American option pricing by C-N (12/7/11)

(1) A visit from Nora Imkeller reveals that while I have some fragments of the code required to do American option pricing for 1-dim diffusions, there's nothing very definitive yet.

So we suppose that the diffusion is $dX_t = \sigma(t,X_t)\,dW_t + \mu(t,X_t)\,dt$, and the objective is to find

$$V(t,x) \equiv \sup_{t \le \tau \le T} E\left[\exp\left(-\int_t^\tau \rho(s,X_s)\,ds\right) g(\tau, X_\tau) \,\Big|\, X_t = x \right].$$

Here, g is the stopping reward function, $\rho \ge 0$ some discount rate. The value V has to solve

$$\mathcal{L}V + \frac{\partial V}{\partial t} - \rho V \le 0, \quad = 0 \quad \text{where} \quad V > g$$

with

$$\mathcal{L} \equiv \tfrac{1}{2}\sigma(t,x)^2 \frac{\partial^2}{\partial x^2} + \mu(t,x)\frac{\partial}{\partial x}.$$

(2) If we do some grid approximation, and write $L^{(n)}$ for some finite-difference approximation to $\mathcal{L}$ at $t = t_n$, with $V^{(n)}$ the approximation to the value at $t=t_n$, then the Crank-Nicolson methodology says

$$\frac{1}{2}\left(L^{(n)}V^{(n)} + L^{(n+1)}V^{(n+1)}\right) - \frac{1}{2}\left(\rho^{(n)}V^{(n)} + \rho^{(n+1)}V^{(n+1)}\right) + \frac{V^{(n+1)} - V^{(n)}}{t_{n+1} - t_n} \le 0$$

with equality where you continue. The unknowns here are $V^{(n)}$, so what we get is

$$\left(L^{(n)} - \rho^{(n)} - \frac{2}{\Delta t}\right)V^{(n)} \le -\left(\frac{2V^{(n+1)}}{\Delta t} + L^{(n+1)}V^{(n+1)} - \rho^{(n+1)}V^{(n+1)}\right)$$

$$\equiv -\alpha^{(n)}$$

Say, with equality in places where you want to continue, $V^{(n)} = g(t_n, \cdot)$ in the stopping region.

> So $V^{(n)}(\cdot)$ is the solution of an optimal stopping problem for a Markov chain with Q-matrix $L^{(n)}$, discount rate $\rho^{(n)} + 2/\Delta t$, and running reward $\alpha^{(n)}$.

As such, it can be solved efficiently by policy improvement?

(3) For standard American put, where should we place the time + space grids? If $dS = S(\sigma dW + r\,dt)$, then routine calculations give the infinite-horizon solution to be place barrier at

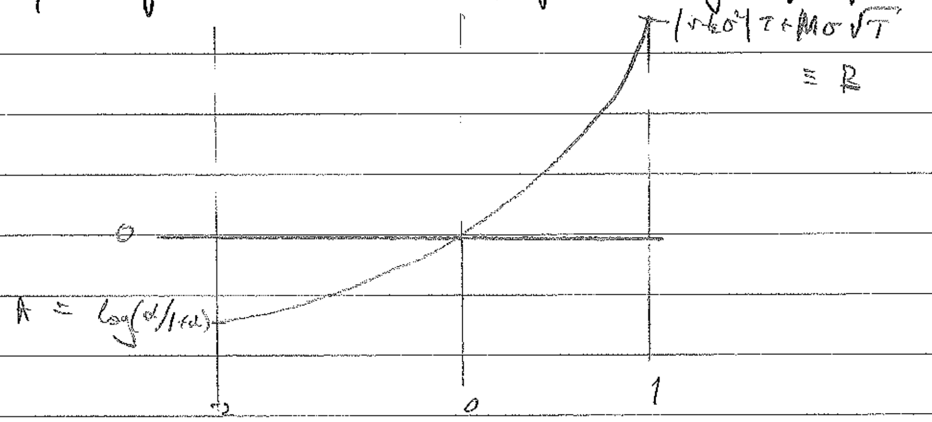
$$S_* = \frac{\alpha K}{1+\alpha} \qquad \left(\alpha = 2r/\sigma^2\right)$$

So for finite horizon we know that the stopping barrier is always above this. In terms of log price $X_t = \log S_t = \log S_0 + \sigma W_t + (r - \frac{1}{2}\sigma^2)t$, we have that $X_t \geq x_* \equiv \log S_*$, so we can restrict the grid points for x to be in $[x_*, \infty)$. Probably makes sense to put a grid point on K as well. How far up do we need to go? If we set

$$x^* = \log K + |r - \tfrac{1}{2}\sigma^2| T + M\sigma\sqrt{T}$$

for $M = 3$ (say) then this is probably not so bad. We might now try to equi-space points in $[-2, 1]$

and try to fit a quadratic $ax^2 + bx$ to match those values



This puts twice as many points between x_* and the log strike as we have above the log strike. Probably more simply, just do piecewise linear!

LAF with point observations (18/7/11)

If we have $dX_t = X_t(\sigma\, dW_t + \mu\, dt)$ as the intensity process for some counting process, then the variational analysis leads to the second-order non-linear ODE

$$x\ddot{x} - \dot{x}^2 - \sigma^2 x^3 = 0 \qquad (*)$$

to be solved for the path between observation points. Maple reckons it can find a solution, but it seems not to be in a very convenient form. Let's see what we can do for it, beginning by rewriting (*) as

$$0 = \frac{\ddot{x}}{x} - \left(\frac{\dot{x}}{x}\right)^2 - \sigma^2 x = \frac{d}{dt}\left(\frac{\dot{x}}{x}\right) - \sigma^2 x$$

As this suggests we write $v = \log x$, giving

$$\ddot{v} - \sigma^2 e^v = 0$$

$$= \dot{v}\frac{d\dot{v}}{dv} - \sigma^2 e^v$$

$$\Rightarrow \quad d(\tfrac{1}{2}\dot{v}^2) = \sigma^2 e^v\, dv \quad \Rightarrow \quad \tfrac{1}{2}\dot{v}^2 = \sigma^2(e^v + b^2) \qquad \text{for some const } b$$

$$\Rightarrow \quad \frac{dv}{dt} = \sqrt{2\sigma^2(e^v + b^2)}$$

$$\Rightarrow \quad t + c = \int \frac{dv}{\sqrt{2\sigma^2(b^2 + e^v)}} = \int \frac{e^{-v/2}\, dv}{\sqrt{2\sigma^2}\,(1 + b^2 e^{-v})^{1/2}}$$

$$= \frac{2}{b\sqrt{2\sigma^2}} \int \frac{du}{\sqrt{1+u^2}} \qquad b e^{-v/2} = u$$

$$= \frac{2}{b\sqrt{2\sigma^2}} \int \frac{\sec^2\theta\, d\theta}{\sec\theta} \qquad u = \tan\theta$$

$$= \frac{2}{b\sqrt{2\sigma^2}} \log(\sec\theta + \tan\theta) = \frac{2}{b\sqrt{2\sigma^2}} \log\left(u + \sqrt{1+u^2}\right)$$

$$\Rightarrow \quad u + \sqrt{1+u^2} = \xi = \exp\left(\frac{b\sqrt{2\sigma^2}}{2}(t+c)\right)$$

$$\Rightarrow \quad 1 + u^2 = \xi^2 - 2\xi u + u^2 \quad \Rightarrow \quad 2\xi u = \xi^2 - 1 \Rightarrow u = \frac{\xi - \xi^{-1}}{2}$$

$$\Rightarrow \quad u = \sinh\left(\frac{b}{2}\sqrt{2\sigma^2}\,(t+c)\right)$$

$$\Rightarrow \quad \boxed{x = e^v = (b/u)^2 = b^2 \operatorname{cosech}^2\left(\frac{b}{2}\sqrt{2\sigma^2}\,(t+c)\right)} = \frac{2A^2}{\sigma^2 \sinh^2(A(t+c))}$$

$$\dot{x} = -b^3\sigma\sqrt{2}\, \coth\left(\frac{b\sigma}{\sqrt{2}}(t+c)\right) \operatorname{cosech}^2\left(\frac{b\sigma}{\sqrt{2}}(t+c)\right)$$

In the LAF application, we have to choose the solution to match given values of x, $\dot{x} \equiv p$ at known times. So the game is to find $\alpha \equiv A(t+c)$ and A so that

$$\left.\begin{aligned} x &= \frac{2A^2}{\sigma^2 \sinh^2 \alpha} \\ p &= \frac{-4A^3 \cosh(\alpha)}{\sigma^2 \sinh^3 \alpha} \end{aligned}\right\}$$

Hence we have $\boxed{p \tanh\alpha + 2Ax = 0}$ but this can only be solved if $|p| > 2Ax$.

To solve this, we use

$$\frac{1}{\sinh^2\alpha} = \frac{1}{\tanh^2\alpha} - 1$$

so that from the first equation

$$x = \frac{2A^2}{\sigma^2}\left(\left(\frac{p}{2Ax}\right)^2 - 1\right) = \frac{p^2}{2\sigma^2 x^2} - \frac{2A^2}{\sigma^2}$$

So for this to be soluble for A we would require

$$\boxed{p^2 > 2\sigma^2 x^3}$$

and then we can find A, and thence α.

Alternatively if $\boxed{p^2 < 2\sigma^2 x^3}$ we can use a different solution to the ODE, viz

$$x_t = \frac{2A^2}{\sigma^2 \cos^2(A(t+c))} \qquad p = \frac{4A^3 \sin\alpha}{\sigma^2 \cos^3\alpha} = (2A\tan\alpha)x$$

which gives

$$\boxed{p = +2Ax\tan\alpha}$$

and then from the equation for x we get

$$x = \frac{2A^2}{\sigma^2}\left(1 + \tan^2\alpha\right) = \frac{2A^2}{\sigma^2}\left(1 + \left(\frac{p}{2Ax}\right)^2\right) = \frac{2A^2}{\sigma^2} + \frac{p^2}{2\sigma^2 x^2}$$

This allows us to find A, and thence α.

A question of Ezequiel Antar (9/8/11)

(1) This is a simplified version of a question Ezequiel asks about. Suppose we have agents $j = 1, \dots, J$, agent j receiving a cumulative cashflow z^j_t, where

$$dz^j_t = a^j_t \cdot dW_t + b^j_t \, dt$$

where W is a d-dimensional Brownian motion. Can we construct a market of financial assets which would allow the agents to diversify away the risk in the z^j while maintaining zero net supply?

(2) Suppose we create a market of d assets,

$$dS^i_t = S^i_t \left(dW^i_t + v^i_t \, dt \right) \qquad i = 1, \dots, d$$

for some previsible process v yet to be identified: then the agents would be free to trade in these assets, generating wealth processes w^j_t with

$$dw^j_t = \theta^j_t \cdot dS_t - c^j_t \, dt + dz^j_t$$

(assume $r = 0$ for simplicity). The SPD process ζ satisfies $d\zeta = (-v\,dW)\,\zeta$, and summing over the agents and using market clearing yields

$$d\bar{w}_t = -\bar{c}\,dt + \bar{a}\,dW + \bar{b}\,dt \qquad \left(\bar{w}_t = \sum_j w^j_t, \text{ etc} \right)$$

Hence

$$d\left(\zeta_t \bar{w}_t \right) = \zeta_t \left\{ (\bar{b}_t - \bar{c}_t - \bar{a}_t \cdot v_t)\,dt + \bar{a}_t \cdot dW_t \right\} - \zeta \bar{w}\, v \cdot dW$$

$$\Rightarrow \boxed{\zeta_t \bar{w}_t + \int_0^t \zeta_s (\bar{c}_s - \bar{b}_s + \bar{a}_s \cdot v_s)\,ds = \bar{w}_0 + \int_0^t \zeta_s (\bar{a}_s - \bar{w}_s v_s) \cdot dW_s}$$

In this, $\bar{a}$, $\bar{b}$ are given; once v is chosen, we know ζ and up to multipliers λ_j we have

$$\bar{c}_t = \sum_j I_j (t, \lambda_j \zeta_t)$$

This is the familiar impossible situation where there is a relation between some random variable + the stochastic integrand in the stochastic integral representation of that RV ... is there a BSDE formulation that might help?

Also expect that v is increasing;

and $V(h,w) = \left(\frac{h}{w}\right)^{1-R} v(x) \cdot w^{1-R} = w^{1-R} x^{R-1} v(x)$

so we'd expect $x^{R-1} v(x)$ to be decreasing

Households and firms (11/8/11)

(1) This modelling story arises from a note of Michael Li + Wei Xiong; I don't entirely agree with their modelling assumptions, but I think something may be done along these lines.

Suppose households can lend their money to banks and receive interest at rate r_t, they can consume, and they get paid dividends by the firms, who in turn borrow from the banks at rate $r_t + \Delta_t$. We want to tell some equilibrium story. Let h_t denote the wealth at time t of the households, w_t the wealth at time t of the firms. The wealth evolutions are

$$dh_t = (r_t h_t - c_t + \varepsilon_t)\, dt$$

$$dw_t = (r_t + \Delta_t)(w_t - \theta_t \cdot 1)\, dt + \theta_t \cdot (\sigma dW + \mu dt) - \varepsilon_t\, dt$$

and we will want a solution where markets clear:

$$h_t + w_t - \theta_t \cdot 1 = 0$$

The objective of the households is

$$E \int_0^\infty e^{-\rho t} U(c_t)\, dt$$

and of the firms is

$$E\left[\int_0^\infty e^{-\rho t} U'(c_t)\, \varepsilon_t\, dt\right] \qquad \text{(maximize shareholder value)}$$

(2) For simplicity, let's have $U'(x) = x^{-R}$. If we take the value

$$V(h,w) = \sup E\left[\int_0^\infty e^{-\rho t}\left(U(c_t) + \beta \varepsilon_t U'(c_t)\right) dt \;\middle|\; h_0 = h,\ w_0 = w\right]$$

then we expect the scaling relation $V(h,w) = h^{1-R} v(x)$, $x \equiv w/h$, and then get HJB

$$0 = \sup\Big[U(c) + \beta\varepsilon U'(c) - \rho V + (rh - c + \varepsilon) V_h + \big((r+\Delta)(w - \theta\cdot 1) + \mu\cdot\theta - \varepsilon\big) V_w$$
$$+ \tfrac{1}{2} |\sigma^T \theta|^2 V_{ww} \Big]$$

so

$$0 = \sup\Big[U(\tilde c) + \beta\tilde\varepsilon U'(\tilde c) - \rho v + (r - \tilde c + \tilde\varepsilon)\big((1-R)v - x v'\big)$$
$$+ \big((r+\Delta)(x - \tilde\theta\cdot 1) + \mu\cdot\tilde\theta - \tilde\varepsilon\big) v' + \tfrac{1}{2} |\sigma^T \tilde\theta|^2 v'' \Big]$$

where $\tilde{c} \equiv c/h$, $\tilde{\varepsilon} \equiv \varepsilon/h$, $\tilde{\theta} = \theta/h$. Optimizing over $\tilde{\theta}$ will give

$$0 = (\mu - (r+\Delta)1)\, v' + \sigma\sigma^T \tilde{\theta}\, v''$$

By market clearing we should have $\tilde{\theta}\cdot 1 = 1+x$, so we learn that

$$\boxed{\tilde{\theta} = -(\sigma\sigma^T)^{-1}(\mu - (r+\Delta)1)\, v'/v''}$$

where $r+\Delta$ is hereby determined as a function of x by market clearing.

Optimizing over $\tilde{c}$ gives us

$$\boxed{U'(\tilde{c}) + \beta\,\tilde{\varepsilon}\,U''(\tilde{c}) = (1-R)v - xv'}$$

and the optimization over $\tilde{\varepsilon} \geq 0$ requires

$$\beta U'(\tilde{c}) + (1-R)v - xv' - v' \leq 0$$

(3) <u>What do we expect?</u> Generally it's better to have money in w than in h, so what I expect is that we just do occasional transfers from w to h, so expect that $\varepsilon_t\,dt$ is actually $d\varepsilon_t$, singular wrt Lebesgue. So expect that we pay no dividends while $x \leq x^*$, and then pay dividends at x^* to keep $x_t \leq x^*$. So should have

$$\begin{cases} U'(\tilde{c}) = (1-R)v - xv' & \forall x \leq x^* \\ \beta U'(\tilde{c}) + (1-R)v - xv' - v' \leq 0 & \forall x \leq x^*, \quad < \text{ for } x < x^* \end{cases}$$

Combining these tells us that

$$(1+\beta)\left\{(1-R)v - xv'\right\} \leq v'$$

The HJB reads (for $d=1$ just to keep it simple)

$$0 = \tilde{U}\big((1-R)v - xv'\big) - \rho v + r\big((1-R)v - xv'\big) + (r+\Delta)xv' - \tfrac{1}{2}\sigma^2\tilde{\theta}^2 v''$$

$$= \tilde{U}\big((1-R)v - xv'\big) - \rho v + r(1-R)v + \Delta x v' - \frac{\sigma^2}{2}(1+x)^2 v''$$

if we insist that the loans market clears, whereupon

$$r = \mu - \Delta + \sigma^2(1+x)\, v''/v'$$

To summarize then, we seek $v(\cdot)$ to satisfy

$$\boxed{0 = \max\Big\{ (1+\beta)\big((1-R)v - xv'\big) - v', \quad \tilde{U}\big((1-R)v - xv'\big) - \rho v + r(1-R)v + \Delta x v' - \frac{\sigma^2}{2}(1+x)^2 v'' \Big\}}$$

where

$$\boxed{r = \mu - \Delta + \frac{\sigma^2 v''(1+x)}{v'}}$$

(4) As a simple-minded first shot, we could try a recursive solution method, where we suppose we have v_n and then we calculate r_n by $\mu - \Delta + \sigma^2 v_n'' .(1+x)/v_n'$; then we select out the places where $(1+\beta)\big((1-R)v_n - xv_n'\big) - v_n' > 0$ and try to solve for v_{n+1} with equality in those places, while elsewhere we try to solve

$$\frac{\sigma^2}{2}(1+x)^2 v'' - \Delta x v' + \rho v = \tilde{U}\big((1-R)v_n - xv_n'\big) - r_n(1-R)v_n$$

(5) Another remark is that we expect v to be increasing, and $x^{R-1}v(x) \equiv g(x)$ to be decreasing (since this is just $V(x^{-1}, 1)$)

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(6) However, this central planner story isn't really how we should be thinking of it. There are two agents, households who control c, and firms who control (θ, ε). These agents have different objectives:

$$E\Big[\int_0^\infty e^{-\rho t} U(c_t)\, dt\Big] \quad \text{for households}$$

$$E\Big[\int_0^\infty e^{-\rho t} U'(c_t)\, d\varepsilon_t\Big] \quad \text{for firms}$$

We expect scaling to hold for the solution, so that the thing which matters for the solution is $x_t \equiv w_t / h_t$, which presumably will be Markovian. So we want an interest rate $r_t = r(x_t)$ such that we can achieve a Nash equilibrium which will also clear the markets.

The optimal value $V(h,w) = h^{1-R}\, v(x)$ for households satisfies

$$0 = \sup_c \Big[U(c) - \rho V + (rh - c)V_h + \big((r+\Delta)w + \theta(\mu - r - \Delta)\big)V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} \Big]$$

$$\therefore\ 0 = \sup_{\tilde{c}} \Big[U(\tilde{c}) - \rho v + (r - \tilde{c})\big\{(1-R)v - xv'\big\} + \big\{(r+\Delta)x + \tilde{\theta}(\mu - r - \Delta)\big\} v' + \tfrac{1}{2}\sigma^2 \tilde{\theta}^2 v'' \Big]$$

where $\tilde{\theta} \equiv \theta/h$. Hence we have

$$\boxed{0 = \tilde{U}\big((1-R)v - xv'\big) - \rho v + r(1-R)v + \Delta x v' + \tilde{\theta}(\mu - r - \Delta)v' + \tfrac{1}{2}\sigma^2\tilde{\theta}^2 v''}$$

with the BC

$$\boxed{(1-R)v - xv' - v' = 0 \qquad \text{at } x = x^*}$$

where x^* is the value at which the firm pays out dividends.

Alongside this we must consider the value $F(h,w) = h^{1-R} f(x)$ of the firm, which satisfies

$$0 = \sup_\theta \Big[F_h\,(rh - c) + F_w\big((r+\Delta)w + \theta(\mu - r - \Delta)\big) + \tfrac{1}{2}\sigma^2\theta^2 F_{ww} - \rho F\Big]$$

$$\therefore 0 = \sup_{\tilde{\theta}} \Big[(r - \tilde{c})\{(1-R)f - xf'\} + \{(r+\Delta)x + \tilde{\theta}(\mu - r - \Delta)\} f' + \tfrac{1}{2}\sigma^2\tilde{\theta}^2 f'' - \rho f\Big]$$

so

$$\boxed{0 = (r - \tilde{c})\{(1-R)f - xf'\} + (r+\Delta)x f' - \tfrac{1}{2}\kappa^2 \frac{f'^2}{f''} - \rho f}\,, \qquad \kappa \equiv \frac{\mu - r - \Delta}{\sigma}$$

$$\tilde{\theta} = -\kappa f'/\sigma f''.$$

along with the information that

$$\boxed{(1-R)f(x) - xf'(x) - f'(x) + U'(\tilde{c}) \le 0\,, \quad \text{equal at } x^*}$$

(7) Maybe better to have $d\varepsilon_t = \varphi_t h_t\, dt$, where $0 \le \varphi_t \le K$, as this will allow us to keep track of what is happening to the right of x^*. The dynamics are

$$\begin{cases} dh_t = (r_t h_t - c_t + \varphi_t h_t)\, dt \\ dw_t = (r_t + \Delta) w_t\, dt + \theta_t\big(\sigma\, dW_t + (\mu - r_t - \Delta)\, dt\big) - \varphi_t h_t\, dt \end{cases}$$

And the HJB equations to solve are

$$0 = \sup \Big[U(c) - \rho V + (rh - c + \varphi h) V_h + \{(r+\Delta)w + \theta(\mu - r - \Delta) - \varphi h\} V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww}\Big]$$

$$\therefore 0 = \sup_{\tilde{c}} \Big[U(\tilde{c}) - \rho v + (r - \tilde{c} + \varphi)\big((1-R)v - xv'\big) + \big((r+\Delta)x + \tilde{\theta}(\mu - r - \Delta) - \varphi\big)v' + \tfrac{1}{2}\sigma^2\tilde{\theta}^2 v''\Big]$$

$$\boxed{\begin{aligned} 0 = \tilde{U}\big((1-R)v - xv'\big) - \rho v + r(1-R)v + \Delta x v' + \tilde{\theta}(\mu - r - \Delta)v' + \tfrac{1}{2}\sigma^2\tilde{\theta}^2 v'' \\ + \varphi\big((1-R)v - xv' - v'\big) \end{aligned}}$$

with $\quad U'(\tilde{c}) = (1-R)v - xv'$

For the firm we have

$$0 = \sup_{\theta,\varphi} \Big[U'(c)\varepsilon - \rho F + (rh - c + \varphi h) F_h + \{ (r+\Delta) w + \theta(\mu - r - \Delta) - \varphi h \} F_w + \tfrac{1}{2}\sigma^2\theta^2 F_{ww} \Big]$$

$$0 = \sup \Big[\varphi U'(\tilde{c}) - \rho f + (r - \tilde{c} + \varphi)\big((1-R)f - xf'\big) + \{ (r+\Delta)x + \tilde{\theta}(\mu - r - \Delta) - \varphi \} f' + \tfrac{1}{2}\sigma^2 \tilde{\theta}^2 f'' \Big]$$

$$\boxed{\begin{aligned} 0 = K\big(U'(\tilde{c}) + (1-R)f - xf' - f'\big)^{+} - \rho f + (r - \tilde{c})\big((1-R)f - xf'\big) + (r+\Delta)xf' \\ - \tfrac{1}{2}\kappa^2 (f')^2 / f'' \qquad \kappa = \frac{\mu - r - \Delta}{\sigma} \end{aligned}}$$

and

$$\boxed{\begin{aligned} \varphi &= K\, I_{\{ U'(\tilde{c}) + (1-R)f - xf' - f' > 0 \}} \\ \tilde{\theta} &= -\kappa\, f' / \sigma f'' \end{aligned}}$$

Market clearing would require $\tilde{\theta} = 1 + x$.

This approach seems to be quite unstable numerically, perhaps no surprise.

(8) How could we get round this? In the region where the firm is paying dividends we must have

$$(1+x) f'(x) + (R-1) f(x) = U'(\tilde{c}(x))$$

and outside that region the equation is

$$\mathcal{L} f = 0$$

where $\mathcal{L} = \tfrac{1}{2}\sigma^2\tilde{\theta}^2 D^2 + \{ (\Delta + \tilde{c})x + \tilde{\theta}(\mu - r - \Delta) \} D - \big(\rho + (R-1)(r - \tilde{c})\big)$. What we could try would be find the (increasing) positive solution to $\mathcal{L} f_0 = 0$ with $f_0(b) = 1$, where b is the right-hand end of the grid. What we then expect is that to the left of some x^* we have $f(x) = A f_0(x)$ for some A, and the be

$$(1+x) f'(x) + (R-1) f(x) = U'(\tilde{c}(x))$$

should be satisfied at x^*. Of course, we can for any x choose A to make hold at x; but we want biggest solution, so let's have

$$A = \sup_x \frac{U'(\tilde{c}(x))}{(1+x) f_0'(x) + (R-1) f_0(x)}$$

First numerical examples put the sup at one end or other of the range . . .

(9) We can extract some boundary information when $x=0$, that is, $w=0$. In this case, the firm doesn't exist, $\theta = w = 0$ and the market clearing condition cannot hold. So the wealth equations as presented must be incorrect – we have to have the possibility that the households will pure hoard cash (so maybe this means $r=0$) For the situation where the households have $\dot{h}_t = -c_t$, the optimization for H is a simple deterministic tale:

$$c_t = h_0 \frac{\rho}{R} e^{-\rho t/R}, \qquad \int_0^\infty e^{-\rho t} U(c_t)\,dt = h_0^{1-R} \frac{R}{\rho} U(\rho/R)$$

Allowing for cash hoarding, we have $\theta = w$ when $r \leq 0$ because then there will be no lending

(10) Suppose we have local time dividends which immediately force x below x^* if $x > x^*$ ... how would we extend v into (x^*, ∞)? The usual kind of story gives us that

$$V(h,w) = h^{1-R}\left(\frac{1+x}{1+x^*}\right)^{1-R} v(x^*) \qquad \text{if } x = w/h \geq x^*$$

Hence we get

$$\boxed{U'(\tilde{c}) = (1-R)v - xv' = v' = \frac{1-R}{1+x}\left(\frac{1+x}{1+x^*}\right)^{1-R} v(x^*)}$$

and this tells us how to interpret $\tilde{c}$ in (x^*, ∞).

Hedge fund manager again (26/8/11)

The expressions on pp 44–45 of WN XXXII for the profiles aren't quite right and turn out to be easier once corrected.

Start off at time 0 with atom $\varphi(w_0)$ at w_0. When the level drops to $\underline{w}$, we take away from that atom a quantity $(1-p)(\varphi(w_0)-\varphi(\underline{w}))$, which is what would be removed if the profile φ extended all the way. So at the end the atom at w_0 is

$$\varphi(w_0) - q(\varphi(w_0)-\varphi(\underline{w})) = p\varphi(w_0) + q\varphi(\underline{w}).$$

Otherwise, the profile is $\varphi'(x)$ for $\underline{w} \leq x \leq w$, and $p\varphi'(x)$ for $w \leq x \leq \bar{w}$. Hence the total AUM will be

$$p\varphi(w_0) + q\varphi(\underline{w}) + p(\varphi(\bar{w})-\varphi(w)) + \varphi(w) - \varphi(\underline{w})$$

$$= p(\varphi(w_0)-\varphi(\underline{w})) + p\varphi(\bar{w}) + q\varphi(w)$$

As for the performance contribution, it will be

$$\{p\varphi(w_0) + q\varphi(\underline{w})\}(w-w_0)^+ + \int_{\underline{w}}^{w} \varphi'(x)(w-x)\,dx$$

and the integral is $\int_{\underline{w}}^{w} \varphi'(x)\left(\int_x^w dv\right)dx = \int_{\underline{w}}^{w} dv\,(\varphi(v)-\varphi(\underline{w}))$

$$= \int_{\underline{w}}^{w} \varphi(v)\,dv - (w-\underline{w})\varphi(\underline{w}).$$

$$\frac{dp}{p} = (I - 1p^T)\,dY - (I - 1p^T)\,dY\,dY^T p$$

$$= (I - 1p^T)(dY - dY\,dY^T p)$$

Evolution of proportions (5/9/11)

(1) Suppose we have continuous positive semimartingales X^i_t, $i=1,\dots,N$ such that

$$dX^i_t = X^i_t \, dY^i_t$$

and we set $\bar{X}_t = \sum_{i=1}^N X^i_t$, $p^i_t = X^i_t/\bar{X}_t$. How does p evolve? This can be done as follows

(i)
$$d(1/\bar{X}_t) = -\frac{1}{\bar{X}_t^2} d\bar{X}_t + \frac{d\bar{X}\,d\bar{X}}{\bar{X}_t^3}$$

$$= \frac{1}{\bar{X}_t}\left\{-\sum_i p^i_t \, dY^i_t + p_t^T dY_t dY_t^T p_t\right\} = \frac{1}{\bar{X}_t}\left\{-p_t\cdot dY_t + p_t\cdot dY_t dY_t^T p_t\right\}$$

(ii)
$$d\left(X^i_t/\bar{X}_t\right) = dp^i_t$$

$$= X^i_t \, d(1/\bar{X}_t) + \frac{dX^i_t}{\bar{X}_t} + dX^i_t \, d(1/\bar{X}_t)$$

$$= p^i_t\left\{-p\cdot dY + p\cdot dY dY^T p\right\} + p^i_t \, dY^i_t - p^i_t \, e_i \cdot dY dY^T p$$

$$\boxed{\therefore\ dp^i_t = p^i_t\left[dY^i - p\cdot dY + p\cdot dYdY^T p - e^i dYdY^T p\right].}$$

If it helps, we can do a few more calculations and derive

$$d(\log p_i) = dY^i - p\cdot dY + \tfrac{1}{2} p^T dYdY^T p - \tfrac{1}{2} e_i^T dYdY^T e_i$$

Similarly, we find that

$$dp^i_t = p^i_t\left[dY^i_t - p_t\cdot dY_t\right] - d\bar{X}_t \, dp^i_t/\bar{X}_t$$

which could also be derived directly.

(2) This question arose in a paper of Löhne & Rudloff that I was asked to discuss. It's a paper about transaction costs and it seemed to me like you could investigate some more interesting questions. For instance, suppose you have multiple log-Brownian assets which all have to be traded through the cash account at small proportional cost, how does this impact the standard Merton problem? What would be the impact on the log growth rate under optimal investment if we do no consuming?

Maybe this last question is a bit closer to tractable. So let's suppose that

$$dX^i_t = X^i_t\left(\sigma_{ij} \, dW^j_t + \mu^i dt + d\nu^i_t\right) \qquad i=1,\dots,N$$

describes the evolution of the wealth in the risky assets, where ν^i are FV processes determining the movement of funds between the assets, and the wealth in the cash account goes as

$$dX_t^0 = rX_t^0\,dt - \sum_1^N X_t^i\,d\nu_t^i - \varepsilon \sum_1^N X_t^i |d\nu_t^i|$$

$$= X_t^0\left[r\,dt - \sum_1^N \frac{p_t^i\,d\nu_t^i}{p_t^0} - \varepsilon \sum_1^N \frac{p_t^i}{p_t^0}|d\nu_t^i|\right]$$

Suppose now our objective is to

$$\max E^x\left[\int_0^\infty \rho e^{-\rho t}\log(\bar{X}_t)\,dt\right] \equiv V(x) = \log\bar{x} + v(p)$$

by the usual scaling story. HJB gives

$$0 = \sup\left[\rho\log(\bar{X}) - \rho V(x) + \mathcal{L}V(x)\right]$$

$$= \sup\left[-\rho v(p) + \mathcal{L}V(x)\right]$$

together with the inequalities for the derivatives. If we look at $V(X_t)$ we see

$$dV(X_t) = d\log\bar{X}_t + d\,v(p_t)$$

$$= p_t\cdot dY_t - \tfrac12 p_t^T\,dY_t\,dY_t^T p_t + Dv(p_t)\cdot dp_t + \tfrac12 dp_t^T D^2v(p_t)\,dp_t$$

$$\doteq p_t\cdot\mu\,dt + p_t\cdot d\nu_t + rp^0\,dt - p\cdot d\nu - \varepsilon p\cdot|d\nu|$$
$$- \tfrac12 p^T A p\,dt + Dv\cdot dp + \tfrac12 dp^T D^2v(p)\,dp \qquad A \equiv \begin{pmatrix} 0 & 0 \\ 0 & \sigma\sigma^T \end{pmatrix}$$

$$= (p_t\cdot\mu + rp^0)\,dt - \varepsilon p\cdot|d\nu| - \tfrac12 p^T A p\,dt + Dv\cdot dp + \tfrac12 dp^T\cdot D^2v(p)\,dp$$

If we use the notation $M = (I - 1p^T)A(I - p1^T)$ then the final qv term becomes $\frac12 p^i p^j M_{ij} D_{ij}v(p)\,dt$. In fact, we can unify the notation somewhat by setting

$$d\nu^0 \equiv -\left\{\sum_1^N p^j d\nu^j + \varepsilon\sum_1^N p^j|d\nu^j|\right\}/p^0, \qquad \mu^0 \equiv r$$

so that we have

$$dp/p = (I - 1p^T)\left[dY - dY\,dY^T p\right]$$

$$= \left[I - 1p^T\right]\cdot\left\{\begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}dW + \mu\,dt + d\nu - Ap\,dt\right\}$$

so

$$Dv\cdot dp \doteq p^i D_i v\,(\delta_{ij} - p^j)\left\{(\mu^j - (Ap)_j)\,dt + d\nu^j\right\}$$

so that all in all we have

$$dV(X_t) \doteq \Big\{ p_t \cdot \mu - \tfrac{1}{2} p_t \cdot A p_t + p_i D_i v \,(\delta_{ij} - p^j)\,(\mu^j - (Ap)_j) + \tfrac{1}{2} p^i p^j (D_{ij} v) M_{ij} \Big\} dt$$

$$- \varepsilon\, p \cdot |d\bar{\nu}| + p_i D_i v\,(\delta_{ij} - p^j)\, d\bar{\nu}^j$$

and we want this to be ρv. What do we learn from that?

The guess is that we have

$$v(p) \approx a + \tfrac{1}{2}(p-\pi) Q (p-\pi)$$

where $z \equiv p - \pi$ is going to be small. So if we take an expansion of the drift terms in $V(X_t)$ out to second order in z, what we see is complicated a bit by the fact that M depends on p. If we set $M^0 \equiv (I - 1\pi^T) A (I - \pi 1^T)$ and $B \equiv A(I - \pi 1^T)$ then what we find is that

$$M = M^0 - 1 z^T B - B^T z 1^T + (z^T A z)\, 1 1^T.$$

Thus the terms in M will contribute

$$\tfrac{1}{2} p^i p^j (D_{ij} v) M_{ij} = \tfrac{1}{2}(\pi_i + z_i)(\pi_j + z_j)\, Q_{ij} \left(M^0_{ij} - B_{ki} z_k - B_{lj} z_l + z_k A_{kl} z_l \right)$$

$$= \tfrac{1}{2} \pi_i Q_{ij} \pi_j M^0_{ij} + \Big\{ z_i \pi_j Q_{ij} M^0_{ij}$$

$$+ \tfrac{1}{2} \pi_i \pi_j Q_{ij} \left(-B_{ki} z_k - B_{lj} z_l \right) \Big\}$$

$$+ \Big\{ \tfrac{1}{2} z_i z_j Q_{ij} M^0_{ij} - z_i \pi_j Q_{ij} (B_{ki} z_k + B_{lj} z_l) + \tfrac{1}{2} \pi_i \pi_j Q_{ij} z_k z_l A_{kl} \Big\}$$

$$+ O(z^3)$$

The terms from Dv will give

$$(Qz)_i\, (\pi_i + z_i)\, (\delta_{ij} - \pi_j - z_j)\left(\mu_j - (A\pi)_j - (Az)_j \right)$$

$$= (Qz)_i \Big[\pi_i (\delta_{ij} - \pi_j)(\mu_j - (A\pi)_j) + z_i (\delta_{ij} - \pi_j)(\mu_j - (A\pi)_j) - \pi_i z_j (\mu_j - (A\pi)_j)$$

$$- \pi_i (\delta_{ij} - \pi_j)(Az)_j \Big] + O(z^3).$$

So we now have expressions for the drift components of $dV(X_t)$ up to quadratic in z and the job is to compare coefficients. What we find is:

z^0] $\rho a = \pi\cdot\mu - \frac{1}{2}\pi\cdot A\pi + \frac{1}{2}\pi_i\pi_j M^0_{ij} Q_{ij}$

z^1] $0 = \mu_i - A_{ij}\pi_j + Q_{ki}\{\pi_k(\delta_{kj}-\pi_j)(\mu_j - (A\pi)_j)\}$

$$+ \pi_i Q_{ij} M^0_{ij} - \frac{1}{2}\pi_\ell\pi_m Q_{\ell m}(B_{i\ell} + B_{im})$$

z^2]: $\frac{1}{2}\rho Q_{ij} = -\frac{1}{2}A_{ij} + \{Q_{ij}(\delta_{ik}-\pi_k)(\mu_k - (A\pi)_k) - Q_{mj}\pi_m(\delta_{m\ell}-\pi_\ell)A_{\ell i} - Q_{\ell i}\pi_\ell(\mu_j - (A\pi)_j)\}^S$

$$+ \frac{1}{2}Q_{ij}M^0_{ij} - (\pi_\ell Q_{i\ell}B_{j\ell} + \pi_\ell Q_{i\ell}B_{j\ell})^S + \frac{1}{2}A_{ij}\pi_k\pi_\ell Q_{k\ell}$$

Where $(\ )^S$ denotes the symmetric part of the matrix. Need to find π, Q. As a first guess, we might try some iterative scheme, where we make a first guess $\pi^{(0)}$ for π, then find Q from z^2], pass this into z^1] to update π and continue ...

Before jumping in, let's notice that we are thinking of $\rho>0$ very small, and in that case we expect the optimised objective to look something like

$$\log\bar{x} + \frac{\alpha}{\rho} + v_0(p)$$

since we expect $E\log(\bar{X}_t)$ to grow like αt, and there will be some transitory effect for the initial portfolio p and initial wealth $\bar{x}$ scales everything, but in the end it's large-t behaviour αt that takes charge. So we could just jump to the limit $\rho=0$ immediately, slightly simplifying the equations for the second-order bits.

<u>But here's a problem</u>: even if we could identify π, Q, α from the above equations, they do not depend on ε in any way ... and this has to be wrong ...

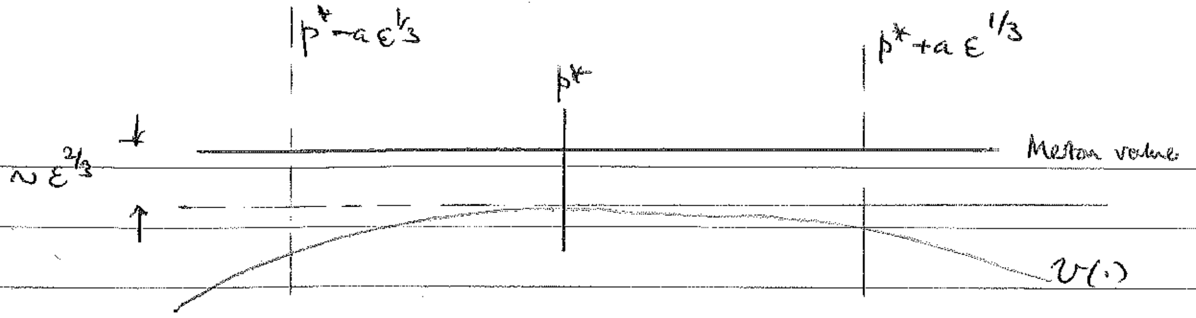
However, I think we can find a workaround. The thing is that if

$$\sup\{-\rho f(x) + \mathcal{L}f(x) + \rho\log\bar{x}\} \geq 0$$

then we know that $f(x_0) \leq V(x_0)$, and reversing the inequality works too. So if we find the quadratic, and determine the no-trade region by finding where the gradient condition bites, then by shifting a up or down a bit so that the inequality holds everywhere we can get bounds on the value.

<u>But there seems to be another problem</u>: Can we confirm that the size of the no-trade region will be $O(\varepsilon^{1/3})$ by such analysis? If not, it is hopeless.

If we look at the picture for $v(p)$ in the standard 1-risky-asset case, we see something like



The change of slope in $v(\cdot)$ between $p^* - a\varepsilon^{1/3}$ and $p^* + a\varepsilon^{1/3}$ has to be $O(\varepsilon)$ and the change of slope is to leading order $-2a\,\varepsilon^{1/3}\,v''(p^*)$, so this suggests we must have

$$\boxed{-v''(p^*) \sim \varepsilon^{2/3}}$$

Could we impose such scaling on Q??

Comparing with the standard Davis-Norman problem, we see a similar issue arising; the resolution in DN is the requirement that the value is C^2 at the boundaries ...

Firms - banks - households differently? (17/9/11)

1) Our current attempts at this really struggle with the dynamic programming; perhaps there can be some way around this? Let's work in continuous time; this may necessitate a rougher treatment of default, but that may be OK. To set some notation:

C_t = consumption rate — L_t = rate of working

K_t = firms' capital — D_t = nominal value of firms' debt

p_t = price level for goods — Δ_t = nominal value of household deposits

w_t = wage rate — R_t = interest charged on loans

I_t = investment rate — r_t = interest charged on deposits

Z_t = random multiplier of (net) production

l_t = rate of issuance of new loans to firms.

The firm has a production function $f(K,L)$ homogeneous of degree 1, concave incr in both arguments. Households gain utility at rate $U(C_t, L_t)$, discounting at rate ρ. Let's define the gross profit rate at time t

$$\boxed{q_t = Z_t f(K_t, L_t) - (w_t L_t + R_t D_t)/p_t} \qquad (0)$$

measured in units of consumption good. Then we have the usual equation

$$\boxed{C_t + I_t = Z_t f(K_t, L_t)} \qquad (1)$$

and evolution equations:

$$\dot{K}_t = I_t - (\delta + \varphi(q_t)) K_t \qquad (2)$$

$$\dot{D}_t = l_t - \varphi(q_t) D_t \qquad (3)$$

$$\dot{\Delta}_t + p_t C_t = r_t \Delta_t + w_t L_t + \{ p_t (q_t - I_t) + l_t \} \qquad (4)$$

for evolution of capital, debt + deposits, where $\varphi(\cdot)$ is smooth, decreasing to 0, representing the rate at which firms die as a function of overall profitability. The evolution of Δ involves

$$\boxed{a_t \equiv p_t (q_t - I_t) + l_t} \qquad (5)$$

which is the rate that the firms pay dividends. This may seem odd; firms increase borrowing via l, only to pay the borrowed money out immediately as dividends?! I think of it rather that the firm wants to invest more, and uses the new borrowings to support investment, but wants to maintain the dividends to keep share value up – more later

By combining (0) and (4) we get

$$\boxed{\dot{\Delta}_t + R_t D_t = \ell_t + r_t \Delta_t} \quad (6)$$

which has the natural interpretation as the bank's cash inflow equals its cash outflow; total cash in the bank remains unchanged, as you would expect in a model where money has infinite speed.

2) As well as these natural evolution equations, we have certain optimality conditions. Let us suppose the stock at time t is priced at $S_t = p_t K_t$, and that debt is collateral constrained:

$$D_t \le b S_t$$

for some $b \in (0,1)$ fixed. Then we have by various simple arguments the optimality equations

$$w_t U_C + p_t U_L = 0 \quad (7)$$

$$p_t Z_t f_L \ge w_t, \quad \text{equal if } D_t < b S_t \quad (8)$$

$$Z_t f_K \ge R_t + \delta + \varphi(q_t), \quad \text{equal if } D_t < b S_t \quad (9)$$

$$e^{-\rho t + \int_0^t r_s ds} U_c(C_t, L_t)/p_t \quad \text{is a martingale} \quad (10)$$

For (7), think that H could do ε more units of work, get paid εw_t for it, and consume at additional rate $\varepsilon w_t / p_t$; optimality requires there to be no benefit from this change.
For (8), think that the firm could employ labour at additional rate ε at additional cost εw_t, resulting in additional output rate $Z_t f_K$ of goods, $p_t Z_t f_K$ of cash. Of course, if he were fully borrowed, he wouldn't be able to borrow the money for this even if it would help.
For (9) we argue similarly that firms could borrow εp_t to acquire additional capital ε to be used for short time Δt, at the end of which must repay $\varepsilon p_t (R_t \Delta t + 1)$ and the depreciation $(\delta + \varphi(q_t))\varepsilon \Delta t$ on the capital, but meantime the improved output has generated extra $p_t Z_t f_K(K_t, L_t) \cdot \varepsilon \Delta t$.
For (10), the households could reduce current consumption by ε for time Δt, thus saving $\varepsilon p_t \Delta t$, which would be invested until later time T when it would be worth

$$\varepsilon p_t \Delta t \, \exp\left(\int_t^T r_s ds\right) / p_T$$

in time-T consumption goods. In expectation, this cannot at optimality improve the expected objective, whence the result.

3) The bank will be required to keep at least a fraction α of deposits in the vault at all times. So we shall have for Money supply M, held in the vaults, that

$$\alpha \Delta \leq M \leq \Delta$$

At the same time, we have the bound

$$D \leq (1-\alpha) \Delta$$

on the total borrowings of firms: this is in addition to the leverage constraint $D_t \leq b S_t = b p_t K_t$. I think the constraints in terms of M will be what stops it all degenerating into indeterminacy: otherwise, you could just scale all cash quantities by the same positive number and get another solution.

4) The final piece of the story relates to the investment behaviour of the households. Their total wealth at time t is $\Delta_t + S_t$. Now suppose that we have

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t + \mu_t \, dt$$

Then if $\tilde{S}_t$ is the price of the stock where we reinvest the dividend process a, we can show that

$$\frac{d\tilde{S}}{\tilde{S}} = \frac{dS}{S} + \frac{a}{S} dt \equiv \sigma_t dW + \tilde{\mu}_t \, dt \quad \text{, say.}$$

As an ansatz, let's suppose that the proportion of wealth invested in the risky asset shall be given by the Merton proportion

$$\boxed{\frac{S_t}{S_t + \Delta_t} = \pi_t = \frac{\tilde{\mu}_t - r_t}{\gamma \sigma_t^2}}$$

where $\gamma > 0$ is the cce of RRA of the hypothetical Merton agent.

TC problem approximation (19/9/11)

(1) Let's return to the problem started on pp 13–17 and see how it all works out in the case of a single risky asset:

$$\begin{cases} dx_t = rx_t\,dt - d\nu - \varepsilon\,|d\nu| \\ dy_t = y_t(\sigma\,dW_t + \mu\,dt) + d\nu \end{cases}$$

If the aim is to maximize the objective

$$E\Big[\int_0^\infty \rho e^{-\rho t}\log(\bar{X}_t)\,dt\Big] = \log(x+y) + v(z) \equiv V(x,y)$$

where $z \equiv y/(x+y)$ is the proportion of wealth $\bar{X} \equiv x+y$ in the risky asset, then we shall have

$$Y_t \equiv e^{-\rho t}V(x_t,y_t) + \int_0^t \rho e^{-\rho s}\log(x_s+y_s)\,ds$$ is a supermart etc

We obtain ($d\eta \equiv d\nu/(x+y)$)

$$dz = \sigma z(1-z)\,dW + z(1-z)(\mu-r-\sigma^2 z)\,dt + d\eta + \varepsilon z\,|d\eta|$$

and

$$dY \doteq -\rho v\,dt + \{\mu z + r(1-z) - \tfrac{1}{2}\sigma^2 z^2\}\,dt - \varepsilon\,|d\eta|$$

$$(*) \qquad + \Big\{\tfrac{1}{2}\sigma^2 z^2(1-z)^2 v'' + z(1-z)(\mu-r-\sigma^2 z)v'\Big\}\,dt + v'(z)\,d\eta + \varepsilon z v'(z)|d\eta|$$

(2) I've stuck this into Maple, and substituted in the quadratic guess $v(z) = \alpha - \frac{1}{2}q(b-z)^2$ into the story, and set the drift to zero (out to the first two powers of z).

What happens is that the three equations are determinate: the coefficients α, b, q are uniquely fixed by the requirement that the quadratic vanishes to this order. The problem with this is that we then have no freedom to adjust the quadratic to match boundary conditions...

(3) Doing some rough calculations with my code in COURSES/OT/MUNICH/COMPUTING and fiddling around with the transaction costs gives some interesting information.

In the computations, the idea was that

$$V(x,y) = y^{1-R} f(x/y) = y^{1-R} g(\log(x/y))$$

$$= (x+y)^{1-R}v(p)$$ where we let p denote the proportion of wealth in bank

Thus $$v(p) = (1-p)^{1-R} f\Big(\frac{p}{1-p}\Big) = (1-p)^{1-R} g\Big(\log\Big(\frac{p}{1-p}\Big)\Big)$$

Writing $\log(p/(1-p)) = y$, we find that

$$v'' \propto (1+e^y)^2 g''(y) + (1+e^y)(2Re^y - 1 - e^y)g' - R(1-R)e^{2y}g(y)$$

if we evaluate v'' at $1-\pi_M$ and plot $\log(-v'')$ against $\log(\varepsilon)$ we get an almost perfect straight line with slope as near as you can reasonably hope to $2/3$!! This tells us pretty clearly that

$$\boxed{v''(1-\pi_M) \doteqdot A\,\varepsilon^{2/3}}$$

This means that the idea that just approximating by a quadratic in the continue region and setting $\mathcal{L}v = 0$, and pulling out the leading terms and setting equal to zero cannot work

(4) OK, well let's look at the singular parts of the evolution (*), namely,

$$-\varepsilon(1-zv'(z))\,|d\eta| + v'(z)\,d\eta$$

$$= \{-\varepsilon(1-zv'(z)) \pm v'(z)\}\,|d\eta|$$

which must be ≤ 0 for both signs, and there should be C^2 matching at both ends of the no-trade interval (a,b). How does the C^2 matching condition look?

If we start at $z > b$, we immediately do a trade of size

$$\theta = (z-b)\,\frac{x+y}{1-\varepsilon b}$$

moving to new wealth

$$x+y-\varepsilon\theta = (x+y)\,\frac{1-\varepsilon z}{1-\varepsilon b}$$

So the value function in $z > b$ is

$$\log(x+y) + v(z) = v(b) + \log(x+y) + \log\frac{1-\varepsilon z}{1-\varepsilon b} \Rightarrow \boxed{v'(z) = -\frac{\varepsilon}{1-\varepsilon z}}$$

By similar reasoning, the value in $z < a$ is

$$v(z) = v(a) + \log\frac{1+\varepsilon z}{1+\varepsilon a} \Rightarrow \boxed{v'(z) = \frac{\varepsilon}{1+\varepsilon z}}$$

Thus we can see that outside $[a,b]$ we always have

$$\boxed{v''(z) = -v'(z)^2}$$

which we use to do the C^2 matching.

So suppose we try an approximate value function of the form

$$\boxed{v(z) = \alpha - \tfrac{1}{2}q(z-\beta)^2}$$

Clearly we find $\{a,b\}$ by matching the C^1 condition, then hope to get q, β from C^2.

Find b by solving

$$v'(z) = -\varepsilon(1-zv'(z))$$

which is

$$-q(z-\beta) = -\varepsilon\{1+zq(z-\beta)\}, \quad \text{that is,} \quad \varepsilon q z^2 - zq(1+\beta\varepsilon) + q\beta + \varepsilon = 0$$

so

$$b = \frac{q(1+\beta\varepsilon) - \sqrt{q^2(1+\beta\varepsilon)^2 - 4\varepsilon q(\beta q+\varepsilon)}}{2\varepsilon q}$$

$$= \frac{2(\beta q+\varepsilon)}{q(1+\beta\varepsilon) + \sqrt{q^2(1+\beta\varepsilon)^2 - 4\varepsilon q(\beta q+\varepsilon)}}$$

More neatly, if we set $b-\beta = t$, then t solves

$$-qt = -\varepsilon\{1+qt(\beta+t)\}$$

whence

$$\boxed{\beta + t = b = \beta + \frac{2\varepsilon}{q(1-\varepsilon\beta) + \sqrt{q^2(1-\varepsilon\beta)^2 - 4\varepsilon^2 q}}}$$

Similarly to find $a = \beta - t$ we match $v'(z) = \varepsilon(1-zv'(z))$ to obtain

$$\boxed{a = \beta - \frac{2\varepsilon}{q(1+\varepsilon\beta) + \sqrt{q^2(1+\varepsilon\beta)^2 - 4\varepsilon^2 q}}}$$

Asking Maple to match the C^2 condition at a, b and thereby solve for q, β gives the solution $\beta = 0$, $q = 4\varepsilon^2$; nothing like what we expect. So there has to be an inequality in the C^2 condition at one boundary or the other... To get an upper bound, we'd expect v'' not to increase as we leave $[a,b]$. For a lower bound, we require a lower bound on $\mathcal{L}v$ globally? The key thing is this. If we write $(*)$ as

$$dY \doteq (\mathcal{L}v - \rho v + Q(z))dt + \{-\varepsilon(1-zv'(z)) \pm v'(z)\}\,|d\nu|$$

$$\left[Q(z) = -\tfrac{1}{2}\sigma^2 z^2 + \mu z + r(1-z)\right]$$

we have to look at

$$z \mapsto \max\{\mathcal{L}v - \rho v + Q(z),\ -\varepsilon(1-zv'(z)) + v'(z),\ -\varepsilon(1-zv'(z)) - v'(z)\} \equiv H(z)$$

If this is zero, we've got solution. If ≤ 0 everywhere, then v is an upper bound, if ≥ 0 everywhere then v is a lower bound. The real issue seems to be establishing an upper bound for $\mathcal{L}v - \rho v$ in the regions where one of the first-derivative conditions is 0.

Now in $z > b$ we have $v(z) = v(b) + \log\left(\frac{1-\varepsilon z}{1-\varepsilon b}\right)$ so we can calculate in this region

$$\mathcal{L}v - \rho v + Q(z) = -\tfrac{1}{2}\sigma^2 z^2 + \mu z + r(1-z) - \rho v(b) + R(z,\varepsilon) = Q(z) - \rho v(b) + R(z,\varepsilon)$$

where the remainder term $R(z,\varepsilon)$ satisfies $|R(z,\varepsilon)| \leq K\varepsilon$ for all z, some K.
Now the quadratic $Q(z)$ is maximised at $z = \pi_M$, the Merton proportion; indeed,

$$Q(z) = r + \tfrac{1}{2}|\kappa|^2 - \tfrac{1}{2}\sigma^2 (z - \pi_M)^2 \qquad [\kappa \equiv (\mu - r)/\sigma]$$

If we assume (plausibly) that $\pi_M \in (a,b)$, then the maximal value of Q on (b,∞) will be $r + \frac{1}{2}|\kappa|^2 - \frac{1}{2}\sigma^2(b-\pi_M)^2$, so if we have

$$\boxed{\rho v(b) \geq r + \tfrac{1}{2}|\kappa|^2 - \tfrac{1}{2}\sigma^2(b-\pi_M)^2 + K\varepsilon}$$

then we shall have $\mathcal{L}v - \rho v + Q(z) \leq 0$ for $z \geq b$. A similar argument on the other side will give

$$\rho v(a) \geq r + \tfrac{1}{2}|\kappa|^2 - \tfrac{1}{2}\sigma^2(a-\pi_M)^2 + K\varepsilon$$

as a sufficient condition to guarantee $\mathcal{L}v - \rho v + Q(z) \leq 0$ for $z \leq a$. We therefore just need to deal with the middle region.

Let's propose to place a, b symmetrically about π_M: $\boxed{a = \pi_M - \delta, \quad b = \pi_M + \delta}$ for some $\delta > 0$, and define the value function v in (a,b) by

$$v(z) = \alpha - \tfrac{1}{2} q (z - \pi_M)^2$$

for some α, q to be discovered. The inequality above for v will require

$$\boxed{\rho(\alpha - \tfrac{1}{2} q \delta^2) \geq r + \tfrac{1}{2}|\kappa|^2 - \tfrac{1}{2}\sigma^2\delta^2 + K\varepsilon}$$

We can allow a discontinuity of v' at a, b provided the right derivative is no bigger than the left derivative there; these become the conditions

$$\boxed{\frac{\varepsilon}{1+\varepsilon a} \geq q\delta \qquad , \qquad -q\delta \geq \frac{-\varepsilon}{1-\varepsilon b}}$$

so we see these are both satisfied if $q\delta \leq \varepsilon/(1+\varepsilon a)$. Provided these inequalities are satisfied, the v constructed is an upper bound for the value. We want to make this upper bound small. We also need to ensure

$$\sup_{a \leq z \leq b} \left[\mathcal{L}v(z) - \rho v(z) + Q(z) \right] \leq 0$$

How can we achieve this? If we look at $\mathcal{L}v(z) - \rho v(z) + Q(z)$, writing

$z = \pi_M + t$, we get (assuming the quadratic form for v) $\quad [A \equiv r + \frac{1}{2}\kappa^2]$

$$\mathcal{L}v(z) - \rho v(z) + Q(z) = A - \rho\alpha - \tfrac{1}{2}\kappa^2(\pi-1)^2 q$$
$$- \sigma^2\pi(2\pi-1)(\pi-1)qt - \tfrac{1}{2}\sigma^2 t^2$$
$$+ O(qt^2)$$

Now we need to have

(1) $\boxed{\rho(\alpha - \tfrac{1}{2}q\delta^2) \geq A - \tfrac{1}{2}\sigma^2\delta^2 + K\varepsilon}$ for control outside $[a,b]$

(2) $q\delta \leq \dfrac{\varepsilon}{1+\varepsilon a}$ (so $\boxed{q\delta = \varepsilon + O(\varepsilon^2)}$) for control at $\{a,b\}$

and maximising over t inside $[a,b]$, we take $t = -\pi(2\pi-1)(\pi-1)q$ and get

(3) $\boxed{\rho\alpha \geq A - \tfrac{1}{2}\kappa^2(\pi-1)^2 q + \tfrac{1}{2}\sigma^2\pi^2(\pi-1)^2(2\pi-1)^2 q^2 + O(qt^2)}$

Now we expect q, δ both go to zero as $\varepsilon \downarrow 0$; looking at (1) we'd think for a low bound on α we'd want large δ; looking at (3), we'd think we want large q, and looking at (2) we see we can't have both. So the obvious thing is to set

$$\tfrac{1}{2}\sigma^2\delta^2 = \tfrac{1}{2}\kappa^2(\pi-1)^2 q$$

with $q = \varepsilon/\delta$, therefore

$$\boxed{\delta^3 = \left(\frac{\kappa(\pi-1)}{\sigma}\right)^2 \varepsilon}$$

with $q = \varepsilon/\delta$. Hence we have $\delta \propto \varepsilon^{1/3}$, $q \propto \varepsilon^{2/3}$, and

$\boxed{\rho\alpha \sim A - \tfrac{1}{2}\sigma^2(\kappa(\pi-1)/\sigma)^{4/3}\varepsilon^{2/3}}$

For lower bounds, the arguments are similar but easier. Outside $[a,b]$, $H(z) \geq 0$ automatically because one of the first-derivative terms there is zero. To get the jump of the gradient at $\{a,b\}$ going the right way, we'll demand

$$q\delta = \frac{\varepsilon}{1-\varepsilon b} = \varepsilon + O(\varepsilon^2)$$

and inside the interval, the worst $t \in [-\delta, \delta]$ to choose will be one of the endpoints giving the requirement

$$0 \leq A - \rho\alpha - \tfrac{1}{2}\kappa^2(\pi-1)^2 q - \sigma^2\pi|2\pi-1|.(1-\pi)q\delta - \tfrac{1}{2}\sigma^2\delta^2 + o(\varepsilon)$$

To get the best possible bound for α (that is, α high) we want to make

$$\tfrac{1}{2}\kappa^2(\pi-1)^2 q + \tfrac{1}{2}\sigma^2\delta^2$$

as small as we can while keeping $q\delta = \varepsilon$. This leads easily to

$$\delta^3 = \frac{\kappa^2(\pi-1)^2}{2\sigma^2}\,\varepsilon$$

This gives a bound

$$\boxed{\rho\alpha \leq A - \tfrac{1}{2}\sigma^2\left(\frac{\kappa(1-\pi)}{\sigma}\right)^{4/3}\varepsilon^{2/3}\left\{2^{1/3} + 2^{-2/3}\right\} + o(\varepsilon)}$$

as a lower bound (which is indeed lower than the upper bound!)

Least-action filtering: going a bit further (24/9/11)

We have already seen the LAF story where we need to minimise

$$\int_0^T \psi(s, x_s, p_s)\, ds + \varphi(x_0)$$

But now, let's view T as a parameter of the problem, and write $x_t = \tilde{x}(t/T)$, $p_t = \tilde{x}'(t/T)\, T^{-1}$, so that the objective can be rewritten as

$$\int_0^1 \Psi(u, \tilde{x}_u, \tilde{x}_u' ; T)\, du + \varphi(\tilde{x}_0)$$

As in the original analysis (now writing $X(u,T) = \tilde{x}_u = x(uT)$) we get the conditions

$$\begin{cases} 0 = D_x \varphi(X_0) - D_p \Psi(0, X_0, \dot{X}_0 ; T) \\ 0 = D_x \Psi - (D_{tp}\Psi) - (D_{px}\Psi)\dot{X} - (D_{pp}\Psi)\ddot{X} \\ 0 = (D_p \Psi)(1, X_1, \dot{X}_1 ; T) \end{cases}$$

Now the key thing here is that this would have to hold $\forall T$, so we could differentiate wrto T and get a PDE for X; this might not be very easy to deal with, but once we had found solution for some T, we would in principle be able to evolve that forward. Of course, it can only give a local minimum, and it's clear that as T varies continuously the LA path may experience a jump.

An equilibrium example (6/10/11)

When we solve an equilibrium problem, it's most common to assume a representative agent or a central planner; examples where agents have different state-price densities yet agree on the price of the stock are not so easy to construct; let's see what we can do

Suppose we have a one-dimensional diffusion X with generator $\mathcal{G}$, and suppose that the output process is $\delta(X_t)$. We'll suppose there are two agents, both with time preference rate ρ, and we'll look for an equilibrium where agent i has SPD $\zeta_t^i = e^{-\rho t} h_i(X_t)$. To have agreement on the price of the stock, it must be that

$$v(x) = \frac{1}{h_i(x)} E^x\left[\int_0^\infty e^{-\rho t} h_i(X_t)\delta(X_t)\,dt\right]$$

$$= \frac{1}{h_i(x)} R_\rho(h_i\delta)(x)$$

is the same function for $i=1,2$. Hence

$$(\rho - \mathcal{G})(h_i v) = h_i\delta$$

so if we write $h_i v = f_i$, we seek two solutions f to

$$\boxed{(\mathcal{G}-\rho)f + f\,\frac{\delta}{v} = 0} \qquad (*)$$

which must be strictly positive, v positive.

Let's suppose we look for one solution f_0; then construct the other from that in the usual way. If we have a diffusion in $(-1,1)$ with generator

$$\mathcal{G} = \tfrac{1}{2}(1-x^2)^2 D^2$$

and try $\boxed{f_0(x) = (1-x^2)^\alpha}$ for some $0<\alpha<1$, we find $(*)$ says

$$-\alpha\{1-(1-2\alpha)x^2\} + \frac{\delta}{v} - \rho = 0$$

$$\therefore \quad \boxed{\frac{\delta}{v} = \rho + \alpha(1-(1-2\alpha)x^2) > 0}$$

Now we write the other solution as $f_0\varphi$, where

$$\tfrac{1}{2}\varphi'' f_0 + \varphi' f_0' = 0 \qquad \therefore (\log\varphi')' = -2(\log f_0)'$$

$$\Rightarrow \boxed{\varphi' = f_0^{-2}}$$

Now provided that $0 < \alpha < 1/2$, f_0^{-2} is integrable, and we may take

$$\varphi(x) = \int_{-1}^{x} f_0(y)^{-2}\, dy$$

to give a different solution $f_1 = \varphi f_0$ which is positive in $(-1,1)$

There remains the issue of whether this can fit an equilibrium; we need

$$h_i(X_t) = u_i'(c_i(X_t)) = f_i(X_t)/v(X_t)$$

$$\Rightarrow \quad c_j(X_t) = I_j\left(f_j(X_t)/v(X_t) \right)$$

$$= I_j\left(f_j(X_t) \, \frac{\psi(X_t)}{\delta(X_t)} \right)$$

where $\psi(x) \equiv \rho + \alpha(1-(1-2\alpha)x^2)$. We therefore need

$$\delta(x) = c_1(x) + c_2(x) = I_1\left(f_1(x) \, \frac{\psi(x)}{\delta(x)} \right) + I_2\left(f_2(x) \, \frac{\psi(x)}{\delta(x)} \right)$$

We might take $I_1(y) = I_2(y) = y^{-1/R}$, perhaps with $R > 1$, and then we would get

$$\delta(x) = \delta(x)^{1/R} \left\{ \left(\frac{\psi(x)}{f_1(x)} \right)^{1/R} + \left(\frac{\psi(x)}{f_2(x)} \right)^{1/R} \right\}$$

which will determine a unique $\delta(x) > 0$ for the dividend process!

Optimal investment with model uncertainty (29/10/11)

(1) Suppose we're in a complete market setting, and we have J different advisors who each think that the LR martingale is Λ^j_t $(j=1,\dots,J)$ and now the overall objective is

$$\max \; E\Big[\sum_j \alpha_j \Lambda^j_T \, U(w^j_T)\Big]$$

where $U(x) = -e^{-x}$. So what will do is to split initial wealth w_0 as a sum $w_0 = \sum_{j=1}^J w^j_0$ of the initial wealths to be assigned to each agent, each of whom maximizes his own objective. From the point of view of the central planner, any $w_T \equiv \sum_{j=1}^J w^j_T$ can be achieved which obeys the budget constraint $E w_T = w_0$, so we will have the usual terminal condition

$$\alpha_j \Lambda^j_T \, U'(w^j_T) = \lambda \zeta_T$$

so what we learn from this is that

$$\boxed{\frac{\Lambda^j_T}{\Lambda^k_T} = \frac{\alpha_k U'(w^k_T)}{\alpha_j U'(w^j_T)} = \frac{\alpha_k}{\alpha_j} \exp\left(w^j_T - w^k_T\right)}$$

What this suggests is that <u>the relative credibility of the different hypotheses can be judged from the success of the individual advisors</u> (models)!

(2) Some remarks are needed here:

(i) As with the log-likelihood story, some geometric downweighting of performance in the past would probably be a good idea...

(ii) While it's tempting to think that we might get hold of the optimal investment policy by a sequence of myopic one-step optimizations

$$\max \; E_{t-1}\{-\exp(\theta \cdot \Delta X_t)\}$$

it's not correct to do this; the value at time t will certainly look like

$$V^j_t(w) = -\exp(-w - \xi^j_t)$$

but the point is that ξ^j_{t+1} won't generally be independent of the increment ΔX_{t+1}

Dual of Cobb-Douglas production function (29/10/14)

Sometimes might need this; it's a simple calculation but let's just record it.
We have for some $0 < \beta < 1$, $A > 0$

$$f(K,L) = A\, K^{\beta} L^{1-\beta}$$

which is concave increasing, and to calculate the dual we seek for $x, y \geq 0$

$$\sup_{K,L} \left[f(K,L) - xK - yL \right]$$

Calculus tells us

$$\frac{\beta f}{K} = x, \quad \frac{(1-\beta) f}{L} = y$$

so that

$$\frac{K}{L} = \frac{\beta y}{(1-\beta) x}$$

Hence we take $K = t\beta y$, $L = t(1-\beta)x$ for some t which we need to find.
We have

$$\sup_t \left[f(t\beta y, t(1-\beta)x) - t\beta xy - t(1-\beta)xy \right]$$

$$= \sup_t \; t \left[A\, (\beta y)^{\beta} ((1-\beta)x)^{1-\beta} - xy \right]$$

So what we find is that $\tilde{f} = 0$ if $[\cdots] \leq 0$, else $\tilde{f} = +\infty$. So the region where $\tilde{f} = 0$ is where

$$\boxed{\left(\frac{x}{\beta}\right)^{\beta} \left(\frac{y}{1-\beta}\right)^{1-\beta} \geq A}$$

or more succinctly,

$$\boxed{x^{\beta}\, y^{1-\beta} \geq f(\beta, 1-\beta)}$$

If the profitability of the shadow firm were < 0, we could reduce r a little + still keep the shadow firm non-profitable, and this would (presumably) lower R and make things better overall.

The shadow firm (29/10/11)

(1) In the model of firm-bank-household that I'm working on with Pawel, there was an argument that if the household reduced consumption a bit and the additional capital was used to boost production in the future, then there should (to leading order) be no change in the household's objective. Both Pawel + I distrust this story, as it seems to assume that the households could *make* the firm put the little extra goods into capital, and it's not clear why this might be.

(2) As an alternative, we may consider what happens if the household chooses to set up a *shadow firm*. This we imagine applies capital k_t at time t (funded by some withdrawals from the household's deposits, therefore charged interest at rate r_t) and employs labour λ_t at time t, charged at wage rate w_t. This shadow firm has to bear the full cost of depreciation and defaults, so the rate at which it generates cash will be

$$p_t Z_t f(k_t, \lambda_t) - w_t \lambda_t - (r_t + \delta + \varphi(\tilde{q}_t)) p_t k_t$$

$$= p_t \left[Z_t f(k_t, \lambda_t) - \theta_t \lambda_t - (r_t + \delta + \varphi(\tilde{q}_t)) k_t \right]$$

The whole expression is homogeneous of degree 1 in (λ, k). Here, $\tilde{q}$ denotes the profitability of the shadow firm. When we maximise over (k, λ), we get

$$p_t Z_t \ \tilde{f}\left(\frac{r_t + \delta + \varphi(\tilde{q}_t)}{Z_t}, \ \frac{\theta_t}{Z_t} \right)$$

So for this to be just on the edge of profitability, we will have (using the C-D form $f(K,L) = A K^\beta L^{1-\beta}$) from the previous page that we must have

$$\left(r_t + \delta + \varphi(\tilde{q}_t)\right)^\beta \ \theta_t^{1-\beta} = Z_t f(\beta, 1-\beta)$$

But profitability is exactly zero for this shadow firm: $\tilde{q}_t = 0$! Hence we shall have the condition

$$\boxed{\left(r_t + \delta + \varphi(0)\right)^\beta \ \theta_t^{1-\beta} = Z_t f(\beta, 1-\beta)}$$

This could be useful. However, since this shadow firm borrows at $r < R$, it will always be better? We will have to treat defaults properly, because the shadow firm pays all default costs, the real firm pays only some... But real firm pays R only on D...

$$C_1 \equiv \sqrt{c^2 + 2r}$$

$$C_2 \equiv \sqrt{c^2 + 2(r-\delta)}$$

$$G(a,b;T,c) \equiv G(b;T,c) - G(a;T,c)$$

American options with log-linear barriers (10/11/11)

(1) Let's firstly remark on the CDF of a drifting Brownian first-passage density:

$$\frac{d}{dt}\left[e^{2ac}\,\bar{\Phi}\left(c\sqrt{t}+\frac{a}{\sqrt{t}}\right) + \Phi\left(c\sqrt{t}-\frac{a}{\sqrt{t}}\right)\right] = \frac{a\, e^{-(a/\sqrt{t}-c\sqrt{t})^2/2}}{\sqrt{2\pi t^3}}$$

So for $a>0$, we deduce that the CDF of the first passage time to a is

$$\mathbb{P}^c[H_a \le t] = e^{2ac}\,\bar{\Phi}\left(c\sqrt{t}+\frac{a}{\sqrt{t}}\right) + \Phi\left(c\sqrt{t}-\frac{a}{\sqrt{t}}\right).$$

Let's write $\psi(t;a,c) \equiv P^c[H_a \le t]$ which can be easily evaluated from (↑).

(2) Now let's suppose $S_0 = 1$, strike K, and we stop when $X_t \equiv \log S_t = \sigma(B_t+\mu t)$ hits a line $a+bt$, which has to remain below $\log K = k$ all the way through $[0,T]$. Set

$$\tau \equiv \inf\{t : \sigma(B_t+\mu t) = a+bt\} \wedge T$$

$$= \inf\{t : B_t + \underbrace{(\mu - b/\sigma)}_{\equiv c}t = \underbrace{a/\sigma}_{-\alpha<0}\} \wedge T$$

and we stop at τ, receiving $e^{-r\tau}(K-e^{X_\tau})^+$. Then the option is valued at

$$\int_0^T e^{-rt}\,\frac{\alpha\, e^{-(\alpha+ct)^2/2t}}{\sqrt{2\pi t^3}}\,dt\,(K-e^{a+bt}) + \int_{-\alpha}^{\infty}\{p_T(0,y)-p_T(-2\alpha,y)\}\,e^{cy-c^2T/2-rT}\,(K-e^{\sigma y+bT})^+\,dy$$

$$= Ke^{(c_1-c)\alpha}\,\psi(T;\alpha,-c_1) - e^{a}\,e^{(c_2-c)\alpha}\,\psi(T;\alpha,-c_2)$$

$$+\int_{-\alpha}^{\beta}\{p_T(0,y)-p_T(-2\alpha,y)\}\,e^{cy-c^2T/2-rT}\,(K-e^{\sigma y+bT})\,dy$$

where $c_1 \equiv \sqrt{c^2+2r}$, $c_2 \equiv \sqrt{c^2+2(r-b)}$, $\beta = \frac{1}{\sigma}(\log K - bT)$. If we now define

$$G(q;T,\lambda) \equiv \int_{-\infty}^{q} p_T(0,y)\,e^{\lambda y}\,dy = e^{\lambda^2T/2}\,\Phi\big((q-\lambda T)/\sqrt{T}\big)$$

then it can all be expressed in terms of that: we see

$$K\,e^{(c_1-c)\alpha}\,\psi(T;\alpha,-c_1) - e^{a+\alpha(c_2-c)}\,\psi(T;\alpha,-c_2)$$

$$+\, e^{-c^2T/2-rT}\Big[K\,G(-\alpha,\beta;T,c) - K\,G(\alpha,\beta+2\alpha;T,c)\,e^{-2\alpha c}$$

$$-\,e^{bT}\Big(G(-\alpha,\beta;T,c+\sigma) - G(\alpha,\beta+2\alpha;T,c+\sigma)\,e^{-2\alpha(c+\sigma)}\Big)\Big].$$

Rolling Geske model (13/11/11)

(i) In the Geske model, we regard the stock as an option on the firm value $V_t = V_0 e^{X_t}$, where $X_t = \sigma W_t + (r - \frac{1}{2}\sigma^2) t$. The trouble with this story is that there is a fixed time horizon, so the solution isn't time homogeneous. As a way to fix this, we can propose an exponential horizon, so that

$$S_t = E_t \left[\int_t^\infty \lambda e^{-\lambda(u-t) - r(u-t)} (V_u - K)^+ \, du \right] = h(X_t)$$

for some function h to be identified. Clearly, $h(x) \to 0$ $(x \to -\infty)$ and $h(x) \sim e^x$ $(x \to \infty)$.

(ii) To solve, notice

$$h(X_t) e^{-(\lambda+r)t} + \int_0^t \lambda e^{-(\lambda+r)s} (V_s - K)^+ \, ds \quad \text{is a martingale,}$$

so

$$\tfrac{1}{2}\sigma^2 h'' + (r - \tfrac{1}{2}\sigma^2) h' - (\lambda + r) h + \lambda (e^x - K)^+ = 0$$

If $Q(z) \equiv \frac{1}{2}\sigma^2 z^2 + (r - \frac{1}{2}\sigma^2) z - (\lambda + r)$ is the auxiliary polynomial, it has roots $-\alpha < 0$, $\beta > 1$ and if $k \equiv \log K$ we can solve h in the two regions

$$h(x) = \begin{cases} e^x - \dfrac{\lambda K}{\lambda + r} + A e^{-\alpha(x-k)} & (x \geq k) \\ B e^{\beta(x-k)} & (x \leq k) \end{cases}$$

Using C^1 condition at $x = k$, we get equations

$$\frac{r}{\lambda + r} K + A = B, \qquad K - \alpha A = \beta B$$

solved by

$$A = \frac{K}{\alpha + \beta} \left(\frac{\lambda + r - \beta r}{\lambda + r} \right), \qquad B = \frac{K(\lambda + r + \alpha r)}{(\alpha + \beta)(\lambda + r)}$$

Can show that $A > 0$.

(iii) Cute enough, but it just shifts the modelling problem somewhere else: the stock price is a time-homogeneous diffusion, but the stock pays some weird dividend process...

... and in fact it turns out to require a negative dividend!!

Variations on a MV OU theme (25/11/11)

(i) If we give ourselves a MV OU process

$$dX_t = \sigma\, dW_t - A X_t\, dt$$

we have

$$X_t = e^{-tA} X_0 + \int_0^t e^{(s-t)A} \sigma\, dW_s$$

so that

$$X_t \sim N\left(e^{-tA} X_0,\ \int_0^t e^{-uA} \sigma\sigma^T e^{-uA^T} du \right)$$

If we assume wlog that $A = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$ then the covariance matrix is

$$\frac{1 - e^{-(\alpha_i + \alpha_j)t}}{\alpha_i + \alpha_j} (\sigma\sigma^T)_{ij} \equiv \frac{1 - e^{-(\alpha_i+\alpha_j)t}}{\alpha_i + \alpha_j} V_{ij}, \quad \text{say.}$$

In the special 2x2 case $A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \equiv \alpha I + N$ we get $\exp(-tA) = e^{-\alpha t}(I - tN)$

(ii) Let's see what this gives for modelling futures prices, where we'll assume that the spot price S_t evolves as $\exp(-v \cdot X_t)$ for some fixed $v \in \mathbb{R}^n$. We would find that

$$F_{tT} = \exp\left\{ -v \cdot e^{-\tau A} X_t + \tfrac{1}{2} v \cdot \Sigma_\tau v \right\}$$

where $\tau \equiv T - t$, and $\Sigma_\tau \equiv \int_0^\tau e^{-uA} V e^{-uA^T} du$. This gives a variety of futures curves we could see.

(iii) For yield curves, if $r_t \equiv v \cdot X_t$, we need to work out the law of $\int_t^T X_u\, du$, which will of course be Gaussian with mean

$$(1 - \exp(-\tau A))\, A^{-1} X_0$$

and variance equal to the variance of

$$\int_0^\tau dt \int_0^t e^{(s-t)A} \sigma\, dW_s = \int_0^\tau \left(I - e^{-(\tau - s)A} \right) A^{-1} \sigma\, dW_s$$

so the variance is $\int_0^\tau \left(I - e^{-uA}\right) A^{-1} V (A^{-1})^T \left(I - e^{-uA^T}\right) du$.

The Nelson-Siegel yield curve story comes from using $A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ and ignoring the term due to the variance. It can be shown quite easily that there's arbitrage, but it seems the story is well known: Filipović has a note on it in 1999 (though it's rather clumsily done there); there's an NBER paper of Christensen, Diebold + Rudebusch also dealing with this, but that too seems to make too much of a meal of things

It seems that we can make a fairly clean finish if we write

$$B(t,T) = \exp(-Y_{tT}) = \exp\left[-Z_1(t)(T-t) - Z_2(t)e^{-\gamma(T-t)} - Z_3(t)e^{-\gamma(T-t)}(T-t)\right]$$

and then use the fact that $\exp(-\int_0^t r_s ds - Y_{tT})$ must be a martingale whatever T. If we just suppose we want a model with continuous semimartingales, we can now argue that $Z_1(t)$ is the long rate at time t, so by the result of Dybvig-Ingersoll-Ross it has to be non-decreasing. We shall have that $(\tau = T-t)$

$$r_t dt + dY_{tT} - \tfrac{1}{2} d\langle Y_{\cdot T}\rangle_t \quad \text{is a martingale}$$

$$= r\,dt + (T-t)dZ_1 - Z_1 dt + e^{-\gamma\tau}dZ_2 + \gamma e^{-\gamma\tau}Z_2 dt + \tau e^{-\gamma\tau}dZ_3$$
$$+ e^{-\gamma\tau}(\gamma\tau - 1)Z_3 dt - \tfrac{1}{2}e^{-2\gamma\tau}\left\{ d\langle Z_2\rangle + 2\tau d\langle Z_2, Z_3\rangle + \tau^2 d\langle Z_3\rangle \right\}$$

If we look at the slope wrto T of $-\log B(t,T)$, the dominant term for very big T will be Z_1, so Z_1 is a martingale; but Z_1 was non-decreasing, so Z_1 is constant. Also, by looking at

$$\lim_{T\downarrow t} \frac{1}{T-t} Y_{tT} = r_t$$

we see that $r_t = Z_1 + Z_3$, so we find more simply that

$$e^{-\gamma\tau}\left[dZ_2 + \tau dZ_3 + \gamma Z_2 dt + \gamma\tau Z_3 dt\right] \qquad (*)$$
$$- \tfrac{1}{2}e^{-2\gamma\tau}\left[d\langle Z_2\rangle + 2\tau d\langle Z_2, Z_3\rangle + \tau^2 d\langle Z_3\rangle\right]$$

is a martingale, so we can clear out a factor of $e^{-\gamma\tau}$ to clean it up:

$$dZ_2 + \tau dZ_3 + \gamma(Z_2 + \tau Z_3 dt) - \tfrac{1}{2}e^{-2\gamma\tau} d\langle Z_2 + \tau Z_3\rangle$$

is a martingale. Take differences for $T=N$, $T=2N$, and let $N\to\infty$ after dividing by N, to learn that $dZ_3 + \gamma Z_3\, dt$ must be a martingale. Therefore

$$dZ_2 + \gamma Z_2 dt - \tfrac{1}{2}e^{-2\gamma\tau} d\langle Z_2 + \tau Z_3\rangle$$

is a martingale; letting $T\to\infty \Rightarrow dZ_2 + \gamma Z_2 dt$ is a martingale, hence $d\langle Z_2 + \tau Z_3\rangle$ is a martingale, hence $Z_2 + \tau Z_3$ is <u>constant</u>; therefore Z_2 and Z_3 are constants. This only works if $Z_2 + \tau Z_3 = 0 \;\; \forall\tau \therefore Z_2 = Z_3 = 0$ (see (*)) ... only possibility is <u>constant</u> riskless rate!

Simulating a CIR process (3/12/11)

(i) This follows on from WN XXVIX, p25, where we set out some plan for simulating from the CIR SDE

$$dX = \sigma\sqrt{X}\, dW + (a - \beta X)\, dt.$$

We have the scale function s characterized by

$$s'(x) = x^{-\alpha} e^{\lambda x} \qquad \alpha \equiv 2a/\sigma^2, \quad \lambda \equiv 2\beta/\sigma^2$$

and zero is inaccessible iff $\alpha \geq 1$. The diffusion Y in natural scale, $Y \equiv s(X)$, solves

$$dY = g(Y)\, dW$$

where $g = (\sigma s') \circ s^{-1}$

(ii) What we observe is that $y \mapsto g(y)$ has a unique minimum in $(0, \infty)$. One point we may care to be careful over is placing the subdivision points so as to keep the covariance roughly constant over intervals. In terms of x, we care about $\sigma(x)\, s'(x) \propto x^{\frac{1}{2}-\alpha} e^{\lambda x}$, which is minimised at $x_{min} = (\alpha - \frac{1}{2})/\lambda$.

We also see that g is growing roughly exponentially, which may help.

Deterministic FBH model (10/12/11)

(i) Suppose we try to optimize

$$\int_0^\infty e^{-\rho t}\, U(C_t, L_t)\,dt$$

where we have a conventional story

$$\dot{K}_t = I_t - \delta K_t = f(K_t, L_t) - C_t - \delta K_t$$

for production, where f is homogeneous of degree 1. Then the value function $V(K)$ must satisfy

$$0 = \sup_{C,L}\left[-\rho V(K) + V'(K)\{ f(K,L) - C - \delta K \} + U(C,L) \right]$$

$$= -\rho V(K) - \delta K V'(K) + \sup_{C,L}\left[V'(K)\left(f(K,L) - C\right) + U(C,L) \right]$$

The essence of the problem is to work out this dual function as explicitly as we can.

(ii) Suppose

$$f(K,L) = A K^\beta L^{1-\beta}$$

and consider for $y > 0$ the problem

$$\sup_{C,L}\left\{ y\left(f(K,L) - C\right) + U(C,L) \right\} \equiv \varphi(y,K),$$

say. Then we get the non-linear first-order ODE

$$0 = -\rho V - \delta K V' + \varphi(V', K)$$

which we can solve in the usual way. With the Cobb-Douglas form for f, we have

$$\varphi(y,K) = \sup_{C,L}\left[q L^{1-\beta} - yC + U(C,L) \right]$$

where $q \equiv y A K^\beta$, and for some U it may be possible to solve this explicitly.

(iii) The BCs are potentially problematic, and the obvious attempt to solve numerically goes awry. Propose that at the boundaries we assume that $I = \delta K$, so that the level of capital never changes, and then we have $C = f(K,L) - \delta K$, with L being chosen to maximise $U(C,L)$ subj to $C = f(K,L) - \delta K$.

(iv) We could seek an optimal steady state:

$$\max U(C,L) \quad \text{s.t.} \quad f(K,L) = \delta K + C.$$

Fourier analysis of BM (15/12/11)

Must be as old as the hills, but if we set

$$\xi_k = X_k + iY_k = \int_0^{2\pi} e^{ikt}\, dB_t$$

then we get

$$E\left[\xi_k \bar{\xi}_n\right] = 2\pi\, \delta_{kn}$$

and $\xi_{-k} = \bar{\xi}_k$. Hence we deduce the $\xi_{|k|}$ are independent (that is, if $|k| \neq |n|$ then X_k, Y_k is indept of X_n, Y_n); and $E X_k^2 = E Y_k^2 = \pi$ for $k \neq 0$, $E X_k Y_k = 0$ and finally ξ_0 is a zero mean Gaussian with covariance 2π. Hence we would have formally that

$$dB_t = \xi_0 + \sum_{k \geq 1} 2(X_k \cos kt - Y_k \sin kt)$$

so that again formally we would see

$$B_t = t\xi_0 + 2\sum_{k\geq 1}\left(X_k \frac{\sin kt}{k} - Y_k \frac{1-\cos kt}{k}\right)$$

Does this look viable as a simulation methodology?! Yes, it's not too bad...! See SOLO/BMFS.

<u>Diverse beliefs for CRRA agents (4/1/12)</u>

Just take agent j to have $\mathcal{U}_j(c) = E^j\left[\int_0^\infty e^{-\rho_j t} U(c_t)\,dt\right]$ where $U'(x) = x^{-R}$ and let's see what we can work out. We have

$$\sum_j I\left(\nu_j \zeta_t e^{\rho_j t}/\Lambda_t^j \right) = \delta_t$$

$$= \zeta_t^{-1/R} \sum \left(e^{-\rho_j t}\Lambda_t^j/\nu_j \right)^{1/R}$$

$$\Rightarrow \quad \zeta_t = \delta_t^{-R} \left\{ \sum \left(e^{-\rho_j t}\Lambda_t^j/\nu_j \right)^{1/R} \right\}^R$$

Suppose we write $y_t \equiv \sum \left(e^{-\rho_j t}\Lambda_t^j/\nu_j \right)^{1/R}$; then $\quad (d\Lambda_t^j = \alpha_t^j \Lambda_t^j\, dX_t)$

$$dy_t = \sum_j \left(e^{-\rho_j t}\Lambda_t^j/\nu_j \right)^{1/R} \left\{ -\frac{\rho_j}{R}dt + \frac{\alpha^j dX}{R} - \frac{\alpha_j^2}{2R}dt + \tfrac{1}{2}\left(\alpha_j/R\right)^2 dt \right\}$$

so that

$$\frac{dy_t}{y_t} = \sum_j q_t^j \left\{ \frac{\alpha^j dX}{R} - \frac{\rho_j}{R}dt + \frac{\alpha_j^2}{2R^2}(1-R)\,dt \right\}$$

where $\quad q_t^j \equiv \left(e^{-\rho_j t}\Lambda_t^j/\nu_j \right)^{1/R} \Big/ \sum_i \left(e^{-\rho_i t}\Lambda_t^i/\nu_i \right)^{1/R}$. Thus if we were to have

$$d\delta_t = \delta_t \sigma_t \left(dX_t + \alpha_t^* \,dt \right)$$

we should be able to find the SDE for ζ. Indeed, some fairly straightforward calculations lead to

$$\frac{d\zeta}{\zeta} = \left(\bar{\alpha} - \sigma R \right) dX - \left[\bar{\rho}_t + \frac{R-1}{2R} v_t + \sigma_t R\left(\alpha_t^* + \bar{\alpha}_t \right) - \tfrac{1}{2}\sigma_t^2 R(1+R) \right] dt,$$

where $v_t = \sum_j q_t^j (\alpha_t^j - \bar{\alpha}_t)^2$, $\bar{\alpha}_t = \sum_j q_t^j \alpha_t^j$;

$$\equiv -\kappa_t\, dX_t - r_t\, dt,$$

so $\kappa_t = \sigma_t R - \bar{\alpha}_t$.

Notice: q_t^j is the fraction of consumption that agent j gets.

How about CARA preferences? This should also be OK, if we assume this time

$$d\delta_t = \sigma_t (dX_t + \alpha^*_t \, dt)$$

then the FOC for optimality is

$$e^{-\rho_j t} \Lambda^j_t \; e^{-\gamma_j c^j_t} = \nu_j \zeta_t$$

so we deduce that

$$\log \zeta_t = -\Gamma \; \delta_t - \bar\rho t + \sum p_j \log \Lambda^j_t$$

with $\Gamma^{-1} = \sum 1/\gamma_j$, $p_j = \Gamma/\gamma_j$, $\bar\rho = \sum p_j \rho_j$ as usual. As $d\zeta_t = \zeta_t(-\kappa_t dX_t - r_t dt)$ we get

$$-\kappa_t dX_t - (r_t + \tfrac12 \kappa_t^2)\, dt = -\Gamma(\sigma_t dX_t + \sigma_t \alpha^*_t \, dt) - \bar\rho\, dt + \sum p_j \left(\alpha^j_t dX_t - \tfrac12 (\alpha^j_t)^2 dt \right)$$

$$\Rightarrow \quad \boxed{\begin{aligned} \kappa_t &= -\bar\alpha_t + \Gamma \sigma_t \\ r_t &= \bar\rho + \Gamma \sigma_t (\alpha^*_t + \bar\alpha_t) - \tfrac12 \Gamma^2 \sigma_t^2 - \tfrac12 v_t \end{aligned}}$$

where $v_t = \sum p_j (\alpha^j_t - \bar\alpha_t)^2$

Utilities bounded below (7/1/12)

(1) I've asked Roman Muraviev to think about the fact that if we were to have some utility for terminal wealth which was defined on $\mathbb{R}$, being short 10^{90} USD is no worse than being short 10^{60} USD at the end of the period; so really we should think of utilities as bounded below and therefore not concave. We ought really only to work with utilities bounded above, as we know, so if the wealth process evolves as

$$dw_t = rw_t\,dt + \theta_t(\sigma\,dW_t + (\mu - r)\,dt)$$

in the usual fashion, we lose no generality in taking $r=0$ (just a redefinition of U). If also $\mu=0$, then we would increase θ massively so that we would achieve arbitrarily close to $\sup_x U(x)$; and if $\mu \neq 0$, we can clearly do at least as well. What stops this? Seems to me that you would need to have some random investigation of your affairs if you were behaving like this, with intensity $\theta^2 g(w)$, say, where g is decreasing non-negative, and if this happened you would be stopped from any further trading and thrown in jail, utility $-K \equiv \inf U(x)$. So what your reward is will be

$$E\left[U(W_T)\, I_{\{T<\tau\}} - K\, I_{\{\tau \le T\}} \right]$$

where τ is the review time (of course, you don't get jailed if your wealth at time τ is positive, but further reviews may happen later). Then the HJB for the value function $V(t,w)$ will be

$$0 = \sup_\theta \left[V_t + (rw + \theta(\mu - r))V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} - \theta^2 g(w)(K+V) \right]$$

with $V(T,\cdot) = U(\cdot)$, which we shall suppose is given to us. Notice that $V+K \geq 0$, so we shall always want $\frac{1}{2}\sigma^2 V_{ww} - (V+K) \leq 0$ ... so some convexity of V might be allowed! Probably nothing but numerics here.

(2) Could we get rid of the time dependence? One way would be to take $\mu = r = 0$, so that we have in effect an optimal stopping problem for BM which is being knocked down to $-K$ at rate $g(W_t)$. We therefore want $V(w)$ such that $V \geq U$ and $\frac{1}{2}\sigma^2 V'' = (V+K)g(w)$. This looks a fairly straightforward question.

(3) Alternatively, we could tell a story with running consumption, so that

$$dW_t = rW_t\,dt + \theta_t(\sigma\,dW_t + (\mu - r)\,dt) - c_t\,dt$$

Suppose we take g to be zero to the right of 0, and constant a below $-b$. Then the ODE

$$\tfrac{1}{2}\sigma^2 \psi'' - g\psi = 0$$

has increasing solution $\psi_+(x) = \exp(\sqrt{2a/\sigma^2}\, x)$ $\qquad x \le -b$

decreasing solution $\psi_-(x) = \text{const}$ $\qquad x \geq 0$

which should make it a bit easier to work ...

At the top end we have BC in:

$$a v + b v' = Y$$

We are writing $v = \psi_- h(\varphi) - K$, so $v' = \psi_-' h(\varphi) + \psi_- \varphi' h'(\varphi)$

and the BC says

$$(a\psi_- + b\psi_-')\, h(\varphi) + b\psi_- \varphi' h'(\varphi) = Y + aK.$$

as usual, but with review according to intensity $\theta^2 g(w)$ as before. If you get caught with negative wealth, then you are subjected to imprisonment, which we'll say is utility $-K\rho$ for ever. So the objective is

$$\max_{\theta, c} E\left[\int_0^\tau e^{-\rho t} U(c_t)\,dt - K\, e^{-\rho\tau} \,\Big|\, W_0 = w\right] = v(w)$$

Then the HJB equation is

$$\boxed{0 = \sup_{\theta, c}\left[-\rho v + (rw + \theta(\mu - r) - c)\, v' + \tfrac{1}{2}\sigma^2\theta^2 v'' + U(c) - \theta^2 g\,(K + v)\right]}$$

One feature of this would be that as $w \to \pm\infty$, $v'(w) \to 0$, so $c \to \infty$! So if there is large negative wealth, you consume madly!

[OOPS! this just repeats WN $\underline{\text{XXXII}}$ p 22 ...]

Suppose we set $f(x) = v(x) + K$, and we now try to represent $\boxed{f(x) = \psi(x)\, h(\varphi(x))}$. Then $f' = \psi' h(\varphi) + \psi\,\varphi' h'(\varphi)$, $f'' = \psi'' h + 2\psi'\varphi' h' + \psi(\varphi'' h' + \varphi'^2 h'')$, so we shall have

$$\tfrac{1}{2}\sigma^2 f'' - g f = \tfrac{1}{2}\sigma^2\left\{\psi(\varphi')^2 h'' + [2\psi'\varphi' + \psi\varphi'']\, h' + \psi'' h\right\} - g\psi h$$

so if $2\psi'\varphi' + \psi\varphi'' = 0$, that is, $\psi^2\varphi'$ is constant, and if $\boxed{\tfrac{1}{2}\sigma^2\psi'' - g\psi = 0}$ then we have that the terms in θ^2 reduce to

$$\theta^2 \times \left(\tfrac{1}{2}\sigma^2\, \psi(\varphi')^2 h''\right)$$

So if we took $\psi = \psi_-$, the decreasing positive solution to $\tfrac{1}{2}\sigma^2\psi'' - g\psi = 0$, and then notice that the other solution can be expressed as ψq, where $q''\psi + 2q'\psi' = 0$ see that $\varphi = q$, and therefore we could take $\boxed{\varphi = \psi_+/\psi_-}$

Does this help? It may be easier to characterise when $\sup_\theta$ is finite $(h'' < 0)$ but will make it harder to see when $\sup_c$ is finite, since

$$f' = \psi_-'\, h(\psi_+/\psi_-) + \frac{\lambda}{\psi_-}\, h'(\psi_+/\psi_-)$$

— when would that be positive ...?

Dividend policy with production (9/1/12)

(i) Suppose we have output $K_t Z_t$ at time t, where $dZ_t = Z_t(\sigma\, dW_t + \mu\, dt)$, and capital K_t at time t goes like

$$\dot{K}_t = I_t - \delta K_t$$

with $K_t Z_t = C_t + I_t$. Now if we suppose that the objective of the firm's management is to maximize

$$E\left[\int_0^\infty e^{-\rho t} U(C_t)\, dt\right]$$

where U is CRRA (recall that investors don't really want bang-bang dividend streams, they prefer smoother. Let $V(K,Z)$ be the value function; scaling tells us that $V(K,Z) = U(K) f(Z)$, and from HJB we get

$$0 = \sup_c \left[-\rho V + U(c) + (I - \delta K) V_K + \mu Z V_Z + \tfrac{1}{2}\sigma^2 Z^2 V_{ZZ} \right]$$

$$= \sup_c \left[-\rho V + U(c) + (KZ - \delta K - c) V_K + \mu Z V_Z + \tfrac{1}{2}\sigma^2 Z^2 V_{ZZ} \right]$$

so $c = K f^{-1/R}$

$$= U(K)\left[-\rho f + \frac{\tilde{U}(V_K)}{U(K)} + (Z-\delta)(1-R) f + \mu Z f' + \tfrac{1}{2}\sigma^2 Z^2 f'' \right]$$

$$= U(K)\left[-\rho f + \tilde{U}(f)(1-R) + (Z-\delta)(1-R) f + \mu Z f' + \tfrac{1}{2}\sigma^2 Z^2 f'' \right]$$

So this one we should be able to solve numerically, but probably no other way.

(ii) An interesting variant would suppose that the firm has a loan to repay, so that $KZ = C + I + a$, where $a > 0$ constant is the interest repayments to be made. But it's hard to see how we would deal with the possibility things go broke

FBH: another example (10/1/12)

1) Our first numerical example didn't seem to behave very sensibly, perhaps because of the additive structure of U. Let's try the following form for U:

$$U(C,L) = -\frac{C^{-\varepsilon}}{(\bar{L}-L)^{\nu}}$$

for positive $\varepsilon, \nu, \bar{L}$. This is strictly concave, incr in C, decr. in L. Keep $f(K,L) = AK^{\beta}L^{1-\beta}$.

We have the equations

$$Zf_L = \theta$$

$$0 = \theta U_c + U_L \quad \Rightarrow \quad \boxed{C = \theta(\bar{L}-L)\varepsilon/\nu}$$

This time, we would probably be better to work with L as a state variable than θ. Recall that L is supposed FV, so we'll suggest the form

$$dL = L(\mu_L dt + \eta_L dA)$$

and then as before

$$\theta = Z A(1-\beta)(K/L)^{\beta} \equiv k_1 Z K^{\beta} L^{-\beta} \qquad [k_1 = A(1-\beta)]$$

so that

$$\frac{d\theta}{\theta} = \sigma dW + \left\{\mu + \beta\left(\frac{I}{K} - \delta - \varphi\right) - \beta\mu_L\right\}dt - \beta\left(\frac{\gamma}{K} + \eta_L\right)dA$$

and hence

$$dC = \frac{\varepsilon}{\nu}\left\{(\bar{L}-L)d\theta - \theta dL\right\}$$

$$= C\left\{\frac{d\theta}{\theta} - \frac{dL}{\bar{L}-L}\right\}$$

$$= C\Big[\sigma dW + \left\{\mu + \beta\left(I/K - \delta - \varphi\right) - \beta\mu_L - \frac{L\mu_L}{\bar{L}-L}\right\}dt$$

$$- \beta\left(\frac{\gamma}{K} + \eta_L\right)dA - \frac{L}{\bar{L}-L}\eta_L dA\Big]$$

Hence

$$\frac{dU_c}{U_c} = \frac{-\varepsilon' dC}{C} + \nu\frac{dL}{\bar{L}-L} + \frac{\varepsilon'(1+\varepsilon')}{2}\sigma^2 dt \qquad (\varepsilon' \equiv 1+\varepsilon)$$

$$= -\varepsilon'\sigma dW - \varepsilon'\left\{\mu + \beta(I/K - \delta - \varphi) - \beta\mu_L - \frac{L\mu_L}{\bar{L}-L}\right\}dt + \frac{\nu L\mu_L}{\bar{L}-L}dt + \frac{\sigma^2}{2}\varepsilon'(1+\varepsilon')dt$$

$$+ \varepsilon'\beta\left(\frac{\gamma}{K} + \eta_L\right)dA + \frac{\varepsilon' L\eta_L}{\bar{L}-L}dA + \frac{\nu L\eta_L}{\bar{L}-L}dA$$

Comparing coefficients gives

$$\boxed{\begin{aligned} \sigma_u &= -\varepsilon'\sigma \\ \mu_u &= \frac{(\nu+\varepsilon')L\mu_L}{\bar{L}-L} - \varepsilon'\mu - \varepsilon'\beta\left(\tfrac{I}{K}-\delta-\varphi\right) + \varepsilon'\beta\mu_L + \varepsilon'(1+\varepsilon')\sigma^2/2 \\ \eta_u &= \frac{(\varepsilon'+\nu)L\eta_L}{\bar{L}-L} + \varepsilon'\beta\left(\frac{\gamma}{K}+\eta_L\right) \end{aligned}}$$

But remember $\eta_u = \gamma/\tilde{S}$, so we can deduce η_L.

2) Overall, it seems we could benefit from dealing with a more general approach, taking $f(K,L) = K\ h(L/K)$ with h not yet explicit, and the state variables (Z,K,D,L,π). Then the first equation

$$Z f_K = \theta = Z h'(L/K)$$

gives θ explicitly in terms of known variables, and if we set $x = L/K$, then we shall have

$$dx = x\left\{ \mu_L\, dt + \eta_L\, dA - \left(\tfrac{I}{K}-\delta-\varphi\right)dt + \frac{\gamma\, dA}{K}\right]$$

$$= x\left\{\left(\mu_L+\delta+\varphi-I/K\right)dt + \left(\eta_L+\gamma/K\right)dA\right\} \equiv x\left\{\mu_x\,dt + \eta_x\, dA\right\},$$

and hence

$$\frac{d\theta}{\theta} = \frac{dZ}{Z} + \frac{h''(x)}{h'(x)}\,dx.$$

The second equation $\theta U_c + U_L = 0$ needs to be re-expressed as $U_c = F(\theta, L)$ for some F to be made explicit in any particular example. Then we have

$$dU_c = F_\theta\, d\theta + \tfrac{1}{2}F_{\theta\theta}\, d\langle\theta\rangle + F_L\, dL$$

$$\Rightarrow \frac{dU_c}{U_c} = \frac{F_\theta}{F}\,\theta\left\{\sigma\, dW + \mu\, dt + \frac{x h''(x)}{h'(x)}\left(\mu_x dt + \eta_x dA\right)\right\} + \frac{1}{2}\frac{F_{\theta\theta}}{F}\theta^2\sigma^2\, dt$$

$$+ \frac{LF_L}{F}\left\{\mu_L\, dt + \eta_L\, dA\right\}$$

$$= \frac{\sigma\theta F_\theta}{F}dW + \left\{\frac{\mu\theta F_\theta}{F} + \frac{\sigma^2\theta^2F_{\theta\theta}}{2F} + \frac{L\mu_L F_L}{F} + \frac{\theta x F_\theta h''}{F h'}\mu_x\right\}dt$$

$$+ \left\{\frac{\theta F_\theta\, x h''\eta_x}{F h'} + \frac{\eta_L LF_L}{F}\right\}dA$$

By comparing coefficients, we learn that

$$\boxed{\begin{aligned} \sigma_u &= \frac{\sigma\theta F_\theta}{F} \\ \mu_u &= \frac{\mu\theta F_\theta}{F} + \frac{\sigma^2\theta^2 F_{\theta\theta}}{2F} + \frac{L\mu_L F_L}{F} + \frac{\theta x F_\theta h''}{F h'}\mu_x \\ \eta_L &= \gamma\left\{\frac{1}{S} - \frac{\theta x F_\theta h''}{K F h'}\right\} \Big/ \left\{\frac{\theta F_\theta x h''}{F h'} + \frac{L F_L}{F}\right\} \end{aligned}}$$

Using the fact that $\eta_u = \gamma/S$

Possible examples for U

(i) $U(c,L) = -\frac{c^{-\varepsilon}}{(\bar{L}-L)^\nu} \Rightarrow U_c \propto \theta^{-\varepsilon'}(\bar{L}-L)^{-\nu-\varepsilon'}$ $(\varepsilon' \equiv 1+\varepsilon)$

(ii) Try $U(c,L) = g(c/L^a)$ for some $a \geq 1$ (if we double L, we would want to at least double c in order to stay as happy). This would give

$$\theta = a\,c/L, \quad U_c = \frac{1}{L^a}g'(\theta/aL^{a-1}) \equiv F(\theta,L)$$

(iii) We could modify this to have a subsistence level C_* for consumption,

$$U(c,L) = g((c-c_*)/L^a)$$

Then F is the same, but the consumption gets pushed up by C_*.

(iv) Our original choice

$$U(c,L) = -\frac{c^{-\varepsilon}}{\varepsilon} - \alpha L^\nu$$

is not obviously stupid in terms of the preferences it generates: We will get

$$U_c = F(\theta,L) \propto L^{\nu-1}/\theta$$

Possible examples for h

(i) Of course, $h(y) = y^{1-\beta}$ is the Cobb-Douglas story; $yh''/h' = -\beta$

(ii) Cobb-Douglas gave trouble with the fourth version of U above, apparently because h was unbounded. So we would try

$$h(y) = 1 - e^{-\nu y}; \quad h''(y)/h'(y) = -\nu$$

(iii) If you want infinite derivative at 0, you could use $h(y) = 1 - e^{-\alpha y^\nu}$, yielding

$$\frac{h''}{h'} = -(1-\nu)y^{-1} - \alpha\nu y^{-(1-\nu)} \qquad (0<\nu<1)$$

Consumption drawdown again (12/2/12)

1) It seems when I first looked at this I may have missed some of the subtleties. Arun has gone into this question also. How it goes is this. The agent wants

$$V(w,\bar c) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \,\Big|\, w_0 = w, \bar c_0 = c\right] \qquad \bar c_t = \sup_{s\le t} c_s$$

under the constraint $c_t \ge b\bar c_t$ for all t, where $b\in(0,1)$ is fixed. As usual, U is CRRA. Scaling gives $V(w,\bar c) = \bar c^{1-R} v(x)$, $x \equiv w/\bar c$, and we can get the HJB in the scaled-out form as usual

$$0 \ge -\rho v + rxv' - (\kappa v')^2/2v'' + \sup_{b\le z\le 1}\{U(z) - zv'\}$$

with condition

$$0 \ge (1-R)v - xv'$$

from the derivative wrt $\bar c$. We know that if wealth drops to b/r $(\bar c = 1)$, then all we can do is riskless investment and consume the interest at rate b, so

$$v(b/r) = U(b)/\rho$$

But how does v look to the right of b/r? Finite slope? Finite second derivative?

2) Suppose $\bar c = 1$, and write $y_t \equiv w_t - b/r$ for the excess wealth of the agent. We're interested in very small y; if y is very small, there is little chance we will ever get to raise $\bar c$, or even consume at $\bar c$, so we can just ignore that possibility and try to solve for the value function f:

$$0 = -\rho f + rxf' - (\kappa f')^2/2f'' + \sup_{c\ge 0}\{U(b+c) - cf'\}.$$

Let's write

$$h(\lambda) = \sup_{c\ge 0}\{U(b+c) - \lambda c\}$$

$$= \begin{cases} \tilde U(\lambda) + b\lambda & \text{if } \lambda \le U'(b) \\ U(b). & \text{if } \lambda \ge U'(b) \end{cases}$$

If we consider the form of the problem where we absorb the budget constraint, and initial excess wealth is ε, we have

$$\max E\int_0^\infty \left(e^{-\rho t} U(b+c_t) - \lambda \zeta_t c_t\right)dt + \lambda\varepsilon$$

$$= E\int_0^\infty e^{-\rho t} h(\lambda e^{\rho t}\zeta_t)\,dt + \lambda\varepsilon$$

$$= (R_\rho h)(\lambda) + \lambda\varepsilon$$

where $R_\rho^{-1} = \rho - (\rho - r) z\dfrac{d}{dz} - \dfrac{\kappa^2}{2} z^2 \dfrac{d^2}{dz^2}$.

So we need to work out $R_\rho h$, and then see what the dual problem looks like for ε small. It's clear that we must have

$$(R_\rho h)(y) = \begin{cases} \frac{U(b)}{\rho} + A(y/\tilde{b})^{-\alpha} & (y \geq \tilde{b} \equiv U'(b)) \\ \gamma_M^{-1}\tilde{U}(y) + \frac{by}{r} + B(y/\tilde{b})^{\beta} & (y \leq \tilde{b}) \end{cases}$$

for some constants A, B which we find by C^1 join at $\tilde{b} \equiv U'(b)$.

Now to work out the value near zero of this approximating problem, we have to find

$$\inf \{ R_\rho h(y) + \varepsilon y \} = \inf_{y > \tilde{b}} [R_\rho h(y) + \varepsilon y]$$

$$= \inf_{y > \tilde{b}} \left[\frac{U(b)}{\rho} + A(y/\tilde{b})^{-\alpha} + \varepsilon y \right] \Rightarrow y/\tilde{b} = \left(\alpha A/\varepsilon\tilde{b}\right)^{1/\alpha+1}$$

$$= \frac{U(b)}{\rho} + (1+\alpha)\left(\frac{\tilde{b}}{\alpha}\right)^{\alpha/\alpha+1} \varepsilon^{\alpha/\alpha+1} A^{1/(\alpha+1)}$$

So this tells us that <u>the derivative of v at $(b/r)+$ must be infinite</u>.

3) Using this, we shall have that the solution we seek to the original problem must look like $U(b)/\rho + A_0(z/\tilde{b})^{-\alpha} - bz/r$ for $z \geq \tilde{b} = U'(b)$, and then like $\gamma_M^{-1}\tilde{U}(z) + A_1(z/\tilde{b})^{-\alpha} + B_1(z/\tilde{b})^{\beta}$ for $U'(1) \leq z \leq \tilde{b}$.

But here's a better way to make it. Suppose we look at the dual equation

$$(*) \quad \tfrac{1}{2}\kappa^2 z^2 J'' + (\rho - r) z J' - \rho J + \tilde{U}_b(z) = 0$$

and try to solve it in $(0, \tilde{b}]$ with smooth paste to $U(b)/\rho$ at $z = \tilde{b}$: then the actual solution we want will be $g(z) + A_0 z^{-\alpha}$ for some A_0, where g solves $(*)$ with $g(\tilde{b}) = U(b)/\rho$, $g'(\tilde{b}) = 0$. We find

$$g(z) = \begin{cases} \gamma_M^{-1}\tilde{U}(z) + A_1 z^{-\alpha} + B_1 z^{\beta}, & U'(1) \equiv 1 \leq z \leq \tilde{b} \equiv U'(b) \\ \frac{U(1)}{\rho} - \frac{z}{r} + A_2 z^{-\alpha} + B_2 z^{\beta} & 1 \geq z. \end{cases}$$

So we want that A_0 is set so that for some z we find $(1-1/R)J(z) = zJ'(z)$ that is,

$$(1-1/R)[g(z) + A_0 z^{-\alpha}] = z[g'(z) - \alpha A_0 z^{-\alpha-1}]$$

that is

$$\left((1-1/R) + \alpha\right) A_0 z^{-\alpha} = z g'(z) - (1-1/R) g(z)$$

Utility bounded below: variable change (10/2/12)

(1) There was this equation for the problem studied on pp 42-43 for utility bounded below:

$$0 = \sup\left[-\rho v + U(c) + (rx + \theta(\mu - r) - c)v' + \theta^2\left(\tfrac{1}{2}\sigma^2 v'' - g(v+K)\right) - g(v+K)\right]$$

(Note that as stated on pp 42-43 the problem is ill posed: once you get to negative wealth you just set $\theta = 0$ and keep on borrowing! Bringing in the additional killing at rate g independently of θ stops this).

Now we let $\psi_\pm$ be the increasing/decreasing solutions to $\frac{1}{2}\sigma^2 f'' = gf$, and then we transform the equation by writing $v = f - K$, with $f \geq 0$, and

$$f(x) \equiv \psi_-(x)\, h(\varphi(x)) \equiv \psi_-(x)\, h(y)$$

~~with new variable~~

$$y \equiv \varphi(x) \equiv \psi_+(x)/\psi_-(x).$$

Notice that $\varphi' = (\psi_- D\psi_+ - \psi_+ D\psi_-)/\psi_-^2 \equiv \lambda/\psi_-^2$ where λ is constant. The ODE looks like

$$-(\rho+g)f + \rho K + U(c) + (rx + \theta(\mu-r)-c)f' + \theta^2\left(\tfrac{1}{2}\sigma^2 f'' - gf\right)$$

$$= -(\rho+g)\psi_- h + (\rho K + U(c)) + (rx + \theta(\mu-r)-c)(\psi_-' h + \psi_-\varphi' h') + \tfrac{1}{2}\sigma^2\theta^2 \lambda^2 h''/\psi_-^3$$

$$= -\left[(\rho+g)\psi_- - (rx+\theta(\mu-r)-c)\psi_-'\right]h + (\rho K + U(c)) + (rx+\theta(\mu-r)-c)\psi_-\varphi' h' + \frac{1}{2}\frac{\sigma^2\theta^2\lambda^2}{\psi_-^3} h''$$

(2) It seems when we do the policy improvement that we get some rather crinkly functions at times. We would probably do better to smooth values. If we are given values a_i at times t_i we want to get (z_i) to

$$\min \sum_{i=1}^n (z_i - a_i)^2 + \gamma \sum_{i=2}^n \left(\frac{z_i - z_{i-1}}{t_i - t_{i-1}}\right)^2 (t_i - t_{i-1})$$

So we get

$$0 = z_i - a_i + \gamma\left\{\frac{z_i - z_{i-1}}{t_i - t_{i-1}} - \frac{z_{i+1} - z_i}{t_{i+1} - t_i}\right\}$$

$$= -a_i - \frac{\gamma z_{i-1}}{t_i - t_{i-1}} + \left(z_i + \gamma\left(\frac{z_i}{t_i - t_{i-1}} + \frac{z_i}{t_{i+1} - t_i}\right)\right) - \frac{\gamma z_{i+1}}{t_{i+1} - t_i}$$

(3) Maybe best is to suppose that $g(w) = \gamma \ \forall w \leq \underline{w}$, and that below $w = w_*$ we come out of the asset entirely, and just consume at infinite rate. This could be augmented by discovery rate depending on c, so that the HJB looks like

$$0 = \sup\Big[-\rho v + (rw + \theta(\mu - r) - c)v' + \tfrac{1}{2}\sigma^2\theta^2 v'' - g(w)(1+\theta^2 + \alpha c)(v+K) + U(c) \Big]$$

So if we suppose that for $w \leq w_*$ (assuming $w_* \leq \underline{w}$) we come out of the asset, ignore r, just consume at fixed rate c, the objective will be

$$-\frac{K\lambda}{\lambda+\rho} + \frac{U(c)}{\lambda+\rho} \qquad \left[\begin{aligned}\lambda &= g(w)(1+\alpha c)\\ &= \gamma(1+\alpha c)\end{aligned}\right]$$

Which we maximise over choice of c:

$$-K + \frac{\rho K + U(c)}{\rho + \gamma(1+\alpha c)}$$

This is what the value will be at w_*

Joint law of (I, X, S) (23/2/12)

Suppose we have a symmetric simple random walk M on the grid $\varepsilon\mathbb{Z}$ which gets stopped at some stopping time, and define $I_n = \inf_{t \le n} M_t$, $S_n = \sup_{t \le n} M_t$, with $I = I_\infty$, $S = S_\infty$, and $X = M_\infty$. I'd like to be able to characterize all possible joint laws of (I, X, S); but as a step to that, suppose I'm given the joint law of (I, X, S), how much can we find out about probabilities of hitting levels etc?

(i) Suppose $a > 0$, $b > 0$. Then by stopping at $H_b \wedge H_{-a}$ we get from OST

$$0 = M_0 = b\, P[H_b < H_{-a}] - a\, P[H_{-a} < H_b] + E[M_\infty : H_b = H_{-a} = \infty]$$

$$= b\, P[H_b < H_{-a}] - a\, P[H_{-a} < H_b] + \underbrace{E[X : S < b,\ I > -a]}_{\text{known}}$$

and since we also know $P[H_b = H_{-a} = \infty] = P[S < b,\ I > -a]$, we can deduce $P[H_b < H_{-a}]$, $P[H_{-a} < H_b]$.

(ii) If we now start the martingale at H_{-a} if $H_{-a} < H_b$, and stop at H_b, then OST gives

$$-a\, P[H_{-a} < H_b] = b\, P[H_{-a} < H_b < \infty] + E[M_\infty : H_{-a} < H_b = \infty]$$

$$= b\, P[H_{-a} < H_b < \infty] + E[X : S < b,\ I \le -a].$$

Hence

$$\boxed{\text{we can deduce } P[H_{-a} < H_b < \infty],\ P[H_b < H_{-a} < \infty]}$$

(iii) We also have

$$P[H_{-a} < H_b < \infty,\ H_b < H_{-a-\varepsilon}] = P[H_{-a} < H_b < \infty] - P[H_{-a-\varepsilon} < H_b < \infty] \quad \text{(known)}$$

$$= P[H_b < \infty, \text{ and when we hit } b, \text{ the inf is } -a]$$

$$= P[S \ge b,\ I = -a] - P[H_b < H_{-a} < \infty = H_{-a-\varepsilon}]$$

$$= P[S \ge b,\ I = -a] - P[I = -a, \text{ first hit } b \text{ before hit } -a].$$

So we can deduce

$$\boxed{P[H_b < H_{-a} \mid I = -a],\ P[H_{-a} < H_b < \infty \mid I = -a]}$$

So in particular we know $\mathcal{L}(S(H_{-a}) \mid I = -a)$. We could also have this from

$$P[H_b < H_{-a} < \infty] - P[H_{b+\varepsilon} < H_{-a} < \infty] = P[S(H_{-a}) = b,\ H_{-a} < \infty] \ ?$$

Now this latter gives $\mathcal{L}\left(S(H_{-a}) \mid I \leq -a\right)$.

Interesting questions

1) Suppose there's a liquid market in calls/puts, and you want to make a static hedge of a European contingent claim $Y = Y_T$. What would you do? Presumably you would just replicate $E[Y_T | S_T]$. But suppose now you were allowed to change position at a stopping time τ - what would you do? Can you answer the analogous question where you are allowed n changes of position?