Asset dynamics with trend (10/6/14).

(i) Let's look again at the situation studied in WN XXXVI, pp 47, 69. The idea is that agent j thinks that the dividend process X in a CARA world evolves as

$$dX_t = dW_t + \mu^j_t\, dt$$

where

$$\mu^j_t = \varepsilon_j \int_{-\infty}^{t} \lambda_j e^{\lambda_j (s-t)} (X_t - X_s)\, ds$$

(standardizing to $\sigma = 1$ wmlog). As before,

$$d\mu^j_t = \varepsilon_j\, dX_t - \lambda_j \mu^j_t\, dt$$

$$= \varepsilon_j \sigma\, dW_t + \alpha_j \mu^j_t\, dt \qquad (\alpha_j \equiv \varepsilon_j - \lambda_j)$$

(ii) The LR martingale for agent j becomes

$$\Lambda^j_t = \exp\left[\int_0^t \mu^j_s\, dX_s - \tfrac{1}{2} \int_0^t (\mu^j_s)^2\, ds \right]$$

and if we assume all agents are CARA, with common ρ, γ, we get

$$e^{-\rho t} \Lambda^j_t\, e^{-\gamma c^j_t} = \beta_j \zeta_t$$

and so market clearing gives us

$$\boxed{\begin{aligned} \log \zeta_t &= \text{const} + \tfrac{1}{J} \sum \log \Lambda^j_t - \rho t - \gamma X_t / J \\ &= \text{const} - \rho t - \gamma J^{-1} X_t + \int_0^t \bar{\mu}_s\, dX_s - \tfrac{1}{2} \int_0^t \bar{\mu}_s^2\, ds \\ &\qquad - \tfrac{1}{2J} \int_0^t \sum_j (\mu^j_s - \bar{\mu}_s)^2\, ds \end{aligned}}$$

What does this say? If we want to do a pricing calculation, it's like the BM has acquired drift $\bar{\mu}_t \equiv J^{-1} \sum_1^J \mu^j_t$, and the discounting is happening at the extra rate

$$\rho t + \tfrac{1}{2J} \int_0^t \sum (\mu^j_s - \bar{\mu}_s)^2\, ds.$$

Note that X will be a Gaussian process still!

If $b_j \equiv \log \beta_j$, $\bar{b} = \frac{1}{J} \sum b_j$ we see

$$\gamma c^j_t = \frac{\gamma}{J} X_t + (\xi^j_t - \bar{\xi}_t) - (b_j - \bar{b}) \qquad j = 1, \dots, J$$

We can interpret

$$d\xi^0 = \mu^0 dX - \tfrac{1}{2}(\mu^0)^2 dt$$

if we wish, so that ξ is a $(J+1)$-dimensional process - it's convenient to do this, with $\theta^0 = 0$ by convention.

Set

$$F(\varphi, \theta, a; \mu) \equiv E\left[\int_0^\infty \exp\{-at + \varphi \cdot \mu_t + \theta \cdot \xi_t\} dt \;\middle|\; \mu_0 = \mu \right]$$

We're interested in F and its derivatives when $\theta = (0, \frac{1}{J}, \frac{1}{J}, \dots, \frac{1}{J})$, $\varphi = (-\frac{\gamma}{J}, 0, 0, \dots, 0)$, $a = \rho$ in the φ-direction e_0, and the θ-direction $[0; e_j - J^{-1}\mathbf{1}]$.

Note that if we shift the path of X by a constant K, all this means is that we add K/J to each agent's consumption for ever, and the effect of that can be seen as a change of $(b_j - \bar{b})$; thus wlog we can restrict to $X_0 = 0$ for the calculations.

(iii) Can we get wealth process, individual wealths from this? The key seems to be to compute

$$E \exp\left(\beta X_t + \theta \cdot \log \Lambda_t \right)$$

for $t \geq 0$, $\beta \in \mathbb{R}$, $\theta \in \mathbb{R}^J$. What we expect is that

$$E\left[\exp(\beta X_t + \theta \cdot \log \Lambda_t) \mid X_0, \mu_0 \right] = V(t, X_0, \mu_0)$$

$$= \exp\left[\tfrac{1}{2} \, z \cdot A(t) z + z \cdot b(t) + c(t) \right]$$

where $z = [X_0 \, ; \, \mu_0]$ and A, b, c are to be found. We know that

$$M_t = \exp\left[\theta \cdot \left\{ \int_0^t \mu_s^j \, dX_s - \tfrac{1}{2} \int_0^t (\mu_s^j)^2 ds \right\} \right] V(T-t, X_t, \mu_t)$$

must be a martingale. If we make the convention $\varepsilon_0 = 1$, $\lambda_0 = 0$, then the equation for μ_t^j extends to the interpretation $\mu_t^0 = X_t$. Thus we may have a $(J+1)$-process $\mu_t = [X_t, \mu_t^1, \dots, \mu_t^J]$, and enquire about

$$E\left[\exp(\varphi \cdot \mu_t + \theta \cdot \log \Lambda_t) \mid \mu_0 = \mu \right] = \exp\left\{ \tfrac{1}{2} \mu \cdot A(t) \mu + \mu \cdot b(t) + c(t) \right\}$$

If we write $\xi_t^j = \log \Lambda_t^j = \int_0^t \mu_s^j \, dX_s - \frac{1}{2} \int_0^t (\mu_s^j)^2 ds$ then we have the expression

$$M_t = \exp\left\{ \theta \cdot \xi_t + \tfrac{1}{2} \mu_t \cdot A(T-t) \mu_t + \mu_t \cdot b(T-t) + c(T-t) \right\}$$ is a mart.

$$\equiv \exp Y_t$$

Any. When you do the Itô calculus, what I find is that

$$0 = \tfrac{1}{2} \left(\mu \cdot (\theta + A\varepsilon) + b \cdot \varepsilon \right)^2 - (\lambda \mu) \cdot A \mu - \tfrac{1}{2} \mu \dot{A} \mu + \tfrac{1}{2} \varepsilon \cdot A \varepsilon - b \cdot (\lambda \mu) - \dot{b} \cdot \mu - \dot{c}$$

Hence we get the relations (understanding $\lambda \equiv \operatorname{diag}(\lambda)$ where necessary)

$$\boxed{\begin{aligned} 0 &= \tfrac{1}{2} (\theta + A\varepsilon)(\theta + A\varepsilon)^T - \tfrac{1}{2}(\lambda A + A \lambda) - \tfrac{1}{2} \dot{A} - \operatorname{diag}(\theta)/2 \\ 0 &= (b \cdot \varepsilon)(\theta + A\varepsilon) - \lambda b - \dot{b} \\ 0 &= \tfrac{1}{2} (b \cdot \varepsilon)^2 + \tfrac{1}{2} \varepsilon \cdot A \varepsilon - \dot{c} \end{aligned}}$$

$A(0) = 0$

$b(0) = \varphi$

$c(0) = 0$.

Anything we can do here? Numerics of course.

If you just earn spend from cash, then your optimal objective with initial wealth w_0 will be

$$\gamma_0^{-R}\, U(w_0)$$

where

$$\gamma_0 = R^{-1}\left[\rho + (R-1)r\right].$$

Merton liquidity problem again (12/6/14)

Seems like we may be OK just to do a change of one of the variables, but the form of the diffeomorphism needs to be chosen with care. Let's work with the variables $x \equiv Y$, $y \equiv H$, as that's what Elena is using. We'll replace y with the new variable

$$\eta = \eta(x,y) = y\,\frac{y+bx}{y+ax}$$

where $0<a<b$. Then ($b-a \equiv \varepsilon$)

$$\frac{\partial\eta}{\partial x} = \frac{\varepsilon y^2}{(y+ax)^2}, \qquad \frac{\partial^2\eta}{\partial x^2} = -2\varepsilon a\frac{y^2}{(y+ax)^3}$$

$$\frac{\partial\eta}{\partial y} = 1 - \frac{\varepsilon a x^2}{(y+ax)^2}$$

We then see that $\partial\eta/\partial y = 1$ at $x=0$, and $\partial\eta/\partial x = \varepsilon$ at $x=0$. If we write $F(x,y) = g(x,\eta)$ we shall have

$$\boxed{\begin{aligned} (F_Y \equiv)\quad F_x &= g_x + g_\eta \frac{\partial\eta}{\partial x} \\ (F_{YY}\equiv)\quad F_{xx} &= g_{xx} + 2g_{x\eta}\frac{\partial\eta}{\partial x} + g_{\eta\eta}\left(\frac{\partial\eta}{\partial x}\right)^2 + g_\eta\frac{\partial^2\eta}{\partial x^2} \\ (F_H\equiv)\quad F_y &= g_\eta\frac{\partial\eta}{\partial y} \end{aligned}}$$

and for the BC $\beta F_x = F_y$ at $x=0$ we demand that $\partial\eta/\partial x = \beta$ at $x=0$, which then guarantees that the BC for g will be

$$\boxed{g_x = 0 \quad \text{at } x=0}$$

So we need $\boxed{\varepsilon \equiv \beta}$ and $\lim_{x\to\infty} \eta/y = (a+\beta)/a$. The HJB PDE is now

$$\boxed{\begin{aligned} 0 = U(c) - \tilde{\rho} g &+ \tfrac{1}{2}\sigma^2 x^2\left(g_{xx} + 2g_{x\eta}\frac{\partial\eta}{\partial x} + g_{\eta\eta}\left(\frac{\partial\eta}{\partial x}\right)^2\right) - (h + hf(c/h) + c + \alpha x)\, g_x \\ &+ \left[\tfrac{1}{2}\sigma^2 x^2\frac{\partial^2\eta}{\partial x^2} - (h + hf(c/h) + \alpha x + c)\frac{\partial\eta}{\partial x} + h\frac{\partial\eta}{\partial y}\right] g_y \end{aligned}}$$

If $q = \sqrt{(\eta - bx)^2 + 4a\eta x}$, we have

$$\frac{\partial}{\partial \eta}\left(\frac{\partial \eta}{\partial x}\right) = \frac{4\varepsilon x a\,(\eta - bx + q)}{q\,(\eta - bx + 2ax + q)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right) = \frac{-4\varepsilon\eta a\,(\eta - bx + q)}{q\,(\eta - bx + 2ax + q)^2}$$

$$\frac{\partial}{\partial \eta}\left(\left(\frac{\partial \eta}{\partial x}\right)^2\right) = \frac{8\varepsilon^2 a x\,(\eta - bx + q)^3}{q\,(\eta - bx + 2ax + q)^4}$$

for the calculation of $\tilde{\mu}$.

Let's write this as

$$U(c) - \tilde{\rho} g + D_i(A_{ij} D_j g) + \tilde{\mu}_i D_i g$$

where $\tilde{\mu}_i = \mu_i - D_j A_{ji}$, and develop it further. The variational form of the problem is

$$0 = \int_\Omega v \{ U(c) - \tilde{\rho} g + D_i(A_{ij} D_j g) + \tilde{\mu}_i D_i g \} dx$$

$$= \int_\Omega \left[\mathrm{div}(v A_{ij} D_j g) - D_i v \, A_{ij} D_j g + v(\tilde{\mu}_i D_i g + U(c) - \tilde{\rho} g) \right] dx$$

$$= \int_{\partial\Omega} v A D g \cdot dn + \int_\Omega \left(v(\tilde{\mu} \cdot Dg + U(c) - \tilde{\rho} g) - \langle Dv, A Dg \rangle \right) dx$$

Now the contributions on the boundary amount to zero on the parts where we have Dirichlet boundary conditions, but on the parts where we have Neumann boundary conditions, at $x = 0$ and $x = x_{max}$, there remains a contribution.
The covariance matrix A is in fact rank 1:

$$A = \frac{1}{2} \zeta \zeta^T, \qquad \zeta^T = \left(1, \frac{\partial \eta}{\partial x}\right) \sigma x$$

So along the left boundary we shall collect (recalling $g_x = 0$ on the Neumann boundary)

$$\int_{\{x=0\}} -\frac{1}{2} v \cdot \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \right) d\eta = -\frac{1}{2} \int_{\{x=0\}} v \cdot \beta \frac{\partial g}{\partial \eta} d\eta$$

and along the right boundary we collect

$$\int_{\{x=x_{max}\}} \frac{1}{2} v \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \right) d\eta = \int_{\{x = x_{max}\}} \frac{1}{2} v \frac{\partial g}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} d\eta$$

Remarkably, it appears possible to put such things directly into fenics !!

Since in fact the covariance vanishes at $x=0$, the boundary contribution here would also vanish, so this needs to be dealt with carefully

Joint distributions with given marginals (14/7/14)

Suppose we have $X \sim \mu$, $Y \sim \nu$ on possibly different spaces, not assumed to be vector spaces. The simplest joint law for (X,Y) we could propose which has the given marginals, but is there a natural way to make (a family of) joint distributions where X, Y are dependent?

(i) Suppose we have some function $\varphi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{++}$ such that

$$\iint \varphi(x,y)\ \mu(dx)\ \nu(dy) = 1$$

Now suppose that

$$\varphi_X(x) \equiv \int \varphi(x,y)\,\nu(dy), \quad \varphi_Y(y) \equiv \int \varphi(x,y)\,\mu(dx)$$

are both bounded. Then if we propose a new measure

$$\left(1 + \varepsilon\{\varphi(x,y) - \varphi_X(x)\varphi_Y(y)\}\right)\mu(dx)\,\nu(dy)$$

Then provided

$$\varepsilon\, \bar{\varphi}_X\, \bar{\varphi}_Y \le 1$$

where $\bar{\varphi}_X \equiv \sup \varphi_X(x)$, $\bar{\varphi}_Y \equiv \sup \varphi_Y(y)$, we shall have defined a dependent joint distribution with the given marginals.

(ii) Could this be done multiplicatively, that is, construct density

$$\exp\left[\psi(x,y) - \psi_X(x) - \psi_Y(y)\right]$$

from given ψ in such a way that the integrals of the density w/o x (or y) are all 1 ?

If we look at the case where the two sets on which the probabilities are defined are in fact finite, then it appears (numerically) that this always works, though I do not have a proof.

(iii) To amplify, if $A = (a_{ij})$ is a $N \times M$ matrix of strictly positive entries, we seek strictly positive x, y such that

$$\begin{cases} \sum_j x_i a_{ij} y_j = M & \forall i = 1, \dots, N \\ \sum_i x_i a_{ij} y_j = N & \forall j = 1, \dots, M \end{cases}$$

Recursively calculating

$$x_i^{(n)} = M \Big/ \sum_j a_{ij} y_j^{(n)}, \qquad y_j^{(n+1)} = N \Big/ \sum_i a_{ij} x_i^{(n)}$$

appears to converge rapidly & stably, but why?
[seems that Ruschendorff had done quite a bit of this before. His paper "Construction of multivariate distributions with given marginals" Ann Inst Stat Math 37, 225–233, 1985, does (i) and in "Closedness of sum spaces and the generalized Schrödinger problem" (with W. Thomsen) TPA 42, 483–494, 199? he does ~~some~~ stuff on topologies to allow one to argue about limits, but it seems less conclusive –]

The Merton illiquidity problem again (22/7/14)

1) Could we do something with a different choice of Φ? Suppose we try

$$\boxed{\Phi(x) = -\log\frac{x+\beta}{\beta} + \tfrac{1}{2}\gamma x^2 + \nu x}$$

which is clearly convex, zero at $x=0$, has a singularity at $x=-\beta$, so we get

$$\Phi'(x) = -\frac{1}{x+\beta} + \gamma x + \nu$$

which suggests that we need $\nu = 1/\beta$ in order to have $\Phi'(0)=0$.

Finding the inverse to Φ' will require solving

$$0 = \gamma x - \frac{1}{x+\beta} + \frac{1}{\beta} - y$$

or equivalently

$$\gamma x^2 + x\left(\beta\gamma + \frac{1}{\beta} - y\right) - \beta y = 0$$

So for a root $> -\beta$ we need

$$x = \frac{-(\beta\gamma + 1/\beta - y) + \sqrt{(\beta\gamma + 1/\beta - y)^2 + 4\beta\gamma y}}{2\gamma}$$

$$= -\beta + \frac{(z+\beta\gamma) + \sqrt{(\beta\gamma+z)^2 + 4\gamma}}{2\gamma}$$

where we write $z \equiv y - 1/\beta$. This shows that the root is always $> -\beta$, and increases with z.

It has the advantage that the story is simpler to handle, but it still doesn't seem to run very well

2) Is there a dual formulation? Suppose we introduce dual variables $z, w \geq 0$

$$\boxed{z = F_Y, \quad w = F_H / F_Y = F_H / z}$$

and the dual value function

$$\boxed{J(z,w) = F(Y,H) - zY - zwH}$$

From this we find that

$$\boxed{\begin{aligned} J_z &= -Y - wH \\ J_w &= -zH \end{aligned}} \qquad \Rightarrow J = F + zJ_z$$

and hence that

$$\boxed{wJ_w - zJ_z = zY}$$

Differentiating $z = F_Y$ tells us that

$$1 = F_{YY}\frac{\partial Y}{\partial z} + F_{YH}\frac{\partial H}{\partial z}, \qquad 0 = F_{YY}\frac{\partial Y}{\partial w} + F_{YH}\frac{\partial H}{\partial w}$$

so that $$1 = F_{YY}\left\{\frac{\partial Y}{\partial z} - \frac{\partial H/\partial z}{\partial H/\partial w}\cdot\frac{\partial Y}{\partial w}\right\} \qquad (*)$$

Differentiating the relations at the top of the page gives

$$J_{zz} = -\frac{\partial Y}{\partial z} - w\frac{\partial H}{\partial z}, \qquad J_{zw} = -\frac{\partial Y}{\partial w} - H - w\frac{\partial H}{\partial w}$$

$$J_{wz} = -H - z\frac{\partial H}{\partial z}, \qquad J_{ww} = -z\frac{\partial H}{\partial w}$$

Hence

$$\left.\begin{aligned} z^2\frac{\partial H}{\partial z} &= -zH - zJ_{zw} = J_w - zJ_{zw} \\ z^2\frac{\partial H}{\partial w} &= -zJ_{ww} \end{aligned}\right\} \Rightarrow \boxed{\frac{\partial H/\partial z}{\partial H/\partial w} = -\frac{J_w - zJ_{zw}}{zJ_{ww}}}$$

We also get

$$\frac{\partial Y}{\partial z} = -J_{zz} - w\frac{\partial H}{\partial z} = -J_{zz} - \frac{w}{z^2}\left(J_w - zJ_{wz}\right)$$

$$\frac{\partial Y}{\partial w} = -J_{zw} - H - w\frac{\partial H}{\partial w} = -J_{zw} + \frac{J_w}{z} + \frac{w}{z}J_{ww}$$

We now substitute into $(*)$ to learn that

$$\boxed{1 = F_{YY}\left\{-J_{zz} + \frac{\left(J_w - zJ_{wz}\right)^2}{z^2 J_{ww}}\right\}}$$

We can return all of this to the original HJB equation to read off the dual

HJB equation:

$$\boxed{\begin{aligned} 0 = \tilde{U}(z) - \tilde{\rho}(J - zJ_z) - \alpha\,(wJ_w - zJ_z) + \tfrac{1}{\varepsilon} z\,\Phi(w-1) \\ + \frac{\sigma^2}{2}\,\frac{(wJ_w - zJ_z)^2}{-z^2 J_{zz} + (J_w - zJ_{wz})^2/J_{ww}} \end{aligned}}$$

Noticing that (with $q>0$) $\inf\{\frac{1}{2}qt^2 - \beta t\} = -\beta^2/2q$, we can express the final term as

$$\inf_t \left\{ \tfrac{1}{2}\left(+z^2 J_{zz} - (J_w - zJ_{wz})^2/J_{ww}\right) t^2 - t\sigma(wJ_w - zJ_z) \right\}$$

$$= \inf_{s,t} \left[\frac{t^2}{2}\left(z^2 J_{zz} + 2\left\{ \frac{s^2}{2} J_{ww} - (J_w - zJ_{wz})s \right\}\right) - \sigma t (wJ_w - zJ_z) \right]$$

$$= \inf_{s,t} \left[\frac{z^2 t^2}{2} J_{zz} + \frac{s^2 t^2}{2} J_{ww} - s t^2 (J_w - zJ_{wz}) - \sigma t (wJ_w - zJ_z) \right]$$

This allows us to write the dual HJB equation as

$$\boxed{0 = \tilde{U}(z) + \frac{z}{\varepsilon}\Phi(w-1) + \mathcal{L}_1 J + \inf_{s,t} \mathcal{L}_2 J(\cdot \mid s,t)}$$

where $\mathcal{L}_1$ is the first-order control-independent operator

$$\mathcal{L}_1 = (\tilde{\rho} + \alpha) z D_z - \alpha w D_w - \tilde{\rho}$$

and $\mathcal{L}_2 = \mathcal{L}_2(\cdot \mid s,t)$ depends on s,t as

$$\boxed{\mathcal{L}_2 \equiv \frac{t^2}{2}\left(z^2 D_{zz} + 2sz D_{wz} + s^2 D_{ww} \right) - st^2 D_w - \sigma t w D_w + \sigma t z D_z}$$

Notice that the diffusion here is (as you would expect) of rank 1.

3) What can we get from P-L? The primal problem is to

$$\max E\left[\int_0^\infty e^{-\tilde{\rho}t} U(c_t)\,dt\right]$$

under the dynamics

$$\begin{cases} dY = \sigma Y dW + (-\alpha Y - h - h f(\epsilon h) - c)\,dt & \times z \quad , \quad dz = z(a\,dW + b\,dt) \\ dH = h\,dt & \times \eta \qquad \eta \text{ is FV ?} \end{cases}$$

So the PL version is

$$\sup E \int_0^\infty \{ e^{-\tilde{\rho}t} U(c_t) + z(-\alpha Y - h - h f(\epsilon h) - c) + Yzb + Yza\sigma + \eta h \}dt + H_t d\eta_t$$
$$+ Y_0 z_0 + H_0 \eta_0$$

$$= \sup E \int_0^\infty \{ e^{-\tilde{\rho}t} \tilde{U}(e^{\tilde{\rho}t} z_t) + zY(-\alpha + b + a\sigma) + (\eta - z)h - zh f(\epsilon h) \}dt + H d\eta$$
$$+ Y_0 z_0 + H_0 \eta_0$$

so we get dual feasibility $-\alpha + b + a\sigma \leq 0$, $d\eta \leq 0$, and complementary slackness would suggest that η only decreases when $H = 0$. Optimizing over h gets us

$$E \int_0^\infty \left\{ e^{-\tilde{\rho}t} \tilde{U}(e^{\tilde{\rho}t} z_t) + \frac{z_t}{\varepsilon} \Phi\left(\frac{\eta_t - z_t}{z_t}\right) \right\} dt + Y_0 z_0 + H_0 \eta_0$$

which we would now try to minimize over dual-feasible controls. If we assume the dual-feasible condition $-\alpha + b + a\sigma \leq 0$ always holds with equality, then the dual control will be (a, η) and

$$b = \alpha - a\sigma$$

In fact, if we write $\begin{cases} z_t \equiv e^{-\tilde{\rho}t} \zeta_t \\ \eta_t \equiv e^{-\tilde{\rho}t} \zeta_t w_t \end{cases}$ ← turns out this must be a supermartingale

and develop this, we end with exact same equations as on the previous page — which is reassuring but not informative

Recall Fundamental Theorem of Statistics: if $Z \sim N(\mu, V)$ then

$$E \exp\left(-\tfrac{1}{2} Z \cdot QZ\right) = \det(I+QV)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\mu \cdot Q(I+VQ)^{-1}\mu\right]$$

Joint laws with given marginals: the MVN case (10/8/14)

Suppose we have

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(0, \begin{pmatrix} V_{XX} & V_{XY} \\ V_{YX} & V_{YY} \end{pmatrix}\right)$$

with joint density $f(x,y)$ and we want to find exponential-quadratic functions φ, ψ such that

(i) $$\int f(x,y)\,\varphi(x)\,\psi(y)\,dy = G(x-\mu_X, \Sigma_X) \qquad \forall x$$

(ii) $$\int f(x,y)\,\varphi(x)\,\psi(y)\,d \quad = G(y-\mu_Y, \Sigma_Y) \qquad \forall y$$

where $G(z,V)$ is the density at z of a $N(0,V)$ distribution. Let's notice that

$$f(x,y) = G(x, V_{XX})\, G(y-Kx, V_{Y|X})$$

where $K = V_{YX}V_{XX}^{-1}$, $V_{Y|X} = V_{YY} - KV_{XY}$. This simplifies (i) quite a bit. Write

$$\boxed{\varphi(x) = \lambda \exp\left(-\tfrac{1}{2}(x-a)A(x-a)\right), \qquad \psi(y) = \exp\left(-\tfrac{1}{2}(y-b)\cdot B(y-b)\right)}$$

so that (i) now says

$$\frac{G(x-\mu_X,\Sigma_X)}{G(x,V_{XX})\varphi(x)} = \int G(y-Kx, V_{Y|X})\exp\left(-\tfrac{1}{2}(y-b)\cdot B(y-b)\right)dy$$

$$= \int G(z+b-Kx, V_{Y|X})\exp\left(-\tfrac{1}{2}z\cdot Bz\right)dz$$

$$= \det(I+BV_{Y|X})^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}(Kx-b)B(I+V_{Y|X}B)^{-1}(Kx-b)\right)$$

$$= \left(\frac{\det Q}{\det B}\right)^{\frac{1}{2}}\exp\left[-\tfrac{1}{2}(Kx-b)Q(Kx-b)\right] \qquad \boxed{Q \equiv B(I+V_{Y|X}B)^{-1}}$$

Thus

$$\varphi(x) = \left(\frac{\det B\,\det V_{XX}}{\det Q\,\det \Sigma_X}\right)^{\frac{1}{2}}\exp\left[-\tfrac{1}{2}(x-\mu_X)\cdot\Sigma_X^{-1}(x-\mu_X) + \tfrac{1}{2}x\cdot V_{XX}^{-1}x + \tfrac{1}{2}(Kx-b)Q(Kx-b)\right]$$

Hence

$$\boxed{\begin{aligned} A &= \Sigma_X^{-1} - V_{XX}^{-1} - K^TQK \\ Aa &= \Sigma_X^{-1}\mu_X - K^TQb \end{aligned}} \qquad \boxed{\begin{aligned} B &= \Sigma_Y^{-1} - V_{YY}^{-1} - \tilde{K}^T\tilde{Q}\tilde{K} \\ Bb &= \Sigma_Y^{-1}\mu_Y - \tilde{K}^T\tilde{Q}a \end{aligned}}$$

$\tilde{K} \equiv V_{XY}V_{YY}^{-1}$, $\tilde{Q} = A(I+V_{X|Y}A)^{-1}$

Glueing measures together (16/8/14)

The aim here is to understand how we might sensibly construct some measure on a (large) finite product space from measures which are defined only on some of the factors. We can try to understand this in various steps.

(1) Suppose m_1, m_2 are (probability) measures defined on the same measurable space. How would we define some measure $m_1 \Diamond m_2$ which somehow combines the features of both? Obvious choice would be $\frac{1}{2}(m_1 + m_2)$, but here this is less useful, because we might have the situation where $m_i = \lambda_i \times \mu_i$ is a product measure on some product space, and we would not want $m_1 \Diamond m_2$ to lose that product structure – indeed, we would want

$$m_1 \Diamond m_2 = (\lambda_1 \Diamond \lambda_2) \times (\mu_1 \Diamond \mu_2) \qquad (1)$$

One thing that would preserve this would be to define $m_1 \Diamond m_2$ with a density

$$f \propto \sqrt{f_1 f_2} \qquad (2)$$

where f_i is the density of m_i with respect to some reference measure. Notice that the choice of the reference measure is unimportant, and f is integrable, by Cauchy-Schwarz. Such a definition would preserve the desirable property (1).

The thing that really makes this interesting is the following observation:

$$\boxed{\begin{array}{l} \min_m \{ H(m|m_1) + H(m|m_2) \} \\ \text{is achieved by } f \text{ of the form (2)} \end{array}} \qquad (3)$$

Why? If m has density f, then

$$H(m|m_1) + H(m|m_2) = \int \left\{ f \log \frac{f}{f_1} + f \log \frac{f}{f_2} \right\} dx$$

so if we optimize over f pointwise we learn that

$$2 \log f - 2 - \log f_1 - \log f_2 = 0$$

which is solved by f of the form (2)!!

(2) Now suppose that m_i is defined on $X_i \times Z$, $i = 1, 2$, where $X_1 \neq X_2$. How would we make a measure m on $X_1 \times X_2 \times Z$ that accomodated m_1 and m_2? We can't calculate $H(m|m_1)$ because m and m_1 are defined on different products,

but we can calculate $H(m_{X_1 \times Z} | m)$ where $m_{X_1 \times Z}$ is the marginal distribution of m on $X_1 \times Z$.

So let's write

$$f_i(x_i, z) = f_i(z)\, k_i(x_i | z),$$

where $f_i(z)$ is the density of the Z-marginal of m_i. We'll express the combination measure as

$$\varphi(z)\, q_1(x_1|z)\, q_2(x_2|z)$$

and then we get (with slight notational abuse)

$$H(m|m_1) + H(m|m_2) = \int_{X_1 \times Z} \varphi(z) q_1(x_1|z) \log\left(\frac{\varphi(z)\, q_1(x_1|z)}{f_1(z)\, k_1(x_1|z)}\right) dz\, dx_1$$

$$+ \int_{X_2 \times Z} \varphi(z)\, q_2(x_2|z) \log\left[\frac{\varphi(z)\, q_2(x_2|z)}{f_2(z)\, k_2(x_2|z)}\right] dz\, dx_2$$

We also have the constraints

$$\int q_i(x_i|z)\, dx_i = 1 \quad \forall z, \quad i = 1, 2,$$ absorbed with multipliers $\lambda_i(z)$, giving a Lagrangian form of the problem

$$\sum_{i=1}^{2} \int_{X_i \times Z} \left\{ \varphi(z)\, q_i(x_i|z) \log\left[\frac{\varphi(z)\, q_i(x_i|z)}{f_i(z)\, k_i(x_i|z)}\right] - \lambda_i(z) q_i(x_i|z) \right\} dz\, dx_i$$

$$+ \int (\lambda_1(z) + \lambda_2(z))\, dz$$

Now if we take the optimization pathwise over $q_i(x_i|z)$ what we conclude is that $q_i(x_i|z) = k_i(x_i|z) \times$ function of z; integrating over x_i tells us that the function of z is identically 1, and

$$\boxed{q_i(x_i|z) = k_i(x_i|z)}$$

So for the optimization over $\varphi(z)$, noting that $\int q_i(x_i|z)\, dx_i = 1$, we have the functional

$$\sum_{i=1}^{2} \int_Z \varphi(z) \log\left[\varphi(z)/f_i(z)\right] dz$$

containing everything to do with φ. This is the same problem as (1), with the same solution

$$\boxed{\varphi(z) \propto \sqrt{f_1(z)\, f_2(z)}}$$

(3) In a very general setting, we might have sets I_j $(j=1,\dots,J)$ of the index set and densities $g_j(x(I_j))$, and we aim to find some density $f(x)$ on the whole product space so as to

$$\min \sum_{j=1}^{J} \int f_j(x(I_j)) \log\left[\frac{f_j(x(I_j))}{g_j(x(I_j))} \right] dx(I_j) \qquad (4)$$

where

$$f_j(x(I_j)) = \int f(x)\, dx(I_j^c) \qquad (5)$$

is the marginal of f on the I_j factors. We can do a Lagrangian form of this, with multiplier $\lambda_j(x(I_j))$ for the constraint (5), and this then gives us terms

$$\int f_j(x(I_j)) \left\{ \log\left[\frac{f_j(x(I_j))}{g_j(x(I_j))} \right] - \lambda_j(x(I_j)) \right\} dx(I_j)$$

involving $f_j(x(I_j))$, which will be optimized when

$$\boxed{f_j(x(I_j)) \propto g_j(x(I_j)) \exp\left(\lambda_j(x(I_j))\right)}$$

and then we have the terms involving f, which will just be

$$\int f(x) \sum \lambda_j(x(I_j))\, dx$$

so we would expect that

$$\boxed{\sum_j \lambda_j(x(I_j)) \geq 0}$$

Maybe correct, but not so very transparent... If we assume the inequality is an equality everywhere (which it will be if f is strictly positive) then if a variable ξ appears in just one I_j (I_k, say) then $\lambda_k(x(I_k))$ does not depend on that variable, so the conditional law of ξ given what's outside will be the same as the g_k-conditional law of ξ given what's outside

(4) One special case is particularly important, namely, that in which all the distributions are MVN. If we look at one of the summands in (4), and suppose that g is a $N(m, \Sigma)$ density, f is a $N(\mu, V)$ density, then the relative entropy is the integral

$$\frac{1}{2}\int \frac{\exp -\frac{1}{2}(x-\mu)\cdot V^{-1}(x-\mu)}{(2\pi)^{d/2}\sqrt{\det V}}\left[\log\det\Sigma - \log\det V - (x-\mu)\cdot V^{-1}(x-\mu) + (x-m)\cdot\Sigma^{-1}(x-m)\right]dx$$

$$= -\frac{1}{2}\log\det\Sigma^{-1}V - \frac{1}{2}\operatorname{tr}(I) + \int\frac{e^{-\frac{1}{2}z\cdot V^{-1}z}}{(2\pi)^{d/2}\sqrt{\det V}}\ \frac{1}{2}(z+\mu-m)\cdot\Sigma^{-1}(z+\mu-m)\ dz$$

$$= -\frac{1}{2}\log\det\Sigma^{-1}V - \frac{d}{2} + \frac{1}{2}\operatorname{tr}(\Sigma^{-1}V) + \frac{1}{2}(\mu-m)\Sigma^{-1}(\mu-m).$$

Now the problem we will be faced with will be to take a sum of such terms (with μ, V unknown) and choose μ, V to minimize. It will be high dimensional, but the objective is nice and convex, so there's a chance we can do it.

(5) (1/9/14) If we had reference measures on the same set of variables, $Q_i \sim N(m_i, \Sigma_i)$ for $i=1,\dots,M$, then the objective to be minimized will be

$$\frac{1}{2}\sum_{i=1}^{M}\left\{-\log\det(\Sigma_i^{-1}V) - \frac{d}{2} + \operatorname{tr}(\Sigma_i^{-1}V) + \frac{1}{2}(\mu-m_i)\Sigma_i^{-1}(\mu-m_i)\right\}$$

so in terms of V we need to minimize

$$-M\log\det V + \operatorname{tr}\left(\left(\sum\Sigma_i^{-1}\right)V\right)$$

achieved when $\boxed{V^{-1} = \frac{1}{M}\sum_1^M \Sigma_i^{-1}}$ ~~consensus precision is the mean of the individual precisions!~~

and $\boxed{V^{-1}\mu = \frac{1}{M}\sum_{i=1}^{M}(\Sigma_i^{-1})m_i}$ by usual stuff

(6) What if the reference measures were on different subsets of the variables? Well, we could extend the individual precisions Σ_i^{-1} to all the variables using a small multiple of the identity, do the same analysis as in (5), and come to the conclusion that

$$V^{-1} = \frac{1}{M}\sum_{i=1}^{M}(\bar\Sigma_i)^{-1}$$

where $\bar\Sigma_i^{-1}$ is just Σ_i^{-1} with zeros in all the rows + columns of variables that model i does not speak about. The combination of the means happens analogously!

... or rather, it doesn't – the trick of extending Σ_i^{-1} to all variables falls down with the $\log\det(\Sigma_j^{-1}V_j)$ term – the parts of V corresponding to I_j^c can't be ignored...

Liquidity problem again (19/8/14)

Getting good pictures remains problematic. It seems that the finite-difference evaluation of F_H, F_Y can be very delicate, and that the new policies which result are often very rough. Maybe we can help things by insisting that $h \geq \underline{h}$, that is, there is a bound to the rate at which we can sell. If we write $\tilde{f}(t) \equiv t f(t)$, and have

$$\Phi(a) = \sup\{at - \tilde{f}(t)\}$$

as before, the optimal choice of h comes from

$$\sup_h \left[h F_H - (h + h f(\varepsilon h)) F_Y \right] = \frac{F_Y}{\varepsilon} \sup_h \left[\varepsilon h \frac{F_H - F_Y}{F_Y} - \varepsilon h f(\varepsilon h) \right]$$

So if we insist $h \geq \underline{h}$ we get that this constrained optimization is

$$\Phi_0(a) = \sup_{t \geq \varepsilon \underline{h}} \left[at - \tilde{f}(t) \right]$$

Diffusion approximation again (29/8/14)

Let's look again at the situation on p63 of WN XXXVII, and introduce a drift into the BM, so that we now see the process

$$x_t = B_t + F(t) \equiv B_t + \int_0^t f(s)ds$$

for some deterministic C^1 function f. As before, we may consider

$$q_n(x) = \sum_{j=1}^{2^n} (\Delta x_j^n)^2 \quad = \xi_n \quad \text{for short}$$

where we write

$$\Delta x_j^n \equiv x(j2^{-n}) - x((j-1)2^{-n}).$$

If we now consider going from n to $n+1$, we may write

$$\Delta x_j^n = \Delta_1 + \Delta_2 = \{x(j2^{-n}) - x((j-\tfrac12)2^{-n})\} + \{x((j-\tfrac12)2^{-n}) - x((j-1)2^{-n})\}$$

and we see that

$$\begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix} \sim N\left(\begin{pmatrix}m_1\\ m_2\end{pmatrix}, \begin{pmatrix}h & 0\\ 0 & h\end{pmatrix}\right)$$

where $h = 2^{-n-1}$, $m_1 = F(j2^{-n}) - F((j-\tfrac12)2^{-n})$, $m_2 = F((j-\tfrac12)2^{-n}) - F((j-1)2^{-n})$.

Thus we get

$$\begin{pmatrix}\Delta_1\\ \Delta_1+\Delta_2\end{pmatrix} \sim N\left(\begin{pmatrix}m_1\\ m_1+m_2\end{pmatrix}, \begin{pmatrix}h & h\\ h & 2h\end{pmatrix}\right)$$

and so

$$(\Delta_1 | \Delta_1+\Delta_2) \sim N\left(m_1 + \tfrac12(\Delta_1+\Delta_2 - m_1 - m_2), \tfrac12 h\right)$$
$$= N\left(\tfrac12(\Delta_1+\Delta_2+m_1-m_2), \tfrac12 h\right)$$

and so

$$E\left(\Delta_1^2 + \Delta_2^2 \,|\, \mathcal{G}_n\right) = \tfrac14\left\{(\Delta_1+\Delta_2+m_1-m_2)^2 + (\Delta_1+\Delta_2+m_2-m_1)^2\right\} + h$$
$$= \tfrac12(\Delta_1+\Delta_2)^2 + \tfrac12(m_1-m_2)^2 + h.$$

Hence

$$E\left[\xi_{n+1} | \mathcal{G}_n\right] = \tfrac12 \xi_n + \tfrac12 + \tfrac12 \sum_{j=1}^{2^n}\left(F(j2^{-n}) - 2F((j-\tfrac12)2^{-n}) + F((j-1)2^{-n})\right)^2$$

From this we see that

$$E\left[(\xi_{n+1}-1) | \mathcal{G}_n\right] = \tfrac12(\xi_n - 1) + \tfrac12 \sum_{j=1}^{2^n}\left(F(j2^{-n}) - 2F((j-\tfrac12)2^{-n}) + F((j-1)2^{-n})\right)^2$$

so that

$$E\left[2^{n+1}(\xi_{n+1}-1)\,\middle|\,\mathcal{G}_n\right] = 2^n(\xi_n-1) + \underbrace{\tfrac{1}{2}2^{n+1}\sum_1^{2^n}\left(F(j2^{-n}) - 2F((j-\tfrac{1}{2})2^{-n}) + F((j-1)2^{-n})\right)^2}_{\equiv\, b_n,\ \text{say.}}$$

Notice that

$$b_n \doteq \tfrac{1}{2}2^{n+1}\sum_{j=1}^{2^n} h^4\, F''((j-\tfrac{1}{2})2^{-n})^2$$

$$= 2^{-2n}\sum_{j=1}^{2^n} 2^{-n} F''((j-1/2)2^{-n})^2 \sim 2^{-2n}\int_0^1 F''(s)^2\,ds$$

What we see is that

$$M_n = 2^n(\xi_n-1) - \sum_{j=1}^{n-1} b_j \quad \underline{\text{is a martingale}}$$

We can do some more calculations using the FTS to learn that

$$E_{\Delta,M}\exp\{-\tfrac{1}{2}\lambda(\Delta_1^2+\Delta_2^2)\} = (1+\lambda h)^{-1/2}\exp\left[-\frac{1}{4}\frac{\lambda}{1+\lambda h}\left((\Delta_1+\Delta_2)^2(1+\lambda h) + (m_1-m_2)^2\right)\right]$$

~~so when we stick it all together~~

$$E\left[\exp(-\tfrac{1}{2}\lambda\xi_{n+1})\,\middle|\,\mathcal{G}_n\right] = (1+\lambda h)^{-2^{n-1}}\exp\left[-\frac{\lambda}{4}\xi_n - \frac{\lambda}{4(1+\lambda h)}2^{-n}b_n\right]$$

Choosing $\lambda = \alpha 2^{n+2}$, we find from this that $\left(Z_n \equiv 2^n(\xi_n-1)\right)$

$$E\left[\exp(-\alpha Z_{n+1})\,\middle|\,\mathcal{G}_n\right] = (1+2\alpha)^{-2^{n-1}}\exp\left[-\alpha Z_n - \frac{\alpha\, b_n}{(1+2\alpha)} + 2^n\alpha\right]$$

[Differentiating w.r.t. α at $\alpha=0$ gives

$$E\left[Z_{n+1}\,|\,\mathcal{G}_n\right] = Z_n + b_n \quad \text{as it should.}$$]

If we write

$$\varphi_n \equiv \log E\exp(-\alpha Z_n)$$

we deduce that

$$\varphi_{n+1} = -2^{n-1}\log(1+2\alpha) + \varphi_n - \frac{\alpha}{1+2\alpha}b_n + 2^n\alpha$$

$$= 2^n\left\{\alpha - \tfrac{1}{2}\log(1+2\alpha)\right\} + \varphi_n - \frac{\alpha}{1+2\alpha}b_n$$

Note that $Z_0 = \xi_0 - 1 = x_1^2 - 1$ and $x_1 \sim N(F(1), 1)$ so by FTS we deduce

$$e^{\varphi_0} = \frac{e^{\alpha}}{\sqrt{1+2\alpha}} \exp\left\{ - \frac{\alpha \mu^2}{1+2\alpha} \right\} \qquad \mu \equiv F(1)$$

so

$$\varphi_0 = \alpha - \frac{\alpha F(1)^2}{1+2\alpha} - \frac{1}{2}\log(1+2\alpha)$$

Hence

$$\boxed{\varphi_n = 2^n \left(\alpha - \tfrac{1}{2} \log(1+2\alpha) \right) - \frac{\alpha}{1+2\alpha} \left\{ F(1)^2 + \sum_{j=0}^{n-1} b_j \right\}}$$

Thus it seems that as $n \to \infty$ there is no limit in distribution, so the martingale doesn't converge.

If we let α_n depend on n as $\alpha_n = 2^{-n/2}\beta$, we get

$$\varphi_n \to \beta^2$$

So we can deduce that

$$\boxed{2^{n/2}(\xi_n - 1) \xrightarrow{\mathcal{D}} N(0, 2)}$$

Model combination (1/9/14)

It appears that in the model combination story, constructing the reference measure Q with density g will actually be quite straightforward. We then need to find a measure P with density f such that the law of the I_j-variables has density f_j for each j. The problem therefore is

$$\min \int f \log\left(\frac{f}{g}\right) dx \qquad \text{s.t.} \qquad f_j(x(I_j)) = \int f(x)\, dx(I_j^c)$$

and if we make a Lagrangian form of this problem, we shall find that

$$f(x) \propto g(x) \exp\left\{ \sum \lambda_j (x(I_j)) \right\}$$

for multiplier functions λ_j to be found. This will likely be difficult in general, but for a centred Gaussian it seems we can get somewhere. We will have the precision M of the reference measure, and need to add some block-diagonal Λ to M so that the covariance $(M+\Lambda)^{-1}$ is correct in each of the diagonal blocks.

Let's just try to understand how this could be done in the simplest setting. We're given a partitioned precision matrix

$$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and we want to find symmetric X, Y such that $\bar{J} \equiv \begin{pmatrix} A+X & B \\ C & D+Y \end{pmatrix}$ when inverted has the correct covariance on the diagonals. We can do rotations for each of the blocks and reduce to the case of I as the target covariance. So we require

$$\begin{cases} A+X - B(D+Y)^{-1}C = I \\ D+Y - C(A+X)^{-1}B = I \end{cases} \Rightarrow D+Y = I + C(A+X)^{-1}B$$

Therefore $B(D+Y)^{-1}C = B(I + C(A+X)^{-1}B)^{-1}C$. Note that $C = B^T$. If we use the SVD $B = RLS$ (L is diagonal, R, S, orthogonal), the equation for $Z \equiv A+X$ will be

$$I = Z - RLS(I + S^TL^TR^TZ^{-1}RLS)^{-1}S^TL^TR$$

so $$I = R^TZR - L(I + L^TR^TZ^{-1}RL)^{-1}L^T$$

<u>However, this will only work for two.</u>

Momentum + equilibrium (9/9/14)

(i) Looking again at the model Keiichi + Dave studying, we have

$$\begin{cases} dX_t = dW_t + \mu_t\,dt \\ d\mu_t = \varepsilon\, dX_t - \lambda \mu_t\,dt \end{cases}$$

So if we set $\tilde{\mu}_t \equiv \exp(\lambda t)\,\mu_t$ we get

$$d\tilde{\mu}_t = \varepsilon e^{\lambda t} dX_t$$
$$= \varepsilon\left\{ e^{\lambda t} dW_t + \tilde{\mu}_t\,dt \right\}$$

so that $\quad d(e^{-\varepsilon t}\tilde{\mu}_t) = \varepsilon e^{\lambda t} dW_t\, e^{-\varepsilon t}$

and therefore

$$\tilde{\mu}_t = e^{\varepsilon t}\left\{ \tilde{\mu}_0 + \int_0^t \varepsilon e^{(\lambda-\varepsilon)s}\, dW_s \right\}$$

and hence

$$X_t - X_0 = W_t - W_0 + \int_0^t e^{(\varepsilon-\lambda)v}\left\{ \mu_0 + \int_0^v \varepsilon e^{(\lambda-\varepsilon)s}\, dW_s \right\} dv$$

$$= W_t - W_0 + \mu_0 \frac{e^{(\varepsilon-\lambda)t}-1}{\varepsilon-\lambda} + \int_0^t \varepsilon\, \frac{e^{(\varepsilon-\lambda)(t-s)}-1}{\varepsilon-\lambda}\, dW_s$$

$$= \mu_0 \frac{e^{(\varepsilon-\lambda)t}-1}{\varepsilon-\lambda} + \int_0^t \left\{ 1 + \frac{\varepsilon}{\varepsilon-\lambda}\left(e^{(\varepsilon-\lambda)(t-s)}-1\right) \right\} dW_s$$

$$= \mu_0 \frac{e^{(\varepsilon-\lambda)t}-1}{\varepsilon-\lambda} + \frac{1}{\varepsilon-\lambda}\int_0^t \left(\varepsilon e^{(\varepsilon-\lambda)(t-s)} - \lambda\right) dW_s$$

If $\varepsilon > \lambda$, then agent's model predicts EX_t will explode exponentially (?)
If $\varepsilon < \lambda$, the agent's model predicts $EX_t \to \mu_0/(\lambda-\varepsilon)$
If $\varepsilon = \lambda$, agent's model says $EX_t = \mu_0 t$: this seems most sensible ~~say~~
but

$$\varepsilon \leq \lambda$$

would be OK. Somehow, $\varepsilon > \lambda$ must be excluded.

(ii) What is the covariance structure of the process X? If we write $k \equiv (\lambda - \varepsilon)$, then some straightforward calculations give us for $0 \leq s \leq t$

$$\boxed{\operatorname{cov}(X_s, X_t) = \varepsilon^2 e^{-k(t-s)} \frac{1-e^{-2ks}}{2k} - \varepsilon\lambda\left(1+e^{-k(t-s)}\right)\frac{1-e^{-ks}}{k} + \lambda^2 s}$$

As we let $\lambda \to \varepsilon$, we get for the limit case

$$\boxed{\operatorname{cov}(X_s, X_t) = \frac{(1+\varepsilon s)^3 - 1}{3\varepsilon} + (t-s)\frac{(1+\varepsilon s)^2 - 1}{2}}$$

(iii) If we look through the theory of the diverse-beliefs equilibrium, we find the SPD implies a change to a measure L, where

$$dL_t = L_t(\bar{\mu}_t - b)\, dX_t \qquad [b = \gamma/J]$$

and we have

$$d\mu = \varepsilon\, dX - \bar{\lambda}\mu\, dt$$

$$= \varepsilon\left(dW + (\bar{\mu} - b)\,dt\right) - \bar{\lambda}\mu\, dt$$

$$= \varepsilon\, dW - \varepsilon b\, dt - \left(\bar{\lambda} - \frac{\varepsilon 1}{J}\right)\mu\, dt$$

So if we consider the matrix $\bar{\lambda} - \varepsilon 1/J \equiv \bar{\lambda} - \varepsilon v^T$, say, for stability we would have to insist that

<u>all eigenvalues of</u> $\bar{\lambda} - \varepsilon v^T$ <u>have positive real part</u>

How does this happen? If $(\bar{\lambda} - \varepsilon v^T)x = q\, x$ for some q, we will have

$$(\lambda - qI)x = \varepsilon v^T x$$

so $\quad (\lambda_j - q)x_j = \varepsilon_j \; v^T x$

so we'd have $\quad \sum v_j x_j = \frac{1}{J}\sum x_j = v \cdot x = \sum \frac{\varepsilon_j}{(\lambda_j - q)} \cdot \frac{1}{J} \cdot v \cdot x$

which requires $\quad \boxed{\sum \frac{\varepsilon_j}{\lambda_j - q} = J}$

This eigenvalue equation for q has exactly J real roots, all positive iff

$$\boxed{\sum \varepsilon_j / \lambda_j < J}$$

(iv) Let's therefore assume this condition. We have

$$d\mu = \varepsilon\, dW - B\mu\, dt - \varepsilon b\, dt$$

$$= \varepsilon\, dW - B(\mu + h)\, dt$$

where $Bh = \bar{\lambda} h - \varepsilon \bar{h} = \varepsilon b$. So what has to happen is that for some ν

$$h_j = \nu \varepsilon_j / \lambda_j$$

where ν is fixed by the condition

$$\nu - \nu J^{-1} \sum \varepsilon_j / \lambda_j = \gamma / J \equiv b$$

Hence $\boxed{\nu = \dfrac{\gamma}{J - \sum \varepsilon_j / \lambda_j}}$

and this fixes h. If now $z \equiv \mu + h$, we see that

$$dz = \varepsilon\, dW - Bz\, dt$$

$$\Rightarrow \quad e^{tB} z_t - z_0 = \int_0^t e^{sB} \varepsilon\, dW_s$$

so $\boxed{\mu_t + h = e^{-tB}\left[\mu_0 + h + \int_0^t e^{sB} \varepsilon\, dW_s\right]}$

and

$$X_t - X_0 + bt = W_t - W_0 + \int_0^t \bar{\mu}_s\, ds \qquad (v = J^{-1} 1)$$

$$= W_t - W_0 - t\, v \cdot h + \int_0^t v \cdot e^{-sB} \left\{ (\mu_0 + h) + \int_0^s e^{uB} \varepsilon\, dW_u \right\} ds$$

$$= W_t - W_0 - t\, v \cdot h + v (I - e^{-tB}) B^{-1} (\mu_0 + h) + v \cdot \int_0^t (I - e^{-(t-u)B}) B^{-1} \varepsilon\, dW_u$$

$$\int^{b} q(a,x,b')\,db' = \frac{2b(b-x)}{(b-a)^2}, \qquad \int_{a} q(a',x,b)\,da' = \frac{-2a(x-a)}{(b-a)^2}.$$

and in fact

$$\boxed{\int_{x^+}^{b} q(a,x,b')\,db' = \frac{2b(b-x)}{(b-a)^2}, \qquad \int_{a}^{x\wedge 0} q(a',x,b)\,da' = \frac{-2a(x-a)}{(b-a)^2}}$$

Provided $a+\varepsilon < 0 < b+\varepsilon$, we get

$$\boxed{q(a+\varepsilon, x+\varepsilon, b+\varepsilon)(b-a)^3 = (b-a)^3\, q(a,x,b) + 2\varepsilon(2x-a-b)}$$

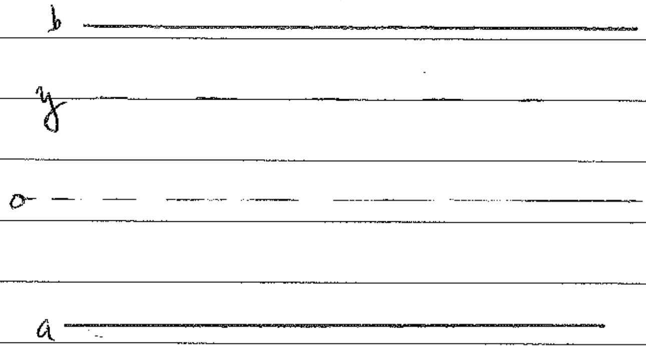
if it helps.

Green's function for (I, X, S) (29/9/14)

(i) Suppose we want to find

$$E\left[\int_0^\infty \varphi(X_t)\, I_{\{S_t \leq b,\, I_t \geq a\}}\, dt\right]$$

for some $a<0<b$, bounded measurable φ. This requires us to find the occupation measure for the BM started at 0, until exit from $[a,b]$. For $y>0$, we get



$$E\, L^y_{H_a \wedge H_b}$$

$$= \frac{-a}{y-a} \cdot \left\{ \frac{1}{2(b-y)} + \frac{1}{2(y-a)} \right\}^{-1}$$

$$= \frac{-2a}{y-a} \; \frac{(b-y)(y-a)}{b-a}$$

$$= \frac{-2a(b-y)}{b-a}.$$

Likewise, for $y<0$ we obtain

$$E\, L^y_{H_a\wedge H_b} = \frac{2b\,(y-a)}{b-a}.$$

Hence

$$E\left[\int_0^\infty \varphi(X_t)\, I_{\{S_t\leq b,\, I_t\geq a\}}\, dt\right] = \int_0^b \frac{-2a(b-y)}{b-a}\,\varphi(y)\,dy + \int_a^0 \frac{2b(y-a)}{b-a}\,\varphi(y)\,dy$$

If we differentiate w.r.t. a, b, we get

$$\int_0^b \frac{2\left((a+b)y - 2ab\right)}{(b-a)^3}\,\varphi(y)\,dy + \int_a^0 \frac{2\left((a+b)y - 2ab\right)}{(b-a)^3}\,\varphi(y)\,dy$$

which tells us the joint density of the occupation measure for (I, X, S): it is

$$\boxed{q(a,x,b) = \frac{2(a+b)x - 4ab}{(b-a)^3}} \qquad \begin{array}{l} a<0<b \\ x\in(a,b) \end{array}$$

$$I_1 = \int_{a_0}^{b_0} \frac{2(x_0 \wedge y - a_0)(b_0 - x_0 \vee y)}{b_0 - a_0} \, \varphi(a_0, y, b_0) \, dy$$

$$= \int \frac{2\,(x_0 \wedge y - a_0)^+ \,(b_0 - x_0 \vee y)^+}{b_0 - a_0} \, \varphi(a_0, y, b_0) \, dy$$

(ii) What would happen if instead we were starting with $S = b_0$, $I = a_0$, $X = x_0$?
In that case, there will be three contributions to

$$E \int_0^\infty \varphi(I_t, X_t, S_t)\,dt = I_1 + I_2 + I_3$$

where

$$I_1 = \int_{x_0}^{b_0} \frac{2(x_0 - a_0)(b_0 - y)}{b_0 - a_0}\, \varphi(a_0, y, b_0)\, dy$$

$$+ \int_{a_0}^{x_0} \frac{2(b_0 - x_0)(y - a_0)}{b_0 - a_0}\, \varphi(a_0, y, b_0)\, dy$$

is the contribution we collect prior to $H_{a_0} \wedge H_{b_0}$,

$$I_2 = \frac{x_0 - a_0}{b_0 - a_0} \iiint q(i, y, s)\, \varphi\big((i + b_0) \wedge a_0,\ y + b_0,\ s + b_0\big)\, di\, dy\, ds$$

which is what we get after exiting $[a_0, b_0]$ at b_0, and correspondingly

$$I_3 = \frac{b_0 - x_0}{b_0 - a_0} \iiint q(i, y, s)\, \varphi\big(i + a_0,\ y + a_0,\ b_0 \vee (s + a_0)\big)\, di\, dy\, ds$$

Can this lead to a characterization of the laws m of (I, X, S) at some finite stopping time by comparison of the potentials??

We can write

$$I_3 = \frac{b_0 - x_0}{b_0 - a_0} \iiint q(i - a_0, y - a_0, s - a_0)\, \varphi(i, y, s \vee b_0)\, di\, dy\, ds$$

Overall, we are going to fix some $i < 0 < s$, $y \in [i, s]$, and then integrate $m(da_0, dx_0, db_0)$ and see what contribution we get to the Green's function at (i, y, s). Clearly, when $a_0 < y$, or $b_0 > s$, we won't get any contribution. We'll get a contribution when $a_0 = i$ and $b_0 = s$, coming from I_1; we'll get a contribution when $a_0 = i$ and $b_0 < s$, coming from I_2; we'll get a contribution when $b_0 = s$ and $a > i$, from I_3; and finally we will get contributions when $a_0 > i$ and $b_0 < s$ from both I_2 and I_3.

Let's make these contributions more explicit. With $a_0 = i$ and $b_0 = s$, we collect

$$\boxed{\int_{x \in [i, s]} m(di, dx, ds) \cdot \frac{2(x \wedge y - i)(s - x \vee y)}{s - i}}$$

For the second contribution, where $a_0 = i$, we obtain

$$\boxed{\iint_{\{i<x_0<b_0,\ b_0\in(0,s)\}} m(di, dx_0, db_0)\ \frac{x_0-i}{b_0-i} \int q(i', y-b_0, s-b_0)\ I_{\{i-b_0<i'<(y-b_0)\wedge 0\}}\ di'}$$

The third contribution, where $b_0 = s$, will be likewise

$$\boxed{\iint_{\{a_0<x_0<s,\ a_0\in(i,0)\}} m(da_0, dx_0, ds)\ \frac{s-x_0}{s-a_0} \int q(i-a_0, y-a_0, s')\ I_{\{(y-a_0)^+<s'<s-a_0\}}\ ds'}$$

and finally we shall have

$$\boxed{\iiint_{\{i<a_0<0,\ 0<b_0<s,\ x_0\in[a_0,b_0]\}} m(da_0, dx_0, db_0) \left\{ \frac{x_0-a_0}{b_0-a_0}\, q(i-b_0, y-b_0, s-b_0) + \frac{b_0-x_0}{b_0-a_0}\, q(i-a_0, y-a_0, s-a_0) \right\}}$$

Notice the simplification

$$\frac{x-a}{b-a}\, q(i-b, y-b, s-b) + \frac{b-x}{b-a}\, q(i-a, y-a, s-a)$$

$$= \frac{2(i+s)(x+y) - 4is - 4xy}{(s-i)^3}$$

$$= q(i-x, y-x, s-x).$$

So the last contribution becomes

$$\boxed{\iiint_{\{i<a_0<0,\ 0<b_0<s,\ x_0\in[a_0,b_0]\}} m(da_0, dx_0, db_0)\ q(i-x_0, y-x_0, s-x_0)}$$

Exploiting the simplification facing p 24 for the indefinite integrals, we find the second contribution is

$$\iint_{\{i<x_0<b_0,\ b_0\in(0,s)\}} m(di, dx_0, db_0)\ \frac{2(x_0-i)(y-i)}{(s-i)^2}$$

and the third contribution is

$$\iint_{\{a_0 < x_0 < s,\ a_0 \in (i,0)\}} m(da_0, dx_0, ds) \quad \frac{2(s-x_0)(s-y)}{(s-i)^2} .$$

So all together, with $i < 0 < s$, $y \in [i,s]$ held fixed, the post-stopping Green function at (i,y,s) will be

$$\boxed{\begin{aligned} &\int_{x \in [i,s]} m(di, dx, ds) \quad \frac{2(x\wedge y - i)(s - x \vee y)}{s-i} \\ &+ \iint_{\{i < x_0 < b_0,\ b_0 \in (0,s)\}} m(di, dx_0, db_0) \quad \frac{2(x_0 - i)(y-i)}{(s-i)^2} \\ &+ \iint_{\{a_0 < x_0 < s,\ a_0 \in (i,0)\}} m(da_0, dx_0, ds) \quad \frac{2(s-x_0)(s-y)}{(s-i)^2} \\ &+ \iiint_{i<a_0<0,\ 0<b_0<s,\ x_0 \in [a_0,b_0]} m(da_0, dx_0, db_0)\ q(i-x_0, y-x_0, s-x_0) \end{aligned}}$$

So: if m_0 is the joint law of (I_τ, X_τ, S_τ) for some a.s. finite stopping time τ, this measure on (i,y,s) should be dominated by $q(i,y,s)\, di\, dy\, ds$.

(iii) Consider the special case where we have a density ρ for the measure m. Then
(?) only the final term in the above expression will be present, and our condition becomes

$$\iiint_{\{i<a<0,\ 0<b<s,\ x\in[a,b]\}} \rho(a,x,b)\, q(i-x, y-x, s-x)\, da\, dx\, db \le q(i,y,s),$$

for all i, y, s. Now exploiting the relation facing p24, and the fact that ρ integrates to 1, we turn this into the condition

$$\iiint_{\{\dots\}} \rho(a,x,b) \frac{2x(2y-i-s)}{(s-i)^3}\, da\, dx\, db \ge 0 \qquad \forall i, y, s.$$

But we can make sense of this, because $2y - i - s$ is negative when $y < \frac{1}{2}(i+s)$, positive when $y > \frac{1}{2}(i+s)$, so the desired inequality holds $\forall i, y, s$ iff

$$\boxed{\iiint_{\{i<a<0,\ 0<b<s,\ x\in[a,b]\}} x\,p(a,x,b)\ da\,dx\,db = 0 \quad \forall\ i<0,\ s>0}$$

(?) But is this correct? Don't the other terms also contribute? Yes, they do.

Production, consumption and trading (13/10/14)

Suppose there are J agents. Each agent j produces a bundle k_j of goods. He has preferences over consumption bundles θ given by

$$\theta \mapsto \theta \cdot m_j - \tfrac{1}{2}\theta \cdot V_j \theta \equiv U_j(\theta)$$

(so he likes more, but there's diminishing returns – a bit crude, but let's see where it goes)

Let p be the vector of prices. Then agent j will attempt to

$$\max U_j(\theta) \quad \text{s.t.} \quad p \cdot \theta \leq b_j \equiv p \cdot k_j$$

so we take Lagrangian form, and find agent j will want

$$\theta_j = V_j^{-1}\left(m_j - \frac{p \cdot (V_j^{-1} m_j - k_j)}{p \cdot V_j^{-1} p}\, p \right).$$

If we now try market clearing condition, we would find

$$\sum \theta_j = \sum k_j = \sum V_j^{-1}\left(m_j - \frac{p \cdot (V_j^{-1} m_j - k_j)}{p \cdot V_j^{-1} p}\, p \right)$$

or in other words

$$\boxed{\sum \frac{p \cdot (V_j^{-1} m_j - k_j)}{p \cdot V_j^{-1} p} V_j^{-1} p = \sum (V_j^{-1} m_j - k_j)}$$

Do we have many/any solutions p?

Numerics show that there can in fact be many different solutions.
This is in effect a (generalization to many agents of a) classical Edgeworth box; as is well known, there are in general many points on the contract curve, and this is an illustration of that happening!

Approach to multi-agent economies (20/10/14)

(i) The approach to be explored here is where we have a number of agents $j=1,\dots,J$ where each agent has a state $\xi_j(t)$ at time $t \in [0,\infty)$. There are some N goods which are inputs to/outputs from some linear production technology, so if an agent chooses an activity vector $\theta = (\theta_m)_{m=1}^{M}$ the net output of goods will be at rate $A\theta$, where the matrix A is $N \times M$, and may in principle depend on the agent (perhaps different agents farm different types of soil ...) as indeed may the number M of available activities. It seems natural to insist on $\theta \geq 0$.

(ii) Let's suppose there's a price vector $p(t)$ for the N goods at time t. Agent j has a demand function d_j for the goods, $d_j(\xi_j, p)$, and the prices should clear the market of produced goods

$$(1) \qquad \sum_{j=1}^{J} d_j(\xi_j(t), p(t)) = \sum_{j=1}^{J} A_j\, \theta_j(\xi_j(t), p(t))$$

where the activities θ_j chosen will depend on the state of the agent, and on the price vector. The states of agents will evolve as

$$(2) \qquad \frac{d\xi_j(t)}{dt} = \Phi_j(\xi_j(t), p(t))$$

So what we will end up with is some ODE for the prices and the states, but of course this is too general to be helpful yet – we need more detail on the functions $d_j, \theta_j, \Phi_j \dots$

(iii) At present, I envisage that the state of the agent j should be a vector of available quantities of labour of different types $L(t)$, along with a vector of available capital goods. Seems like we need a distinction between capital goods (farms, factories, tractors, ...) which are durable, and consumption goods (food, clothing, heating, electricity, ...) which are essentially to be consumed as they are produced. The state of agent j will also include a vector $K(t)$ of quantities of the durable capital goods available at time t. What the agent will be trying to do will be to

$$(3) \qquad \max_{\theta(t) \geq 0} \; p(t) \cdot (A\,\theta(t) - I(t)) \quad \text{s.t.} \quad \begin{aligned} \lambda\, \theta(t) &\leq L(t) - \tau(t) \\ \kappa\, \theta(t) &\leq K(t) \end{aligned}$$

where $I(t)$ is the rate of investing in capital goods ($\dot{K}(t) = -\Delta K(t) + I(t)$, where

Δ is a diagonal matrix of depreciation rates) and $\tau(t)$ is the rate at which labour of the various types is being used for training, which impacts labour as

$$\dot{L}(t) = \Lambda L(t) + B\tau(t)$$

where Λ is the matrix governing the evolution of the population without any training effort, and λ is the matrix telling us what quantities of labour of different types will be required for the different activities which can be conducted, with κ being the matrix telling us what amounts of capital of different types will be needed for different activities.

Notice that we will have to have $1^T B = 0$; training does not change the total size of the population.

Once the agent has computed his optimal θ, he will have proceeds $p\cdot(A\theta - I)$ to spend on the consumption goods, so he will aim to maximize some objective subject to the budget constraint. If he consumes a bundle x let's say his utility will be

$$x\cdot m - \tfrac{1}{2}x\cdot Qx$$

where $m \gg 0$, and Q is PDS, so the consumption optimization is

$$\max \quad x\cdot m - \tfrac{1}{2}x\cdot Qx \qquad \text{s.t.} \quad p\cdot x \le p\cdot(A\theta - I) = b, \text{ say,}$$

(we could say that $m_j = 0$ for j corresponding to a durable good). This we can solve fairly explicitly:

$$x^* = Q^{-1}\left(m - \left(\frac{p\cdot Q^{-1}m - b}{p\cdot Q^{-1}p}\right)^+ p\right).$$

(iv) The issue here is that if this is all we cared about, then the choice of I and τ would be 0 - why would we impose a more restrictive constraint?! So now we have to think how I, τ would be determined, which requires a long term view.

Maybe we could consider what we would try to do in steady state, so we would try to

$$\max \quad p\cdot(A\theta - I)$$

subj to $\lambda\theta \le L - \tau$, and $\kappa\theta \le K$, where $I = \Delta K$, and $\alpha L = \Lambda L + B\tau$, where α is the population growth rate, $1^T\Lambda = \alpha 1$. We will also require $\theta \ge 0$, $K \ge 0$, as well as $L, \tau \ge 0$. The last two are harder to deal with; for $K \ge 0$ we just set

$I = \Delta K$. Without more structure on Λ, B, it's not clear how to proceed, so let's just take the special (but interesting) situation where you have skill levels $0, 1, \dots, n$ and the form of Λ is

$$\Lambda = \begin{bmatrix} -\mu+\lambda & \lambda & \lambda & \lambda & \cdots \\ 0 & -\mu & 0 & 0 & \cdots \\ 0 & 0 & -\mu & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

where $\alpha \equiv \lambda - \mu > 0$ is the net growth rate of the population. We could suppose that you need people at skill level k to train up those at skill level $k-1$, so

$$B = \begin{bmatrix} 0 & -\beta_1 & 0 & 0 & \cdots \\ 0 & \beta_1 & -\beta_2 & 0 & \cdots \\ 0 & 0 & \beta_2 & -\beta_3 & \cdots \\ 0 & 0 & 0 & \beta_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

As the steady-state equation $\alpha L = \Lambda L + B\tau$ would tell us

$$\begin{cases} \lambda L_j = \beta_j \tau_j - \beta_{j+1} \tau_{j+1} & (j \geq 1) \\ \lambda L_0 = \lambda 1 \cdot L - \beta_1 \tau_1 & (j = 0) \end{cases}$$

where $\beta_{n+1} = 0$. This would say

$$\boxed{\tau_j = \beta_j^{-1} \left(\lambda L_j + \beta_{j+1} \tau_{j+1} \right)} \qquad (j \geq 1)$$

by looking at the equations for $j \geq 1$, and summing all these over j would give the equation at $j = 0$ as a ~~consistent~~ check. Thus it's clear that if we choose the $L_j \geq 0$, subject to $\Sigma L_j = 1$ for normalization, we can work out a training schedule τ which would achieve L as the steady state profile. However, it is not obvious that we would be able to ensure $\tau_j \leq L_j$. It would be OK if $\lambda \leq \beta_n$, and then we have

$$\beta_j \tau_j = \lambda (L_j + \dots + L_n)$$

but it's not clear this will hold for general $L \geq 0$. So in the end, the best we can do is to state the problem as the LP at the foot of the previous page.

High hopes and equilibrium price (30/10/14)

(i) Phil D asks what would be the price of the asset in a representative agent equilibrium where the preferences are given by our high hopes paper:

$$E \int_0^\infty e^{-\rho t}\, G(c_t, L_t)\, dt$$

where

$$G(c,L) = \begin{cases} (1-p)\, U(L) + p\, U(c), & c \geq L \\ (1-\tilde{p})\, U(L) + \tilde{p}\, U(c) & c \leq L \end{cases}$$

where $\tilde{p} \equiv p(1+K)$. We'll assume that consumption is the dividend process x_t, assumed to be log Brownian:

$$dx_t / x_t = \sigma\, dW_t + b\, dt.$$

We would want to calculate

$$V(x, \ell) = \sup E\left[\int_0^\infty e^{-\rho t} G(x_t, L_t) \,\middle|\, x_0 = x, L_0 = \ell \right]$$

where the sup is taken over increasing processes L.

(ii) Let's just focus now on the case of CRRA utility, $U'(x) = x^{-R}$. Noticing that $G(\lambda c, \lambda L) = \lambda^{1-R} G(c, L)$ for $\lambda > 0$, we must have by scaling that

$$V(x, \ell) = \ell^{1-R}\, v(x/\ell) \equiv \ell^{1-R}\, v(z).$$

The optimization statement is that

$$e^{-\rho t} V(x_t, L_t) + \int_0^t e^{-\rho s} G(x_s, L_s)\, ds \quad \text{is a supermartingale}$$

and a martingale under optimal control, so this gives the HJB

$$0 = G(x, L) - \rho V(x, L) + b x V_x + \tfrac{1}{2}\sigma^2 x^2 V_{xx}$$

$$= L^{1-R}\left[G(z, 1) - \rho v(z) + b z v'(z) + \tfrac{1}{2}\sigma^2 z^2 v''(z) \right] \qquad (1)$$

as well as

$$V_L = L^{-R}\left[(1-R)\, v(z) - z v'(z) \right] \leq 0. \qquad (2)$$

If v is C^1 at $z=1$, then because it satisfies (1) it will automatically be C^2

If we consider the quadratic $F(t) \equiv \frac{1}{2}\sigma^2 t(t-1) + bt - \rho$, with roots $-\alpha < 0 < \beta$, we are able to express the solution more explicitly. We shall have

$$v(z) = \frac{(1-p)U(1)}{\rho} - \frac{pU(z)}{F(1-R)} + Az^{-\alpha} + Bz^{\beta} \qquad (z \geq 1)$$

$$= \frac{(1-\tilde{p})U(1)}{\rho} - \frac{\tilde{p}\,U(z)}{F(1-R)} + \tilde{A}z^{-\alpha} + \tilde{B}z^{\beta} \qquad (z \leq 1) \qquad (3)$$

and what we will expect is that there is some critical z_* > 1 such that the function $z^{R-1}v(z)$ increases to a constant value to which it is smooth pasted at z_*. So if we require a C^2 fit at 1, and the C^2 pasting to a constant, we obtain conditions in total to fix the 5 unknowns

(iii) Working the conditions a bit, the function $h(z) \equiv z^{R-1}v(z)$ has derivatives

$$h'(z) = -\frac{(1-p)}{\rho} z^{R-2} + A(R-1-\alpha)z^{R-2-\alpha} + B(R-1+\beta)z^{R-2+\beta}$$

$$h''(z) = -\frac{(1-p)(R-2)}{\rho} z^{R-3} + A(R-1-\alpha)(R-2-\alpha)z^{R-3-\alpha} + B(R-1+\beta)(R-2+\beta)z^{R-3+\beta}$$

giving the conditions at $z = z_*$:

$$0 = -\frac{(1-p)}{\rho} + A(R-1-\alpha)z^{-\alpha} + B(R-1+\beta)z^{\beta}$$

$$0 = -\frac{(1-p)}{\rho}(R-2) + A(R-1-\alpha)(R-2-\alpha)z^{-\alpha} + B(R-1+\beta)(R-2+\beta)z^{\beta}$$

Solving for $A' \equiv A(R-1-\alpha)z^{-\alpha}$, $B' \equiv B(R-1+\beta)z^{\beta}$ gives

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \frac{1-p}{\rho(\alpha+\beta)} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \qquad (4)$$

From the smooth fit of v and v' at 1 we shall have the conditions

$$\begin{pmatrix} 1 & 1 \\ -\alpha & \beta \end{pmatrix}\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\alpha & \beta \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} pKU(1)/\rho + pKU(1)/F(1-R) \\ pK/F(1-R) \end{pmatrix}$$

leading to

$$\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} + \frac{1}{\beta+\alpha}\begin{pmatrix} \beta & -1 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} pKU(1)\left(\frac{1}{\rho} + \frac{1}{F(1-R)}\right) \\ pK/F(1-R) \end{pmatrix}$$

$$= \begin{pmatrix} A \\ B \end{pmatrix} + \frac{pK\,U(1)}{(\beta+\alpha)F(1-R)}\begin{pmatrix} \beta & -1 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} F(1-R)/\rho + 1 \\ 1-R \end{pmatrix} \qquad (5)$$

[Then there is the second derivative condition

$$\alpha(\alpha+1)A + \beta(\beta-1)B - \frac{pR}{F(1-R)} = \alpha(\alpha+1)\tilde{A} + \beta(\beta-1)\tilde{B} - \frac{\tilde{p}R}{F(1-R)}$$

or equivalently

$$\begin{pmatrix} \alpha(\alpha+1) & \beta(\beta-1) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha(\alpha+1) & \beta(\beta-1) \end{pmatrix}\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} - \frac{pKR}{F(1-R)}.$$

This is a redundant condition: v solves (1) by construction, so $C^1 \Rightarrow C^2$ as $G(\cdot,1)$ is cts.]

(iv) Let's think what would happen for tiny x. In effect, we will never raise L, and the objective will be roughly

$$V(x,L) \approx \frac{1-\tilde{p}}{\rho}U(L) + \tilde{p}\,U(x)\,E\left[\int_0^\infty e^{-\rho t}\,\delta_t^{1-R}\,dt \,\middle|\, \delta_0 = 1\right]$$

$$= \frac{1-\tilde{p}}{\rho}U(L) + \tilde{p}\,U(x)\int_0^\infty \exp\{-\rho t + (1-R)(b-\tfrac{1}{2}\sigma^2)t + \tfrac{1}{2}\sigma^2(1-R)^2 t\}\,dt$$

$$= \frac{1-\tilde{p}}{\rho}U(L) + \tilde{p}\,U(x)\int_0^\infty \exp\{tF(1-R)\}\,dt$$

from which we learn that

- For the problem to be well posed, must have $\boxed{F(1-R) < 0}$
- Hence $-\alpha < 1-R < b$, and $\boxed{v(x) \sim \frac{1-\tilde{p}}{\rho}U(L) - \frac{\tilde{p}}{F(1-R)}U(x) \qquad (x \downarrow 0)}$

Since we know $-\alpha < 1-R < \beta$, inspection of the form (3) of the solution tells us that we can only have this asymptotic at ∞ to if

$$\tilde{A} = 0. \qquad (6)$$

This gives us a way to fix z_*. Looking at (5) we see that

$$0 = \tilde{A} = A + \frac{pK\,u(1)}{\rho(\alpha+\beta)F(1-R)}\left\{\beta F(1-R) + \beta\rho - \rho(1-R)\right\}$$

$$= \frac{A'}{R-1-\alpha}\, z^{-\alpha} + \frac{pK}{\rho(\alpha+\beta)F(1-R)}\left\{\frac{\beta(F(1-R)+\rho)}{1-R} - \rho\right\}$$

So from (4) we shall have

$$\frac{(1-p)\beta}{\rho(\alpha+\beta)(\alpha+1-R)}\, z^{\alpha} = \frac{pK}{\rho(\alpha+\beta)F(1-R)}\left\{\beta(-\tfrac{1}{2}\sigma^2 R + b) - \rho\right\} \qquad (7)$$

and now we observe that $\alpha + 1 - R > 0$ if the problem is well posed, and

$$0 = F(\beta) = \tfrac{1}{2}\sigma^2\beta(\beta-1) + b\beta - \rho = \tfrac{1}{2}\sigma^2\beta^2 + \beta(-\tfrac{1}{2}\sigma^2 + b) - \rho$$

$$= -\rho + \beta(-\tfrac{1}{2}\sigma^2 R + b) + \tfrac{1}{2}\sigma^2\beta^2 - \tfrac{1}{2}\sigma^2\beta(1-R)$$

$$\Rightarrow \rho - \beta(-\tfrac{1}{2}\sigma^2 R + b) = \tfrac{1}{2}\sigma^2\beta\,(\beta - (1-R)) > 0 \qquad (8)$$

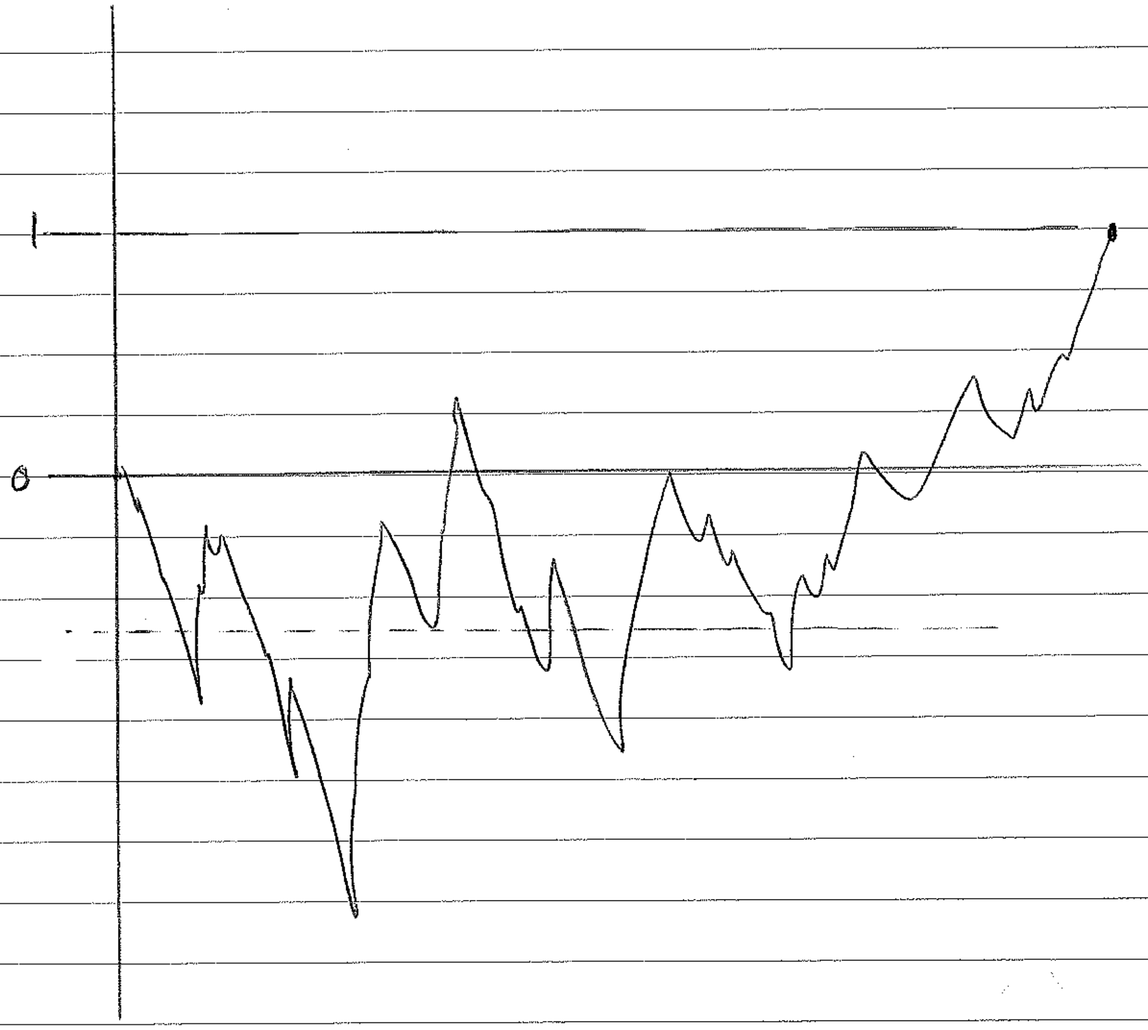
Hence both sides of (7) are positive, and (7) does indeed define the value of z_* via

$$\boxed{z_*^{\alpha} = \frac{pK\,(\alpha+1-R)}{\beta(1-p)\,F(1-R)}\left\{\beta(-\tfrac{1}{2}\sigma^2 R + b) - \rho\right\}} \qquad (9)$$

(Would we always have $z_* > 1$? Using (8) and the fact that $pK > 1-p$, we get a lower bound from (9): with $t \equiv 1-R$,

$$z_*^{\alpha} > \frac{(\alpha+t)}{\beta\;F(t)}\cdot\{t-\beta\}\,\frac{\sigma^2\beta}{2} = 1$$

So in fact we are assured that z_* defined by (9) will be > 1.)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Ray-Knight run backwards (5/11/14)

(i) Suppose we start BM at 0 and run until $\tau \equiv \inf\{t : W_t = 1\}$. If we have $Z_x \equiv L_\tau(x)$, then we know that $(Z_{1-y})_{y \geq 0}$ solves the usual RK SDE ... but what does it behave like if we don't reverse the direction?!

Of course, we shall have that the starting level

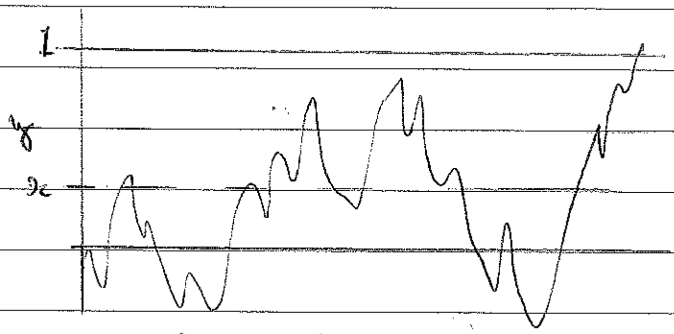
$$\underline{X} \equiv \inf\{ W_t : 0 \leq t \leq \tau \}$$

will be random, $P[\underline{X} \leq a] = 1/(1-a)$ for $a < 0$. But if we are told the value of $\underline{X}$, how does it evolve above that?

(ii) Maybe the first case to study is when $0 < x < y < 1$, we're told $Z_x = \ell$, and want to find the dist^n of Z_y.

Let $\delta \equiv y - x > 0$ and then the rate of excursions up to y which don't reach 1 will be

$$\mu = \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{1-x} \right) = \frac{1-y}{2(1-x)\delta}$$



Given $Z_x = \ell$, there will be a $P(\ell\mu)$ number of these things. Each one contributes an exponential with rate $\beta = \frac{1}{2(1-y)} + \frac{1}{2\delta}$ to local time at y. Finally, there is the last excursion out from x which goes all the way to 1, which contributes the same dist^n. So the density of $L_\tau(y)$ will be

$$v \mapsto \sum_{n \geq 0} \frac{(\mu\ell)^n}{n!} e^{-\mu\ell} \quad (\beta v)^n e^{-\beta v} \; \beta / \Gamma(n+1)$$

$$= \exp\{-\mu\ell - \beta v\} \, \beta \sum_{n \geq 0} \frac{(\mu\ell\beta v)^n}{n! \, \Gamma(n+1)}$$

$$\equiv \exp\{-\mu\ell - \beta v\} \, \beta \, I_0\left(\sqrt{2\mu\ell\beta v}\right).$$

We can also calculate the LT of $L_\tau(y)$:

$$E\left[\exp - \lambda L_\tau(y) \,\middle|\, L_\tau(x) = \ell \right] = \sum_{n \geq 0} \frac{(\mu\ell)^n e^{-\mu\ell}}{n!} \left(\frac{\beta}{\beta + \lambda} \right)^{n+1}$$

$$= \frac{\beta}{\beta + \lambda} \exp\left[-\frac{\mu\ell\lambda}{\beta + \lambda} \right].$$

(iii) For $x<y<0$, we know that there must have been some local time accrued at level y before we reach x, an exponential of rate $\frac{1}{2(1-y)} + \frac{1}{2(y-x)} \equiv \beta$. Accordingly, the LT will be

$$\left(\frac{\beta}{\beta+\lambda}\right)^2 \exp\left[-\frac{\mu l \lambda}{\beta+\lambda}\right]$$

(iv) For $x<0<y<1$, we find that with probability $p = -x/(y-x)$ we reached y before we got to x, and in that case we collected some local time. The LT this time looks like

$$\frac{\beta}{\beta+\lambda}\left(1-\frac{\lambda p}{\beta+\lambda}\right)\exp\left[-\frac{\mu l \lambda}{\beta+\lambda}\right].$$

(v) Let's consider what happens when $x<0$, and we fix $y=0$. We have

$$\beta \equiv \beta(x) = \frac{1}{2}\left(1-\tfrac{1}{x}\right)$$

$$\mu = \mu(x) = {}^{-1}/_{2(1-x)x}$$

so we will expect to find that if we define

$$\varphi(x,l) \equiv \left(\frac{1-x}{1-x-2\lambda x}\right)^2 \exp\left[-\frac{\lambda l}{(1-x)(1-x-2\lambda x)}\right]$$

then $\varphi(x,Z_x)$ should be a martingale. If we think that the generator will be

$$\mathcal{G} = 2z\frac{\partial^2}{\partial z^2} + b(x,z)\frac{\partial}{\partial z} + \frac{\partial}{\partial x}$$

then we can do some calculations (Maple) and learn that

$$\boxed{b(x,z) = 4 - \frac{2z}{1-x}} \qquad (x\le 0)$$

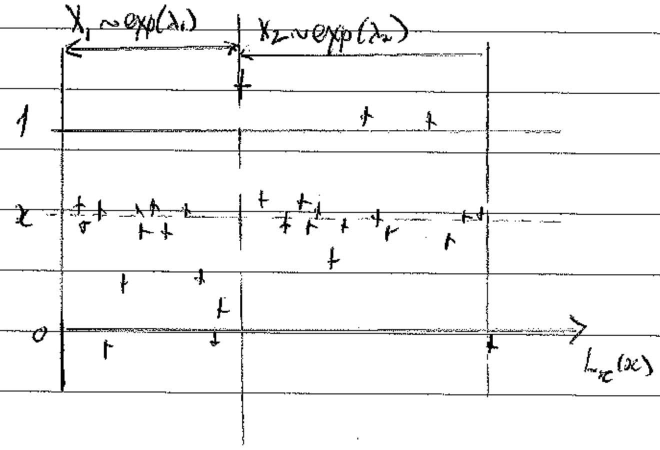
(vi) If we now fix some $y\in(0,1)$ and define for $0<x<y$

$$\psi(x,z) = \frac{\beta}{\beta+\lambda}\exp\left(-\frac{\mu\lambda z}{\beta+\lambda}\right)$$

and likewise argue that $\psi(x,Z_x)$ should be a martingale, then we learn that

The drift for $x \in (0,1)$ will be

$$\boxed{b(x,z) = 2 - \frac{2z}{1-x}}$$



Let $X_1 \sim \exp(\lambda_1)$ be local time recorded up to the first crossing of 1, $X_2 \sim \exp(\lambda_2)$ the rest of the local time recorded, where
$\lambda_1 = 1/2(1-x)$, $\lambda_2 = 1/2x$

Now given $X_1 + X_2 = \ell$ we shall have the law of X_1 has density

$$(\lambda_2-\lambda_1)\frac{\exp(\lambda_2-\lambda_1)v}{\exp((\lambda_2-\lambda_1)\ell)-1} \qquad (\mu \equiv \lambda_2-\lambda_1).$$

So the mean of X_1 given $X_1+X_2=\ell$ will be
$$\frac{\mu\ell e^{\mu\ell} - e^{\mu\ell} + 1}{\mu(e^{\mu\ell}-1)}.$$

We can similarly work out

$$E\left[e^{-yL_\infty(1)} \,\middle|\, L_\infty(x)=\ell\right] = \int_0^\ell \frac{\mu e^{\mu v}}{(e^{\mu\ell}-1)} \sum_{n\geq 0} \frac{(\lambda_1(\ell-v))^n}{n!} e^{-\lambda_1(\ell-v)} \left(\frac{\lambda_1}{\lambda_1+y}\right)^{n+1} dv$$

$$= \frac{\lambda_1}{\lambda_1+y}\frac{\mu}{e^{\mu\ell}-1}\int_0^\ell e^{\mu v}\exp\left(-\lambda_1(\ell-v) + \frac{\lambda_1^2(\ell-v)}{\lambda_1+y}\right) dv$$

$$= \frac{\lambda_1}{\lambda_1+y}\frac{\mu}{e^{\mu\ell}-1}\int_0^\ell \exp\left\{\mu v - \frac{\lambda_1 y(\ell-v)}{\lambda_1+y}\right\} dv$$

$$= \frac{\lambda_1}{\lambda_1+y}\frac{\mu}{e^{\mu\ell}-1} e^{-\lambda_1 y\ell/(\lambda_1+y)} \frac{e^{\nu\ell}-1}{\nu} \qquad \left[\nu = \mu + \frac{\lambda_1 y}{\lambda_1+y}\right]$$

High Ropes story (7/11/14)

(i) In our original account, we had an objective

$$E\left[\int_0^\infty e^{-\rho t} G(c_t, L_t)\, dt\right]$$

where L had to be non-decreasing. $G(\cdot, L)$ was concave increasing, and $G(c,\cdot)$ was unimodal, but had no particular convexity properties. Seems like the key to making the approach work is the scaling assumption

$$\boxed{G(c,L) = L^{1-R}\, g(c/L)}$$

where g is concave increasing. We have

$$\begin{aligned}\tilde{G}(y,L) &\equiv \sup\{G(c,L) - cy\} \\ &= L^{1-R} \sup\{g(z) - yL^R z\} \\ &= L^{1-R}\, \tilde{g}(yL^R).\end{aligned}$$

The first stage of the optimization was to think of L_t as given, then optimize over consumption streams satisfying the budget constraint. This leads to the problem

$$V(z,L) \equiv \sup_{L_t \uparrow} E\left[\int_0^\infty e^{-\rho t}\, \tilde{G}(\Lambda_t, L_t)\, dt \,\Big|\, \Lambda_0 = z,\ L_0 = L\right]$$

where $d\Lambda = \Lambda\{(\rho - r)dt - \kappa\, dW\}$. We have a scaling relation for $\tilde{G}$:

$$\boxed{\tilde{G}(\lambda^{-R} y, \lambda L) = (\lambda L)^{1-R}\, \tilde{g}(yL^R) = \lambda^{1-R}\, \tilde{G}(y,L)}$$

so that $\tilde{G}(\Lambda_t, L_t) = L_0^{1-R}\, \tilde{G}(L_0^R \Lambda_t,\ L_t/L_0)$, and hence

$$\boxed{V(z,L) = L^{1-R}\, V(zL^R, 1) \equiv L^{1-R}\, v(zL^R)}$$

The optimization problem we encounter here is dealt with using MPOC as usual:

$$e^{-\rho t} V(\Lambda_t, L_t) + \int_0^t e^{-\rho s}\, \tilde{G}(\Lambda_s, L_s)\, ds$$ is a supermartingale and a martingale under optimal control.

Working through the Itô calculus, we shall discover the equations

The condition $V_L \leq 0$ is easily shown to be

$$(1-R)\, v(x) + R x v'(x) \leq 0$$

or again:

$x^{\frac{1}{R}-1} v(x)$ nonincreasing

If $Q = \frac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho$

we have auxiliary quadratic now is $Q(t+b) = Q(t) + b\kappa^2 t + b(\rho - r) + \frac{1}{2}\kappa^2 b(b-1)$

with $b = 1-R'$

$$\max\left[V_L,\ \tfrac{1}{2}\kappa^2 x^2 v''(x) + (\rho - r)x v'(x) - \rho v(x) + \tilde{g}(x) \right] = 0 \qquad (1)$$

Notice this says $V(z,\cdot)$ is nonincreasing, so this also means that

$$L \mapsto (zL^R)^{R'-1}\, v(zL^R) \text{ is nonincreasing}$$

where $R' \equiv 1/R$; Hence if we define

$$\boxed{h(x) \equiv x^{R'-1} v(x)} \qquad (2)$$

then we conclude that h must be non increasing. The second order ODE for optimality can be expressed

$$\boxed{\begin{aligned} &\tfrac{1}{2}\kappa^2 x^2 h''(x) + (\rho - r + \kappa^2(1-R'))x h'(x) \\ &\quad + \left[\tfrac{1}{2}\kappa^2(1-R')(-R') + (\rho - r)(1-R') - \rho\right] h(x) + x^{R'-1}\tilde{g}(x) = 0 \end{aligned}}$$

Notice that the coefficient of $h(x)$ is $-\gamma_M \equiv -R^{-1}\left[\rho + (R-1)(r + \kappa^2/2R)\right]$, and in order that Merton problem be well posed we must have that this is negative, so the homogeneous equation has two roots $-\alpha' < 0 < \beta'$

For this to make sense, we want the resolvent applied to $\tilde{g}$ to be well defined. So if $-\alpha < 0 < \beta$ are roots of $Q(t)$, we have $\beta > 1$, and $Q(1-R') = -\gamma_M < 0$, so $-\alpha < 1-R' < \beta$

Finiteness of the resolvent will require

$$\boxed{\int^{\infty} |\tilde{g}(y)|\, y^{-\beta-1}\, dy < \infty, \quad \int_{0+} |\tilde{g}(y)|\, y^{\alpha-1}\, dy < \infty.}$$

(ii) Phil has proposed the following example:

$$\boxed{G(c,L) = U(L) + \frac{U'(L)}{U_1'(L)}\left\{ U_1(c) - U_1(L) \right\}}, \quad U_1(x) = \frac{x^{1-R_1}}{1-R_1}$$

where $R_1 > R > 0$. For this example,

$$\boxed{g(x) = \frac{R - R_1}{(1-R)(1-R_1)} + \frac{x^{1-R_1}}{1-R_1}} \qquad (3)$$

and easy calculations lead to the fact

$$\frac{\partial G}{\partial L} = \frac{R_1 - R}{1-R_1} L^{-R}\left[\left(\frac{c}{L}\right)^{1-R_1} - 1\right]$$

which shows that $\begin{cases} \frac{\partial G}{\partial L} > 0 & \text{if } L < c \\ \frac{\partial G}{\partial L} < 0 & \text{if } L > c \end{cases}$

So for fixed c, the maximization over L happens at $L = c$, and the maximized value of $G(c,L)$ is just $U(c)$.

For this example,

$$\tilde{g}(y) = \frac{R-R_1}{(1-R)(1-R_1)} + \tilde{U}_1(y) = \frac{R-R_1}{(1-R)(1-R_1)} - \frac{y^{1-1/R_1}}{1-1/R_1} \qquad (4)$$

We need to calculate the resolvent R_ρ applied to $\tilde{g}$ which turns out to be

$$\frac{R-R_1}{\rho(1-R_1)(1-R)} - \frac{\tilde{U}_1(y)}{Q(1-1/R_1)} \equiv v_0(y), \text{ say}, \qquad (5)$$

provided $Q(1-1/R_1) < 0$. Notice that since $R_1 > R$, we have $1-1/R < 1-1/R_1 < 1 < \beta$ so if the original Merton problem is well posed with CRRA R then this condition will certainly be satisfied.

The value function we seek will have the form

$$v(x) = V_0(x) + Ax^{-\alpha} + Bx^{\beta} \qquad (6)$$

for some A, B, where $-\alpha < 0 < \beta$ are the roots of $Q(t) \equiv \frac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho$. We also know that v should be convex decreasing; since $\beta > 1$, the dominant term for large x is Bx^β, so decreasing convex forces $B = 0$. Using this

$$\frac{d}{dx}\left(x^{1/R-1} v(x)\right) = x^{1/R-2}\left[\frac{R-R_1}{\rho R(1-R_1)} + \frac{(1-1/R_1)^{-1}}{Q(1-1/R_1)}\frac{R_1-R}{RR_1} x^{1-1/R_1} + A\underbrace{\left(\tfrac{1}{R}-\alpha-1\right)}_{<0} x^{-\alpha}\right] \qquad (7)$$

Two cases to consider here:

<u>$R_1 \in (0,1)$</u> Here, the constant is negative, the middle term is positive convex decreasing to 0, and the third term tends to zero, and is dominant for small x.

$R_1 > 1$ This time, the constant is positive, the middle term is negative, convex, decreasing to $-\infty$, and dominant at ∞.

Usual arguments tell us that v (equivalently, h) must be C^2 at x_*. Using this, we can do some algebra (Maple) to deduce that

$$\boxed{\begin{aligned} A &= \left(\frac{1}{R}-\frac{1}{R_1}\right)\left[(\alpha+1-\tfrac{1}{R})(\alpha+1-\tfrac{1}{R_1})\rho\left(\frac{-\alpha Q(1-1/R_1)}{\rho(\alpha+1-1/R_1)}\right)^{\alpha R_1/(1-R_1)}\right]^{-1}, \qquad (8)\\ x_* &= \left(\frac{-\alpha Q(1-1/R_1)}{\rho(\alpha+1-1/R_1)}\right)^{-R_1/(1-R_1)}. \end{aligned}}$$

(iii) Equilibrium pricing. Suppose we have a representative agent model where the dividend process (= consumption process) satisfies

$$dx = x(\sigma\, dW + \mu\, dt)$$

To work out prices, we first need to solve the problem

$$F(x,L) = \sup_{L\uparrow} E\left[\int_0^\infty e^{-\rho t} G(x_t, L_t)\, dt \;\middle|\; x_0 = x,\ L_0 = L\right]$$

By scaling, we shall have

$$F(x,L) = L^{1-R} f(x/L)$$

and the thing to look at will be

$$e^{-\rho t} F(x_t, L_t) + \int_0^t e^{-\rho s} G(x_s, L_s)\, ds \qquad \text{is a supermartingale etc}$$

So the conditions are $(z \equiv x/L)$

$$\boxed{\max\left[F_L,\ -\rho f + \mu z f'(z) + \tfrac{1}{2}\sigma^2 z^2 f''(z) + g(z)\right] = 0}$$

This is very similar to the equations for V at the top of p 41 !! The quadratic

$$q(t) \equiv \tfrac{1}{2}\sigma^2 t(t-1) + \mu t - \rho$$

has two roots $-a < 0 < b$, and provided $q(1-R_1) < 0$, that is, $-a < 1-R_1 < b$, we shall have a well-posed problem. What we expect is that there is some critical value z_* such that $z_t = x_t/L_t$ never exceeds z_*: once z gets up to z_* we will raise L in a local time fashion to hold $z \le z_*$. The second-order ODE to solve has general solution

$$f(z) = \frac{R-R_1}{\rho(1-R)(1-R_1)} - \frac{U_1(z)}{q(1-R_1)} + Az^{-a} + Bz^{b}$$

The other thing to notice is that for tiny values of z, you will in effect not raise L for a very long time, so for small z we would expect

$$f(z) \sim \text{const} + \text{const}\; U_1(z)$$

The fact that $-a < 1-R_1$ means that we have to have $\boxed{A=0.}$ The derivative condition in the HJB equation means

$$z^{R-1} f(z) \text{ is non-decreasing}$$

so

$$z^{R-1} f(z) = \frac{R-R_1}{\rho(1-R)(1-R_1)} z^{R-1} - \frac{z^{R-R_1}}{(1-R_1)q(1-R_1)} + Bz^{b+R-1} \equiv \varphi(z)$$

is nondecreasing. At some point z_* we shall have $\varphi'(z_*) = \varphi''(z_*) = 0$, which gives two conditions to determine B and z_*. If we write

$$k_1 = \frac{R-R_1}{\rho(1-R)(1-R_1)}, \qquad k_2 = \frac{-1}{(1-R_1)q(1-R_1)}$$

then we find

$$\boxed{\begin{aligned} z_* &= \left(\frac{(R-1)k_1 b}{(b+R_1-1)(R_1-R)k_2}\right)^{1/1-R_1} = \left(\frac{-bq(1-R_1)}{(b+R_1-1)\rho}\right)^{1/1-R_1} \qquad (9)\\ B &= \frac{(R-R_1)\, z_*^{1-R_1-b}}{b(b+R_1-1)q(1-R_1)} \end{aligned}}$$

How would the SPD be? The conventional tale tells us it's

$$\boxed{\zeta_t = G'(x_t, L_t)e^{-\rho t} = e^{-\rho t} L_t^{-R} g'(x_t/L_t) - e^{-\rho t} x_t^{-R_1} L_t^{R_1-R}} \qquad (10)$$

So the bits that don't involve L here contribute

$$\exp\{-\rho t - R_1\sigma W_t - R_1(\mu - \tfrac12\sigma^2)t\}$$

$$= \exp\{-\sigma R_1 W_t - \tfrac12\sigma^2 R_1^2 t\}\exp\{-\rho t + \tfrac12\sigma^2R_1^2 t + (\tfrac12\sigma^2-\mu)R_1 t\}$$

which gives us a risk premium of σR_1, and a riskless rate of

$$\rho + \mu R_1 + \tfrac12\sigma^2 R_1(1+R_1).$$

(19/11/14) (iv) Verification. We constructed (pp 41–43) a function v defined separately in $(0,x_*]$ and $[x_*,\infty)$ which satisfies $\mathcal{L}v+\tilde g = 0$ in $[x_*,\infty)$, $(1/R - 1)v + xv' = 0$ in $(0,x_*]$ and which is C^2 at x_*. We will also need to know that the variational characterization at the top of p 41 holds, that is,

(a) $\mathcal{L}v + \tilde g \le 0$ in $(0,x_*]$

(b) $h(x) \equiv x^{1/R-1}v(x)$ is non-increasing in $[x_*,\infty)$

(b)

Using the expression (7) for h' in $[x_*,\infty)$ we multiply by $x^{1/R_1 - 1/R + 1}$ to find that

$$x^{1-\nu}h'(x) = \frac{R-R_1}{R(1-R_1)}\left[\frac{1}{\rho}x^{1/R_1-1} - \left(\frac{-1}{Q(1-1/R_1)}\right)\right] + A\left(\frac1R - 1-\alpha\right)x^{1/R_1-1-\alpha} \quad (11)$$

where $\nu = 1/R - 1/R_1 > 0$. We know this will be zero at $x = x_*$ by construction.

Case 1: $0<R_1<1$. Solving (11) for $h'(x) = 0$ will be the same as solving

$$\underbrace{\frac{R_1-R}{\rho R(1-R_1)}x^{1/R_1-1}}_{\text{concave, pos}} - \underbrace{A\left(\frac1R-1-\alpha\right)x^{1/R_1-1-\alpha}}_{\text{concave, neg}} = \frac{R_1-R}{R(1-R_1)}\left(\frac{-1}{Q(1-1/R_1)}\right) \quad (12)$$

Consider the slope of the LHS, $x^{1/R_1-2}(a_1 + a_2x^{-\alpha})$. This can only change sign once in $\mathbb{R}^+$, and it does so when

$$\frac{R_1-R}{\rho R R_1} = \left(\frac1R - 1-\alpha\right)\left(\frac1{R_1}-1-\alpha\right)Ax^{-\alpha}$$

which (see (8)) happens exactly at $x = x_*$. The LHS of (12) is dominated at infinity by the first (positive) term, so the LHS of (12) must be increasing in $[x_*,\infty)$, hence (from (11)) $x^{1-\gamma} h'(x)$ is negative throughout (x_*, ∞) as required.

<u>Case 2:</u> <u>$R_1 > 1$</u> Re-express (12) as

$$\underbrace{\frac{R_1 - R}{\rho R R_1} \; \frac{x^{1/R_1 - 1}}{1/R_1 - 1}}_{\text{concave, neg}} - \underbrace{A\left(\frac{1}{R} - 1 - \alpha\right) x^{\frac{1}{R} - 1 - \alpha}}_{\text{concave, neg}} = -\frac{R_1 - R}{R(R_1 - 1)} \left(\frac{-1}{Q(1 - 1/R_1)} \right)$$

As before, the slope of the left-hand side is zero just once, at x_*, and the first term is again dominant as $x \to \infty$, so the LHS will be increasing throughout (x_*, ∞), so will be greater than the RHS throughout (x_*, ∞). Going back to (11), we learn that $h'(x) < 0$ in (x_*, ∞).

(a) Now we'll look at what happens in $(0, x_*]$. For brevity, let's write

$$B_1 = -1/Q(1 - 1/R_1), \qquad B = -1/Q(1 - 1/R)$$

both positive. We have in $[x_*, \infty)$

$$v(x) = \frac{R - R_1}{\rho(1-R)(1-R_1)} + B_1 \tilde{u}_1(x) + A x^{-\alpha}$$

so if we consider

$$v'(x_*) = -B_1 x_*^{-1/R_1} - \alpha A x_*^{-\alpha - 1}$$

$$= x_*^{-1} \left[-B_1 \frac{\alpha}{\rho(\alpha + 1 - 1/R_1)} \cdot \frac{1}{B_1} - \alpha \frac{R_1 - R}{\rho R R_1 (\alpha + 1 - 1/R)(\alpha + 1 - 1/R_1)} \right]$$

$$= \frac{\alpha x_*^{-1}}{\rho R R_1 (\alpha + 1 - 1/R)(\alpha + 1 - 1/R_1)} \left[-R R_1 (\alpha + 1 - 1/R) - R_1 + R \right]$$

$$= -\frac{\alpha x_*^{-1} \; R R_1}{\rho R R_1 (\alpha + 1 - 1/R)} < 0.$$

Thus we conclude that $v''(x_*) = -v'(x_*)/x_* R > 0$, and

$$\boxed{v(x_*) = x_* v'(x_*)/(1 - 1/R) \quad \begin{array}{l} \text{is } \underline{\text{positive}} \text{ if } 0 < R < 1 \\ \text{is } \underline{\text{negative}} \text{ if } R > 1 \end{array}}$$

Now let's consider $\mathcal{L}v + \tilde{g}$ in $(0, x_*]$. We have

$$\varphi(x) \equiv \mathcal{L}v + \tilde{g} = \quad Q(1-1/R)\,v + \frac{R-R_1}{(1-R)(1-R_1)} + \tilde{u}_1(x)$$

$$= -\frac{1}{B}\left(\frac{x}{x_*}\right)^{1-1/R} v(x_*) + \frac{R-R_1}{(1-R)(1-R_1)} - \frac{x^{1-1/R_1}}{1-1/R_1}$$

. The dominant term near zero is the first, and this tends to $-\infty$ if $R_1 \in (0,1)$

Notice $\varphi'(x) = -\frac{1}{B}\, v'(x_*)\left(\frac{x}{x_*}\right)^{-1/R} - x^{-1/R_1}$

and just as before, there can only be one place where $\varphi'(x) = 0$. We see that

$$x_*\varphi'(x_*) = \left[\frac{1}{B}\,\frac{\alpha}{\rho(\alpha+1-1/R)} - \frac{\alpha}{B_1\,\rho(\alpha+1-1/R_1)}\right]$$

But we shall have that $-B^{-1} = \frac{1}{2}K^2(1-1/R-\beta)(1-1/R+\alpha)$, so

$$x_*\varphi'(x_*) = \frac{K^2}{2}\left[-\frac{\alpha}{\rho}(1-1/R-\beta) + \frac{\alpha}{\rho}(1-1/R_1-\beta)\right]$$

$$= \frac{K^2}{2}\,\frac{\alpha}{\rho}\left(-\frac{1}{R_1}+\frac{1}{R}\right) > 0.$$

Thus φ is increasing at x_*-, and is zero at x_*. If $R_1 \in (0,1)$, then the limit of φ at zero will be $-\infty$. If $R_1 > 1$, the final term in the expression for φ tends to zero at zero, and the first tends to zero if $R>1$, or to $-\infty$ if $R<1$. So if $R_1 > R > 1$, we have

$$\varphi(x) \to -\frac{R_1-R}{(R_1-1)(R-1)} < 0 \qquad (x \downarrow 0)$$

otherwise $\varphi(x) \to -\infty$ $(x\downarrow 0)$. Since φ is increasing at x_*-, we see therefore that is impossible that φ could be above $0=\varphi(x_*)$ anywhere in $(0, x_*)$, because otherwise there would have to be at least two zeros of φ', which we know cannot happen.

Prices + beliefs in a simple economy (19/11/14)

The development of this story in continuous time leads us to a dynamic which is in large part linear, but gets a quadratic twist at some point. It's not very transparent in the original discrete-time setting, so let's see whether we can get things done in cts time.

(i) How does the Kalman filter look? (quote this from WN $\overline{XXXIV}$ p 16). The state evolution is

$$dX = AX dt + dM$$
$$dY = CX dt + dN$$

and if $d\nu = dY - C\hat{X} dt$ is the innovations martingale, we have in the steady state KF that

$$d\hat{X} = A\hat{X} dt + K d\nu$$

where

$$K = (V^T C + \Sigma_{XY}) \Sigma_{YY}^{-1}$$

and V solves

$$0 = VA^T + AV - (VC^T + \Sigma_{XY}) \Sigma_{YY}^{-1} (CV + \Sigma_{YX}) + \Sigma_{XX}$$

(ii) Our story with just one stock has $Y_t = [S_t ; \delta_t]$ and

$$dY_t = C\hat{X}_t dt + d\nu_t$$

so we might stack everything into the single vector $Z_t \equiv [\hat{X}_t ; Y_t]$ which now evolves as

$$dZ = \sigma dW + MZ dt$$

with $S = k_1 Z$, $\delta = k_2 Z$ for simplicity. We shall have

$$\Sigma \equiv \sigma\sigma^T = \begin{pmatrix} K \\ I \end{pmatrix} \Sigma_{YY} \begin{pmatrix} K \\ I \end{pmatrix}^T, \qquad M = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$$

in terms of earlier notation. Now suppose an investor gets to invest in the stock and a bank account with fixed riskless rate r. Then wealth evolves like

$$dw = (rw - c) dt + \theta (dS + \delta dt)$$

$$= (rw - c) dt + \theta (k_1 \cdot dZ + k_2 \cdot Z dt).$$

We'll suppose that the agent wants to achieve

$$V(w, z) \equiv \sup E\left[\int_0^\infty e^{-\rho t} U(c_t) dt \;\middle|\; w_0 = w, Z_0 = z\right]$$

where $U(x) = -\exp(-\gamma x)$. We shall guess (and then verify) the form of the function V to be

$$V(w,z) = -\exp\left[-\gamma r w - \tfrac{1}{2} z \cdot q z - b \cdot z - \lambda\right]$$

where q, b, λ are to be determined. The HJB equation for the problem will be

$$0 = \sup\Big[U(c) - \rho V + (rw - c + \theta k_2 \cdot z) V_w + V_{z_i} (Mz)_i$$
$$+ \tfrac{1}{2}\theta^2 k_1 \Sigma k_1 V_{ww} + V_{wz_i} \theta (\Sigma k_1)_i + \tfrac{1}{2} V_{z_i z_j} \Sigma_{ij}\Big]$$

Notice $V_w = -\gamma r V$

$$V_z = -(qz + b) V, \qquad V_{zw} = \gamma r (qz + b) V$$

$$V_{zz} = \{(qz+b)(qz+b)^T - q\} V$$

so substituting back into the HJB equation gives

$$0 = \sup\Big\{U(c) + V\Big\{-\rho - \gamma r(rw - c + \theta k_2 \cdot z) - (qz+b)\cdot Mz$$
$$+ \tfrac{1}{2}(\gamma r \theta)^2 k_1 \Sigma k_1 + \gamma r \theta (qz+b)\cdot(\Sigma k_1)$$
$$+ \tfrac{1}{2}(qz+b)\cdot \Sigma (qz+b) - \tfrac{1}{2} \mathrm{tr}(q\Sigma)\Big\}$$

Maximizing over c leads to

$$\boxed{\gamma c = -\log r + \gamma r w + (\tfrac{1}{2} z \cdot q z + b z + \lambda)}$$

and hence $\gamma(rw - c) = \log r - (\tfrac{1}{2} z \cdot q z + b z + \lambda)$, and

$$0 = \sup V\Big[r - \rho - r(\log r - \tfrac{1}{2} z \cdot q z - b \cdot z - \lambda) - \gamma r \theta k_2 \cdot z$$
$$- (qz+b)\cdot Mz + \tfrac{1}{2}(\gamma r\theta)^2 k_1 \Sigma k_1 + \gamma r \theta (qz+b)\cdot(\Sigma k_1)$$
$$+ \tfrac{1}{2}(qz+b)\Sigma(qz+b) - \tfrac{1}{2}\mathrm{tr}(q\Sigma)\Big]$$

Optimal θ therefore satisfies

$$\boxed{\gamma r \theta \; k_1 \Sigma k_1 = k_2 \cdot z - (qz+b)\Sigma k_1}$$

and HJB is

$$\boxed{\begin{aligned} 0 = r - \rho - r(\log r - \tfrac{1}{2} z \cdot q z - b \cdot z - \lambda) - \tfrac{1}{2}\frac{(k_2 \cdot z - (qz+b)\Sigma k_1)^2}{k_1 \Sigma k_1} \\ - (qz+b)\cdot Mz + \tfrac{1}{2}(qz+b)\Sigma(qz+b) - \tfrac{1}{2}\mathrm{tr}(q\Sigma) \end{aligned}}$$

Weakly continuous preferences (21/11/14)

Talking to Phil D, there's a paper of Hindy + Huang (Econometrica 60, 781–801, 1992) where they consider the problem of preferences on consumption processes which are expressed in terms of cumulative consumption (C_t), an adapted $\mathbb{R}$-process, increasing from value $C_{0-}=0$. Whatever the preferences, you would like that if pathwise we see $C^{(n)} \Rightarrow C$ then we'd want $U(C^{(n)}) \to U(C)$, which wouldn't happen with standard von Neumann–Morgenstern preferences. Let's look at the deterministic setting first.

(i) Suppose we want to define some concave increasing map U on the set of measures m on $[0,1]$ which is going to be weakly continuous. Any such U would be an infimum of linear functionals, we guess, so we'd expect

$$U(m) = \inf_\theta \left\{ \alpha(\theta) + \int_0^1 \varphi(s;\theta)\, m(ds) \right\}$$

where $\theta \in \Theta$ is some parameter, and $\varphi(\cdot\,;\theta)$ is bounded continuous non-negative for all θ. We'd also think that our choice of m would be constrained by overall wealth

$$\int_0^1 m(ds) = w$$

(might want a state-price density, but this could be absorbed into m by redefining φ – not exactly, but let's see where it goes).

The problem we want to tackle will be

$$\max U(m) \qquad \text{s.t.} \quad \int_0^1 dm = w$$

or equally $\max v$ s.t. $v \le \alpha(\theta) + \int_0^1 \varphi(s;\theta)\, m(ds)$,

$$\int_0^1 dm = w$$

So if we express in Lagrangian terms,

$$\max \left[v + \int \lambda(d\theta) \left\{ \int_0^1 \varphi(s;\theta)\, m(ds) + \alpha(\theta) - v \right\} + y\left(w - \int_0^1 m(ds)\right) \right\}$$

so for dual feasibility, $\lambda \ge 0$, $\int \lambda(d\theta) = 1$, and we get

$$\max \left[yw + \int \alpha(\theta)\,\lambda(d\theta) + \int_0^1 \left\{ \int \varphi(s;\theta)\,\lambda(d\theta) - y \right\} m(ds) \right]$$

which reveals the further dual feasibility condition

$$y \geq \int \varphi(s;\theta)\lambda(d\theta) \qquad \forall s$$

and so the dual problem will be

$$\boxed{\min \left\{ yw + \int \alpha(\theta)\lambda(d\theta) \right\} \quad \text{subj to } \lambda \geq 0, \int \lambda(d\theta) = 1 \text{ and } y \geq \int \varphi(s;\theta)\lambda(d\theta) \ \forall s}$$

We can equally express this as

$$\boxed{\min_{\lambda \geq 0, \int d\lambda = 1} \ \sup_s \int \{\alpha(\theta) + w\varphi(s,\theta)\} \lambda(d\theta)} \qquad (*)$$

(ii) How would it look for a random world? We might propose

$$U(C) = E\left[\inf_\theta \left\{ \alpha(\theta) + \int_0^1 \varphi(s,\theta)\, dC_s \right\} \right]$$

subject to C increasing adapted, $E\left[\int_0^1 \zeta_s\, dC_s\right] = w$. By absorbing the optionality of C, expressed and characterized by

$$E\left[\int_0^1 Z_s\, dC_s\right] = E\left[\int_0^1 {}^oZ_s\, dC_s\right] \qquad \forall \text{ bdd processes } Z$$

this says $0 = E\left[Z_1 C_1 - \int_0^1 E_s(Z_1)\, dC_s\right] = E\left[\int_0^1 (M_1 - M_s)\, dC_s\right]$

where $M_s \equiv E_s Z_1$ is a bounded martingale, so we could take the Lagrangian form

$$\max E\left[\inf_\theta \left\{ \alpha(\theta) + \int_0^1 \varphi(s;\theta)\, dC_s \right\} + \int_0^1 (M_1 - M_s)\, dC_s + y\left(w - \int_0^1 \zeta_s\, dC_s\right) \right]$$

and now try to solve pathwise, as in (i).

(iii) We can alternatively express the problem (*) as

$$\min_\lambda \ \max_\pi \int \pi(ds) \int \lambda(d\theta) \{\alpha(\theta) + w\varphi(s,\theta)\}$$

where λ, π are restricted to probability measures — so it's a two person zero sum game. But that does not make it particularly simple to solve

Partially-observed drifting BM (27/11/14)

Suppose there is some process

$$S_t = S_0 + \mu t + \sigma W_t$$

where σ is known, $\sigma\sigma^T \equiv a$, but μ is not, having some Gaussian prior. At times $t_0 < t_1 < t_2 < \dots$, we observe

$$Y_i = C_i S_{t_i}$$

where C_i is a matrix which selects out some of the components of S, and the matrices C_i are known, can vary from one observation to another. Write

$$Z_t \equiv \begin{pmatrix} \mu \\ S_t \end{pmatrix}$$

for the state variable, and $\varphi_n(y_1, \dots, y_n; z_{n+1})$ for the joint density of $(Y_1, Y_2, \dots, Z_{n+1})$, where $Z_i = Z(t_i)$ for short. The claim is that φ_n has the form

$$(*) \quad \varphi_n(y_1, \dots, y_n; z_{n+1}) = \lambda_n(y_1, \dots, y_n) \exp\left[-\tfrac{1}{2}(z-\hat{z}) V^{-1} (z-\hat{z}) \right] (2\pi)^{-d/2} (\det V)^{-1/2}$$

for some $\hat{z}$ which depends on the observations, and V which depends on the sequence of C_i, but not on observed values.

If we now partition

$$Z_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$$

into unobserved and observed values, the exponential in $(*)$ can be expressed as

$$\exp\left[-\tfrac{1}{2}(y-\hat{y}) V_{YY}^{-1} (y-\hat{y}) - \tfrac{1}{2}(x-\hat{x}-K(y-\hat{y}))(V^{-1})_{XX}(x-\hat{x}-K(y-\hat{y})) \right]$$

where as usual $K = V_{XY} V_{YY}^{-1}$. So upon observing $Y_{n+1} = y_{n+1}$, we get that

$$\varphi_n(y_1, \dots, y_n; z_n) = \tilde{\lambda}_{n+1}(y_1, \dots, y_{n+1}) \exp\left[-\tfrac{1}{2}(x-\hat{x}-K(y-\hat{y}))(V^{-1})_{XX}(x-\hat{x}-K(y-\hat{y})) \right] (\det V)^{-1/2} (2\pi)^{-d/2}$$

where $\tilde{\lambda}_{n+1}(y_1, \dots, y_{n+1}) = \lambda_n(y_1, \dots, y_n) \exp\left(-\tfrac{1}{2}(y_{n+1}-\hat{y}) V_{YY}^{-1} (y_{n+1}-\hat{y}) \right)$. This gives us

$$\varphi_n = \tilde{\lambda}_{n+1}(y_1, \dots, y_{n+1}) \; \gamma(x-\hat{x}, V_{XX}-V_{XY}V_{YY}^{-1}V_{YX}) \, \frac{\det(V_{XX}-V_{XY}V_{YY}^{-1}V_{YX})^{\frac{1}{2}}}{(\det V)^{\frac{1}{2}}} (2\pi)^{\text{some power}}$$

where $\gamma(x, V)$ is MVN density. Thus if we define

$\tilde{x} = \hat{x} + K(y_{n+1} - \hat{y})$

$$\lambda_{n+1}(y_1, \dots, y_{n+1}) = \tilde{\lambda}_{n+1}(y_1, \dots, y_{n+1}) \left(\frac{\det V_{XX} - V_{XY} V_{YY}^{-1} V_{YX}}{\det V} \right)^{\frac{1}{2}}$$

we shall have that after seeing y_{n+1} we have

$$\varphi_n(y_1, \dots, y_n ; z_n) = \lambda_{n+1}(y_1, \dots, y_{n+1})\, \gamma\left(\begin{pmatrix} x - \tilde{x} \\ y - y_{n+1} \end{pmatrix}, \underbrace{\begin{pmatrix} V_{XX} - V_{XY} V_{YY}^{-1} V_{YX} & 0 \\ 0 & 0 \end{pmatrix}}_{\bar{V}_{n+1}} \right)$$

in a formal sense. If we step forward $h = t_{n+2} - t_{n+1}$

$$\begin{pmatrix} \mu \\ S \end{pmatrix} \mapsto \begin{pmatrix} \mu \\ S + h\mu + \sigma \Delta W_h \end{pmatrix} = \begin{pmatrix} I & 0 \\ h & I \end{pmatrix} \begin{pmatrix} \mu \\ S \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma W_h \end{pmatrix}$$

so the mean of Z_{n+2} will be

$$\begin{pmatrix} I & 0 \\ h & I \end{pmatrix} \begin{pmatrix} \tilde{x} \\ y_{n+1} \end{pmatrix}$$

with covariance

$$\begin{pmatrix} I & 0 \\ h & I \end{pmatrix} \bar{V}_{n+1} \begin{pmatrix} I & h \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & ha \end{pmatrix}$$

This tells us how to step from just before time $n+1$ to just before time $n+2$.

Equilibrium pricing with a derivative (22/12/14)

(i) Let's consider a market with J agents, all CARA agents with $\gamma = 1$, and suppose that the value X of some asset at time 1 is symmetric two-sided exponential, where agent j thinks the parameter of the exponential is α_j.

Let's suppose that the positive and negative parts of X are separately tradeable at prices π_+, π_-. An agent trying to optimize investment will attempt to

$$\min E \exp\{ -\theta_+ (X^+ - \pi_+) - \theta_- (X^- - \pi_-) \}$$

$$= \min e^{\theta_+ \pi_+ + \theta_- \pi_-} \int \frac{\alpha}{2} e^{-\alpha |x| - \theta_+ x^+ - \theta_- x^-} \, dx$$

$$= \min e^{\theta_+ \pi_+ + \theta_- \pi_-} \left\{ \frac{1}{\alpha + \theta_+} + \frac{1}{\alpha + \theta_-} \right\} \cdot \frac{\alpha}{2}$$

Differentiate to find the optimal demands

$$\left. \begin{aligned} 0 &= \frac{\pi_+}{\alpha + \theta_+} + \frac{\pi_+}{\alpha + \theta_-} - \frac{1}{(\alpha + \theta_+)^2} \\ 0 &= \frac{\pi_-}{\alpha + \theta_+} + \frac{\pi_-}{\alpha + \theta_-} - \frac{1}{(\alpha + \theta_-)^2} \end{aligned} \right\} \Rightarrow \frac{\alpha + \theta_+}{\alpha + \theta_-} = \sqrt{\pi_- / \pi_+}$$

Hence

$$0 = \pi_+ + \pi_+ \sqrt{\pi_- / \pi_+} - \frac{1}{\alpha + \theta_+} \Rightarrow \boxed{\begin{aligned} \alpha + \theta_+ &= \frac{1}{\sqrt{\pi_+} (\sqrt{\pi_+} + \sqrt{\pi_-})} \\ \alpha + \theta_- &= \frac{1}{\sqrt{\pi_-} (\sqrt{\pi_+} + \sqrt{\pi_-})} \end{aligned}}$$

Now suppose that the asset itself is available in quantity J, and the derivative is in zero net supply; the market clearing condition is

$$\Sigma \{ \theta_+^j + \theta_-^j \} = 0, \qquad \Sigma \, \theta_+^j = J$$

where

$$\boxed{\begin{aligned} \bar{\alpha} + 1 &= \frac{1}{\sqrt{\pi_+} (\sqrt{\pi_+} + \sqrt{\pi_-})} \\ 2 \bar{\alpha} &= 1 / \sqrt{\pi_+} \sqrt{\pi_-} \end{aligned}}$$

To solve this, we will need the condition

$$\bar{\alpha} > 1$$

(otherwise the agents could not collectively be induced to hold the asset) and then we find

$$\sqrt{\pi_+/\pi_-} = \frac{\bar{\alpha}-1}{\bar{\alpha}+1} < 1$$

and after some algebra

$$\boxed{\pi_- = \frac{\bar{\alpha}+1}{2\bar{\alpha}(\bar{\alpha}-1)}, \quad \pi_+ = \frac{\bar{\alpha}-1}{2\bar{\alpha}(\bar{\alpha}+1)}}$$

The price of the stock would then be

$$\boxed{\pi_+ - \pi_- = -\frac{2}{\bar{\alpha}^2-1}} \qquad (*)$$

(ii) Has the introduction of the derivative into this market changed the price of the stock? If we just could trade the stock at price π, the optimization would be

$$\min_\theta e^{\theta\pi} \int \tfrac{\alpha}{2} e^{-\alpha|x| - \theta x}\,dx = \min_\theta e^{\theta\pi}\left\{\frac{1}{\alpha+\theta} + \frac{1}{\alpha-\theta}\right\}\cdot\frac{\alpha}{2}$$

$$= \min\; e^{\theta\pi}\,\frac{\alpha^2}{\alpha^2-\theta^2}$$

After a few calculations, we get optimal θ would be

$$\boxed{\theta = -\frac{\sqrt{1+\alpha^2\pi^2}-1}{\pi}} \qquad (**)$$

Could the value $\pi = -2/(\bar{\alpha}^2-1)$ found at $(*)$ actually be the market-clearing π here? If we do market clearing on $(**)$, what we see is that

$$\pi = 1 - \frac{1}{J}\sum \sqrt{1+\alpha_j^2\pi^2} > 1 - \sqrt{1+\bar{\alpha}^2\pi^2} \qquad \text{(if } \alpha_j \text{ not all the same)}$$

so if π_0 denotes the market clearing price without the derivative, we have learned that

$$\sqrt{1+\bar{\alpha}^2\pi_0^2} > 1 - \pi_0$$

However, if we take the value from $(*)$

$$\pi = -\frac{2}{\bar{\alpha}^2 - 1}$$

and evaluate the two sides of the inequality, we get on the left

$$\sqrt{1 + \frac{4\bar{\alpha}^2}{(\bar{\alpha}^2-1)^2}} = \frac{\bar{\alpha}^2+1}{\bar{\alpha}^2-1}$$

and on the right

$$1-\pi = 1 + \frac{2}{\bar{\alpha}^2-1} = \frac{\bar{\alpha}^2+1}{\bar{\alpha}^2-1}$$

which is the <u>same</u> value.

So the conclusion is

if the α_j are not all the same, the equilibrium price of the asset in the market with the derivative X^- is <u>different</u> from the equilibrium price of the asset in the market without the derivative X^-.

(This was already done WN $\overline{\text{XXXVI}}$, p48).

Optimal investment with tax on capital gains (30/12/14)

(i) Suppose you have to pay tax at rate τ on your profits, so the wealth equation is

$$dw_t = (rw_t - c_t)\,dt + \theta_t(\sigma\, dW_t + (\mu - r)\,dt) - \tau\, d\bar{w}_t \qquad (\tau \in (0,1))$$

where $\bar{w}_t = \sup_{s \le t} w_s$. If we have objective

$$V(w, \bar{w}) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \;\Big|\; w_0 = w,\ \bar{w}_0 = \bar{w}\right]$$

for CRRA utility U, we should get the scaling relationship $V(\lambda w, \lambda \bar{w}) = \lambda^{1-R} V(w, \bar{w})$ hence

$$V(w, \bar{w}) = \bar{w}^{1-R}\, v(x) \qquad \text{where} \quad x \equiv w/\bar{w} \in [0,1].$$

By the MPOC, we'll have

$$Y_t \equiv e^{-\rho t} V(w_t, \bar{w}_t) + \int_0^t e^{-\rho s} U(c_s)\,ds$$

is a supermartingale etc, so

$$e^{\rho t} dY_t \doteq U(c_t)\,dt - \rho V\,dt + (rw - c + \theta(\mu - r))V_w\,dt + \tfrac{1}{2}\sigma^2\theta^2 V_{ww}\,dt - \tau V_w\, d\bar{w} + V_{\bar{w}}\, d\bar{w}$$

so we shall have

$$0 = \sup\left[U(c) - \rho V + (rw - c + \theta(\mu - r))V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww}\right]$$

and $V_{\bar{w}} - \tau V_w = 0$ when $w = \bar{w}$. So the HJB gets re-expressed as

$$\begin{cases} 0 = \sup\ \bar{w}^{1-R}\left[U(s) - \rho v + (rx - s + y(\mu - r))v' + \tfrac{1}{2}\sigma^2 y^2 v''\right] \\ 0 = (1-R)v(x) - x v'(x) - \tau v'(x) \qquad \text{at } x = 1. \end{cases}$$

Optimizing in the first gives

$$\boxed{\begin{array}{ll} 0 = \tilde{U}(v') - \rho v + r x v' - \tfrac{1}{2}\kappa^2 (v')^2/v'' & (0 \le x \le 1) \\ (1+\tau)v'(1) = (1-R)v(1) & \end{array}}$$

(ii) Let's now convert to dual variable $z = v'(x)$, $J(z) = v(x) - xz$, so

$$0 = \tilde{U}(z) + \tfrac{1}{2}\kappa^2 z^2 J'' + (\rho - r) z J' - \rho J \qquad (z \ge z_*)$$

$$(1+\tau) z_* = (1-R)\left(J(z_*) - z_* J'(z_*)\right),$$

where we know $J'(z_*) = -1$, and J is convex decreasing. If we set

$$Q(t) = \tfrac{1}{2} R^2 t(t-1) + (\rho - r)t - \rho$$

with roots $-\alpha < 0$, $\beta > 1$, we have $Q(1-\tfrac{1}{R}) = -\gamma_M < 0$ if the problem is well posed, and eliminating the infeasible part of the general solution leads to

$$J(z) = -\frac{\tilde{U}(z)}{Q(1-\frac{1}{R})} + A\left(z/z_*\right)^{-\alpha}$$

The condition $J'(z_*) = -1$ will give

$$\boxed{A = \alpha^{-1}\left\{ z_* - z_*^{1-1/R}\big/\gamma_M \right\}}$$

The other boundary condition say

$$(1+r)z_* = (1-R)\left\{ J(z_*) + z_* \right\}$$

so $\quad (R+r)z_* = (1-R)\,J(z_*) = \dfrac{(1-R)\,\tilde{U}(z_*)}{\gamma_M} + \dfrac{1-R}{\alpha}\left\{ z_* - \dfrac{z_*^{1-1/R}}{\gamma_M} \right\}$

After some algebra,

$$\boxed{z_* = \left\{ \frac{\alpha + 1 - \frac{1}{R}}{\gamma_M\left(\alpha + 1 - \frac{1}{R} + \alpha r/R\right)} \right\}^R}$$

Notice that if $r \to 0$, $z_* \to \gamma_M^{-R}$, and consequently $A \to 0$, which is as you would expect.

Preferences affected by previous consumption (31/12/14)

Suppose we have standard wealth dynamics $dw = (rw - c)\,dt + \theta\,(\sigma dW + (\mu - r)dt)$ and the agent has objective

$$V(w, \bar{c}) \equiv \sup E\left[\int_0^\infty e^{-\rho t} g(c_t, \bar{c}_t)\, dt \,\Big|\, w_0 = w, \bar{c}_0 = \bar{c} \right]$$

where

$$g(c, \bar{c}) = U(c)\, (c/\bar{c})^{\nu}$$

with $U'(x) = x^{-R}$, and ν having the same sign as $1-R$. Here, we think of $\bar{c}$ as one of

$$(i)\ \bar{c}_t = \sup_{s \le t} c_s \qquad (ii)\ \bar{c}_t = \int_{-\infty}^t \lambda e^{\lambda(s-t)} c_s\, ds$$

both capturing a notion of how things were in the past. So we get utility from consumption, but if current c is low relative to $\bar{c}$ this leads to dissatisfaction.

The usual scaling story gives us

$$V(w, \bar{c}) = \bar{c}^{1-R}\, v(x), \qquad x \equiv w/\bar{c}$$

so it reduces to a one-variable problem.

First example HJB is

$$0 = \sup\left[g(c, \bar{c}) - \rho V + (rw - c + \theta(\mu - r)) V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} \right]$$

$$= \sup \bar{c}^{1-R} \left[\frac{s^{1-R+\nu}}{1-R} - \rho v + (rx - s + y(\mu - r)) v' + \tfrac{1}{2}\sigma^2 y^2 v'' \right]$$

so $$s_* = \left(\frac{(1-R)v'}{1-R+\nu} \right)^{1/\nu - R}$$ at optimum, and

$$0 = -\frac{1}{\nu} \left(\frac{(1-R)v'}{\nu - R + 1} \right)^{(\nu - R + 1)/(\nu - R)} - \rho v + r x v' - \frac{\kappa^2}{2} (v')^2 / v''$$

We shall also require

$$V_{\bar{c}} = \bar{c}^{-R} \{ (1-R) v - x v'(x) \} \le 0$$

with equality only when we choose to raise $\bar{c}$, that is, when $s_* = 1$. So we expect that there is a threshold x_* such that for $x < x_*$ we do not raise $\bar{c}$, but as x gets up to x_* we will gradually nudge $\bar{c}$ upward. We deduce the boundary conditions

$$\boxed{ v'(x_*) = \frac{\nu - R + 1}{1 - R}, \qquad x_* v'(x_*) = (1-R) v(x_*) }$$

Recasting this in dual form gives

$$0 = -\frac{1}{\gamma}\left(\frac{(1-R)z}{(\gamma+1-R)}\right)^{(\gamma+1-R)/(\gamma-R)} + \mathcal{L}J$$

where $\mathcal{L} = \frac{\kappa^2}{2} z^2 D^2 + (\rho - r) z D - \rho$, so the general solution will be

$$J(z) = \frac{1}{\gamma\, Q((\gamma+1-R)/(\gamma-R))}\left(\frac{(1-R)z}{\gamma+1-R}\right)^{(\gamma+1-R)/(\gamma-R)} + A\left(\frac{z}{z_*}\right)^{-\alpha} + B\left(\frac{z}{z_*}\right)^{\beta}.$$

In this example, it's not immediately obvious that $w \mapsto V(w, \bar{c})$ is concave, so the convex duality game may not work..

If we restrict to $\boxed{R > 1}$ and write $\boxed{\gamma = \lambda(1-R)}$ for some $\lambda > 0$, then

$$\frac{\gamma+1-R}{\gamma-R} = 1 + \frac{1}{\lambda(1-R)-R} = 1 - \frac{1}{R+\lambda(R-1)} > 1 - \frac{1}{R} > 0$$

so things look more credible here. For convex decreasing J, we need $B = 0$, and

$$\boxed{z_* = v'(x_*) = 1 + \lambda}$$

$$x_* v'(x_*) = -z_* J'(z_*) = (1-R)\, v(x_*) = (1-R)\left[J(z_*) - z_* J'(z_*)\right]$$

$$\text{so} \quad \boxed{(R-1) J(z_*) = R\, z_* J'(z_*) = R(1+\lambda) J'(z_*)}$$

This last condition fixes A.

Second example We have $d\bar{c} = \lambda(c - \bar{c})\, dt$, $V(w, \bar{c}) = \bar{c}^{1-R} v(x)$, with $x = w/\bar{c}$ so HJB is

$$0 = \sup\left[U(c)\left(\frac{c}{\bar{c}}\right)^{\nu} - \rho V + (rw - c + \theta(\mu - r)) V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + \lambda(c - \bar{c}) V_{\bar{c}}\right]$$

$$= \sup \bar{c}^{1-R}\left[s^{\nu} U(s) - \rho v + (rx - s + y(\mu - r)) v' + \tfrac{1}{2}\sigma^2 y^2 v'' + \lambda(s-1)\{(1-R)v - xv'\}\right]$$

so

$$0 = rxv' - \frac{\kappa^2}{2}\frac{(v')^2}{v''} + \lambda\{(R-1)v + xv'\} - \rho v$$
$$+ \sup\left[\frac{s^{1+\nu-R}}{1-R} - s\left(v' + \lambda(R-1)v + \lambda x v'\right)\right]$$

Optimal s turns out to be

$$s_* = \left(\frac{1-R}{1-R+\nu} \underbrace{\left(v'(1+\lambda x) + \lambda(R-1)v \right)}_{\equiv\, b \text{ for short}} \right)^{1/\nu - R}$$

and returning that to the HJB leads to

$$\boxed{0 = \left(\frac{(1-R)b}{1-R+\nu} \right)^{(\nu+1-R)/(\nu-R)} \frac{R-\nu}{1-R} + rxv' - \frac{1}{2}\kappa^2 \frac{(v')^2}{v''} + \lambda(R-1)v + \lambda x v' - \rho v}$$

In terms of dual variables,

$$b = z + \lambda\left((R-1)J - RzJ' \right)$$

So the dual form of the HJB is

$$\boxed{0 = \left(\frac{(1-R)(z + \lambda((R-1)J - RzJ'))}{1-R+\nu} \right)^{(\nu+1-R)/(\nu-R)} \frac{R-\nu}{1-R} + \frac{1}{2}\kappa^2 z^2 J'' + (\rho - r - \lambda R) z J' - (\rho - \lambda(R-1)) J}$$

Probably OK numerically, but that will be all. If $\lambda = 0$ it reduces to a studied problem (standard Merton problem)

Deterministic agents. (21/1/15)

(i) Suppose an agent can hold n different types of good, and can supply m different types of labour. Suppose his holdings of the different types of goods evolve as

$$dx_t^i = (I_t^i - \delta_i x_t^i)\, dt$$

where I_t^i is the rate of investing in good i. Suppose wage rate w^j holds for labour of type j, + good i costs p_i, so we would in the simplest story have

$$p \cdot I_t = w \cdot L_t$$

where $L_t = (L_t^1, \dots, L_t^m)$ is the rate of supplying the different types of labour at time t.

(ii) Let's also suppose that the agent (household, really) has preferences given by

$$\int_0^\infty e^{-\rho t} U(x_t, L_t)\, dt$$

where U is concave increasing in x, concave increasing in L. Can we figure out what he will do?

(iii) Suppose that I_t can be negative as well as positive, so in effect value can be costlessly and immediately switched between goods. If

$$V(x) = \sup\left[\int_0^\infty e^{-\rho t} U(x_t, L_t)\, dt \;\middle|\; x_0 = x\right]$$

we would expect the HJB equation

$[\Delta = \mathrm{diag}(\delta_i)]$

$$0 = \sup\left[-\rho V + DV(I - \Delta x) + U(x, L)\right]$$

to hold, where we max over I for which $p \cdot I = w \cdot L$. If we are able to switch costlessly between goods, we will expect

$$V(x) = h(p \cdot x)$$

since $z \equiv p \cdot x$ is the agent's mark-to-market wealth, and now the HJB says more simply

$$0 = \sup\left[-\rho h(z) + h'(z)\{p \cdot I - p\Delta x\} + U(x, L)\right]$$

$$= \sup\left[-\rho h(z) + h'(z)\{w \cdot L - p\Delta x\} + U(x, L)\right].$$

(iv) Any tractable examples? We might try a separable form

$$U(x, L) = -\prod x_i^{-\alpha_i} \varphi(L)$$

where $\varphi > 0$ is convex increasing. In the HJB, we have to

$$\max \; -h'(z)\, p\,\Delta x - \prod x_i^{-\alpha_i} \varphi \qquad \text{s.t. } x \cdot p = z$$

so Lagrangian is

$$\max \left[-h'(z)\, p\,\Delta x - \prod x_i^{-\alpha_i} \varphi + \lambda (z - x \cdot p) \right]$$

so $0 = -\delta_i p_i h'(z) - \frac{\alpha_i}{x_i} U - \lambda p_i$

so $\frac{\alpha_i}{p_i x_i} = \frac{-1}{U} \left\{ \lambda + \delta_i h'(z) \right\}$

Looks like we could only discover λ (and therefore x_i) numerically ...
The Pontryagin-Lagrange approach is not more successful

(v) Another idea might be to get $U(x_t, L_t)$ to grow as fast as possible, but this is stupid. OR we could try to identify the best steady state — but this would be the same whatever ρ, so it's not so attractive.

CARA utility optimization - an honest story? (16/2/15)

(i) We have the problem of

$$V(w) = \max \; E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \;\Big|\; w_0 = w\right]$$

with wealth dynamics $dw = (rw - c)\,dt + \theta\,(\sigma dW + (\mu - r)dt)$ and with CARA utility $U(x) = -\exp(-\gamma x)$. "The solution" is easily seen to be $V(w) = \text{const}\, e^{-\gamma r w}$ but what problem has this actually solved?!

(ii) Suppose we introduce an independent $\exp(\eta)$ time T and try to solve

$$V(w) = \sup \; E\left[\int_0^T e^{-\alpha t} U(c_t)\,dt + e^{-\alpha T} A\, U(r w_T) \;\Big|\; w_0 = w\right]$$

for some constant A yet to be determined. The objective can therefore be written as

$$V(w) = \sup \; E\left[\int_0^\infty \left\{ e^{-(\alpha+\eta)t} U(c_t) + \eta A\, e^{-(\alpha+\eta)t} U(r w_t)\right\} dt \;\Big|\; w_0 = w\right]$$

so the HJB is

$$0 = \sup\left[U(c) - (\alpha+\eta)V + \eta A\, U(rw) + \left((rw - c) + \theta(\mu - r)\right)V' + \tfrac{1}{2}\sigma^2\theta^2 V''\right]$$

and we think that $V(w) = -B e^{-\gamma r w}$. We would then need

$$0 = \tilde{U}(V') + \eta A\, U(rw) - (\alpha+\eta)V + rwV' - \tfrac{1}{2}\kappa^2 (V')^2/V''$$

$$= rBe^{-\gamma r w}\left\{-1 + \log\left(B r e^{-\gamma r w}\right)\right\} - \eta A e^{-\gamma r w} + (\alpha+\eta)Be^{-\gamma r w} + \gamma r w B e^{-\gamma r w} + \tfrac{1}{2}\kappa^2 B e^{-\gamma r w}$$

$$= e^{-\gamma r w}\left[-rB + rB\log(rB) - \eta A + (\alpha+\eta)B + \tfrac{1}{2}\kappa^2 B\right] \qquad (1)$$

(iii) If we compare with the base case where we don't have killing, and we think the solution is given by $V(w) = -Be^{-\gamma r w}$, then the value of B for this would be determined by solving

$$0 = r\{\log(rB) - 1\} + \rho + \kappa^2/2$$

which is what we would get from (1) if we required $A = B$ (so identifying α there with ρ here). The optimal consumption would be

$$\boxed{c^* = rw - \frac{1}{\gamma}\log(rB)\ ,\quad \theta^* = \kappa/\gamma r\sigma .}$$

(iv) Still seems hard to finish off. Let's go back to the original formulation, and consider

$$Y_t \equiv \zeta_t w_t + \int_0^t \zeta_s c_s ds$$

We can do some calculations and find

$$dY_t = \zeta_t\left(\sigma\theta_t - \kappa w_t\right) dW_t$$

so Y is always a local martingale. One fact will assist us.

<u>Proposition</u> Define $X_t = \log \zeta_t = -\kappa W_t - (r + \frac{1}{2}\kappa^2)dt$. Then for $x > 0$

$$P\left[\sup_t X_t > x\right] = \exp\left(-\left(1 + \frac{2r}{\kappa^2}\right)x\right).$$

Therefore the MGF of X is finite valued up to $1 + 2r/\kappa^2$:

$$\boxed{E \exp(aX_t) = E\left[\zeta_t^a\right] = \frac{1 + 2r/\kappa^2}{1 + 2r/\kappa^2 - a}}$$

Now suppose we seek an optimum under the conditions that for some $B < \infty$

$$\boxed{|\theta_t| \leq B\ ,\quad |c_t - rw_t| \leq B .}$$

Then

$$d\left(e^{\varepsilon W_t - \lambda t}\right) = e^{\varepsilon W_t - \lambda t}\left\{\varepsilon\theta\sigma\left(dW + \kappa dt\right) + \varepsilon(rw - c)dt + \tfrac{1}{2}\varepsilon^2\theta^2\sigma^2 dt - \lambda dt\right\}$$

$$\doteq e^{\varepsilon W_t - \lambda t}\left(-\lambda + \varepsilon\theta_t\sigma\kappa + \varepsilon q_t + \tfrac{1}{2}\varepsilon^2\theta^2\sigma^2\right)dt$$

where $q_t \equiv rw_t - c_t$. Now provided $\lambda \geq \varepsilon\left(B\sigma\kappa + B + \frac{1}{2}\sigma^2\varepsilon B^2\right)$ we have that

$$e^{\varepsilon W_t - \lambda t} \quad \underline{\text{is a supermartingale}}$$

In particular, for each t, all exponential moments of w exist.

Now we go back and inspect the quadratic variation of Y,

$$\langle Y\rangle_t = \int_0^t \zeta_s^2 \left(\sigma \theta_s - \kappa w_s\right)^2 ds$$

so

$$E\,\langle Y\rangle_t = \int_0^t E\,\zeta_s^2 \left(\sigma\theta_s - \kappa w_s\right)^2 ds$$

$$\leq C\int_0^t \sqrt{E\,\zeta_s^4}\,\sqrt{E\left(\sigma^4 B^4 + \kappa^4 w_s^4\right)}\quad ds$$

$< \infty$ for each t.

Thus under the boundedness assumptions on θ, c, we have that Y is always a martingale

Not so, because if $R>1$, the option to declare bankruptcy might be preferable to continuing! $\longrightarrow$

Optimal investment funded by borrowing (19/2/15)

This is an example I tried on the Pt III course, where the wealth dynamics will be

$$dw = (rw - c - \varepsilon)\,dt + \theta\big(\sigma dW + (\mu - r)dt\big)$$

The difference is the presence of $\varepsilon > 0$, constant repayment rate on the loan. Suppose the investor sets some $w_* \geq 0$ as the level at which bankruptcy will be declared: $\tau = \inf\{t : w_t \leq w_*\}$. Suppose the objective is

$$V(w) = \sup E\left[\int_0^\tau e^{-\rho t} U(c_t)\,dt - K e^{-\rho\tau}\right]$$

where $K \geq 0$ is some bankruptcy penalty. We have the HJB

$$0 = \sup\left[U(c) - \rho V + \big(rw - c - \varepsilon + \theta(\mu - r)\big)V' + \tfrac{1}{2}\sigma^2\theta^2 V''\right]$$

$$= \tilde{U}(V') - \rho V + (rw - \varepsilon)V' - \tfrac{\kappa^2}{2}(V')^2/V''$$

so in dual variables

$$\boxed{0 = \tilde{U}(z) - \varepsilon z + \mathcal{L}J}, \qquad \mathcal{L} \equiv \tfrac{1}{2}z^2\kappa^2 D^2 + (\rho - r)zD - \rho$$

The auxiliary quadratic $Q(t) = \frac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho$ has roots $-\alpha < 0$, $\beta > 1$ and the solution to the dual equation is

$$J(z) = \frac{\tilde{U}(z)}{\gamma_M} - \frac{\varepsilon z}{r} + Az^{-\alpha} + Bz^{\beta}$$

Now if wealth is very large, $V(w) \approx V_M(w - \varepsilon/r)$ (use some wealth to pay off the debt and then do Merton – in fact, that argument shows that $V(w) \geq V_M(w - \varepsilon/r)$?) So for small z we should have

$$J(z) \approx \sup\left\{V_M(w - \tfrac{\varepsilon}{r}) - wz\right\} = \frac{\tilde{U}(z)}{\gamma_M} - \varepsilon z/r$$

This tells us we must have $\boxed{A = 0}$ and

$$J(z) = \frac{\tilde{U}(z)}{\gamma_M} - \frac{\varepsilon z}{r} + Bz^{\beta} \qquad \text{for } z \leq z_* = V'(w_*)$$

We don't know yet what B and z_* are.

From (*) we get

$$-\beta K = \left(\beta - 1 + k\right) \frac{\tilde{U}(z_*)}{r_m} - \frac{(\beta - 1)\varepsilon z_*}{r}$$

$$= \frac{\beta \tilde{U}(z_*)}{r_m} - \frac{\beta \varepsilon z_*}{r} + \beta B z_*$$

$$\therefore \beta B z_* = -\frac{(1-k)\tilde{U}(z_*)}{r_m} + \frac{\varepsilon z_*}{r} > 0 \quad \therefore B > 0,$$ J is convex, dev

Two cases arise.

Case 1: $0 < R < 1$. Here we would never stop before wealth hits 0, so we must have that J has a split definition

$$J(z) = \begin{cases} \dfrac{\tilde{U}(z)}{\gamma_M} - \dfrac{\varepsilon z}{r} + B z^{\beta} & \text{for } z \leq z_* \\ -K & \text{for } z \geq z_* \end{cases}$$

with $J'(z_*) = -w_* = 0$. So we get two equations

$$J(z_*) = \frac{\tilde{U}(z_*)}{\gamma_M} - \frac{\varepsilon z_*}{r} + B z_*^{\beta} = -K$$

$$z_* J'(z_*) = (1-R)\frac{\tilde{U}(z_*)}{\gamma_M} - \frac{\varepsilon z_*}{r} + \beta B z_*^{\beta} = 0$$

so combining gives

$$\frac{\varepsilon z_*}{r} - (1-R)\frac{\tilde{U}(z_*)}{\gamma_M} = -\beta K - \beta\frac{\tilde{U}(z_*)}{\gamma_M} + \beta\frac{\varepsilon z_*}{r}$$

which we see is

$$\boxed{\beta K = (\beta-1)\frac{\varepsilon z_*}{r} - (\beta-1+R)\frac{\tilde{U}(z_*)}{\gamma_M}} \qquad (*)$$

Now the RHS as a function of z_* is increasing from $-\infty$ to ∞, so there's a unique z_* to solve it

Case 2: $R > 1$ This time we get the analogous thing. The case where we try to solve $(*)$ is still OK, because the RHS is increasing from 0 at $z_* = 0$ to ∞ at $z_* = \infty$. So once again there is a unique positive root z_*, and this means that the gradient of V doesn't get bigger than z_*. This tells us in particular that the consumption does not get arbitrarily close to 0.

Coming out of stocks as you get older (23/2/15)

I asked the Optimal Investment class why we would want to come out of stocks as we get older. A couple of suggestions came up: change of risk aversion, reduction in income.

(i) For the first, might propose retirement date T

$$\sup E\left[\int_t^T \varphi(s)\, U(c_s)\, ds + F(w_T) \,\middle|\, w_t = w\right] = V(t,w)$$

which would give HJB

$$0 = \sup\left[\varphi(t) U(c) + V_t + (rw - c + \theta(\mu - r)) V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww}\right]$$

$$= \varphi(t)\, \tilde{U}(V_w/\varphi(t)) + V_t + rw V_w - \tfrac{\kappa^2}{2} V_w^2/V_{ww}$$

with $V(T,\cdot) = F(\cdot)$. For interesting results, we will need to go numerical. Choosing

$$r = 0.05, \quad \mu = 0.09, \quad \sigma = 0.25$$

seemed to match our rough ideas

Age	stock	bank
20	120%	-20%
40	60%	40%
60	10%	90%

if we had $U(c) = \sqrt{c}$, $F(w) = -w^{-5}$ $(R = 6)$

If we want to do a Crank-Nicolson scheme, with V^k the value at t_k, V^{k+1} the value at $t_{k+1} = t_k + \Delta t$, we would have

$$0 = \frac{\varphi(t) + \varphi(t+\Delta t)}{2} U(c) + \frac{V^{k+1} - V^k}{\Delta t} + L(V^k + V^{k+1})/2$$

which then becomes

$$\boxed{(2 - \Delta t . L) V^k = (2 + \Delta t . L) V^{k+1} + \Delta t(\varphi(t) + \varphi(t + \Delta t)) U(c)}$$

to be solved for V^k. Maybe worth doing other FD schemes, such as fully implicit.

(ii) Are we getting the correct boundary behaviour when we do this? An alternative would be to fix values at some high/low values of w, and to insist that once we get there we have to come out of the stock entirely. To calculate the value in these circumstances, need to assume $\varphi(t) = e^{-\alpha t}$ to be able to do it closed form.

$$\max \int_0^\tau e^{-\alpha s} U_1(c_s)\, ds + a U_2(w_\tau) \quad s.t. \quad \int_0^\tau e^{-rs} c_s\, ds + e^{-r\tau} w_\tau = w_0$$

is the problem. From Lagrangian

$$e^{-\alpha t} U_1'(c_t) = \lambda e^{-rt}, \quad a U_2'(w_\tau) = \lambda e^{-r\tau}$$

so $\boxed{c_t^{R_1} = e^{(r-\alpha)t}/\lambda, \quad w_\tau^{R_2} = a e^{r\tau}/\lambda}$

Budget constraint says

$$w_0 = \lambda^{-1/R_1} \frac{e^{\beta\tau} - 1}{\beta} + \lambda^{-1/R_2} a^{1/R_2} e^{r\tau(1/R_2 - 1)}$$

where $\beta = (r-\alpha)/R_1 - r$, and the value of the objective is

$$\lambda \left[\frac{\lambda^{-1/R_1}}{1-R_1} \frac{e^{\beta\tau} - 1}{\beta} + \frac{\lambda^{-1/R_2}}{1-R_2} e^{r\tau(1/R_2 - 1)} \right]$$

If we were at some time $t \in (0, T)$, $\tau \equiv T - t$, then we want to

$$\cancel{\max \int_t^T e^{-\alpha s} U_1(c_s) + A U_2(w_T)}$$

$$= \max \; e^{-\alpha t} \left\{ \int_0^\tau e^{-\alpha u} U_1(c_{t+u})\, du + A e^{\alpha t} U_2(w_T) \right\}$$

subject to the budget constraint.

However, this appears to be problematic numerically.

(iii) Maybe there is a simpler way around this. Let's suppose that we decree that consumption must be at some fixed rate q throughout $[0, \tau]$. This will leave

$$w_\tau = e^{r\tau} w_0 - q(e^{r\tau} - 1)/r$$

with objective $\dfrac{1 - e^{-\alpha\tau}}{\alpha} \cdot U_1(q) + a U_2(w_\tau)$ to be optimized over q

$$\equiv b U_1(q) + a U_2(k - \lambda q)$$

$b \equiv \dfrac{1 - e^{-\alpha\tau}}{\alpha}$,

$k \equiv e^{r\tau} w_0$,

$\lambda \equiv (e^{r\tau} - 1)/r$

The FOC is

$$b q^{-R_1} = \lambda a (k - \lambda q)^{-R_2}$$

which can only be solved numerically. Imposing these BCs doesn't seem to work too well –

Optimal investment with employment/unemployment (4/3/15)

When you are employed, you get income $\varepsilon > 0$, when unemployed you get nothing, and you flip between as a Markov chain. If your objective is

$$\sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \,\middle|\, w_0 = w, \xi_0 = j \right] = V_j(w)$$

then the HJB equations would read

$$0 = \sup\left[U(c) - \rho V_1 + (rw - c + \varepsilon + \theta(\mu - r)) V_1' + \tfrac{1}{2}\sigma^2\theta^2 V_1'' + \alpha(V_0 - V_1) \right]$$

$$0 = \sup\left[U(c) - \rho V_0 + (rw - c + \theta(\mu - r)) V_0' + \tfrac{1}{2}\sigma^2\theta^2 V_1'' + \beta(V_1 - V_0) \right]$$

This is quite a neat little example, which seems to be intractable except via numerics

Hunger as incentive (9/3/15)

(i) Back on p 63-65 of WN XXXV, I proposed using hunger as an objective, so that we have

$$\begin{cases} dw_t = rw_t\,dt + \theta_t(\sigma dW_t + (\mu - r)dt) - dC_t \\ dh_t = \varphi(h_t)\,dt - dC_t \end{cases}$$

and aim for

$$\inf E\left[\int_0^\infty e^{-\rho t} f(h_t)\,dt \,\Big|\, h_0 = h,\ w_0 = w\right] \equiv V(w,h).$$

Back there, I made the assumption $\varphi(h) = h$, $f(h) = h^\varepsilon$, but this is unrealistic because if $w > h$ we could immediately avoid all hunger for ever!

(ii) Maybe a better story is to have

$$\varphi(h) = \varepsilon, \quad f(h) = e^{ah}$$

so if we set $y_t \equiv h_t - \varepsilon t$ we have

$$dy_t = -dC_t$$

with objective

$$\inf E\left[\int_0^\infty \exp\{-\rho t + a y_t + a\varepsilon t\}\,dt \,\Big|\, w_0 = w,\ y_0 = y\right]$$

$$= \inf E\left[\int_0^\infty \exp(-\tilde\rho t + a y_t)\,dt \,\Big|\, w_0 = w,\ y_0 = y\right]$$

$[\tilde\rho \equiv \rho - a\varepsilon$, assumed $> 0]$

$$= e^{ay} v(w) \equiv V(w,y)$$

The HJB here will be

$$0 = \min\left\{ \begin{array}{l} \inf\left[-\tilde\rho V + (rw + \theta(\mu - r))V_w + \tfrac12 \sigma^2\theta^2 V_{ww} + e^{ay}\right] \\ -a e^{ay} v - v' e^{ay} \geq 0 \end{array}\right.$$

so

$$\boxed{\begin{cases} 0 = -\tilde\rho v + rwv' - \frac{\kappa^2}{2}(v')^2/v'' + 1 \\ 0 \geq v' + av \end{cases}}$$

is what we got.

(iii) Let's go to the concave dual function $J(z) = \inf_w\{v(w) - wz\}$, which is concave, decreasing, finite-valued only for $z \leq 0$, and non-negative

We get general solution here as

$$J(z) = \frac{1}{\tilde\rho} + A|z|^{-\alpha} + B|z|^{\beta}$$

where $\alpha < 0 < \beta$ solve $Q(t) \equiv \frac{1}{2}\kappa^2 t(t-1) + (\tilde\rho - r)t - \tilde\rho = 0$. For J to be non-negative concave decreasing everywhere, we'd expect $B=0$ as usual. What we expect is that when wealth exceeds w_* we reduce wealth to w_* + buy food:

$$V(w_* + \Delta, y) = V(w_*, y-\Delta) = e^{a(y-\Delta)} v(w_*)$$

so $v(w) = e^{-a(w-w_*)} v(w_*)$ for $w \geq w_*$. The form of the dual value for $z_* \leq z \leq 0$ would then be

$$J(z) = -\frac{z}{a} - z\left(w_* - \frac{1}{a}\log\left(-z/a v(w_*)\right)\right),$$

if it matters. What we shall therefore have is

$$J(z) = \frac{1}{\tilde\rho} + A\left|z/z_*\right|^{-\alpha} \qquad \text{for } z \leq z_*$$

and at $z = z_*$ we will need

$$0 = z_* + a(J - z_* J')$$

$$= z_* + a\left\{\frac{1}{\tilde\rho} + A + \alpha A\right\}$$

which gives us $(1+\alpha)a\,A = -z_* - a/\tilde\rho$. For C^2 join at z_* we will require

$$J''(z_*+) = \frac{1}{a z_*} = J''(z_*-) = \alpha(\alpha+1)A/z_*^2$$

so this implies $\boxed{z_* = a\alpha(\alpha+1)A}$

Thus $\boxed{A = -1/\tilde\rho(1+\alpha)^2, \quad z_* = -\frac{a\alpha}{\tilde\rho(1+\alpha)}}$