\gamma^2 r^2 |\sigma^T \theta|^2}$$

and optimising over θ gives $\theta = (\gamma r)^{-1} (\sigma\sigma^T)^{-1} (\mu - r)$. For a fixed (not necessarily optimal) θ, the objective we get by choosing b optimally will be

$$-(\gamma r)^{-1} \exp\left\{-\gamma r w_0 - \frac{\rho}{r} + 1 - \gamma\theta \cdot (\mu - r) + \tfrac{1}{2}\gamma^2 r |\sigma^T \theta|^2\right\}$$

(ii) Suppose we try doing an initial move from portfolio θ_0 to some other portfolio θ_1, where the move costs us $\varepsilon \sum_i |\Delta\theta_i|$. What's the best move to make?

We have problem well posed iff $Q(1-1/R)<0$, and then $\gamma = -Q(1-R^{-1})$

Optimal investment/consumption with retirement (25/2/08)

There was a submission to BPS 2008 which proposed the problem of choosing (c, θ, τ) to optimise

$$E \int_0^\infty e^{-\rho t}\{ U(c_t) - \lambda I_{t\le\tau} \} dt$$

with wealth dynamics

$$dw_t = rw_t + \theta(\sigma dW_t + (\mu - r)dt) + \varepsilon I_{\{t\le\tau\}} - c_t dt$$

We see τ as the time of retirement, and ε the income stream up to retirement, λ the disutility for working. I didn't go through their paper in detail, but I guess the story runs like this.

If $V = V(w)$ is the value function, then we expect $V(w) = \gamma_M^{-R} U(w)$ for $w \ge w^*$ for some critical w^*. Below w^*, we shall have

$$0 = \sup\Big[U(c) - \lambda - \rho V + (rw + \theta(\mu - r) + \varepsilon - c) V' + \tfrac{1}{2}\sigma^2\theta^2 V'' \Big]$$

$$= \tilde{U}(V') - \lambda - \rho V + (rw + \varepsilon) V' - \tfrac{1}{2}\kappa^2 \, V'^2/V'' \qquad \kappa = \frac{\mu - r}{\sigma}.$$

We do the usual dual variables trick $J = V - zw$, $z = V'$, $J' = -w$, $J'' = -1/V''$ to give

$$0 = \tilde{U}(z) - \lambda - \rho(J - zJ') + (\varepsilon - rJ')z + \tfrac{1}{2} z^2 \kappa^2 J''$$

$$= \tilde{U}(z) - \lambda + \varepsilon z + \tfrac{1}{2}\kappa^2 z^2 J'' + (\rho - r) z J' - \rho J$$

for $z \ge z^*$. For $z \le z^*$ we shall have

$$J(z) = \gamma_M^{-R} \tilde{U}(\gamma_M^R z) = \gamma_M^{-1} \tilde{U}(z)$$

There is a particular solution: if $Q(t) = \tfrac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho$, we have

$$J_0(z) = \frac{-\tilde{U}(z)}{Q(1 - 1/R)} - \frac{\lambda}{\rho} + \frac{\varepsilon z}{r}$$

For large z, the general solution $J_0(z) + Az^{-\alpha} + Bz^{\beta}$ $\qquad (-\alpha < 0 < 1 < \beta$ roots of $Q)$ is dominated by the term in z^β, and for convexity we'd have to have $B \ge 0$. But if $B > 0$, J would eventually be increasing ✗, so $B = 0$, and for large enough z we have

$$J(z) = J_0(z) + Az^{-\alpha} \qquad (z \ge z^*)$$

We do C^1 matching at z^* and deduce z^*, A.

Optimal contracting continued (26/2/08)

(i) Things simplify quite considerably, as DM-S notice. Suppose $g(x)$ is the maximal value of $E[\int_0^\tau e^{-rs}((\mu - rL)ds - dI_s)]$ subject to the constraint that the agent gets

$$\int_0^\tau e^{-\gamma s}(dI_s - \gamma R ds - dW_s) = x$$

If we have reached time t and value ξ_t, then the gap still to cover is $(A - \xi_t)e^{\gamma t}$, and the principal's value is depleted by e^{-rt}, so we get

$$V(t,\xi) = e^{-rt} g(e^{\gamma t}(A - \xi_t))$$

and thus

$$\boxed{\begin{array}{l} 0 = \frac{1}{2} g'' + \gamma(R+z) g' - rg + \mu - rL \\ g' \geq -1 \quad , \quad g(0) = 0 \end{array}}$$

Usual smooth-pasting story at the change-over point z^* will give g is C^2 at z^*, so

$$\boxed{\begin{array}{l} g''(z^*-) = 0, \quad g'(z^*-) = -1, \\ r g(z^*) = \mu - rL - \gamma(R + z^*). \end{array}}$$

This can certainly be done quite simply numerically; a lot easier than a PDE in (t,x).

(ii) But it appears we may be able to do quite a bit more. Suppose that the agent can create drift μ by making exertion at rate $\varphi(\mu)$, where φ is convex increasing. He can still misreport as before, so now his change-of-measure martingale is

$$dZ_t = Z_t (\mu_t - a_t) dW_t$$

and his objective is

$$E\left[Z_\tau \int_0^\tau e^{-\gamma s}(dC_s - \gamma R ds - \varphi(\mu_s) ds)\right].$$

Now suppose

$$\sup_\mu \{ \mu - \varphi(\mu) \} = \mu^* - \varphi(\mu^*),$$

and suppose that the principal plans to pay the agent $\varphi(\mu^*) dt + d\tilde{I}_t$. Let's take as reference measure the measure under which the output is $dY = dW + \mu^* dt$, which the agent can change to $dY = dW + \mu^* dt + (\varepsilon_t - a_t) dt$ by exerting effort $\varphi(\mu^* + \varepsilon_t)$ and by mis-reporting according to schedule a_t. As before, the agent will never save, so his expected value if he follows strategy (a, ε) will be

$$E\left[Z_\tau \int_0^\tau e^{-\gamma s} \{ d\tilde{I}_s + \varphi(\mu^*)ds + a_s\, ds - \gamma R ds - \varphi(\mu^* + \varepsilon_s)\, ds \} \right]$$

$$\leq E\left[Z_\tau \int_0^\tau e^{-\gamma s} \{ d\tilde{I}_s + (a_s - \varepsilon_s)\, ds - \gamma R\, ds \} \right]$$

$$= E\left[Z_\tau \int_0^\tau e^{-\gamma s} \{ d\tilde{I}_s - \gamma R ds - dW_s \} \right]$$

So once again we have got back to a situation where if the principal offers a contract for which

$$\int_0^\tau e^{-\gamma s} \{ d\tilde{I}_s - \gamma R ds - dW_s \} = A \quad a.s.,$$

then the agent does not benefit from departing from $a = 0 = \varepsilon$. The principal gets

$$E\left[\int_0^\tau e^{-rt} \{ (\mu^* - rL)\, dt - \varphi(\mu^*) dt - d\tilde{I}_t \} \right]$$

So it's just the same structure as we got before!

Investment with retirement again (8/3/08)

Back to the problem on p 45, where we have the variational form

$$\left.\begin{array}{ll} \sup_{c,\theta}\left[U(c) - \lambda - \rho V + (rw + \theta(\mu - r) + \varepsilon - c) V' + \tfrac{1}{2}\sigma^2\theta^2 V'' \right] & \\ = \tilde{U}(V') - \lambda - \rho V + (rw + \varepsilon) V' - \tfrac{1}{2}\kappa^2 V'^2/V'' & \leq 0 \\ V_M - V & \leq 0 \end{array}\right\}$$

What we expect is that we will have $V = V_M$ in some region, and in that region we should get a negative value if we stick V_M into the HJB; outside the region where we take $V = V_M$, we solve the HJB, with a C^2 join. Let's look at what we get when we stick $V = V_M$ into the HJB. Everything except the terms in λ, ε will vanish (of course!) to leave

$$-\lambda + \varepsilon\, \gamma_M^{-R} w^{-R}$$

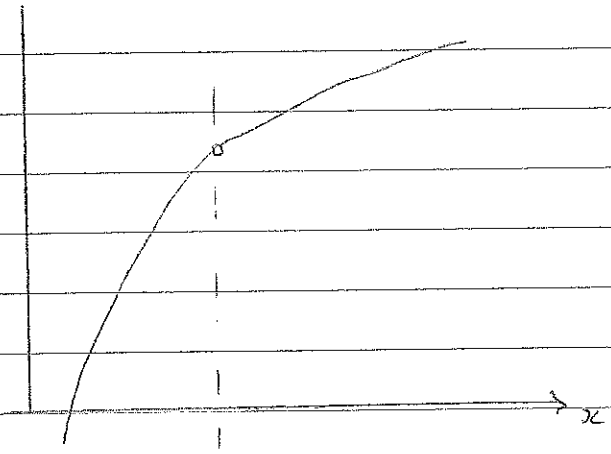
and this will be equal to zero when

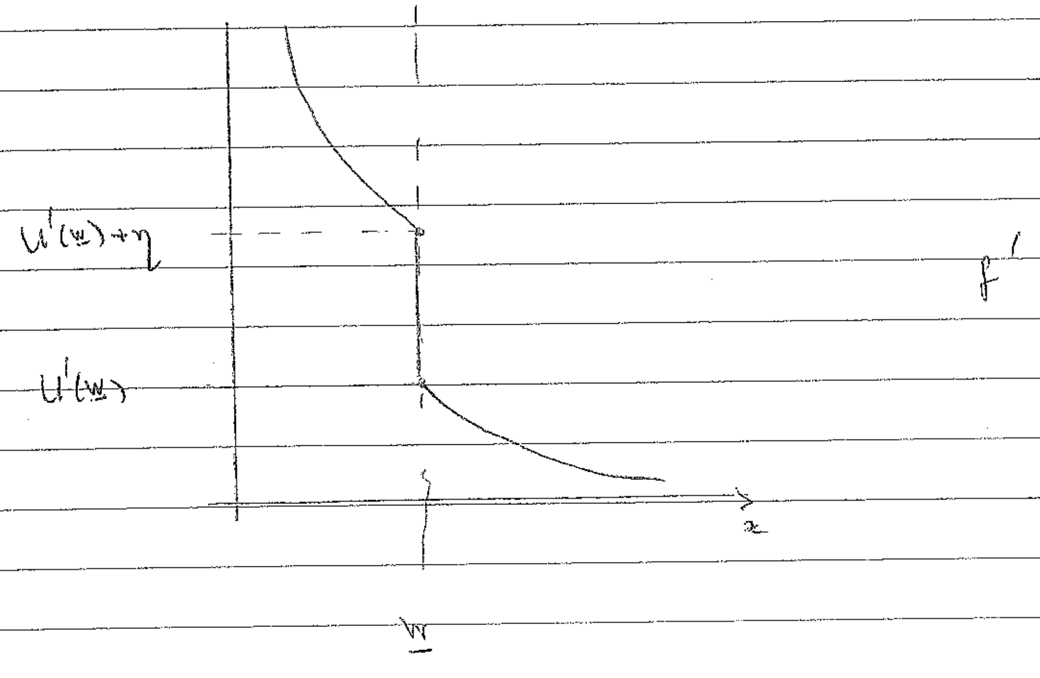
$$\gamma_M w = (\varepsilon/\lambda)^{1/R} \equiv \gamma_M w^*$$

This identifies $z^* = V_M'(w^*) = \lambda/\varepsilon$, and we can solve for the dual value function in $z < z^*$

Note: $E\left[\zeta_T^{\theta} f(W_T)\right] = \int e^{-\kappa\theta x - \theta(\frac{1}{2}\kappa^2 + r)T - x^2/2T} f(x) \frac{dx}{\sqrt{2\pi T}}$

$= \exp\left[\theta T\left(\tfrac{1}{2}\kappa^2(\theta-1) - r\right)\right] \int e^{-(x+\kappa\theta T)^2/2T} f(x) \frac{dx}{\sqrt{2\pi T}}$





<u>Optimal investment with expected-shortfall constraint (5/3/08)</u>

(i) Suppose we have the usual terminal-wealth objective $\max E\, U(w_T)$ with initial wealth w_0, $U'(x) = x^{-R}$, but subject to an expected-shortfall constraint of the form

$$E\left[(w_T - \underline{w}) \wedge 0 \right] \geq -\alpha \qquad (*)$$

How is this solved?

(ii) If $\zeta_t = \exp\{-\kappa W_t - (r + \frac{1}{2}\kappa^2)t\}$, $\kappa \equiv (\mu - r)/\sigma$, then the optimal solution to the Merton wealth problem is with

$$w_T = w_0 \, e^{\gamma T} \, \zeta_T^{-1/R}$$

where $\gamma \equiv R^{-1}(R-1)(r + \kappa^2/2R)$. <u>When would the constraint (*) actually bite?</u> Assuming $\boxed{\kappa > 0}$

$$E\left[(w_T - \underline{w}) \wedge 0\right] = E\left[w_0 e^{\gamma T} \zeta_T^{-1/R} - \underline{w} \;;\; W_T < a_0\right]$$

$$\left[a_0 \equiv \frac{1}{\kappa}\left\{ R \log(\underline{w}/w_0) - \gamma R T - (\tfrac{1}{2}\kappa^2 + r)T \right\} \right]$$

$$= w_0 e^{\gamma T} \exp\left(\frac{T}{R}\left(r + \tfrac{1}{2}\kappa^2(1 + \tfrac{1}{R})\right)\right) P\left[W_T < a_0 - \kappa T/R\right]$$

$$- \underline{w}\, P\left[W_T < a_0\right]$$

and if this is $\geq -\alpha$, then imposing the constraint (*) doesn't change the solution.

(iii) Let's therefore assume that α is small enough that the shortfall constraint has some effect, and consider the Lagrangian form

$$\max E\left[U(w_T) + \lambda(w_0 - \zeta_T w_T) + \eta(\alpha + (w_T - \underline{w}) \wedge 0) \right]$$

which we can easily maximise. If $f(x) \equiv U(x) + \eta\,(x - \underline{w}) \wedge 0$, then f is concave increasing and the optimal choice of w_T will be

$$w_T = (f')^{-1}(\lambda \zeta_T)$$

Thus the maximised Lagrangian is

$$\lambda w_0 + \eta\alpha + E\left[\tilde{u}(\lambda\zeta_T) : \lambda\zeta_T < U'(\underline{w})\right] + E\left[U(\underline{w}) - \lambda\zeta_T \underline{w} : U'(\underline{w}) < \lambda\zeta_T < \eta + U'(\underline{w})\right]$$

$$+ E\left[\tilde{u}(\lambda\zeta_T - \eta) - \eta\underline{w} : \lambda\zeta_T > \eta + U'(\underline{w})\right]$$

Now $\lambda \zeta_T < u'(\underline{w}) \iff W_T > \frac{1}{\kappa}\left[-(\frac{1}{2}\kappa^2 + r)T + \log\lambda - \log u'(\underline{w}) \right] \equiv b$

$\lambda\zeta_T < \eta + u'(\underline{w}) \iff W_T > \frac{1}{\kappa}\left[-(\frac{1}{2}\kappa^2 + r)T + \log\lambda - \log(\eta + u'(\underline{w})) \right] \equiv a$

~~so the value of the Lagrangian is~~

$$\lambda w_0 + \eta\alpha + E\left[\frac{R}{1-R}(\lambda\zeta_T)^{1-1/R} : W_T > b \right] + U(\underline{w}) P[a < W_T < b]$$

$$-\lambda \underline{w}\, E[\zeta_T : a < W_T < b] + E\left[\frac{R}{1-R}(\lambda\zeta_T - \eta)^{1-1/R} : W_T < a \right] - \eta \underline{w} P[W_T < a]$$

$$= \lambda w_0 + \eta\alpha + \tilde{U}(\lambda) E\left[\zeta_T^{1-1/R} : W_T > b \right] + U(w) P[a < W_T < b] - \lambda \underline{w}\, E[\zeta_T : a < W_T < b]$$

$$+ \frac{R}{1-R} E\left[\zeta_T^{1-1/R} (\lambda - \eta\zeta_T^{-1})^{1-1/R} : W_T < a \right] - \eta \underline{w} P[W_T < a]$$

$$= \lambda w_0 + \eta\alpha \overset{+\tilde{U}(\lambda)}{|} e^{-\gamma T} P[W_T > b + \kappa(1-\tfrac{1}{R})T] + U(\underline{w}) P[a < W_T < b] - \lambda \underline{w} e^{-rT} P[a + \kappa T < W_T < b + \kappa T]$$

$$+ \frac{R}{1-R}\exp(-\gamma T) \int_{-\infty}^{a} e^{-(x+\theta\kappa T)^2/2T} \left(\lambda - \eta e^{\kappa x + (\frac{1}{2}\kappa^2 + r)T}\right)^{1-1/R} \frac{dx}{\sqrt{2\pi T}} - \eta \underline{w} P[W_T < a]$$

The term that is written as an integral has to be evaluated numerically, everything else is in terms of Φ. ($\theta = 1 - 1/R$ in integral)

(iv) But we don't have the correct story concerning achievable values of shortfall. If we want to max $E[(W_T - \underline{w}) \wedge 0]$, we shall have $w_T = \underline{w}\, I_{\{\zeta_T < q\}}$ for some q determined by the budget constraint

$$w_0 = \underline{w}\, E[\zeta_T ; \zeta_T \le q] = \underline{w}\, E\left[\zeta_T : W_T \ge \tfrac{1}{\kappa}(-(r + \tfrac{1}{2}\kappa^2)T - \log q)\right]$$

$$= \underline{w}\, e^{-rT} P\left[W_T > \tfrac{1}{\kappa}(-(r + \tfrac{1}{2}\kappa^2)T - \log q) + \kappa T \right].$$

This gives us a maximal value

$$E[(W_T - \underline{w}) \wedge 0] \le -\underline{w}\, P[\zeta_T \ge q] = -\underline{w}\, P\left[W_T \le \tfrac{1}{\kappa}(-(r + \tfrac{1}{2}\kappa^2)T - \log q) \right]$$

History-dependent preferences (12/3/08)

(i) This an example I appear to have cooked up a year ago, but haven't yet written up! The idea is that we

$$\max E\left[\int_0^\infty e^{-\rho t} U(\xi_t)\,dt\right]$$

where $\xi_t = \int_{-\infty}^t e^{\lambda(s-t)} c_s\,ds$, usual wealth equation, CRRA utility, so $d\xi = \lambda(c-\xi)dt$. Scaling gives $V(w,\xi) = \xi^{1-R} v(w/\xi)$. $[x \equiv w/\xi]$

Going through HJB gives

$$0 = \frac{1}{1-R} - (\rho + \lambda(1-R))v + (r+\lambda) x v' - \frac{1}{2}\kappa^2 \frac{v'^2}{v''}$$

with the first-order condition

$$\lambda\big((1-R)v - xv'\big) - v' \leq 0$$

equivalently, $(1+\lambda x)^{R-1} v$ is increasing.

(ii) A second perhaps more natural version of the problem takes objective as above, but $\xi_t = \int_{-\infty}^t e^{\lambda(s-t)} c_s^\alpha\,ds$, for some $\alpha \in (0,1)$. Then the value scales like

$$V(\beta w, \beta^\alpha \xi) = \beta^{\alpha(1-R)} V(w,\xi)$$

so that $V(w,\xi) = \xi^{1-R} V(w\xi^{-1/\alpha}, 1) \equiv \xi^{1-R} v(x)$, $x \equiv w/\xi^{1/\alpha}$.

and HJB is

$$0 = U(1) - \rho v + r x v' - \lambda\left((1-R)v - \frac{1}{\alpha} x v'\right) - \frac{1}{2}\kappa^2 \frac{v'^2}{v''}$$

$$+ \sup_s \left\{ \lambda s^\alpha \left((1-R)v - \frac{1}{\alpha} x v'\right) - s v'\right\}$$

maxed when $\lambda s^{\alpha-1}\left(\alpha(1-R)v - xv'\right) = v'$

Although the form (i) can be tackled via the dual transform, the form (ii) cannot, it seems. However, the rule coming from (i) would be for bang-bang consumption, and this really doesn't seem at all plausible.

$$\varphi(x) = \frac{1}{2} k x^2 \qquad q(x) = x/k$$

$$\psi(x) = \frac{1}{2} x^2/k$$

$$L_D \equiv \begin{cases} \frac{1}{2}\sigma^2\theta^2 D^2 + (\mu(x+\lambda) - \psi(\theta))\, D & \text{if } \theta \in [\lambda, 1] \\ D & \text{if } \theta = 0 \end{cases}$$

$$RHD = \begin{cases} rm & \text{if } \theta \in [\lambda, 1] \\ 0 & \text{if } \theta = 0 \end{cases}$$

$$vol = \theta\sigma$$

$$G = \begin{cases} q(\theta) - rL & \text{if } \theta \in [\lambda, 1] \\ 1 & \text{if } \theta = 0 \end{cases}$$

DeMarzo-Sannikov example again (13/3/08)

Let's suppose that under φ^0 the canonical process X is $\sigma \times$ Brownian motion. As before the agent can misreport (benefits by factor $\lambda \in (0,1)$ if he tries to steal), but now he has some choice to affect the growth rate μ; achieving instantaneous growth rate μ costs effort $\varphi(\mu)$, which we suppose is C^2 convex increasing. Thus the agent's objective is

$$E^0\Big[Z_\tau \int_0^\tau e^{-\gamma s}\big(dI_s + \ell(a_s)\,ds - \gamma R\,ds - \varphi(\mu_s)\,ds\big)\Big] + R$$

where $\ell(x) = (\lambda x)\wedge x$. We have $dZ_t = \sigma^{-2} Z_t(\mu_t - a_t)\,dX_t$. For any previsible process (θ_t) with values in $[\lambda, 1]$, we can bound the agent's objective by

$$\le E^0\Big[Z_\tau \int_0^\tau e^{-\gamma s}\big\{dI_s + \theta_s(a_s - \mu_s)\,ds + (\theta_s\mu_s - \varphi(\mu_s))\,ds - \gamma R\,ds\big\}\Big] + R$$

$$\le E^0\Big[Z_\tau \int_0^\tau e^{-\gamma s}\big\{dI_s - \theta_s\,dX_s + \psi(\theta_s)\,ds - \gamma R\,ds\big\}\Big] + R$$

where $\psi(\theta) \equiv \sup\{\theta\mu - \varphi(\mu)\}$, as we saw before. This bound is attained if the agent doesn't misreport and uses effort $\mu_s = (\varphi')^{-1}(\theta_s)$.

The principal therefore will choose I, τ, θ so that the process

$$\xi_t \equiv \int_0^t e^{-\gamma s}\big\{dI_s - \theta_s\,dX_s + \psi(\theta_s)\,ds - \gamma R\,ds\big\}$$

gets stopped at the first time τ at which it reaches $A_0 - R$, where A_0 is the agent's participation constraint. Of course, the principal wishes to do well for himself, so if we set

$$V(x) = \sup\Big\{E^0\Big[Z_\tau \int_0^\tau e^{-rt}\big\{dX_t - dI_s - rL\,ds\big\}\Big] \;;\; \xi_\tau = x\Big\}$$

then we have (MPOC)

$$Y_t \equiv \int_0^t e^{-rs}\big\{dX_s - dI_s - rL\,ds\big\} + e^{-rt}\,V\big(e^{\gamma t}(A_0 - R - \xi_t)\big)$$

is a supermartingale and a martingale under optimal control ($dX_t = \sigma\,dW_t + \mu_t\,dt$, where $\mu_t \equiv q(\theta_t) \equiv (\varphi')^{-1}(\theta_t)$). Then ($x_t \equiv e^{\gamma t}(A_0 - R - \xi_t)$)

$$e^{rt}\,dY_t = dX_t - dI_t - rL\,dt - rV\,dt + V'(+\gamma x\,dt - e^{\gamma t}d\xi_t) + \tfrac12 V''\theta_t^2\,dt\,\sigma^2$$

$$\doteq \big\{\mu_t - rL - rV + (\gamma x + \gamma R - \psi(\theta_t))V' + \tfrac12\sigma^2\theta_t^2 V''\big\}\,dt - (1+V')\,dI_t$$

~~so our HJB formulation is (with $V(0)=0$)~~

$$\boxed{\max\Big\{-1-V',\ \sup_{\lambda\le\theta\le 1}\Big(q(\theta) - r(L+V) + (\gamma(x+R) - \psi(\theta))V' + \tfrac12\sigma^2\theta^2 V''\Big)\Big\} = 0}$$

Parameter uncertainty done cleanly (2/4/08)

1) In the parameter uncertainty problem, with multiple assets, we see

$$Y_t = W_t + \alpha t$$

where α is unknown, $N(\hat{\alpha}_0, \tau_0^{-1})$ prior. The stocks solve $dS_t = S_t\,\sigma\, dY_t$, and the change-of-measure martingale Z solves

$$dZ_t = -Z_t\,\kappa\, dW_t, \qquad \kappa \equiv \alpha - r\sigma^{-1}1.$$

By the usual Bayesian analysis, the posterior at time t is $N(\hat{\alpha}_t, \tau_t^{-1})$ where

$$\tau_t = \tau_0 + tI, \qquad \tau_t \hat{\alpha}_t = \tau_0 \hat{\alpha}_0 + Y_t$$

and we get

$$d\hat{\alpha}_t = d\hat{\kappa}_t = \tau_t^{-1}\, d\hat{W}_t \equiv \tau_t^{-1}\,(dY_t - \hat{\alpha}_t\, dt).$$

2) To calculate the spd $\hat{\zeta}_t \equiv e^{-rt}\hat{Z}_t$ cleanly, we notice that for any wealth process w_t whose portfolio process is previsible in $\mathcal{Y}_t \equiv \sigma(Y_s : s \le t)$ we shall have that $\zeta_t w_t$ is a martingale in the filtration $\mathcal{G}_t \equiv \mathcal{Y}_t \vee \sigma(\alpha)$, and so $\hat{\zeta}_t w_t$ is a martingale in $\mathcal{Y}_t$, where $\hat{\zeta}_t$ is just the $\mathcal{Y}$-optional projection of ζ. So we need to find the $\mathcal{Y}$-optional projection $\hat{Z}$ of Z. This is a $\mathcal{Y}$-martingale, and as such may be represented as

$$d\hat{Z}_t = \hat{Z}_t\, H_t \cdot d\hat{W}_t$$

for some $\mathcal{Y}$-previsible process H. To identify H, we take any bounded $\mathcal{Y}$-previsible θ, and calculate

$$E\left[\hat{Z}_t \int_0^t \theta_s \cdot d\hat{W}_s\right] = E\left[\int_0^T \hat{Z}_s\, H_s \cdot \theta_s\, ds\right]$$

$$= E\left[Z_t \int_0^t \theta_s \cdot d\hat{W}_s\right]$$

$$= E\left[Z_t \int_0^t \theta_s \cdot \{dW_s + (\alpha - \hat{\alpha}_s)\, ds\}\right]$$

$$= E\left[\int_0^t Z_s\,(-\kappa)\cdot\theta_s\, ds + \int_0^t Z_s\, \theta_s \cdot (\alpha - \hat{\alpha}_s)\, ds\right]$$

$$= E\left[\int_0^t Z_s\, \theta_s \cdot (r\sigma^{-1}1 - \hat{\alpha}_s)\, ds\right]$$

$$= E\left[\int_0^t Z_s\, \theta_s \cdot (-\hat{\kappa}_s)\, ds\right]$$

Hence $\boxed{H_t = -\hat{\kappa}_t \equiv r\sigma^{-1}1 - \hat{\alpha}_t}$

Checks with OI/NOTES/nsoi.pdf.

3) We now have $d\hat{Z}_t = \hat{Z}_t(-\hat{\kappa}_t)\cdot d\hat{W}_t$ so if we look at $z_t \equiv \log \hat{Z}_t$ we get

$$dz_t = -\hat{\kappa}_t\cdot d\hat{W}_t - \tfrac{1}{2}|\hat{\kappa}_t|^2\,dt$$

$$= -\hat{\kappa}_t\cdot\tau_t\,d\hat{\kappa}_t - \tfrac{1}{2}|\hat{\kappa}_t|^2\,dt.$$

Now
$$d\left(\tfrac{1}{2}\hat{\kappa}_t\cdot\tau_t\hat{\kappa}_t\right) = (\tau_t\hat{\kappa}_t)\cdot d\hat{\kappa}_t + \tfrac{1}{2}|\hat{\kappa}_t|^2dt + \tfrac{1}{2}\tau_t^{ij}\,d\hat{\kappa}^i d\hat{\kappa}^j$$

$$= (\tau_t\hat{\kappa}_t)\cdot d\hat{\kappa}_t + \tfrac{1}{2}|\hat{\kappa}_t|^2dt + \tfrac{1}{2}\mathrm{tr}(\tau_t^{-1})\,dt$$

Thus
$$dz_t = -d\left(\tfrac{1}{2}\hat{\kappa}_t\cdot\tau_t\hat{\kappa}_t\right) + \tfrac{1}{2}\mathrm{tr}(\tau_t^{-1})dt$$

so we need to calculate $\int_0^t \mathrm{tr}(\tau_s^{-1})\,ds = \int_0^t \sum_1^d \frac{1}{\lambda_j+s}\,ds$ if λ_j are e-values of τ_0

$$= \log\det\tau_t - \log\det\tau_0.$$

Therefore
$$\boxed{\hat{Z}_t = \left(\frac{\det\tau_t}{\det\tau_0}\right)^{\frac{1}{2}}\exp\left\{-\tfrac{1}{2}\hat{\kappa}_t\cdot\tau_t\hat{\kappa}_t + \tfrac{1}{2}\hat{\kappa}_0\cdot\tau_0\hat{\kappa}_0\right\}}$$

4) If we simplify to $\boxed{\tau_0 = t_0 I}$ the calculation we need to do to get wealth at time $t\in[0,T]$ in terms of the terminal wealth is

$$E\left[\hat{\zeta}_T I(\lambda\hat{\zeta}_T)\,\middle|\,\mathcal{Y}_t\right]$$

$$= \lambda^{-1/R}\left(\frac{t_0+t}{t_0}\right)^{\frac{d}{2}(1-1/R)}\left\{1+\frac{(1-1/R)(T-t)}{t_0+t}\right\}^{-d/2}\exp\left[-\tfrac{1}{2}|\hat{\kappa}_t|^2\frac{(1-1/R)(t_0+T)(t_0+t)}{t_0+t+(1-1/R)(T-t)} + \tfrac{1}{2}(1-1/R)\hat{\kappa}_0\cdot\tau_0\hat{\kappa}_0 - r(1-1/R)t\right]$$

$$\equiv f(\hat{\kappa}_t;t,T,\lambda),\ \text{say}.$$

$$= \hat{\zeta}_t w_t$$

Now we have that $dw_t = \theta_t(d\hat{W}_t + \lambda_t dt)$ for some portfolio θ_t, and this allows us to identify θ.

Indeed,
$$d(\hat{\zeta}_t w_t) = \hat{\zeta}_t(\theta_t - w_t\hat{\kappa}_t)\,d\hat{W}_t$$

$$= f'(\hat{\kappa}_t)\,d\hat{\kappa}_t = f'(\hat{\kappa}_t)\,d\hat{W}_t/\tau_t$$

This calculation appears to be quite unstable. We may instead use the identity

$$\int_0^t dx_1 \int_0^{t-x_1} dx_2 \cdots \int_0^{t-x_1-\cdots-x_{n-1}} dx_n \prod_{j=1}^{n} x_j^{\nu_j} = t^{n+\nu\cdot 1} \prod_j \Gamma(\nu_j+1) / \Gamma(n+\nu\cdot 1+1)$$

, deduce

$$E\left[X_1^{\nu_1} \cdots X_n^{\nu_n}\right] = \prod_j \Gamma(\nu_j+1) \quad \Gamma(n+1)/\Gamma(n+\nu\cdot 1+1)$$

here X is uniform on $\{x : x_i \geq 0, \sum x_i = 1\}$ and obtain moments to expand $E\, e^{\alpha\cdot X}$

If ν_i are integer,

$$E\left[\prod_1^n X_j^{\nu_j}\right] = \left(\prod_1^n \nu_j!\right) n!/(n+\nu\cdot 1)!$$

Now

$$(\alpha\cdot X)^k = \sum \frac{k!}{k_1!\cdots k_n!} \prod (\alpha_j X_j)^{k_j}, \text{ so } E\left[(\alpha\cdot X)^k\right] = \frac{k!\,n!}{(n+k)!} \sum \prod \alpha_j^{k_j}$$

o if $S_k(\alpha) = \sum \left(\prod \alpha_j^{k_j}\right)$ (summed where $\sum k_j = k$, $k_j \geq 0$), this would give a series.

The formal power series

$$\sum_{k\geq 0} t^k S_k(\alpha) = \prod_j \sum_{m\geq 0} (t\alpha_j)^m = \prod_j \frac{1}{1-t\alpha_j}$$

may help, because we could build this by induction on n.

~~The expression for~~ $E\, e^{\alpha\cdot X} = \sum_{k\geq 0} \frac{n!}{(n+k)!} S_k(\alpha)$

or numerator, want

$$E\left[X_p\,(\alpha\cdot X)^k\right] = \frac{k!\,n!}{(n+k+1)!} \sum (1+k_p) \prod \alpha_j^{k_j} = \frac{k!\,n!}{(n+k+1)!} S_{k,p}(\alpha)$$

and $$\sum_{k\geq 0} t^k S_{k,p}(\alpha) = \left(\prod_{j\neq p} \frac{1}{1-t\alpha_j}\right) \sum_{k\geq 0} (1+k)(\alpha_p t)^k = \left(\prod_{j\neq p} \frac{1}{1-t\alpha_j}\right) \frac{1}{(1-\alpha_p t)^2}$$

Hence

$$\theta_t = w_t \hat{\pi}_t + \frac{w_t}{\pi_t} \frac{f'(\hat{\pi}_t)}{f(\hat{\pi}_t)}$$

$$= w_t \hat{\pi}_t \left[1 - \frac{(1-1/R)(t_0+T)}{t_0+t+(1-1/R)(T-t)} \right]$$

gives the portfolio. For the log investor, we just do 'pointwise Merton', but if $R>1$ we will be more careful:

$$\theta_t = w_t \hat{\pi}_t \left[\frac{1}{R} - (1-1/R) \frac{(T-t)/R}{t_0+t+(1-1/R)(T-t)} \right]$$

So if $R>1$ we underinvest relative to log, for $R<1$ we overinvest, and the degree of over/under-investing falls off as $t \uparrow T$.

<u>Universal portfolio calculation (3/4/08)</u>

In calculating this, we need to find

$$\int_0^t a_1 e^{-a_1 x_1} dx_1 \int_0^{t-x_1} a_2 e^{-a_2 x_2} dx_2 \int_0^{t-x_1-x_2} a_3 e^{-a_3 x_3} dx_3 \cdots \int_0^{t-x_1-\cdots-x_{n-1}} a_n e^{-a_n x_n} dx_n$$

$$= P(X_1 + X_2 + \cdots + X_n \le t)$$

Now the LT of $X_1+\cdots+X_n$ is $\prod_i^n \frac{a_i}{a_i+\lambda} = \sum_i^n \frac{c_i}{a_i+\lambda}$, $c_i = a_i \prod_{j\neq i} \frac{a_j}{a_j-a_i}$

Hence $P(X_1+\cdots+X_n < t) = \sum_{i=1}^n (1-e^{-a_i t})\, c_i/a_i$

Similarly,

$$\int_{\{\Sigma x_i = t\}} e^{-a\cdot x}\, dx = \sum_{j=1}^n \frac{e^{-a_j t}}{\prod_{i\neq j}(a_i - a_j)},$$

implying

$$\boxed{\int_{\{1\cdot x = 1\}} e^{a\cdot x}\, dx = \sum_{j=1}^n e^{a_j} \Big/ \prod_{i\neq j}(a_j - a_i)}$$

Finally,

$$\boxed{\begin{aligned}\int_{\{1\cdot x=1\}} x_k e^{a\cdot x}\, dx &= \frac{\partial}{\partial a_k}\left(\int_{1\cdot x=1} e^{a\cdot x} dx\right) \\ &= \frac{e^{a_k}}{\prod_{i\neq k}(a_k-a_i)} + \sum_{\substack{j=1\\ j\neq k}}^n \frac{1}{a_j-a_k}\left\{\frac{e^{a_j}}{\prod_{i\neq j}(a_j-a_i)} - \frac{e^{a_k}}{\prod_{i\neq k}(a_k-a_i)}\right\}\end{aligned}}$$

Splitting into business lines (4/4/08)

Suppose you have N log-Brownian assets, $dS_t = S_t(\Sigma dW_t + A\,dt)$, which get partitioned into J business lines, which are then given to J managers. The managers are CRRA investors, who get compensated at rate $\delta_j w_t\,dt$; it is supposed that they know the growth rates of their own stocks. The residue of what they generate is passed back to their bosses.

1) Story for a BL manager:

$$dw = rw\,dt + \theta\cdot(\sigma_j dW + a_j dt - r1\,dt) - \delta_j w_t\,dt$$

where σ_j is just the rows of Σ corresponding to manager j's assets, a_j the same rows of A. The analysis is standard, and $\theta_t^* = w_t R_j^{-1}(\sigma_j\sigma_j^T)^{-1}(a_j - r1)$. The wealth is log-Brownian:

$$~~dw_t^j = w_t^j\left[\pi_j\cdot(\sigma_j dW + (a_j - r)dt) + (r-\delta_j)dt\right]~~$$

so the wealth process that manager j passes back to his boss is

$$dw_t^j = w_t^j\left[\pi_j\cdot\sigma_j dW + (r-\delta_j + \pi_j\cdot(a_j - r))\,dt\right] = w_t^j\left[\pi_j\cdot\sigma_j dW + \bar{\mu}_j dt\right]$$

The boss is now able to invest in this smaller universe of log-Brownian assets with growth rates $\bar{\mu}_j = r - \delta_j + R_j^{-1}(a_j - r1)\cdot(\sigma_j\sigma_j^T)^{-1}(a_j - r1)$ and covariances $\pi_j\cdot(\sigma_j\sigma_i^T\pi_i) \equiv v_{ij}$. Notice that the growth rate of asset j is $r - \delta_j + R_j v_{jj}$, so by observing the covariation structure of his available business lines, the boss (knowing δ_j, R_j) can work out the $\bar{\mu}_j$, so he can now combine these.

2) Why would the boss want to split things into business lines, other than manageability? It seems that you need to tell some story; if the boss knew A, he would achieve a sub-optimal allocation if he does it via subordinates. So maybe the boss is less knowledgeable about A. Tom + Mathias + I wondered about possible tax effects, but there was never any advantage to taking individual BLs separately. Some risk management story might be told, but then the simplicity of w^j is lost. So we like the story of a boss who doesn't know A.

3) How would the BL managers want to set δ_j to compete with each other? What are the Pareto-efficient choices?

Let's suppose we denote quantities which the boss sees by $\bar{\sigma}$, $\bar{\mu}$, etc, so that he has a wealth process $d\bar{w} = \bar{w}\{\bar{\pi}\cdot\bar{\sigma}\,dW_t + \bar{\pi}\cdot(\bar{\mu}-r)dt + r\,dt\}$

$$\Rightarrow \quad \bar{w}_t = \bar{w}_0 \exp\{\bar{\pi}\cdot\bar{\sigma}W_t + (\bar{\pi}\cdot(\bar{\mu}-r) + r - \tfrac{1}{2}\bar{\pi}\cdot\bar{\sigma}\bar{\sigma}^T\bar{\pi})t\} \equiv \bar{w}_0 e^{\bar{\pi}\cdot\bar{\sigma}W_t + \bar{b}\,dt}$$

where $\bar{\pi} = \bar{R}^{-1}(\bar{\sigma}\bar{\sigma}^T)^{-1}(\bar{\mu}-r)$. This now gets split among the business lines in the proportions $\bar{\pi}$ so what the manager of BL j gets as objective will be

$$U_j(\delta_j \bar{\pi}_j)\ E\int_0^\infty e^{-\rho_j t}\exp\left((1-R_j)(\bar{\pi}\cdot\bar{\sigma}W_t + \bar{b}t)\right)dt$$

$$\bar{\pi} = \bar{R}^{-1} (\bar{\sigma}\bar{\sigma}^{T})^{-1} (\bar{\mu} - r) = \bar{R}^{-1} v^{-1} (\bar{\mu} - r) \qquad (1)$$

$$= U_j(\bar\pi_j \delta_j)\, E\int_0^\infty \exp\{-\rho_j t + (1-R_j)\bar b t + \tfrac12(1-R_j)^2\,\bar\pi\cdot\bar\sigma\bar\sigma^T\bar\pi\, t\}\,dt$$

$$= U_j(\bar\pi_j\delta_j)\Big/\Big\{\rho_j + (R_j-1)\Big(r + \bar\pi\cdot(\bar\mu - r) - \tfrac12 R_j\,\bar\pi\cdot(\bar\sigma\bar\sigma^T)\bar\pi\Big)\Big\}$$

$$= U_j(\bar\pi_j\delta_j)\Big/\Big\{\rho_j + (R_j-1)\Big(r + (\bar\mu - r)\cdot(\bar\sigma\bar\sigma^T)^{-1}(\bar\mu - r)\big(\bar R^{-1} - R_j/2\bar R^2\big)\Big)\Big\}$$

Values of δ enter into $\bar\mu$, but not into $\bar\sigma$. Moreover, the values of δ appear linearly in $\bar\mu = q - \delta$, for $q_j = r + R_j^{-1}(a_j - r)\cdot(\sigma_j\sigma_j^T)^{-1}(a_j - r)$, and also $\bar\pi$ depends on δ. The objective of manager j is

$$U_j(\bar\pi_j\delta_j)\Big/\Big[\rho_j + (R_j-1)\big(r + (\bar R - \tfrac12 R_j)\;\bar\pi\cdot v\,\bar\pi\big)\Big] \qquad (*)$$

where $v \equiv \sigma\sigma^T$. We have $\partial\bar\pi_i/\partial\delta_k = -\frac{1}{\bar R}v^{-1}e_k = -\bar R^{-1}v^{ik}$.

To study Pareto efficient choices of δ, take logs in $(*)$, differentiate wrt δ_j:

$$0 = \frac{1-R_j}{\delta_j} - \frac{1-R_j}{\bar R}\,\frac{v^{jj}}{\bar\pi_j} + \frac{(R_j-1)(2\bar R - R_j)\;|\bar\pi_j|^2}{\bar R\,\big[\rho_j + (R_j-1)(r + (\bar R - \tfrac12 R_j)\,\bar\pi\cdot v\bar\pi)\big]}$$

<u>Special case</u>: $R_j = R$, $\rho_j = \rho$ $\forall j$

Now we have

$$(2)\qquad \frac{1}{\delta_j} = \frac{v^{jj}}{\bar R\,\bar\pi_j} + c_j = \frac{v^{jj}}{\bar R\,\bar\pi_j} + \frac{(2\bar R - R)\;|\bar\pi_j|^2}{\big(\rho + (R-1)(r + (\bar R - \tfrac12 R)\,\bar\pi\cdot v\bar\pi)\big)}$$

Can these be solved consistently? Perhaps; we could try some recursive solution, picking $\delta^{(n)}$ then solving $\pi^{(n)} = \bar R^{-1}v^{-1}(\bar\mu^{(n)} - r)$, then going back to (2) to find the next $\delta^{(n+1)}$. If we choose this approach, there's no need to assume agents are the same; not obvious we shall have convergence.... and it seems from some numerical examples that we typically <u>don't</u>. Nevertheless, the equations (1), (2) can be solved numerically

Interesting questions

1) Angus proposes a meeting model of buyers + sellers who have different beliefs about the price of the commodity. When two meet, they may trade (if both think the proposed price is advantageous) but they certainly modify beliefs. How does this evolve? A drop in the perceived value of the asset would result in a reduction in liquidity, and a slow learning of the new regime ... a nice modelling idea; how to make it concrete??

) Dilip emphasises that classical economic stories of production + consumption contain no role for finance, so not surprisingly conclude that there is no effect of finance on the real economy. What stories could be told that justify a role for money?

) Tom Sargent talks about situations where you could have two different optimal control problems, but the optimally-controlled process behaves the same in each case. This could matter if you got pushed off the optimal trajectory...

) One of the lab's PhD/postdocs say that the following result is known. If $\Omega \subseteq \mathbb{R}^d$ is simply connected, h cts on $\bar{\Omega}$ solves $\mathcal{L}h = 0$ in Ω, where $\mathcal{L}$ is a strictly elliptic operator, and if $h = 0$ on some open subset $\Omega_0 \subset \Omega$, then $h \equiv 0$. Probabilistic proof?

) Jan Obłój has some study with Alex Cox of double touch options; seems like we should be looking for a characterisation of all possible trivariate laws $(M_1, \sup_{s\le 1} M_s, \inf_{s \le 1} M_s)$ for cts martingales M

) Jan also asks this: if you have a Markov chain with values in a finite set $I \subseteq \mathbb{R}$, which is a nearest-neighbour process and a martingale, does the law of X_1 determine the Q-matrix?

) Josef Teichmann points out that for affine processes (in any dimension) it's actually quite easy to calculate $P_t f(\cdot)$ for polynomial f, just by matrix calculations. Would this help for pathwise stochastic optimal control?

) Nice (= hard) question that Walter + Marc spent time on. Suppose $X_t = W_t + \mu t$ and we take a ± 1-valued previsible process H, to form $Y_t = (H \cdot X)_t$. If $(\mathcal{G}_t)$ is the filtration of
can it be that Y is a Brownian motion??

) Question embarked on with Hurd + Grasselli; a number of log-Brownian base assets are split into business lines + given to a manager who knows the growth rates of those assets + manages them in conventional Merton way. They get $\delta w_t\, dt$ compensation, pass all the rest back to their boss, who doesn't know the individual growth rates. How does the boss split investment in the business lines?

) Hazaat has a forthcoming paper in RFS where he exhibits that introducing a derivative into incomplete equilibrium changes prices.