More on coupling of random walks (5/11/92)

Suppose we have two distributions F, G, with $F \leq_{st} G$, and both supported in the compact interval K. Suppose given some map $\varphi: \mathbb{R} \to \mathbb{R}$ which is strictly convex. Then we have the following.

THEOREM Every solution μ to the problem

$$(*) \quad \max \left\{ \iint \varphi(y-x)\, \mu(dx, dy) : \int_y \mu(dx,dy) = F(dx), \int_x \mu(dx, dy) = G(dy), \ \mu(\{(y,x) : y < x\}) = 0 \right\}$$

has the property that

$$(1) \qquad \mu(X=Y) = \int f(t) \wedge g(t) \ m(dt)$$

where m is a measure such that $F(dt) = f(t)\, m(dt)$, $G(dt) = g(t)\, m(dt)$. At least one solution to the problem exists.

Remark Of course, (1) says that $X = Y$ with the largest possible probability given that $X \sim F$, $Y \sim G$.

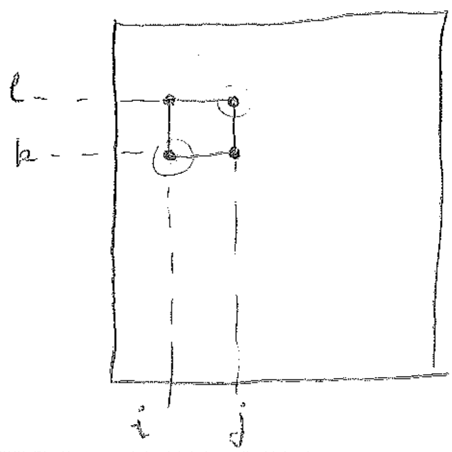
Proof Case (i): F, G are of finite support. Let's suppose that F, G take values in a common finite set $I \subseteq \mathbb{R}$. The optimisation problem is

$$\max \sum_{i \leq j}\sum p_{ij}\ \varphi(j-i) \quad \text{subj to} \quad \sum_j p_{ij} = \alpha_i \quad \forall i \in I$$

$$\sum_i p_{ij} = \beta_j \quad \forall j \in I.$$

and $p_{ij} = 0$ if $i > j$.

Now we can see that in an optimal p, it is impossible to have
$i < j \leq k < l$ and $p_{ik} > 0$, $p_{jl} > 0$.



An algorithm for building this coupling in the discrete case

	$\cancel{p_0}$ $(p_0-q_0)^+$	p_1 $(p_1-q_1)^+$	p_2	p_3	p_4	p_5	p_6
q_6							$p_6 \wedge q_6$
q_5						$p_5 \wedge q_5$	
q_4					$p_4 \wedge q_4$		
q_3				$p_3 \wedge q_3$			
q_2			$p_2 \wedge q_2$				
$(q_1-p_1)^+$ q_1		$p_1 \wedge q_1$					
$(q_0-p_0)^+$ $\cancel{q_0}$	$p_0 \wedge q_0$						

First fill in $p_j \wedge q_j$ along diagonal; then subtract this from the mass remaining in j^{th} row, column. (So now have $(q_j - p_j)^+ = q_j'$ next to row j, for example.)

Now fill in position $(j, j+1)$ with $p_j' \wedge q_{j+1}'$,

and subtract what has been used from the marginal totals. Continue to fill in $(j, j+2)$, etc....

There appears to be nothing one can do in the general case to express this bivariate law nicely in terms of the marginals; simple numerical examples reveal no pattern.

Why? Because, if so, we could reduce p_{ik}, p_{jl} by some $\delta > 0$ while keeping both ≥ 0, and add δ to p_{il}, p_{jk}. This change preserves the marginals, and increases the payoff by

$$\delta \{ \varphi(l-i) - \varphi(l-j) - \varphi(k-i) + \varphi(k-j) \}$$

$$= \delta \int_k^l \{ \varphi'(x-i) - \varphi'(x-j) \} dx$$

$$> 0.$$

This contradicts the assumed optimality of p.

Immediately it follows that

$$p_{ii} = \alpha_i \wedge \beta_i .$$

<u>Case (ii): general support</u>. If now $X \sim F$, $Y \sim G$, both dist^ns supported in compact K, let F_n be the law of $\rho_n(X)$, G_n the law of $\rho_n(Y)$, where

$$\rho_n(x) = 2^{-n} [2^n x],$$

and let

$$V_n \equiv \max \{ E\varphi(Y-X) : Y \geq X, \ Y \sim G_n, \ X \sim F_n \} \tag{2.i}$$

$$V \equiv \max \{ E\varphi(Y-X) : Y \geq X, \ Y \sim G, \ X \sim F \}. \tag{2.ii}$$

Now since the joint law of (X, Y) is concentrated on a compact set, and

$$\mu \longmapsto \iint \varphi(y-x) \, \mu(dx, dy)$$

is a bounded continuous functional of μ, the maximum in the definitions (2) will certainly be attained. Suppose that μ^* is an optimal law for (2.ii), μ_n^* for (2.i). If $(X, Y) \sim \mu^*$, then there's a Lipschitz constant c such that

$$V = E \varphi(Y-X) \leq c \cdot 2^{-n} + E\varphi(\rho_n(Y) - \rho_n(X))$$

$$\leq c \cdot 2^{-n} + V_n$$

so that $\liminf V_n \geq V$, and if $\limsup V_n = V + \varepsilon > V$, there are (X_n, Y_n) such that $X_n \sim F_n$, $Y_n \sim G_n$, $X_n \leq Y_n$, and

$$V_n - \tfrac{\varepsilon}{2} \leq E\varphi(Y_n - X_n) \to E\varphi(Y' - X') \quad \text{down a s/seq.}$$

Thus (X', Y') satisfies the constraints, but $E\,\varphi(Y'-X') \geq V + \varepsilon/2 > V$ $\divideontimes$

Thus $\qquad V_n \longrightarrow V.$

Now let's observe that

$$P\left[X_n^* = j2^{-n} = Y_n^*\right] = \int_{[j2^{-n},(j+1)2^{-n})} f(t)\,dm(t) \wedge \int_{[j2^{-n},(j+1)2^{-n})} g(t)\,dm(t)$$

$$\geq \int_{[j2^{-n},(j+1)2^{-n})} f(t) \wedge g(t)\; m(dt)$$

Hence

$$P\left[X_n^* = Y_n^*\right] \geq \int f(t) \wedge g(t)\; m(dt).$$

Now take a weak limit of (some subsequence of) the μ_n^*. Since the diagonal is a closed set,

$$\mu_\infty^*(X=Y) \geq \limsup P(X_n^* = Y_n^*) \geq \int f(t) \wedge g(t)\; m(dt).$$

The only possibility is that equality holds throughout, since the law of X, Y under $\mu_\infty^* = \text{w-}\lim \mu_n^*$ is F, G.

$$E[Y_t \mid Y_0 = y] = e^{\beta t}\left\{ y + \nu \frac{e^{-\beta t}-1}{-\beta} \right\}$$

$$E[Z_t \mid Z_0 = g] = e^{\beta t}\left\{ g + \gamma \frac{e^{-\beta t}-1}{-\beta} \right\}$$

Invariant density of Z is

$$\pi(x) \propto x^{2\gamma/\alpha^2 - 1} \exp(2\beta x/\alpha^2) = x^{\nu/\lambda - 1} \exp\{-x(\mu-\lambda)/\lambda\mu\}.$$

Invariant law of the IBD chain is

$$\pi_k = \left(\prod_{j=0}^{k-1} \frac{\lambda j + \nu}{\mu(j+1)} \right) \pi_0$$

so that

$$\sum_{k \geq 0} s^k \pi_k = \pi_0 \left(1 - \frac{\lambda s}{\mu}\right)^{-\nu/\lambda} = \left(\frac{\mu-\lambda}{\mu-\lambda s}\right)^{\nu/\lambda}.$$

An interesting result of Peter Clifford (9/11/92)

Peter Clifford takes two point processes on $\mathbb{R}^+$. The first is the point process of death times in an immigration-birth-death process Y. with

$$\begin{cases} \text{immigration at constant rate } \nu \\ \text{death at constant rate } \mu \text{ per individual} \\ \text{birth at constant rate } \lambda \text{ per individual.} \end{cases}$$

The second is a Poisson process whose intensity Z_t at time t is the solution of a stochastic differential equation

(1) $$dZ_t = \alpha\sqrt{Z_t}\, dW_t + (\beta Z_t + \gamma)\, dt.$$

THEOREM. Provided we relate the parameters by

(2) $$\tfrac{1}{2}\alpha^2 = \lambda\mu, \quad \beta = \lambda-\mu, \quad \gamma = \mu\nu$$

and the initial conditions are

(3) $$Z_0 = 0, \qquad Y_0 = 0$$

then the two point processes considered have the same law.

Proof Let's fix $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which is C_K^∞, and calculate

(*) $$E \exp\{- \int_0^\infty \varphi_t \, dN_t\}$$

for the two counting processes N. If we get the same answer, then since φ is arbitrary, the two counting processes have the same law.

(i) Let's begin with the chain, and let's denote by X_t the b+d chain without any immigration, just death rate μ per particle, birth rate λ per particle. If we write N' for the point process of deaths in X, and set

(4) $$f(t) \equiv E\left[\exp - \int_0^\infty \varphi_{t+s}\, dN'_s \,\middle|\, X_0 = 1\right],$$

then

$$f(t) = \int_0^\infty e^{-(\lambda+\mu)s} \{ \lambda f(t+s)^2 + \mu e^{-\varphi_{t+s}} \} \, ds .$$

Hence in the usual way

$$\dot{f}_t - (\lambda+\mu) f_t = - \lambda f_t^2 - \mu e^{-\varphi_t}$$

so that

$$(5) \qquad \boxed{\dot{f}_t - (\lambda+\mu) f_t + \lambda f_t^2 = -\mu e^{-\varphi_t}} \, .$$

This differential equation has a unique solution satisfying $f_t = 1$ for all large enough t.

For the IBD process, we may express (*) simply in terms of f. The times of immigration are the times of an independent Poisson process of rate ν and if these times are $T_1 < T_2 < \dots$, then

$$E \left[\exp\{ - \int_0^\infty \varphi_u \, dN_u \} \mid Y_0 = 0 \right]$$

$$= E \left[\prod_j f(T_j) \mid Y_0 = 0 \right]$$

$$= E \left[\exp\{ - \int_0^\infty \log(1/f_t) \, dC_t \} \mid Y_0 = 0 \right]$$

where C_t counts the number of immigrants;

$$= \exp\left(- \int_0^\infty (1 - f_t) \, \nu \, dt \right)$$

by basic properties of Poisson processes.

Hence we get

$$(6) \qquad \boxed{E \left[\exp\{ - \int_0^\infty \varphi_u \, dN_u \} \mid Y_0 = k \right] = f(0)^k \exp\{ - \nu \int_0^\infty (1 - f_t) \, dt \}}$$

(ii) If we now work on the other point process, N'', say, we have

$$(7) \qquad E \exp\left\{- \int_0^\infty \varphi_t \, dN''_t \right\} = E \exp\left\{ - \int_0^\infty (1-e^{-\varphi_t}) Z_t \, dt \right\}$$

by simple properties of Poisson processes, and thus if

$$A_t \equiv \int_0^t (1-e^{-\varphi_u}) Z_u \, du$$

we have that

$$M_t \equiv E\left[e^{-A_\infty} \mid \mathcal{F}_t \right]$$

$$(8) \qquad = e^{-A_t} \exp\{ -\rho_t Z_t - \theta_t \}.$$

for some functions ρ, θ, I claim. Indeed, by Itô's formula on the righthand side

$$d\left(e^{-A_t} \exp\{-\rho_t Z_t - \theta_t\}\right) = \exp(-A_t - \rho_t Z_t - \theta_t)\Big[-(1-e^{-\varphi_t}) Z_t - \dot\rho_t Z_t - \dot\theta_t$$
$$- (\beta Z_t + \gamma)\rho_t + \tfrac{1}{2}\rho_t^2 \alpha^2 Z_t \Big] dt + d(\text{mart})$$

and so if

$$\boxed{\begin{aligned} &\dot\theta_t + \gamma \rho_t = 0 \\ &\tfrac{1}{2}\alpha^2 \rho_t^2 - \beta\rho_t - \dot\rho_t - (1-e^{-\varphi_t}) = 0 \end{aligned}}$$

we do have that the expression (8) is a local martingale; if $\theta = \rho = 0$ for large enough t, and $\rho \geq 0$, then the expression (8) is bounded, and therefore is a martingale, with terminal value $\exp(-A_\infty)$. Thus

$$(9) \qquad E\left[e^{-A_\infty} \mid \mathcal{F}_0 \right] = M_0 = \exp\left[-\rho_0 Z_0 - \int_0^\infty \gamma \rho_s \, ds \right]$$

With the relations (2) between the parameters, it is easy to show that

$$\tilde f_t \equiv 1 - \gamma \rho_t / \nu$$

solves the same de as f_t, and is equal to 1 for all large enough t $\therefore \tilde f = f$, and the theorem follows.

(iii) (23/11) Peter says that the case where initially both processes are in equilibrium is also of interest. Taking (6) and mixing over the invariant distribution for Y gives

$$E \exp - \int_0^\infty \varphi_u \, dN_u = \left(\frac{\mu - \lambda}{\mu - \lambda f(0)} \right)^{\nu/\lambda} \exp\left\{ -\nu \int_0^\infty (1 - f(t))\,dt \right\}.$$

Mixing (9) with the invariant density

$$\pi(x) = x^{\nu/\lambda - 1} \exp\left\{ - \frac{x(\mu-\lambda)}{\lambda\mu} \right\} \left(\frac{\mu - \lambda}{\lambda\mu} \right)^{\nu/\lambda} / \Gamma(\nu/\lambda)$$

yields

$$E \exp\left(- \int_0^\infty \varphi_u \, dN_u'' \right) = \left(\frac{\mu - \lambda}{\mu - \lambda f(0)} \right)^{\nu/\lambda} \exp\left\{ -\nu \int_0^\infty (1 - f(t))\,dt \right\},$$

as before.

(iv) If one took the IBD chain with $Y_0 \sim \mathcal{P}(z/\mu)$, then

$$E \exp\left\{ - \int_0^\infty \varphi_u \, dN_u \right\} = \exp\left[-(1 - f_0) z/\mu - \nu \int_0^\infty (1 - f_s)\,ds \right]$$

which is what we get in the Cox-process story with $Z_0 = z$.

Some remarks on reciprocal processes (23/11/92)

(i) We are going to consider some process $(X_t)_{0\le t\le 1}$ with values in a Polish space S such that for any $0\le s<t\le 1$,

$$\mathcal{F}_{[s,t]^c} \text{ and } \mathcal{F}_{(s,t)} \text{ are conditionally independent given } \mathcal{F}_{\{s,t\}}$$

(where $\mathcal{F}_I \equiv \sigma(\{X_u : u\in I\})$ and for $0\le s\le t\le u\le 1$

$$P(X_t\in A \mid X_s=x,\ X_u=y) \equiv P(s,x;\ t,A;\ u,y)$$

is the transition function of the process. Conditional on $X_1=y$, the process X is Markovian with transition function

$$P^y_{st}(x,dx') \equiv P(s,x;\ t,dx';\ 1,y).$$

Thus the law of X is a mixture of Markovian laws. However, we can consider the process $Z_t\equiv(X_t, X_1)$ taking values in $\bar{S}\equiv S\times S$ with transition mechanism

$$P\left[X_t\in A, X_1\in B \mid X_s=x, X_1=y\right] = I_B(y)\,P(s,x;\ t,A;\ 1,y).$$

This way, the process X is a function of the Markov process Z, and we find ourselves in the old setting of "Markov functions" Ann Prob 9, 573-582, 1981.

(ii) We have a map $\varphi : \bar{S}\equiv S\times S\to S$ which is projection onto the first component, and some law $\mathbb{P}$ on Ω, the space of $\bar{S}$-valued paths which are right continuous with left limits, such that under $\mathbb{P}$, Z is Markovian with transition semigroup $\{P_{st} : 0\le s\le t\le 1\}$.

Let Φ denote the kernel from $\bar{S}$ to S given by the function φ, and let Λ_t be the kernel from S to $\bar{S}$, given by

$$\Lambda_t(x,A) \equiv \mathbb{P}(Z_t\in A \mid X_t=x).$$

Then if X is Markov, its transition function is given by

(1) $$Q_{st} \equiv \Lambda_s P_{st} \Phi,$$

It is the result of Rogers & Pitman that if μ is the initial law of Z, then X is Markov if for all $0 \le s \le t \le 1$

(2) $$\boxed{\Lambda_s P_{st} = Q_{st} \Lambda_t.}$$

However, we hve have a converse to this result. Define
$$\mathcal{C}_t \equiv \{ P_{tt_1} f_1 \cdots P_{t_{n-1} t_n} f_n : f_j = g_j \circ \varphi,\ g_j \in b\mathcal{S},\ t \le t_1 \le \cdots \le t_n \le n \}.$$

This is a class of bounded measurable functions on $\bar{S}$, and forms the collection of conditional expectations of all cylinder events on the process X from time t onward, given Z_t.

THEOREM. Suppose that for every $0 \le t \le 1$ the property

(*) $$\boxed{\nu_1(g\circ\varphi . f) = \nu_2(g\circ\varphi . f) \ \forall g \in b\mathcal{S},\ f \in \mathcal{C}_t \Rightarrow \nu_1 = \nu_2}$$

holds. Then for all $0 \le s \le t \le 1$, $\psi \in b\bar{\mathcal{S}}$, if X is Markov we have

(3) $$\boxed{\Lambda_s P_{st} \psi = Q_{st} \Lambda_t \psi \qquad \mu P_{0s} \Phi\text{-a.e.}}$$

Proof The Markov property of X says that with $g, h, g_n \in b\mathcal{S}$, $f_i \equiv g_i \circ \varphi$,

(4) $$\mu P_{0s} \Phi\, h\, Q_{st}\, g\, Q_{tt_1} g_1 \cdots Q_{t_{n-1} t_n} g_n$$

$$= \mu P_{0s}\, h\circ\varphi\, P_{st}\, g\circ\varphi\, P_{tt_1} f_1 \cdots P_{t_{n-1} t_n} f_n$$

$$\equiv \mu P_{0s}\, h\circ\varphi\, P_{st} (g\circ\varphi . f) \qquad \text{for short, where}$$

$$f \equiv P_{tt_1} f_1 \cdots P_{t_{n-1} t_n} f_n.$$

$$= \mu P_{0s} \Phi\, h\, \Lambda_s P_{st} (g\circ\varphi f)$$

But
$$Q_{tt_1} g_1 \cdots Q_{t_{n-1} t_n} g_n (x) = E\left[\prod_1^n g_j(X_{t_j}) \,\middle|\, X_t = x \right]$$

$$= \Lambda_t P_{tt_1} f_1 \cdots P_{t_{n-1}t_n} f_n (x)$$

$$= \Lambda_t f(x).$$

Thus (4) says

$$\mu P_{0s} \Phi h \; Q_{st} \, g \, \Lambda_t f = \mu P_{0s} \Phi h \, \Lambda_s P_{st} (g \circ \varphi \cdot f)$$

$$= \mu P_{0s} \Phi h \; Q_{st} \, \Lambda_t (g \circ \varphi \cdot f).$$

Condition (*) now guarantees that

$$\mu P_{0s} \Phi h \, \Lambda_s P_{st} = \mu P_{0s} \Phi h \, Q_{st} \Lambda_t ,$$

and since h is arbitrary the theorem is proved

(iii) The above theorem is true in complete generality: the explicit nature of the transition mechanism of Z is not assumed. However, if we now assume this, it is trivial to prove that (*) holds, because with $f = I_A \circ \varphi$

$$P_{t1} f(x, x') = I_A(x'),$$

and now (*) is obvious!

(iv) If we had a reciprocal transition function and some measure μ on (X_0, X_1) such that X is Markovian, is there anything interesting which can be said about the reciprocal transition function?

(v) Let's now suppose that the reciprocal process has arisen from a Markov process which has a strictly positive transition density w.r.t. some measure m;

(5) $$p(s, x;\, t, x';\, u, y) = p_{st}(x, x')\, p_{tu}(x', y) / p_{su}(x, y)$$

We shall abbreviate $m(dx)$ to dx. Let μ denote the joint law of (X_0, X_1).

<u>When is X a Markov process?</u>

We have

$$\Lambda_s(x, dy) = \frac{\int \mu(dz, dy)\ p(0,z;\ s,x;\ 1,y)}{\iint \mu(dz,dy')\ p(0,z;\ s,x;\ 1,y')}$$

(6)

$$= p_{s1}(x,y)\ \frac{\int \mu(dz,dy)\ p_{0s}(z,x)/p_{01}(z,y)}{\iint \mu(dz,dy')\ p(0,z;\ s,x;\ 1,y')}$$

and

(7) $$q_{st}(x,x') = \int \Lambda_s(x,dy)\ p_{st}(x,x')\ p_{t1}(x',y)/p_{s1}(x,y).$$

The Markov condition (2)-(3) says

$$\Lambda_s(x,dy)\ \frac{p_{st}(x,x')\ p_{t1}(x',y)}{p_{s1}(x,y)} = q_{st}(x,x')\ \Lambda_t(x',dy)$$

so

(8) $$\frac{\Lambda_s(x,dy)}{p_{s1}(x,y)} = \left[\int \Lambda_s(x,dy')\ \frac{p_{t1}(x',y')}{p_{s1}(x,y')}\right]\ \frac{\Lambda_t(x',dy)}{p_{t1}(x',y)}.$$

Holding t, x' fixed for the moment, we have for $s<t$,

(9) $$\boxed{\frac{\Lambda_s(x,dy)}{p_{s1}(x,y)} = f(s,x)\ \eta(dy)}$$

and so substituting back in to the right-hand side of (8) gives for $s \le u < t$

$$\frac{\Lambda_s(x,dy)}{p_{s1}(x,y)} = f(s,x)\ \int \eta(dy')\ p_{u1}(z,y')\ .\ f(u,z)\ \eta(dy)$$

implying

(10) $$f(u,z) = \left\{\int \eta(dy')\ p_{u1}(z,y')\right\}^{-1}$$

and from (7), for $0 \le s \le u < t$

(11) $$\boxed{q_{su}(x,z) = p_{su}(x,z)\ f(s,x)/f(u,z).}$$

Thus the Markov transition function $\{Q_{st}\}$ is an h-transform of $\{P_{st}\}$.

Taking $s=0$ in (9) reveals that

(12) $$\boxed{\Lambda_0(x, dy) = p_{01}(x,y)\ f(0,x)\,\eta(dy)}$$

which is the result of Jamison.

(vi) If we wanted to mix over initial and final values so as to make a Markov process with initial law μ_0, final law μ_1, how would we do it? The Markov process would have to be an h-transform of (P_{st}), with final law μ_1, so define

$$h(s,x) \equiv \int p_{s1}(x,y)\,\mu_1(dy) = f(s,x)^{-1}, \quad \eta(dy) \equiv \mu_1(dy).$$

From (12),

$$\Lambda_0(x,dy) = p_{01}(x,y)\,\mu_1(dy)\,/\,h(0,x)$$

and the joint law of X_0, X_1 is given by

(13) $$\boxed{P(X_0 \in dx,\ X_1 \in dy) = \mu_0(dx)\ \frac{p_{01}(x,y)\,\mu_1(dy)}{h(0,x)}}$$

and

$$\boxed{q_{st}(x,z) = p_{st}(x,z)\ h(t,z)\,/\,h(s,x).}$$

It is now easy to confirm from (6) that (9) holds, and also the Markov condition (8)

(vii) Again assuming the existence of densities $p^y_{st}(x,x')$, $\lambda_s(x,y)$ for the general case, we have the result of Frank Kelly, which tells us about the reversal. Indeed, for $0 \le s \le t \le 1$, if (3) holds then

$$\hat{p}^{\,y}_{ts}(x',x) \equiv \frac{p_{st}(x,x')\ P(X_s = x,\ X_1 = y)}{P(X_t = x',\ X_1 = y)}$$

$$= p^y_{st}(x,x')\,.\ \frac{P(X_s=x)}{P(X_t=x')}\ \frac{\lambda_s(x,y)}{\lambda_t(x',y)} = q_{st}(x,x')\ \frac{P(X_s=x)}{P(X_t=x')}\,.$$

Thus we have

$$\hat{p}^{y}_{ts}(x',x) = \hat{q}_{ts}(x',x)$$

is the same for all y. Thus the time-reversed process is Markovian because of the trivial fact that the time-reversed evolution of X is the same whatever the value y of X_1 !

<u>Coupling random walks again</u> 7/12/92

(i) If μ is a law on $\mathbb{R}$, then we have the following nice dichotomy.

LEMMA. <u>For each</u> $x \in \mathbb{R}$,

<u>either</u> (1.i) $\| \delta_x * \mu^{*n} - \mu^{*n} \| \to 0 \quad (n \to \infty)$,

<u>or</u> (1.ii) $\| \delta_x * \mu^{*n} - \mu^{*n} \| = 2$ <u>for all</u> n.

<u>Proof</u> We certainly have $\| \delta_x * \mu^{*n} - \mu^{*n} \|$ decreases with n. If it's <u>not</u> equal to 2 for all n, we can pick some k such that

$$\| \delta_x * \mu^{*k} - \mu^{*k} \| < 2.$$

If we can prove $\| \delta_x * \mu - \mu \| < 2 \Rightarrow \| \delta_x * \mu^{*n} - \mu^{*n} \| \to 0$, then by using μ^{*k} in place of μ, we get the result we want.

So we lose no generality by assuming that

$$\| \delta_x * \mu - \mu \| < 2.$$

Also, let's suppose that $x > 0$, and, wlog, that $x = 1$. We shall build a pair of random walks $S_n = X_1 + \dots + X_n$, $S'_n = X'_1 + \dots + X'_n + 1$ such that

(2.i) (S_n), (S'_n) are random walks with step distributions μ;

(2.ii) $X_n - X'_n = 1, 0,$ or -1;

(2.iii) $E(X_n - X_n') = 0$;

(2.iv) $P(X_n - X_n' = 0)$ is the same for all n, and is in $(0,1)$.

If we can make such a coupling, then evidently $S_n - S_n'$ eventually hits 0.

The method is simply the Mineka coupling. Writing $\mu_a \equiv \delta_a * \mu$, we have that μ_1 and μ_0 have densities f_1 and f_0 wrt some reference measure m, and

$$(\mu_0 \wedge \mu_1)(\mathbb{R}) \equiv \int (f_0 \wedge f_1)\, dm \quad > 0.$$

Now we shall describe the law of a pair (X, X') which has the property that

$$(X, X') \in L_j \equiv \{(x,y) : y = x + j\} \quad \text{for one of } j = -1, 0, 1$$

and such that X and X' each have law μ. Define

$$P(X \in A,\ (X, X') \in L_j) \equiv \gamma_j(A),$$

where

$$\gamma_{-1} \equiv \tfrac{1}{2}(\mu_0 \wedge \mu_1),$$

$$\gamma_1 \equiv \tfrac{1}{2}(\mu_0 \wedge \mu_{-1}),$$

$$\gamma_0 \equiv \mu_0 - \tfrac{1}{2}(\mu_0 \wedge \mu_1) - \tfrac{1}{2}(\mu_0 \wedge \mu_{-1}) \geq 0.$$

Evidently, $\gamma_0 + \gamma_{-1} + \gamma_1 = \mu_0 \equiv \mu$, so all that is needed is to verify the other marginal. But

$$E\,\psi(X') = \int \psi(x+1)\, \gamma_1(dx) + \int \psi(x)\, \gamma_0(dx) + \int \psi(x)\, \gamma_{-1}(dx)$$

$$= \int \psi(x+1) \cdot \tfrac{1}{2}(\mu_0 \wedge \mu_{-1})(dx) + \int \psi(x)\, \gamma_0(dx) + \int \psi(x-1)\, \tfrac{1}{2}(\mu_0 \wedge \mu_1)(dx)$$

$$= \tfrac{1}{2} \int \psi(x)\, (\mu_1 \wedge \mu_0)(dx) + \int \psi(x)\, \gamma_0(dx) + \tfrac{1}{2} \int \psi(x)\, (\mu_0 \wedge \mu_{-1})(dx)$$

$$= \int \psi(x)\, \mu(dx),$$

as asserted. □

The main point of the lemma is this. In case (1.ii), coupling is evidently impossible. In all other situations, coupling IS possible, and can be achieved by the Mineka coupling!

Of course, this is being a little too rapid; it may be that

$$k = \inf \{ n : \| \delta_x * \mu^{*n} - \mu^{*n} \| < 2 \}$$

is greater than 1. However, this is really a trivial extension; create random walks (S_{nk}), (S'_{nk}) which couple, and use a r.c.d. for the path between $S_{(n-1)k}$ and S_{nk} given $S_{(n-1)k}, S_{nk}$ to fill in the gaps.

(ii) Now here's an example which shows that for the Lévy process story, there's a new phenomenon.

We'll construct a symmetric Lévy process on $\mathbb{R}$, with bounded jumps and no Gaussian component, such that the jumps are all $\pm 2^{-n}$ for some $n \geq 0$ and such that the Kesten criterion

$$(3) \qquad \int_{-\infty}^{\infty} Re \, \frac{1}{1+\psi(t)} \, dt < \infty$$

holds. Here, $E\, e^{i\theta X_t} = e^{-t\psi(\theta)}$,

$$(4) \qquad \psi(\theta) = \sum_{n \geq 0} \mu_n \{ 1 - \cos(2^{-n}\theta) \} \geq 0$$

and μ satisfies the integrability condition

$$(5) \qquad \sum_{n \geq 0} \mu_n (2^{-n})^2 = \sum_{n \geq 0} 4^{-n} \mu_n < \infty .$$

Since the Kesten condition holds, two independent copies of the Lévy process with distinct starting points will couple a.s.; but if we were to excise all small jumps, making X, X' such that $X - X'$ were a compound Poisson process, then the step distribution is lattice, and coupling is not certain: one would have to start on the lattice.

We achieve this example as follows. Fix $1 < \alpha < 2$, and take

$$\mu_n \equiv 2^{n\alpha},$$

so that the integrability condition on μ is satisfied. I claim that for large t, for some constant $c > 0$

$$(6) \qquad \psi(t) \geq c\, t^{\alpha}$$

and thus the Kesten criterion is satisfied.

To prove this, note that

$$\psi(2t) = \sum_{n \geq 0} 2^{n\alpha} \{ 1 - \cos(2^{-n+1} t) \}$$

$$(7) \qquad = 1 - \cos 2t + 2^{\alpha} \psi(t) \geq 2^{\alpha} \psi(t).$$

Hence
$$\psi(2n\pi) = 2^{\alpha} \psi(n\pi),$$

and therefore
$$\psi(2^n \pi) = 2^{n\alpha} \psi(\pi).$$

Now ψ is continuous, non-negative, and zero only at $t = 0$, so
$$\inf \{ \psi(t) : \pi \leq t \leq 2\pi \} \equiv \varepsilon > 0.$$

By (7), $\psi(t) \geq 2^{\alpha} \varepsilon$ for $2\pi \leq t \leq 4\pi$, and by induction

$$\psi(t) \geq 2^{n\alpha} \varepsilon \quad \text{for} \quad 2^n \pi \leq t \leq 2^{n+1} \pi.$$

Now (6) follows, and we have the Kesten criterion (3) for coupling.

Eigenvalue analysis of KR model (31/12/92)

(i) The continuous-time model is

$$\begin{cases} dX_t = \sigma\, dW_t + \alpha(X_t - \xi_t)\,dt - \beta(X_t - \eta_t)\,dt \\ d\xi_t = \lambda(X_t - \xi_t)\,dt \\ d\eta_t = \mu\,(X_t - \eta_t)\,dt \end{cases}$$

$$\boxed{\begin{array}{l} a \equiv -\alpha < 0 \\ b \equiv \beta > 0 \end{array}}$$

where $\alpha, \beta, \lambda, \mu$ are all >0, and typically $\lambda < \mu$.

We can rewrite this as

$$dZ_t = \sigma \begin{pmatrix} dW_t \\ 0 \\ 0 \end{pmatrix} + AZ_t\,dt$$

where $Z_t = (X_t, \xi_t, \eta_t)^T$ and

$$A = \begin{pmatrix} \alpha - \beta & -\alpha & \beta \\ \lambda & -\lambda & 0 \\ \mu & 0 & -\mu \end{pmatrix}.$$

The characteristic polynomial of A is

$$P(t) \equiv \boxed{t\left(t^2 + (\lambda + \mu + \beta - \alpha)t + \lambda\mu - \mu\alpha + \lambda\beta\right).}$$

[One can also define $x_t \equiv X_t - \xi_t$, $y_t \equiv X_t - \eta_t$, and then

$$d\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \sigma dW_t \\ \sigma dW_t \end{pmatrix} + \begin{pmatrix} \alpha - \lambda & -\beta \\ \alpha & -\beta - \mu \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} dt$$

and this explains the quadratic part of the characteristic polynomial.]

Are the roots real?

$$(\lambda + \mu + \beta - \alpha)^2 - 4(\lambda\mu - \mu\alpha + \lambda\beta)$$

$$= (b - a + \mu - \lambda)^2 + 4ab$$

and this is non-negative iff

$$|b - a + \mu - \lambda| \equiv |\beta + \alpha + \mu - \lambda| \geqslant 2\sqrt{\alpha\beta}$$

So if we assume $\boxed{\mu > \lambda}$ we certainly have real roots.

<u>Is the process stable?</u> The condition for stability is that both eigenvalues are negative, so

$$\begin{cases} \lambda+\mu+\beta-\alpha>0 \\ \lambda\mu-\mu\alpha+\lambda\beta>0 \end{cases}$$

which is equivalent to

$$\alpha < \lambda+\mu+\beta, \qquad \alpha < \lambda(\mu+\beta)/\mu \, .$$

But since $\mu>\lambda$, the second inequality implies the first, and the stability condition is simply

$$\boxed{\alpha < \lambda(\mu+\beta)/\mu}$$

Let's write the roots of P as $0, -\gamma_1, -\gamma_2$ with $0<\gamma_1<\gamma_2$.

<u>How far does the X process go when it gets a shock?</u> (assuming $\sigma=0$).

Let's assume $X_0=0$, and write $\tilde{X}(s) \equiv \int_0^\infty e^{-st} X_t \, dt$, so that we get

$$\int_0^\infty e^{-st} \dot{\xi}_t \, dt = s\tilde{\xi}(s) - \xi_0$$

and hence

$$s\tilde{X}(s) = (\alpha-\beta)\tilde{X}(s) - \alpha\tilde{\xi}(s) + \beta\tilde{\eta}(s) \, .$$

But $\quad s\tilde{\xi}(s)-\xi_0 = \lambda\{\tilde{X}(s)-\tilde{\xi}(s)\} \quad \therefore \quad \tilde{\xi}(s) = \dfrac{\lambda\tilde{X}(s)+\xi_0}{s+\lambda}$

Thus

$$s\tilde{X} = (\alpha-\beta)\tilde{X} - \frac{\alpha}{\lambda+s}(\lambda\tilde{X}+\xi_0) + \frac{\beta}{\mu+s}(\mu\tilde{X}+\eta_0),$$

so

$$(\lambda+s)(\mu+s)(s+\beta-\alpha)\tilde{X} = -\alpha\lambda(\mu+s)\tilde{X} + \beta\mu(\lambda+s)\tilde{X} - \alpha(\mu+s)\xi_0 + \beta(\lambda+s)\eta_0,$$

whence

$$s\left(s^2+(\lambda+\mu+\beta-\alpha)s+\lambda\mu-\alpha\mu+\beta\lambda\right)\tilde{X} = \beta(\lambda+s)\eta_0 - \alpha(\mu+s)\xi_0$$

or

$$\boxed{\tilde{X}(s) = \frac{\beta(\lambda+s)\eta_0 - \alpha(\mu+s)\xi_0}{s(s+\gamma_1)(s+\gamma_2)}}.$$

$$
\begin{aligned}
X_{n+1} - X_n = \sigma \varepsilon_{n+1} &+ \alpha \left(X_n - \lambda \sum_{r=0}^{n} \bar{\lambda}^r X_{n-r} - \bar{\lambda}^{n+1} \xi_0 \right) \\
&- \beta \left(X_n - \mu \sum_{r=0}^{n} \bar{\mu}^r X_{n-r} - \bar{\mu}^{n+1} \eta_0 \right)
\end{aligned}
$$

Observe that

$$X_\infty = \lim_{s\downarrow 0} s\tilde{X}(s) = \frac{\beta\lambda\eta_0 - \alpha\mu\xi_0}{\gamma_1\gamma_2} = \frac{\beta\lambda\eta_0 - \alpha\mu\xi_0}{\lambda\mu - \mu\alpha + \lambda\beta}$$

Also

$$\tilde{X}(s) = \frac{A}{s} + \frac{B}{s+\gamma_1} + \frac{C}{s+\gamma_2},$$

where

$$\left.\begin{aligned} A &= (\beta\lambda\eta_0 - \alpha\mu\xi_0)/\gamma_1\gamma_2 \\ B &= \frac{\alpha(\mu-\gamma_1)\xi_0 - \beta(\lambda-\gamma_1)\eta_0}{\gamma_1(\gamma_2-\gamma_1)} \\ C &= \frac{\beta(\lambda-\gamma_2)\eta_0 - \alpha(\mu-\gamma_2)\xi_0}{\gamma_2(\gamma_2-\gamma_1)} \end{aligned}\right\}$$

(ii) Let us now similarly analyse the discrete-time model:

$$\begin{cases} X_{n+1} - X_n = \sigma\varepsilon_{n+1} + \alpha(X_n - \xi_n) - \beta(X_n - \eta_n) \\ \xi_{n+1} - \xi_n = \lambda(X_n - \xi_n) \\ \eta_{n+1} - \eta_n = \mu(X_n - \eta_n) \end{cases}$$

where $0 < \alpha, \beta$, $0 < \lambda < \mu < 1$. Write $\bar{\lambda} \equiv 1-\lambda$, $\bar{\mu} \equiv 1-\mu$, $a = -\alpha < 0$, $b \equiv \beta > 0$.

We have

$$\begin{pmatrix} X_{n+1} \\ \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \sigma\varepsilon_{n+1} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1+\alpha-\beta & -\alpha & \beta \\ \lambda & \bar{\lambda} & 0 \\ \mu & 0 & \bar{\mu} \end{pmatrix}\begin{pmatrix} X_n \\ \xi_n \\ \eta_n \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \sigma\varepsilon_{n+1} \\ \sigma\varepsilon_{n+1} \end{pmatrix} + \begin{pmatrix} \alpha+\bar{\lambda} & -\beta \\ \alpha & -\beta+\bar{\mu} \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

where $x_n \equiv X_n - \xi_n$, $y_n = X_n - \eta_n$. Stability corresponds to the e-values of the matrix

$$\begin{pmatrix} \alpha + \bar{\lambda} & -\beta \\ \alpha & -\beta + \bar{\mu} \end{pmatrix}$$ lying inside unit circle.

Characteristic polynomial is

$$\boxed{Q(t) \equiv t^2 - t(\bar{\mu} + \bar{\lambda} - b - a) + (\bar{\lambda}\bar{\mu} - b\bar{\lambda} - a\bar{\mu})}$$

Does this have real roots? Yes, because

$$(\bar{\mu} + \bar{\lambda} - b - a)^2 - 4(\bar{\lambda}\bar{\mu} - b\bar{\lambda} - a\bar{\mu})$$

$$= \bar{\mu}^2 + \bar{\lambda}^2 + b^2 + a^2 - 2\bar{\lambda}\bar{\mu} - 2\bar{\mu}b + 2\bar{\mu}a + 2\bar{\lambda}b - 2\bar{\lambda}a + 2ab$$

$$= (b - a - \bar{\mu} + \bar{\lambda})^2 + 4ab$$

and $\bar{\lambda} > \bar{\mu}$, $b > 0 > a$, so it's just as for the continuous case. □

Let's define

$$\boxed{\begin{aligned} \varphi_1 &\equiv \bar{\mu} + \bar{\lambda} - b - a \\ \varphi_2 &\equiv b\bar{\lambda} + a\bar{\mu} - \bar{\lambda}\bar{\mu} \end{aligned}}$$

so that $$\boxed{Q(t) = t^2 - \varphi_1 t - \varphi_2}$$

the form in which it is used in Splus.

When is this stable? Stability is the condition that the roots of Q are in $(-1,1)$. Now if this model arose as a discretisation of the continuous model, we would also have both roots >0. The conditions are

$$\boxed{\begin{aligned} \varphi_2 < 0 < \varphi_1 < 2 \\ \varphi_1 + \varphi_2 < 1 \end{aligned}}$$ $\Leftrightarrow$ both roots of Q are in $(0,1)$

In terms of the original parameters, these are

$$(*) \quad \boxed{\bar{\lambda}\bar{\mu} - a\bar{\mu} - b\bar{\lambda} > 0, \quad 0 < a + b + \lambda + \mu < 2, \quad \lambda\mu + a\mu + b\lambda > 0.}$$

Where does X go after an initial shock when $\sigma = 0$?

If
$$A \equiv \begin{pmatrix} 1+\alpha-\beta & -\alpha & \beta \\ \lambda & \bar{\lambda} & 0 \\ \mu & 0 & \bar{\mu} \end{pmatrix}$$

a left eigenvector of eigenvalue 1 is $(\lambda\mu, \; \mu a, \; \lambda\beta)$, so

$$A^n \to \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (\lambda\mu \;\; \mu a \;\; \lambda\beta) \quad (\lambda\mu + \mu a + \lambda\beta)^{-1} \qquad (n \to \infty$$

and thus the limit of X_n with $X_0 = 0, \; \xi_0, \eta_0$ will be simply

$$\boxed{\frac{\mu a \, \xi_0 + \lambda b \, \eta_0}{\lambda\mu + \mu a + \lambda\beta}}$$

Equilibrium of the continuous-time process.

Back to the process (x_t, y_t) on page 17. We have

$$\begin{cases} dx_t^2 = 2x_t(\sigma dW_t - (a+\lambda)x_t \, dt - b y_t \, dt) + \sigma^2 dt \\ d(x_t y_t) = (x_t + y_t)\sigma dW_t - (a+\lambda+b+\mu)x_t y_t \, dt - (b y_t^2 + a x_t^2) \, dt + \sigma^2 dt \\ d(y_t^2) = 2y_t(\sigma dW_t - (b+\mu) y_t \, dt - a x_t \, dt) + \sigma^2 dt \end{cases}$$

So if

$$u \equiv \lim_{t\to\infty} E x_t^2, \qquad w \equiv \lim_{t\to\infty} E x_t y_t, \qquad v \equiv \lim_{t\to\infty} E(y_t^2)$$

we must have

$$\begin{cases} \tfrac{1}{2}\sigma^2 = (a+\lambda)u + b w \\ \sigma^2 = a u + (a+\lambda+b+\mu) w + b v \\ \tfrac{1}{2}\sigma^2 = a w + (b+\mu) v \end{cases}$$

These can be solved in a reasonably simple form;

$$u = \frac{\sigma^2(\mu^2 + \gamma_1\gamma_2)}{2\gamma_1\gamma_2(\gamma_1+\gamma_2)}, \quad w = \frac{\sigma^2(2\lambda\mu + a\mu + b\lambda)}{2\gamma_1\gamma_2(\gamma_1+\gamma_2)}, \quad v = \frac{\sigma^2(\lambda^2+\gamma_1\gamma_2)}{2\gamma_1\gamma_2(\gamma_1+\gamma_2)}$$

Another example from the wonderful world of EMM (7/1/93)

Suppose that $X_t \equiv B_t + \frac{1}{2}t$ is a price process; can we make arbitrage with bounded risk? It seems that we can! Take $\xi_n = -1 + 2^{-n}$ $(n \in \mathbb{Z}^+)$ and now make up a stochastic integral

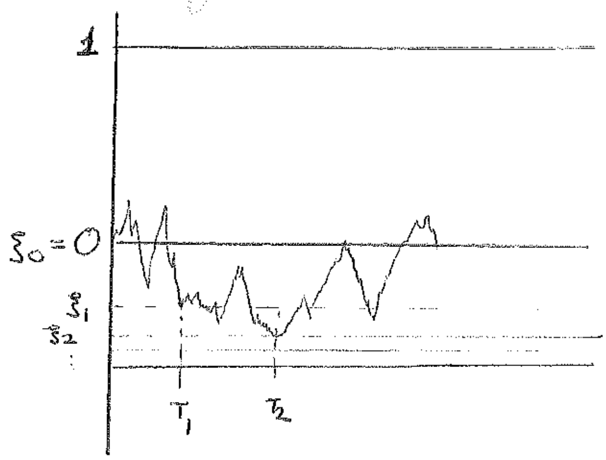
$$Y_t = \int_0^t \theta_u \, dX_u$$

where θ is a simple process. Define $T_0 \equiv 0$,

$$T_{n+1} = \inf\{t > T_n : a_n(X_t - X_{T_n}) = 2 - 2^{-n} \text{ or } -2^{-n-1}\}$$

where $a_0 = 1 > a_1 > a_2 > \dots$ will be chosen shortly. We take

$$\theta_t = a_n \quad \text{on } (T_n, T_{n+1}]$$



Since the scale f^n $s(t) = -e^{-t}$, we have

$$P(Y_{T_{n+1}} = \xi_{n+1} \mid Y_{T_n} = \xi_n)$$

$$= \frac{s\left(\frac{2-2^{-n}}{a_n}\right) - s(0)}{s\left(\frac{2-2^{-n}}{a_n}\right) - s\left(-\frac{2^{-n-1}}{a_n}\right)}$$

$$= \frac{1 - \exp((2^{-n}-2)/a_n)}{\exp(+2^{-n-1}/a_n) - \exp\left(\frac{2^{-n}-2}{a_n}\right)}$$

$$= \frac{\exp\left(-\frac{2^{-n-1}}{a_n}\right) - \exp(-2/a_n)\, e^{2^{-n-1}/a_n}}{1 - \exp(-2/a_n)\, e^{2^{-n-1}/a_n}}$$

$\boxed{\text{PTO}}$

When we take $\boxed{a_n = 2^{-n-1}/\log n^2 \equiv 2^{-n-2}/\log n}$ the sum of the terms is finite, therefore arbitrage.

And yet another! If R is a $BES(3)$ process, $R_0 = 1$, and

$$h(t,x) = \Phi\left(-x/\sqrt{1-t} \right)$$

then $0 \le h \le 1$, and

$$h(t, R_t) = h(0, R_0) + \int_0^t \frac{\partial h}{\partial x}(s, R_s)\, dR_s$$

Rather amazingly,

$$h(t, R_t) \xrightarrow[t \uparrow 1]{} 0 \quad , \text{ even though } h(0,1) > 0, \quad 0 \le h \le 1 \;!!$$

So $BES(3)$, though it's an example of no ELMM, doesn't satisfy NABR, because here we see an arbitrage.

How does a pure-exchange economy work? (10/1/93)

A) Suppose we have d commodities, N agents, the j^{th} agent has utility $U_j(x)$ for endowment $x \in \mathbb{R}^d$. Assume the commodities are continuous. We shall consider continuous trading governed by the principle:

(1) $\boxed{\text{When agents trade together, everyone's utility rises by the same amount.}}$

Thus if agent j's holdings follow trajectory $x_j(t)$, must have

$$\sum_{j=1}^{N} \dot{x}_j(t) \equiv 0 \quad , \quad \left(\dot{x}_j(t), \nabla U_j(x_j(t)) \right) = 1 \quad \text{for all } j.$$

This gives $N+d$ equations for the Nd unknowns $\dot{x}_j(t)$. Thus there are in general many ways of achieving (1), so we take as our next principle:

(2) $\boxed{\text{Utility gains are achieved with minimal transfer of commodity.}}$

Thus we would attempt to $\quad (v_j \equiv \dot{x}_j)$

$$\min \tfrac{1}{2} \sum_{j=1}^{N} c_j^{-1} |v_j|^2 \quad \text{subj to} \quad \sum_{j=1}^{N} v_j = 0 \quad , \quad (v_j, w_j) = 1 \text{ for all } j$$

where

$$w_j \equiv \nabla U_j(x_j),$$

and the c_j are positive constants (perhaps all equal), $\sum c_j = 1$. Lagrangian form:

$$\min_{v} \tfrac{1}{2} \sum c_j^{-1} |v_j|^2 - \lambda \sum v_j + \sum \theta_j (1 - v_j^T w_j),$$

achieved when

$$v_j = c_j (\lambda + \theta_j w_j).$$

Thus

$$\boxed{\lambda = -\sum_{j=1}^{N} c_j \theta_j w_j, \qquad \theta_j = \frac{1 - c_j w_j^T \lambda}{c_j |w_j|^2}}$$

Note one important consequence of this. If we start with allocation (x_j) and (y_j) is allocation which maximises $U_j(y_j) - U_j(x_j)$ s.t. same for all j, then we cannot stop trading at some other allocation (z_j), because $(y_j - z_j)$ would be a feasible direction, and we could move away from z and improve everyone equally.

$$|v_1|^2 = \frac{|w_1 + w_2|^2}{|w_1|^2\,|w_2|^2 - (w_1 \cdot w_2)^2}$$

This gives

$$\left(I - \sum c_j \frac{w_j w_j^T}{|w_j|^2} \right) \lambda = -\sum_1^N w_j / |w_j|^2$$

$$\equiv M \lambda,$$

for brevity. Notice that $My = 0 \Rightarrow y^T M y = 0 \Rightarrow$ all w_j are collinear and lie along y. This is the known (equilibrium) situation where no improvement is possible, but otherwise we can ignore $\det M = 0$, and conclude

$$\boxed{\lambda = -M^{-1} \sum_{j=1}^N w_j / |w_j|^2 \equiv -\left(I - \sum c_j \frac{w_j w_j^T}{|w_j|^2} \right)^{-1} \sum \frac{w_j}{|w_j|^2}}$$

Hence

$$\boxed{v_j = \frac{w_j}{|w_j|^2} + c_j \left(I - \frac{w_j w_j^T}{|w_j|^2} \right) \lambda .}$$

3) In the special case $N = 2$, $c_1 + c_2 = 1$, and

$$M^{-1} = I + \frac{1}{c_1 c_2 (1-\rho^2)} \left\{ c_1^2 z_1 z_1^T + c_1 c_2 \rho (z_1 z_2^T + z_2 z_1^T) + c_2^2 z_2 z_2^T \right\}$$

where $z_j \equiv w_j / |w_j|$, $\rho = z_1^T z_2$. After a bit of algebra, we get

$$\boxed{v_1 = -v_2 = \frac{(w_1 w_2^T - w_2 w_1^T)}{|w_1|^2 |w_2|^2 - (w_1 \cdot w_2)^2} (w_1 + w_2)}$$

Notice that this does not depend on c_1, c_2, slightly surprisingly!

Remark: The minimisation problem we have set ourselves assumes a possible asymmetry among agents (c_j not all same), which is perhaps less natural than assuming an asymmetry among commodities. Then we should try to

$$\min \tfrac{1}{2} \sum c_j^{-1} v_j^T S v_j \quad \text{subj to} \quad \sum v_j = 0, \quad (v_j, w_j) = 1 \text{ for all } j.$$

Here, of course, S is a pds (diagonal) matrix, and the solution is easily

Who trades with whom, and how?

Discrete assets: need a story that has this as limiting form

If we don't assume $V_{tt} = I$ for all t, then the condition is

$$V_{su} = V_{st} V_{tt}^{-1} V_{tu} \qquad \text{for} \quad 0 \le s \le t \le u$$

deduced from the foregoing. If $S \neq I$, we could reduce to that situation by a linear transformation of $\mathbb{R}^d$ and the utilities, so it's not really any more general!

Gauss-Markov processes in $\mathbb{R}^n$ (31/1/93)

Let us try to find the most general process $(X_t)_{t \geq 0}$ which is Gaussian and (time-inhomogeneous) Markov. Also, it would be nice to characterise the processes with continuous paths.

(i) If X is such a process, then so is $a_t + K_t X_t$ for any functions a with values in $\mathbb{R}^n$, K with values in $GL(n,\mathbb{R})$ (or even K which are singular). Thus we lose no generality in assuming

$$\boxed{E X_t = 0 \quad \forall t, \qquad \operatorname{cov}(X_t, X_t) = I \quad \text{for all } t.}$$

(it could be that the covariance of X sometimes singular, but let's ignore this possibility, + characterise a restricted class of processes)

(ii) Let's write

$$V_{st} \equiv V(s,t) \equiv E(X_s X_t^T) \qquad (0 \leq s \leq t)$$

with $V(0,t) \equiv V_t$.

PROPOSITION. For X to be Markovian, it is necessary and sufficient that

$$\boxed{V_{su} = V_{st} V_{tu}} \qquad \text{for all } 0 \leq s \leq t \leq u$$

Proof. We have for $t \leq u$

$$\begin{pmatrix} X_t \\ X_u \end{pmatrix} \sim N\left(0, \begin{pmatrix} I & V_{tu} \\ V_{tu}^T & I \end{pmatrix}\right)$$

so that

$$(X_u | X_t) \sim N(V_{tu}^T X_t, \; I - V_{tu}^T V_{tu}).$$

If X is Markovian, then $E(X_u | \mathcal{F}_t) = V_{tu}^T X_t$, so for $s \leq t$,

$$0 = E X_s (X_u - V_{tu}^T X_t)^T = V_{su} - V_{st} V_{tu},$$

<u>Note</u> If A, B are real symmetric, then $A \leq B \Leftarrow A^2 \leq B^2$

"$\Leftarrow$" Rtp $x^TAx \leq x^TBx$ for all x; so if false, we minimise $x^T(B-A)x$

over $|x|=1$, and find e-vector y, $(B-A)y = -\rho y$, so $Ay = (B+\rho)y$.

Hence
$$y^TA^2y = y^T(B^2 + 2\rho B + \rho^2)y > y^TB^2y \quad ※$$

Surprisingly, the converse is false. To see this, we write $B = A + \Delta$, where $\Delta > 0$, symmetric, and note that
$$B^2 - A^2 = \Delta^2 + A\Delta + \Delta A$$
So if we replaced Δ by $\varepsilon\Delta$, ε very small, for the result to hold it would be necessary that $A\Delta + \Delta A \geq 0$. However, the example
$$A = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with } \rho = 15/16, \text{ say,}$$
is a counter-example.

establishing necessity. However, it is also not difficult to establish sufficiency. Indeed, if we take $s_1 < s_2 < \cdots < s_n \le t$, and consider

$$E \exp\left[\sum \alpha_j X(s_j) + \beta (X_u - V_{tu}^T X_t) \right]$$

$$= \exp\left[\tfrac{1}{2} (\alpha^T, \beta)\, \Sigma \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right]$$

where Σ is the covariance matrix of $(X_{s_1}, \dots, X_{s_n}, X_u - V_{tu}^T X_t)$, then the condition of the Proposition implies that Σ partitions as

$$\Sigma = \left(\begin{array}{c|c} \Sigma_0 & 0 \\ \hline 0 & c \end{array} \right)$$

and so $X_u - V_{tu}^T X_t$ is independent of $(X_{s_1}, \dots, X_{s_n})$, and hence of $\mathcal{F}_t$. That's all. □

(ii) The next result gives a better picture of what the covariance structure is like.

PROPOSITION. (i) <u>If V_{st} is the covariance structure of a Gauss-Markov process with identity variance</u>, <u>then</u>

$$\boxed{V_s V_s^T \equiv Q_s \text{ is nnd, decreasing, } Q_0 = I.}$$

(ii) <u>Given some nnd symmetric decreasing function</u> Q, $Q_0 = I$, <u>then there is a Gauss-Markov process with</u>

$$V_s = Q_s^{1/2}.$$

<u>Proof</u>. (i) Since $\begin{pmatrix} I & V_{st} \\ V_{st}^T & I \end{pmatrix} \ge 0$

by left multiplying by $\begin{pmatrix} V_s & \\ & I \end{pmatrix}$ and right-multiplying by its transpose, we learn that $\begin{pmatrix} V_s V_s^T & V_t \\ V_t^T & I \end{pmatrix} \ge 0$. This implies $V_s V_s^T \ge V_t V_t^T$.

(ii) Set $V_s \equiv Q_s^{1/2}$, and $V_{st} \equiv Q_s^{-1/2} Q_t^{1/2}$ for $s \le t$; in view of the fact that Q decreases, this is well defined even though $Q_s^{-1/2}$ is not in general well defined. We have immediately that

$$V_{su} = V_{st} V_{tu} \qquad 0 \le s \le t \le u,$$

but what we don't yet know is that these things make the covariance structure of some Gaussian process. To show this, take $0 \le t_1 < \dots < t_n$, and consider

$$\begin{pmatrix} I & V_{12} & V_{13} & \cdots & V_{1n} \\ V_{12}^T & I & V_{23} & \cdots & V_{2n} \\ V_{13}^T & V_{23}^T & I & \cdots & V_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ V_{1n}^T & V_{2n}^T & V_{3n}^T & \cdots & I \end{pmatrix} \equiv M,$$

where $V_{ij} \equiv V(t_i, t_j)$, and $V_i = V(0, t_i)$. Then we have that

$$M = S \begin{pmatrix} V_1V_1^T & V_2V_2^T & V_3V_3^T & \cdots & V_nV_n^T \\ V_2V_2^T & V_2V_2^T & V_3V_3^T & \cdots & V_nV_n^T \\ V_3V_3^T & V_3V_3^T & V_3V_3^T & \cdots & V_nV_n^T \\ \vdots & \vdots & & & \\ V_nV_n^T & V_nV_n^T & V_nV_n^T & \cdots & V_nV_n^T \end{pmatrix} S^T \equiv S\tilde{M}S^T$$

where $S = \begin{pmatrix} V_1^{-1} & & & \\ & V_2^{-1} & & \\ & & \ddots & \\ & & & V_n^{-1} \end{pmatrix}$ (The fact that V_j may not be invertible causes no problems: replace Q by $Q + \varepsilon I$ and proceed as we wish, then let $\varepsilon \downarrow 0$ at the end)

But it is easy to prove that $\tilde{M}$ is nnd, so M is nnd, and this definition of V really does give us a Gauss-Markov process.

(iii) Let's now assume that V is C^1, $\dot{V}_t = V_t v_t$. Then

$$E(X_t - X_s \mid \mathcal{F}_s) = (V_{st}^T - I) X_s = (V_t^T (V_s^T)^{-1} - I) X_s$$

$$= (t-s) \quad \dot{V}_s^T (V_s^{-1})^T X_s \; + \; o(t-s)$$

$$= (t-s) \quad v_s^T X_s + o(t-s).$$

Hence $\boxed{X_t - \int_0^t v_s^T X_s \, ds \text{ is a Gaussian martingale.}}$

It is not hard to confirm that the covariance structure is given by

$$\boxed{d\langle X, X^T\rangle_t = -(v_t + v_t^T)}$$

which is nnd because $\dot{Q} = V(v + v^T)V^T$ is non-positive definite.

Thus X solves the SDE

$$\boxed{dX_t = v_t^T X_t \, dt + (-(v_t + v_t^T))^{\frac{1}{2}} \, dB_t}$$

More generally, if we took instead $K_t X_t$, where K was C^1, we would find that X solved

$$(*) \quad \boxed{dX_t = C_t X_t \, dt + \sigma_t \, dB_t}$$

for matrix-valued functions C, σ; and, provided there's no explosion, these things are all Gauss-Markov processes.

(iv) Fixing T, and assuming X satisfies $(*)$, let's compute

$$E\left[\exp\left\{-\tfrac{1}{2}\int_t^T |X_u - \alpha_u|^2 \, du\right\} \,\middle|\, X_t = x\right] \equiv \psi(t, x).$$

The fact that $\exp(-\frac{1}{2}\int_0^t |X_u - \alpha_u|^2 du)\, \psi(t, X_t)$ is a martingale, together with the conjectured form

$$\psi(t, x) = \exp\left(-\tfrac{1}{2}(x - \beta_t)^T Q_t (x - \beta_t) - K_t\right)$$

lead to a system of differential equations for Q, β and K, Model I $\quad dX_t = \sigma_t \, dB_t + C_t X_t \, dt \quad , \quad r_t = \frac{1}{2} |X_t - \alpha_t|^2$

If μ_t is the mean of X_t, V_t its variance, we shall have

$$\dot{\mu}_t = a_t + C_t \mu_t , \quad \dot{V}_t = V_t C_t^T + C_t V_t + \sigma_t \sigma_t^T .$$

Model II $\quad d\tilde{X}_t = \sigma_t \, dB_t + (a_t + C_t \tilde{X}_t) \, dt \quad , \quad r_t \equiv \frac{1}{2} |\tilde{X}_t|^2$

$$\boxed{\begin{aligned} & I + QC + (QC)^T + \dot{Q} - Q\sigma\sigma^T Q^T = 0 \qquad \text{Model I} \\ & (I + QC)\beta = \alpha + Q\dot{\beta} \\ & \tfrac{1}{2}|a|^2 + \dot{\kappa} + \tfrac{1}{2}\beta^T Q \dot{\beta} = \tfrac{1}{2}\beta^T\alpha + \tfrac{1}{2}\beta^T C^T Q \beta - \tfrac{1}{2}\operatorname{tr}(\sigma^T Q \sigma) \end{aligned}}$$

$\boxed{\dot{\kappa} + \tfrac{1}{2}\operatorname{tr}(\sigma^T Q\sigma) + \tfrac{1}{2}|\beta - \alpha|^2 = 0}$ is a simpler form of the last.

(V) Remarks (a) The apparently more general SDE

$$(**) \qquad \boxed{dX_t = (a_t + C_t X_t)\,dt + \sigma_t\, dB_t}$$

can be reduced to the above, because if $\xi_t \equiv EX_t$, we have $\dot{\xi} = a + C\xi$, so

$$d(X - \xi) = C(X - \xi)\,dt + \sigma\, dB$$

(b) The class of processes of the form

$$Y_t = \sigma_t\,(W_t + v_t)$$

is strictly smaller, because $dY = \sigma\, dW + \dot{\sigma} W\, dt + (\dot{\sigma} v + \sigma \dot{v})dt$ $= \sigma\, dW + \dot{\sigma}\sigma^{-1} Y\, dt + \sigma\dot{v}\, dt$ shows that the C of the above model is actually determined by σ! Thus the one-dimensional OU process is impossible.

(Vi) The most general interest-rate process we are going to consider is $r_t = \frac{1}{2}|X_t|^2$, where X is as at $(**)$. So, as above, with

$$\psi(t,x) \equiv E\left[\exp\left(-\tfrac{1}{2}\int_t^T |X_u|^2\, du\right) \Big| X_t = x\right] \equiv \exp\left\{-\tfrac{1}{2}(x - b_t)^T Q_t (x - b_t) - \gamma_t\right\}$$

we obtain similarly

$$\boxed{\begin{aligned} & I + QC + (QC)^T + \dot{Q} - Q\sigma\sigma^T Q^T = 0 \qquad \text{Model II} \\ & (I + QC)\, b = Q(\dot{b} - a) \\ & \dot{\gamma} + \tfrac{1}{2} b^T Q \dot{b} = \tfrac{1}{2} b^T Q a + \tfrac{1}{2} b^T Q C b - \tfrac{1}{2}\operatorname{tr}(\sigma^T Q \sigma) \end{aligned}}$$

The last simplifies to $\boxed{\dot{\gamma} + \tfrac{1}{2}\operatorname{tr}(\sigma^T Q\sigma) + \tfrac{1}{2}|b|^2 = 0}$

(vii) As the one-dimensional example with $\sigma \equiv 1$, $C = 0 \equiv a$ will show, we have

$$Q(t,T) = \tanh(T-t)$$

and the de for b is

$$b_t = \dot{b}_t \tanh(T-t)$$

has the solution $b_t = \lambda \operatorname{cosech}(T-t)$ for every $\lambda \in \mathbb{R}$!! This non-uniqueness is resolved by looking at the third equation, which shows that we are only interested in solutions b such that

$$\boxed{\int^T |b_t|^2 \, dt < \infty .}$$

Now if δ_t is the difference between two such solutions, we have

$$\dot{\delta} = (Q^{-1} + C)\, \delta$$

and $Q(t,T) \simeq (T-t)I$ for t near T, so we get

$$\delta_t^T \dot{\delta}_t = \delta_t^T \left(Q(t,T)^{-1} + C(t,T) \right) \delta_t$$

$$\geq \frac{\rho}{T-t} |\delta_t|^2$$

for all t near enough to T, where $\rho \in (0,1)$. Thus for t very close to T,

$$\frac{d}{dt}\left(\tfrac{1}{2} |\delta_t|^2 \right) \geq \frac{\rho}{T-t} |\delta_t|^2$$

which implies that for t near to T

$$|\delta_t|^2 \geq \frac{c}{(T-t)^{2\rho}}$$

which violates the integrability assumption if $\rho > \frac{1}{2}$. Thus there is at most one solution to the differential equation satisfying also the integrability condition. Indeed, a similar argument proves that if (for example) a is bounded near T and C is bounded near T, then necessarily $b_t \to 0$ $(t \to T-)$.

If $k_t = q_t^{-1} + B$, then we have

$$\dot{k}_t = (q_t^{-1})^{\circ} = -k_t k_t^T + CC^T + \sigma\sigma^T = S - k_t k_t^T$$

Notice also that if $M(t,T) = m(T-t)$, then

$$\dot{m} = m k$$

and so $m^{-1}\dot{m} = k$. Differentiating once more,

$$-m^{-1}\dot{m}m^{-1}\dot{m} + m^{-1}\ddot{m} = \dot{k} = S - k^2 \qquad (\underline{\text{assuming}}\ C = C^T)$$

$$= S - (m^{-1}\dot{m})^2$$

and hence

$$\boxed{\ddot{m} = mS}$$

(viii) Let us now make the simplifying assumption that σ, C <u>are constant</u>, and try to obtain the term structure more explicitly.

We have that

$$\boxed{Q(t,T) = q(T-t),}$$

where

$$1 + qC + C^T q - \dot{q} - q\sigma\sigma^T q = 0 \quad , \quad q(0) = 0.$$

Now

$$\dot{b} = a + (Q^{-1} + C) b,$$

so if

$$\boxed{\dot{M} = -M(Q^{-1} + C),}$$

we have

$$\frac{d}{dt}(Mb) = Ma,$$

and so

$$b(t,T) = M(t,T)^{-1}\left[M(T,T)\, b(T,T) - \int_t^T M(s,T)\, a_s \, ds \right].$$

Now the de. for M has a possible explosion at $t = T$. To handle this more carefully, define $Y(t,T) \equiv M(t,T)\, Q(t,T)^{-1}$, and observe that

$$\begin{aligned}
\frac{d}{dt} Y &= \dot{M} Q^{-1} - M Q^{-1} \dot{Q} Q^{-1} \\
&= \left[-M(Q^{-1}+C) + MQ^{-1}\left(1 + QC + C^T Q - Q\sigma\sigma^T Q\right) \right] Q^{-1} \\
&= MQ^{-1}\left(C^T Q - Q\sigma\sigma^T Q \right) Q^{-1} \\
&= Y\left(C^T - Q\sigma\sigma^T \right),
\end{aligned}$$

so if we suppose that $Y(t,T)$ solves $\boxed{\frac{\partial Y}{\partial t} = Y(C^T - Q\sigma\sigma^T), \quad Y(T,T) = I,}$ then we can express

$$M(t,T) = Y(t,T)\, Q(t,T),$$

and hence

$$\boxed{b(t,T) = -M(t,T)^{-1} \int_t^T M(s,T)\, a_s \, ds.}$$

<u>Note</u>: We have $M(t,T) = m(T-t)$.

$$\dot{q}(t) = \left(q(t)/m(t)\right)^2, \quad \text{where} \quad m(t) \equiv \lambda^{-1} \sinh \lambda t. \longrightarrow$$

$$= \left(\cosh \lambda t - \frac{c}{\lambda} \sinh \lambda t\right)^{-2}$$

(ix) One situation we may be able to do something with is if $\boxed{q_t = e^{\theta t} a}$. Assuming this,

$$b(t,T) = -m(T-t)^{-1}\left(\int_t^T m(T-s)\, e^{\theta s}\, ds\right) a$$

and so if $m_\theta(t) \equiv e^{-\theta t} m(t)$, we have $\dot m_\theta = m_\theta(q^{-1} + C - \theta)$, and

$$e^{-\theta t}\, b(t,T) = -m_\theta(T-t)^{-1}\left(\int_0^{T-t} m_\theta(s)\, ds\right) a.$$

This gives $\boxed{b(t,T) \equiv e^{\theta t}\, b(0,T-t)}$

If we now abbreviate $b(0,t) \equiv \beta(t)$, we obtain (differentiating wrt t)

$$\frac{\partial b}{\partial t}(t,T) = a e^{\theta t} + \left(q^{-1}(T-t) + C\right) b(t,T) = \theta e^{\theta t}\beta(T-t) - e^{\theta t}\beta'(T-t)$$

and so

$$\boxed{a + \left(q^{-1}(s) + C\right)\beta(s) = \theta\beta(s) - \beta'(s)}$$

This reduces the acreage a bit!

(x) Let's now consider the <u>one-dimensional case</u> <u>with σ, c</u> <u>constant</u>. Here we solve

$$\dot q = 1 + 2cq - \sigma^2 q^2 \quad , \quad q(0) = 0$$

$(\lambda^2 = \sigma^2 + c^2)$

which yields

$$\boxed{q(t) = (1-e^{-\rho t})\left[\rho - \alpha\sigma^2(1-e^{-\rho t})\right]^{-1} = (\lambda \coth \lambda t - c)^{-1}}$$

where α, β are the roots of $\sigma^2 t^2 - 2ct - 1 = 0$, $\rho \equiv \sigma^2(\alpha - \beta)$. (In the case of $c=0$, we have $\alpha = \sigma^{-1}$, $\beta = -\sigma^{-1}$, $\rho = 2\sigma$, $q(t) = \frac{1}{\sigma}\tanh \sigma t$) Hence

$$\boxed{\begin{aligned} m_\theta(t) &= e^{(c-\theta-\alpha\sigma^2)t}\,(e^{\rho t} - 1) \\ &= 2e^{-\theta t}\sinh(\rho t/2), \end{aligned}}$$

$$\boxed{\beta(t) = -\frac{e^{\theta t}}{2\sinh \rho t/2}\left\{\frac{1-e^{-(\theta-\rho/2)t}}{\theta - \rho/2} - \frac{1-e^{-(\theta+\rho/2)t}}{\theta+\rho/2}\right\} a}$$

What sort of forward-rate curves can one get here?

$$f_{tT} = -\frac{\partial b}{\partial T}\cdot q\,(x-b) + \tfrac{1}{2}\,(x-b)\quad \dot{q}\,(x-b) \quad + \frac{\partial \gamma}{\partial T}$$

$$= -\frac{\partial b}{\partial T}\cdot q\,(x-b) + \tfrac{1}{2}\,(x-b)\quad \mathring{q}\,(x-b) + \tfrac{1}{2}\sigma^2 q + \tfrac{1}{2}e^{2\theta t}\beta^2$$
$$+\ \theta e^{2\theta T}\int_0^{T-t} e^{-2\theta u}\beta(u)^2\,du$$

The dependence of this on θ, for example, or c, is probably too complicated to use.

(xi) Let's now assume that $\boxed{C \text{ is symmetric.}}$ This gives us that

$$\boxed{\ddot{m} = m\,S} \qquad \text{where } S \equiv \sigma\sigma^T + c\,c^T.$$

If now we write

$$S = R\,\Lambda^2 R^T,$$

where Λ is diagonal, positive-definite, and R is orthogonal, we obtain simply

$$\ddot{m}R = mR\,\Lambda^2$$

and we have the boundary conditions $m(0)=0$, $\dot{m}(0)=I$, so the unique solution is

$$\boxed{m(t) = R\,\Lambda^{-1}\sinh(t\Lambda)\,R^T\ .}$$

Now if we differentiate the representation

$$b(t,T) = -m(T-t)^{-1}\int_t^T m(T-s)\,q_s\,ds$$

of b with respect to T, we want

$$\frac{\partial}{\partial T}\left[-m(T-t)^{-1}\,m(T-s)\right] = \frac{\partial}{\partial T}\left[-R\,\left(\sinh(T-t)\Lambda\right)^{-1}\sinh((T-s)\Lambda)\ R^T\right]$$

$$= R\left\{\left(\sinh(T-t)\Lambda\right)^{-2}\Lambda\Big[-\cosh(T-s)\Lambda\,\sinh(T-t)\Lambda + \cosh(T-t)\Lambda\,\sinh(T-s)\Lambda\Big]\right\}R^T$$

$$= R\,\left(\sinh(T-t)\Lambda\right)^{-2}\Lambda\quad\sinh(t-s)\Lambda\quad R^T$$

$$= -\,m(T-t)^{-2}\quad m(s-t)$$

This now allows us the expression

$$\boxed{\frac{\partial b}{\partial T}(t,T) = -m(T-t)^{-2} \int_t^T M(s-t)\, a_s \, ds.}$$

In the expression for $\gamma(t,T)$, we see (reducing wlog to the one-dimensional case)

$$\tfrac{1}{2}\int_t^T |b(s,T)|^2 ds = \tfrac{1}{2}\int_t^T ds \int_s^T du \int_s^T dv \; a_u a_v \frac{\sinh\lambda(T-u)\sinh\lambda(T-v)}{\sinh^2\lambda(T-s)}$$

$$= \int_t^T du \int_u^T dv \; a_u a_v \sinh\lambda(T-u)\sinh\lambda(T-v) \int_t^u ds \frac{1}{\sinh^2\lambda(T-s)}$$

$$= \int_t^T du \int_u^T dv \, a_u a_v \frac{\sinh\lambda(T-v)\sinh\lambda(u-t)}{\sinh\lambda(T-t)} \cdot \frac{1}{\lambda}.$$

This is easily differentiated wrto T to give

$$\boxed{\begin{aligned} &\int_t^T dv \int_t^v du \; a_u a_v \frac{\sinh\lambda(u-t)\sinh\lambda(v-t)}{\sinh^2\lambda(T-t)} \\ &= \tfrac{1}{2}\operatorname{cosech}^2\lambda(T-t)\left(\int_t^T a_u \sinh\lambda(u-t)\,du\right)^2, \end{aligned}}$$

which is a worthwhile simplification!!

We can also link the partial derivatives of b wrto t and T. Indeed,

$$\frac{\partial}{\partial t}\left[-\frac{1}{\sinh\lambda(T-t)}\int_t^T \sinh\lambda(T-s)\,a_s\,ds\right]$$

$$= -\lambda\int_t^T \frac{\sinh\lambda(T-s)\cosh\lambda(T-t)}{\sinh^2\lambda(T-t)}\, a_s\, ds + a_t$$

$$= -\lambda\int_t^T \frac{\sinh\lambda(t-s) + \sinh\lambda(T-t)\cosh\lambda(T-s)}{\sinh^2\lambda(T-t)}\, a_s\, ds + a_t$$

$$= -\frac{\partial b}{\partial T} - \operatorname{cosech}\lambda(T-t)\int_t^T \dot m(T-s)\, a_s\, ds + a_t\,;$$

so

$$\boxed{\frac{\partial b}{\partial t}(t,T) = -\frac{\partial b}{\partial T}(t,T) - m(T-t)^{-1}\int_t^T \dot m(T-s) a_s\, ds + a_t.}$$

(xii) (4/3/93) Though it is essentially the same, we might try to express the log price as a slightly different quadratic form. Wolfgang suggested this. For Model I, if we write

$$\int_t^T f_{tu}\,du = \tfrac{1}{2}(x-\beta)^T Q (x-\beta) + \kappa = \tfrac{1}{2} x^T Q x - \tilde{\beta}^T x + \tilde{\kappa}$$

we derive the differential equations for $\tilde{\beta} \equiv Q\beta$ and $\tilde{\kappa} = \kappa + \tfrac{1}{2}\beta^T Q \beta$ to be

$$\boxed{\begin{aligned} \frac{d\tilde{\beta}}{dt} &= \dot{Q}\beta + Q\dot{\beta} = (Q\sigma\sigma^T - C^T)\tilde{\beta} - \alpha \\ \frac{d\tilde{\kappa}}{dt} &= -\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}\operatorname{tr}(\sigma^T Q \sigma) + \tfrac{1}{2}\tilde{\beta}^T \sigma\sigma^T \tilde{\beta} \end{aligned}}$$

and for Model II, writing

$$\tfrac{1}{2}(x-b)^T Q (x-b) + \gamma = \tfrac{1}{2} x^T Q x - \tilde{b}\cdot x + \tilde{\gamma}$$

we get for $\tilde{b} \equiv Qb$, $\tilde{\gamma} = \gamma + \tfrac{1}{2} b^T Q b$ the differential equations

$$\boxed{\begin{aligned} \frac{d\tilde{b}}{dt} &= Qa + (Q\sigma\sigma^T - c^T)\tilde{b} \\ \frac{d\tilde{\gamma}}{dt} &= \tilde{b}^T a - \tfrac{1}{2}\operatorname{tr}(\sigma^T Q \sigma) + \tfrac{1}{2}\tilde{b}\sigma\sigma^T\tilde{b} \end{aligned}}$$

They have the virtue of removing some of the singularity at $t=T$ in the de of b, and also allowing a more general boundary condition $b(t,t)$.

(xiii) (9/3/93). Returning to the Model II specification, in the original form on p 29, we have that

$$f_{tT} = -\frac{\partial b}{\partial T} q\,(x-b) + \tfrac{1}{2}(x-b)^2 \dot{q} + \frac{\partial \gamma}{\partial T}$$

$$= -\frac{\partial b}{\partial T} q\,(x-b) + \tfrac{1}{2}(x-b)^2\left(\frac{q}{m}\right)^2 + \tfrac{1}{2}\sigma^2 q(T-t) + \tfrac{1}{2} m^2\left(\frac{\partial b}{\partial T}\right)^2$$

$$= \tfrac{1}{2}\left[m(T-t)\frac{\partial b}{\partial T}(t,T) - (x - b(t,T))\frac{q(T-t)}{m(T-t)}\right]^2 + \tfrac{1}{2}\sigma^2 q(T-t).$$

This actually has quite a lot to say. First, notice that in this set up, we must have

$$\boxed{f_{tT} \geq \tfrac{1}{2}\sigma^2 q(T-t) \qquad \text{for all} \quad 0 \leq t \leq T}$$

Next, since $q(t)/m(t) \to 1$ $(t \downarrow 0)$ and $\frac{\partial b}{\partial T}(t,T)$ remains bounded as $T \downarrow t$, provided that a remains bounded, we find the differential equation in T for b:

$$\boxed{\left(x - b(t,T)\right) \frac{q(T-t)}{m(T-t)} - m(T-t) \frac{\partial b}{\partial T}(t,T) = \left(2f_{tT} - \sigma^2 q(T-t)\right)^{\frac{1}{2}}}$$

Hence

$$\frac{\partial b}{\partial T} + \frac{q(T-t)}{m(T-t)^2}\, b(t,T) = \frac{x\, q(T-t)}{m(T-t)^2} - \frac{1}{m(T-t)}\left(2f_{tT} - \sigma^2 q(T-t)\right)^{\frac{1}{2}}$$

$$= \frac{\partial b}{\partial T} + \frac{\lambda}{\sinh \lambda t\,(\cosh \lambda t - \frac{c}{\lambda}\sinh \lambda t)}\, b(t,T)$$

The integrating factor to multiply by is $q(T-t)$, giving

$$\frac{\partial}{\partial T}\left[q(T-t)\, b(t,T)\right] = x\,\dot{q}(T-t) - \frac{q(T-t)}{m(T-t)}\left(2f_{tT} - \sigma^2 q(T-t)\right)^{\frac{1}{2}}$$

Observe also that $\frac{1}{2}x^2 = r_t = f_{tt}$, so it is OK to define

$$\boxed{q(T-t)\left(x - b(t,T)\right) = \int_t^T \frac{q(u-t)}{m(u-t)}\left(2f_{tu} - \sigma^2 q(u-t)\right)^{\frac{1}{2}}\, du}$$

(assuming $x > 0$). Now everything in this equation except b is known, <u>so this equation defines</u> $b(t,\cdot)$. Knowing this, we know

$$g(T) \equiv -m(T-t)\, b(t,T) = \int_t^T \sinh \lambda(T-s)\; a_s\, ds / \lambda$$

and then

$$\boxed{g''(T) - \lambda^2 g(T) = a_T}$$

$$\varphi_0(s,t) = e^{-\alpha s - \beta t} \sqrt{\frac{\alpha \beta s}{t}}\, I_1\left(2\sqrt{\alpha\beta s t}\right) \sim e^{-(\sqrt{\alpha s} - \sqrt{\beta t})^2} \left(\frac{\beta^2}{4\pi} \quad \frac{\sqrt{\alpha s \beta t}}{\beta^2 t^2} \right)^{\frac{1}{2}}$$

$$\varphi_1(s,t) = e^{-\alpha s - \beta t}\, \beta\, I_0\left(2\sqrt{\alpha \beta s t}\right) \sim e^{-(\sqrt{\alpha s} - \sqrt{\beta t})^2} \left(\frac{\beta^2}{4\pi} \quad \frac{1}{\sqrt{\alpha s \beta t}} \right)^{\frac{1}{2}}$$

as $\beta t \to \infty$.

I've verified that this h satisfies $\mathcal{G} h = 0$ at least when $l_0 < a$, $l_1 < b$, but what actually is this h-transform?

<u>Martin boundaries of simple chain/occupation-time processes</u> (18/3/93)

(i) Consider a finite Markov chain with N states, irreducible Q-matrix Q. If $L_j(t) \equiv \int_0^t I_{\{j\}}(X_s)\,ds$, then the process

$$\xi_t \equiv (X_t, (L_j(t))_{j=1}^N)$$

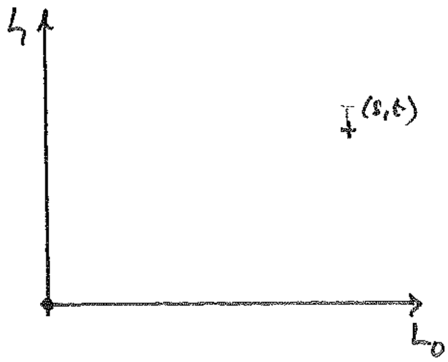
is a Markov process with statespace $\{1,\dots,N\} \times (\mathbb{R}^+)^N$ (or, more generally, $\{1,\dots,N\} \times \mathbb{R}^N$) and generator

$$\mathcal{G}f(i;\underline{l}) = Qf(i;\underline{l}) + \frac{\partial f}{\partial l_i}(i;\underline{l}).$$

What sort of possible limit behaviours are there?

(ii) Take the simplest interesting case $I = \{0,1\}$, $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$. Here we can compute most things. For a first, let's condition the process to pass through $(0; a, b)$. Define for $s,t>0$

$$\varphi_i(s,t) \equiv P^{(i;0,0)}\left[X_{s+t} = 0,\ L_0(s+t) = s\right].$$



Then

$$\varphi_0(s,t) = \sum_{n\geq 1} \frac{(\alpha s)^n}{n!} e^{-\alpha s} \cdot \beta^n t^{n-1} e^{-\beta t}/\Gamma(n)$$

$$= e^{-\alpha s - \beta t}\ \alpha\beta s \sum_{k\geq 0} (\alpha\beta s t)^k / k!\,(k+1)!$$

and

$$\varphi_1(s,t) = \beta e^{-\beta t - \alpha s} + \int_0^t \beta e^{-\beta u}\, \varphi_0(s, t-u)\, du$$

$$= \beta e^{-\alpha s - \beta t} \sum_{k\geq 0} (\alpha\beta s t)^k / k!\, k!$$

$$\equiv \beta e^{-\alpha s - \beta t} f(st), \quad \text{say}.$$

The harmonic function to transform the process to go through (a,b) is

$$h(i; l_0, l_1) = \varphi_i(a - l_0, b - l_1)$$

at least for $l_0 < a$, $l_1 < b$. The h-transform has jump rates

$$\boxed{\tilde{q}_{01} = \frac{\alpha}{\beta}(a - l_0)\frac{f'}{f}((a-l_0)(b-l_1))}, \text{ for example.}$$

A question of Uwe Küchler (29/3/93)

Define the process

$$Y_s \equiv \int_0^1 W_{(t-s)^+}\, dW_t \,, \qquad 0 \leq s \leq 1.$$

We shall prove (i) Y has continuous sample paths;
(ii) the process Y is not an f.v. process.
I conjecture that Y has finite quadratic variation, $\langle Y \rangle_t = \int_0^t (1-u)\,du$.

(i) For $0 \leq s \leq t \leq 1$, we estimate for $p > 2$

$$E\,|Y_s - Y_t|^p \leq c\, E\Big(\int_0^1 (W_{(u-t)^+} - W_{(u-s)^+})^2\, du \Big)^{p/2}$$

$$\leq c\, E \int_0^1 |W_{(u-t)^+} - W_{(u-s)^+}|^p\, du$$

$$= c\, E\Big(\int_t^1 |W_{u-t} - W_{u-s}|^p\, du \Big) + c\, E \int_s^t |W_{u-s}|^p\, du$$

$$= c\,(t-s)^{p/2} \int_t^1 du + \frac{2c}{2+p}\, |t-s|^{1+p/2}$$

$$\leq K\, |t-s|^{p/2}$$

for suitable constants. Now Kolmogorov's Lemma finishes the task.

(ii) Define

$$V_n \equiv \sum_{j=1}^n \big(Y_{j/n} - Y_{(j-1)/n} \big)^2.$$

If Y were an FV process, then $V_n \to 0$ a.s. However, we shall prove that

(a) $\sup_n E\, V_n^2 < \infty$;

(b) $E\, V_n \longrightarrow c > 0 \quad (n \to \infty)$.

The first implies that $(V_n)_{n \geq 1}$ is uniformly integrable, and, if $V_n \xrightarrow{a.s.} 0$, we would have a contradiction of the second.

<u>Proof of (b)</u> Fix $0 < s < t$, and compute

$$E\,(Y_t - Y_s)^2 = E \int_0^1 (W_{(u-t)^+} - W_{(u-s)^+})^2\, du$$

$$= E\left(\int_t^1 (W_{u-t} - W_{u-s})^2 \, du + \int_s^t W_{u-s}^2 \, du \right)$$

$$= (t-s)(1-t) + \tfrac{1}{2}(t-s)^2.$$

Now it follows easily that $E V_n \to \frac{1}{2}$.

Proof of (a)

$$E\left[\left(Y_{j/n} - Y_{(j-1)/n}\right)^2 \left(Y_{k/n} - Y_{(k-1)/n}\right)^2 \right]$$

$$\leq \left\{ E\left(Y_{j/n} - Y_{(j-1)/n}\right)^4 \right\}^{\frac{1}{2}} \left\{ E\left(Y_{k/n} - Y_{(k-1)/n}\right)^4 \right\}^{\frac{1}{2}}$$

So we estimate

$$E\left(Y_t - Y_s\right)^4 \leq c\,(t-s)^2,$$

as in the proof of continuity. Statement (a) now follows.

Remark This same approach proved useful in "Multiple points of Markov processes in a complete metric space" Sém. de Probabilités XXIII, 186–197, 1989.

Pricing via maximisation of expected utility (30/3/93).

Let's consider the problem of obtaining

$$\sup\left\{ E\, U(X_1) : X \text{ is admissible wealth process}, X_0 = 0 \right\}$$

where U is a strictly increasing strictly concave function $U : \mathbb{R} \to \mathbb{R}$, and we shall suppose in addition that

$$\boxed{\lim_{x \to -\infty} \frac{U(x)}{x} = 1, \qquad \lim_{x \to \infty} \frac{U(x)}{x} = \frac{1}{2}.}$$

What are admissible wealth processes? It seems natural (if the class is a vector space) to include all bounded processes which arise as stochastic integrals wrt price processes. As an example, suppose the world contains just one asset,

$$dS = S\,(dB + c\,dt)$$

where $c > 0$ is fixed. We can then make an admissible wealth process by

$$d X_t = (X_t + K)\, \sigma\, dS_t / S_t$$

where K is (large) positive, and $\sigma > 0$ is fixed. At least, this will be admissible so long as we stop at some level $a > 0$. Up to this time, we have

$$(X_t + K) = K \exp(\sigma B_t + \sigma \mu t), \qquad \mu \equiv c - \sigma/2 .$$

Now let's write $Y_t \equiv W_t - \mu t$, $\alpha \equiv \frac{1}{\sigma} \log(1 + a/K)$ and compute

$$E^\alpha \left[e^{-\theta(Y_1 - \alpha)} : H_0 > 1 \right] = \int_0^\infty \{ p_1(\alpha, y) - p_1(-\alpha, y) \} e^{-\mu(y-\alpha) - \mu^2/2} \cdot e^{-\theta(y-\alpha)} dy$$

$$= e^{(\theta+\mu)\alpha - \mu^2/2} \int_0^\infty \{ e^{-(\theta+\mu)y - (y-\alpha)^2/2} - \dots \} \frac{dy}{\sqrt{2\pi}}$$

$$= \exp\left((\theta+\mu)\alpha - \mu^2/2 - \alpha^2/2 \right) \left[f(\theta+\mu-\alpha) - f(\theta+\mu+\alpha) \right],$$

where

$$f(x) \equiv e^{x^2/2} \bar{\Phi}(x),$$

which is a strictly decreasing, strictly convex function.

Now

$$P^0 (X \text{ reaches } a \text{ before time } 1) = P^\alpha (Y \text{ reaches } 0 \text{ before time } 1)$$

$$= 1 - e^{-(\mu+\alpha)^2/2} \left[f(\mu-\alpha) - f(\mu+\alpha) \right],$$

so if we think of keeping $\sigma = c$, $\mu = \frac{c}{2}$ fixed, and think of K as getting big as a is held fixed, we have $\alpha \sim a/K\sigma$, so the probability that X does not reach a before time 1 is $O(\alpha)$. Precisely, as $K \to \infty$

$$P^0 (X \text{ does not reach } a \text{ before time } 1) \sim -2 e^{-\mu^2/2} f'(\mu) \cdot \alpha ,$$

so that

$$E^0 (|X_1| : X \text{ doesn't reach } a \text{ before time } 1) \leqslant \frac{3a}{\sigma} e^{-\mu^2/2} |f'(\mu)| .$$

Now if c is large enough, we have

$$3\sigma^{-1} e^{-\mu^2/2} |f'(\mu)| = 3c^{-1} e^{-c^2/8} |f'(c/2)| \leqslant \frac{1}{4}$$

So if $\tau = 1 \wedge \inf\{t : X_t = a\}$ we have

$$E\, U(X_\tau) = E\big[U(X_1) : \tau = 1\big] + U(a)\, P(\tau < 1)$$

$$\geq -\tfrac{1}{4}a + U(a)\, P(\tau < 1) + o(1)$$

as $K \to \infty$. But since $U(a) \sim \tfrac{1}{2}a$ for large a, this means that we can make arbitrarily large expected utility if c is large enough:

$$\boxed{\sup\{E\, U(X_1) : X \text{ admissible}, X_0 = 0\} = +\infty !!}$$

<u>Pricing a futures contract</u> (1/4/93) (cf book IV p 36)

Suppose we just consider a futures contract of fixed maturity T, and let F_t be the market price of the contract at time $t \in [0,T]$. The way to think of this is that at time t, the holder of the contract hands over an amount F_0 to the financial institution, which puts the money into a pot, and adjusts it as follows; if A_t is the amount in the pot at time t, then

$$dA_t = r_t A_t\, dt + dF_t, \qquad A_0 = F_0.$$

Then at time T, the pot contains

$$A_T = e^{R_T}\Big(A_0 + \int_0^T e^{-R_s}\, dF_s\Big) \qquad \Big(R_t \equiv \int_0^t r_u\, du\Big)$$

$$= e^{R_T}\Big(A_0 + e^{-R_T} F_T - F_0 + \int_0^T r_s e^{-R_s} F_s\, ds\Big)$$

$$= e^{R_T}\Big(e^{-R_T} F_T + \int_0^T r_s e^{-R_s} F_s\, ds\Big).$$

The price you would pay at time 0 to get this would be (E = risk-neutral)

$$\boxed{F_0 = E\Big[e^{-R_T} F_T + \int_0^T r_s e^{-R_s} F_s\, ds\Big]}.$$

When r is deterministic, we find that this is solved by $\boxed{F_t = e^{R_T - R_t} S_t}$, where S is the underlying asset price.

Note that $F_T = S_T$.

We can generalise this analysis to allow for an asset which pays a dividend stream δ;
we get $F_t + \int_0^t \delta_u \, du$ is a martingale

[dividends get paid to the holder of the futures contract.]

$$\therefore \; F_t = E_t\left[S_T + \int_t^T \delta_u \, du \right]$$

If G_t is the forward price at time t, and $P(t,T)$ the price of a bond maturing at T, we have

$$P(t,T)\, G_t = E_t\left[S_T \, e^{-R_T} \right]$$

and so

$$\boxed{G_t = E_t^*\left[S_T \right]} \qquad \text{where} \qquad \frac{dP^*}{dP} = e^{-R_T} / P(0,T).$$

(Eg: if the asset were a zero-coupon bond maturing at time $T' > T$, then

$$G_t = P(t,T') / P(t,T)$$

$$F_t = E_t\left[P(T,T') \right].$$
)

Making it a little more general, if we were to start at time t_0, the amount in the pot at time $t \in [t_0, T]$ will be

$$A_t = e^{R_t - R_{t_0}} A_{t_0} + \int_{t_0}^{t} e^{R_t - R_u} dF_u$$

$$= F_t + e^{R_t} \int_{t_0}^{t} r_u e^{-R_u} F_u \, du.$$

Thus

$$e^{-R(t_0)} F(t_0) = E\left[e^{-R_T} A_T \mid \mathcal{F}(t_0) \right] \qquad \text{if we were to start at } t_0$$

$$= E\left[e^{-R_T} F_T + \int_{t_0}^{T} r_u e^{-R_u} F_u \, du \mid \mathcal{F}(t_0) \right]$$

from which

$$\boxed{e^{-R_t} F_t + \int_0^t r_u e^{-R_u} F_u \, du \equiv F_0 + \int_0^t e^{-R_u} dF_u \quad \text{is a martingale.}}$$

However, this implies that F is a martingale, and, since $F_T = S_T$, we have the simple relation

$$\boxed{F_t = E\left[S_T \mid \mathcal{F}_t \right]}$$

<u>Remarks</u> (i) If we make the dependence on T explicit, it follows that for $t < T < T'$, we must have

$$F(t,T) = E\left[S_T \mid \mathcal{F}_t \right] = E\left[E\left(S_{T'} e^{R_T - R_{T'}} \mid \mathcal{F}_T \right) \mid \mathcal{F}_t \right] \leq F(t, T').$$

However (see p 30 in Hull's book) this need not always happen; the table he reproduces from the WSJ shows examples where it goes the other way, or up + down ... WHY? Because this is assuming that S is the price of a traded asset! (see ⤺)

(ii) In practice, there is the business of a margin held by the bank. This is theoretically irrelevant, since if the buyer only had to put forward the part of F_0, he would be gaining riskless interest on the remainder meantime. In other words, it doesn't matter whether some part of F_0 is nominally under the control of the bank, or the investor.

Markov chains + occupation times: some bounds (1/4/93)

(i) To get some feel for the behaviour of these things, let's fix some state j and consider what we can say about

$$P^{(j,0)}\left[L_j(T) \le t \right]$$

where typically $t = o(T)$. If τ is the inverse to L_j, then under $P^{(j,0)}$ it is a subordinator,

$$E \exp(\lambda \tau_t) = \exp t\psi(\lambda),$$

where

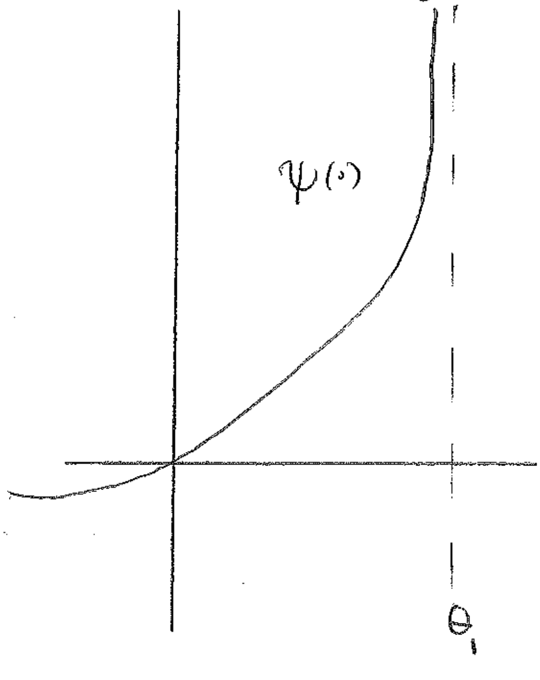
$$\psi(\lambda) = \lambda - \lambda b(\lambda + D)^{-1} 1.$$

[Partition the Q-matrix of the chain as $\begin{array}{cc} \left(\begin{array}{c|c} -q & b \\ \hline c & D \end{array}\right) & \begin{array}{c} j \\ I \setminus j \end{array} \end{array}$, and then the rate (in terms of L_j) of λ-marked excursions away from j is $b\lambda(\lambda - D)^{-1}1$]

Assuming that D is diagonalisable with e-values $0 > -\theta_1 > -\theta_2 > \dots$, we get

$$\psi(\lambda) = \lambda + \sum_i \frac{a_i \lambda}{\theta_i - \lambda},$$

so the mgf is well defined for $\mathrm{Re}\,\lambda < \theta_1$, and the singularity at θ_1 is like $1/x$. We also have $a_1 > 0$.



(ii) An upper bound

$$P\left[L_j(T) \le t\right] = P\left[\tau_t \ge T\right]$$

$$\le e^{-\lambda T} e^{t\psi(\lambda)}$$

for all λ, so that

$$\boxed{\begin{aligned} \log P(\tau_t \ge T) &\le \inf_\lambda \{ t\psi(\lambda) - \lambda T \} \\ &\equiv t\,\psi_*(T/t), \end{aligned}}$$ say.

The λ which minimises in the expression for ψ_* solves $\boxed{\psi'(\lambda) = x}$

(iii) <u>A lower bound</u> If we change the law of τ to P_α, defined by $dP_\alpha/dP = \exp(\alpha\tau_t - t\psi(\alpha))$, the log mgf changes to $\psi(\lambda+\alpha) - \psi(\alpha)$, and

$$E_\alpha \tau_t = t\,\psi'(\alpha) \quad , \quad \mathrm{var}_\alpha(\tau_t) = t\,\psi''(\alpha).$$

Thus

$$P[\tau_t \geq T] = E_\alpha\left[e^{-\alpha\tau_t + t\psi(\alpha)} : \tau_t \geq T\right].$$

So if we pick α, T so that

$$T \equiv T(\alpha) = t\,\psi'(\alpha) - 2\sqrt{t\psi''(\alpha)},$$

we can estimate

$$P(\tau_t \geq T) \geq E_\alpha\left[e^{-\alpha\tau_t + t\psi(\alpha)} ; \ |\tau_t - t\psi'(\alpha)| \leq 2\sqrt{t\psi''(\alpha)}\right]$$

$$\geq \exp\left\{t\psi(\alpha) - \alpha\left(T + 4\sqrt{t\psi''(\alpha)}\right)\right\} \cdot \frac{3}{4}$$

by Chebyshev. If we think of $\alpha \uparrow\uparrow \theta_1$, then the corresponding value of T will be going to infinity, and we get

$$\boxed{t\psi(\alpha) - \alpha T(\alpha) \geq \log P(\tau_t \geq T(\alpha)) \geq t\psi(\alpha) - \alpha T(\alpha) - 2\alpha\sqrt{t\psi''(\alpha)} + \log\left(\tfrac{3}{4}\right).}$$

As $\alpha \uparrow\uparrow \theta_1$, we get $T(\alpha) \sim a_1 t\theta_1/(\theta_1-\alpha)^2$, $\psi''(\alpha) \sim 2a_1\theta_1/(\theta_1-\alpha)^3$. The dominant term in the estimates is the $-\alpha T(\alpha)$, and the term

$$2\alpha\sqrt{t\psi''(\alpha)} \sim 2\alpha\left(\frac{2a_1\theta_1}{(\theta_1-\alpha)^3}\right)^{1/2}$$

is of smaller order. So this gives, with t fixed, the asymptotics of the log of $P(L_j(T) \leq t)$. However, one <u>really</u> wants to be able to let t grow with T, but slowly and then see what happens. Also, what if we replace t by $t+c$ for some constant $c>0$? I believe that

$$P[L_j(T) \leq t] = o\left(P[L_j(T) \leq t+c]\right) \ldots$$

Expected-utility pricing: an observation (2/4/93)

Let's suppose that U' decreases from $+\infty$ to 0, so that we can hope to find an optimal strategy at $(U')^{-1}(\lambda Z)$, where $Z \equiv d\tilde{P}/dP$ is the EMM density. Now for $EU(X)$ to remain bounded, we shall therefore need (if the argument works OK)

$$E\, U(I(Z)) < \infty \qquad\qquad (I^{-1} \equiv U')$$

and the big values of $U(I(Z))$ happen when Z is close to 0. Now

$$E\, U(I(Z)) = \tilde{E}\, \zeta\, U(I(1/\zeta))$$

and we need to consider big values of $\zeta \equiv 1/Z$. The only way we could make this expectation finite whatever positive ζ, $\tilde{E}\zeta^{2} = 1$, would be if U is bounded above!

Markov chains Large deviations: better estimates (6/4/93)

(i) From the estimate on p. 43, we have

$$\log P[\tau_t \geq T] \leq t\, \psi_*(T/t) = t[\psi(\lambda) - \lambda\psi'(\lambda)],$$

where $\psi'(\lambda) = T/t$. Now

$$\psi(\lambda) - \lambda\psi'(\lambda) = -\sum \frac{a_i \lambda^2}{(\theta_i - \lambda)^2},$$

so the relationship of λ to T/t is

$$\frac{T}{t} = \psi'(\lambda) = 1 + \sum \frac{a_i\theta_i}{(\theta_i - \lambda)^2} \leq 1 + \frac{a_1\theta_1}{(\theta_1-\lambda)^2} + c$$

where $c \equiv \sum_{i>1} |a_i|\theta_i/(\theta_i - \theta_1)^2$. We have the corresponding lower-bound

$$\frac{T}{t} \geq 1 + \frac{a_1\theta_1}{(\theta_1-\lambda)^2} - c$$

and hence we get the squeeze

$$\boxed{\frac{a_1\theta_1}{T/t - 1 + c} \leq (\theta_1 - \lambda)^2 \leq \frac{a_1\theta_1}{T/t - 1 - c}}$$

Thus

$$\psi(\lambda) - \lambda\psi'(\lambda) \le -\frac{a_1\lambda^2}{(\theta_1-\lambda)^2} \le -\frac{a_1\lambda_0^2}{(\theta_1-\lambda_0)^2}$$

where

$$\lambda_0 \equiv \theta_1 - \left(\frac{a_1\theta_1}{T/t - 1 - c}\right)^{\frac{1}{2}},$$

so that

$$\boxed{\log P[\tau_t \ge T] \le -t\left(\tfrac{T}{t} - 1 - c\right)\left\{\theta_1 - 2\sqrt{\frac{a_1\theta_1}{T/t-1-c}} + \frac{a_1}{T/t - 1 - c}\right\}}$$

(ii) To deal more carefully with the lower bound, we pick α so that

$$T + 2\sqrt{t\psi''(\alpha)} = t\psi'(\alpha),$$

and then have

$$\log P[\tau_t \ge T] \ge t\psi(\alpha) - \alpha t\psi'(\alpha) + \log(3/4).$$

We have always that $\alpha \ge \lambda$, so we need to get an upper bound for α. The equation for α reads

$$T + 2\sqrt{t\psi''(\alpha)} = t\sum a_i\theta_i/(\theta_i-\alpha)^2 + t \ge t\frac{a_1\theta_1}{(\theta_1-\alpha)^2} + t$$

Now

$$T + 2\sqrt{t\psi''(\alpha)} = T + 2\sqrt{t}\left(\sum\frac{2a_i\theta_i}{(\theta_i-\alpha)^3}\right)^{\frac{1}{2}}$$

$$\le T + 2\sqrt{t}\left(\frac{2a_1\theta_1}{(\theta_1-\alpha)^3}\right)^{\frac{1}{2}} + 2\sqrt{t}\,c'$$

for some unimportant constant. Hence the inequality is

$$\frac{a_1\theta_1}{(\theta_1-\alpha)^2} \le \frac{T}{t} + \frac{2}{\sqrt{t}}\left\{\sqrt{\frac{2a_1\theta_1}{(\theta_1-\alpha)^3}} + c'\right\} - 1$$

$$\le \frac{T}{t} + \frac{2}{\sqrt{t}}\left[\sqrt{\frac{2a_1\theta_1}{(\theta_1-\alpha)^4}}\cdot\sqrt{\theta_1-\lambda_0} + c'\right] - 1$$

yielding

$$(\theta_1-\alpha)^{-2}\left\{a_1\theta_1-\sqrt{8a_1\theta_1(\theta_1-\lambda_0)/t}\right\} \le \frac{T}{t}+\frac{2}{\sqrt{t}}c'-1.$$

Thus $\alpha \le \alpha_0$, where

$$(\theta-\alpha_0)^{-2}\left\{a_1\theta_1-\sqrt{8a_1\theta_1(\theta_1-\lambda_0)/t}\right\} = \frac{T}{t}+\frac{2}{\sqrt{t}}c'-1.$$

Hence

$$\begin{aligned}\log P[\tau_t \ge T] &\ge -t\left(\psi(\alpha)-\alpha\psi'(\alpha)\right)+\log(3/4)\\
&= -t\sum_i \frac{a_i\alpha^2}{(\theta_i-\alpha)^2}+\log(3/4)\\
&\ge -t\sum \frac{a_i\alpha_0^2}{(\theta_i-\alpha_0)^2}+\log(3/4)\\
&\ge -t\frac{a_1\alpha_0^2}{(\theta_1-\alpha_0)^2}-c''.t+\log(3/4)\\
&= -\frac{a_1t\left(\frac{T}{t}+\frac{2c'}{\sqrt{t}}-1\right)}{a_1\theta_1-\sqrt{8a_1\theta_1(\theta_1-\lambda_0)/t}}\left\{\theta_1-\left(\frac{a_1\theta_1-\sqrt{8a_1\theta_1(\theta_1-\lambda_0)/t}}{T/t+2c'/\sqrt{t}-1}\right)^{\frac12}\right\}^2\\
&\qquad -c''t+\log(3/4).\end{aligned}$$

Thus we have estimates

$$\boxed{-\frac{a_1t\lambda_0^2}{(\theta_1-\lambda_0)^2}\ge \log P[\tau_t\ge T]\ge -\frac{a_1t\alpha_0^2}{(\theta_1-\alpha_0)^2}-c''t+\log(3/4).}$$

(iii) Let's now compare the two sides of this bound, writing $\theta_1-\lambda_0\equiv \ell > \theta_1-\alpha_0\equiv a$.
We need to know about

$$\begin{aligned}-\left(\frac{\theta_1}{\ell}-1\right)^2+\left(\frac{\theta_1}{a}-1\right)^2 &= \frac{\theta_1^2}{a^2\ell^2}(\ell^2-a^2)-2\frac{\theta_1}{a\ell}(\ell-a)\\
&= (\ell-a)\left(\frac{\theta_1}{a\ell}\right)^2\left\{\ell+a-\frac{2a\ell}{\theta_1}\right\}.\end{aligned}$$

We have $\ell = \sqrt{r_1} \equiv \left(a_1\theta_1 / (\tfrac{T}{t}-1-c)\right)^{\frac{1}{2}}$

$$a = \sqrt{r_2} \equiv (a_1\theta_1 - \sqrt{s'})^{\frac{1}{2}} / (\tfrac{T}{t} - 1 + \tfrac{2}{\sqrt{t}}c')^{\frac{1}{2}} \qquad \left(s \equiv \frac{8a_1\theta_1(\theta_1-\lambda)}{t}\right)$$

so that

$$\ell - a = \frac{r_1 - r_2}{\sqrt{r_1}+\sqrt{r_2}} = \frac{a_1\theta_1\left(\frac{2}{\sqrt{t}}c' + c\right) + \sqrt{s'}\left(\frac{T}{t}-1-c\right)}{\left(\frac{T}{t}-1-c\right)\left(\frac{T}{t}-1+\frac{2}{\sqrt{t}}c'\right)\left(\sqrt{r_1}+\sqrt{r_2}\right)}$$

Assembling the difference between the two bounds is

$$-a_1t\left(\frac{\theta_1}{\ell}-1\right)^2 + a_1t\left(\frac{\theta_1}{a}-1\right)^2 + c''t - \log(3/4)$$

$$= a_1t\,(\ell-a)\left(\frac{\theta_1}{a\ell}\right)^2\left[\ell+a-\frac{2a\ell}{\theta_1}\right] + c''t - \log(3/4).$$

Now

$$\ell - a \leqslant \frac{c_0 + (c_1 + c_2(T/t)^{3/4})/\sqrt{t}}{(T/t)^{3/2}}$$

and $\quad \ell + a - 2a\ell/\theta_1 \leqslant \text{const.}\left(\frac{T}{t}\right)^{-\frac{1}{2}}$,

so the difference between the bounds is

$$\leqslant \text{const.}\left(\frac{T}{t}\right)^{-2} t\left(c_0 + \frac{1}{\sqrt{t}}\left(c_1 + c_2\left(\frac{T}{t}\right)^{3/4}\right)\right) + c''t - \log(3/4)$$

$$\leq a + bt \quad \text{eventually if } (T/t) \to \infty.$$

(iv) Now let's consider what happens as $T \to \infty$, and $T/t \to \infty$ (we may also allow $t \to \infty$), when we compare the situation for t and for $t+k$, where $k > 0$ is fixed. We certainly have

$$P(\tau_{t+k} \geq T) \geq P(\tau_t \geq T)$$

but I claim that $\boxed{\dfrac{P(\tau_t \geq T)}{P(\tau_{t+k} \geq T)} \to 0,}$ provided $\frac{t^3}{T} \to 0$.

Let's understand why. The gap between the upper and lower estimates on $\log P(\tau_t \geq T)$ is at most $a + bt$ eventually, but if we take the gap between the upper estimates on $\log P(\tau_{t+k} \geq T)$ and $\log P(\tau_t \geq T)$ we see

$$-a_1(t+k)\left(\theta_1\sqrt{\frac{T/_{t+k}-1-c}{a_1\theta_1}}-1\right)^2 + a_1 t\left(\theta_1\sqrt{\frac{T/_t-1-c}{a_1\theta_1}}-1\right)^2$$

$$= -a_1(t+k)\left[\frac{\theta_1}{a_1}\left(\frac{T}{t+k}-1-c\right)-\frac{2\theta_1}{\sqrt{a_1\theta_1}}\left(\frac{T}{t+k}-1-c\right)^{\frac{1}{2}}+1\right]+a_1 t\left[\cdots\right]$$

$$= -a_1\left[\frac{\theta_1}{a_1}\{T-(1+c)(t+k)\}-2\sqrt{\frac{\theta_1}{a_1}}\,\sqrt{t+k}\,\sqrt{T-(1+c)(t+k)}\right]$$

$$+a_1\left[\frac{\theta_1}{a_1}\{T-(1+c)t\}-2\sqrt{\frac{\theta_1}{a_1}}\,\sqrt{t}\,\sqrt{T-(1+c)t}\right]-a_1 k$$

$$= \theta_1 k(1+c)-a_1 k + 2\sqrt{\theta_1 a_1}\left\{\sqrt{t+k}\,\sqrt{T-(1+c)(t+k)}-\sqrt{t}\,\sqrt{T-(1+c)t}\right\}.$$

The interesting bit is the bit in brackets, which equals

$$\frac{k}{\sqrt{t+k}+\sqrt{t}}\sqrt{T-(1+c)(t+k)} + \sqrt{t}\left(\sqrt{T-(1+c)(t+k)}-\sqrt{T-(1+c)t}\right)$$

$$= k\,\frac{\sqrt{T-(1+c)(1+k)}}{\sqrt{t+k}+\sqrt{t}} + \frac{\sqrt{t}\,\left(-k(1+c)\right)}{\sqrt{T-(1+c)(t+k)}+\sqrt{T-(1+c)t}}$$

$$\sim k\,\frac{\sqrt{T}}{\sqrt{t}+\sqrt{t+k}} \to \infty.$$

So provided we have $\boxed{t^3/T \to 0}$, the gap between the upper and lower bounds is negligible compared to the change in the upper bound when we replace t by $t+k$. Thus if we condition the process on $L_j(T) \leq t$, in the limit as $T \to \infty$, $t^3/T \to 0$, the process will jump immediately to state j and never leave it.

Reversing fluid models with finite buffer (8/4/93)

In the work on invariant laws of fluid models with finite buffers, assuming $m(E_-) > m(E_+)$, we got the invariant distn for buffer capacity $a > 0$ was given by

$$P(\xi = 0,\ X = j) = p_-(j), \qquad P(\xi = a,\ X = i) = p_+(i) \qquad (i \in E_+,\ j \in E_-)$$

$$P(\xi \in dx,\ X = j) = \pi_j(x)\,dx \qquad (0 < x < a)$$

where $\pi(x) = p_- Q \exp(x V^{-1} Q) V^{-1} = -p_+ Q \exp[(x-a) V^{-1} Q]\, V^{-1}$, and

$$\begin{cases} p_- = m_-(I - \Pi_+ \Pi_-)\, e^{aG_-} K_-(a) = m_-(I - \Pi_+\Pi_-) K_-(a) \\ p_+ = m_+ (I - \Pi_- \Pi_+)\, e^{aG_+} K_+(a) \end{cases}$$

with

$$K_\pm(a) = [I - \Pi_\mp e^{aG_\mp} \Pi_\pm e^{aG_\pm}]^{-1}, \qquad K_\pm \equiv K_\pm(0).$$

When we reverse things, using the notation Z^* for $z^*_{ij} \equiv m_j z_{ji}/z_i$, we get

$$\hat\Pi_\pm = \Pi^*_\pm, \qquad \hat G_\pm = (K^{-1}_\mp G_\mp K_\mp)^* \qquad \text{(reversal uses } -v \text{ in place of } v\text{!!)}$$

Just for the record,

$$\boxed{\begin{aligned} \hat K_\pm &= (K_\mp)^* \qquad \text{(where } \hat K_+ \equiv (I - \hat\Pi_- \hat\Pi_+)^{-1}, \text{ etc } \dots) \\ \hat K_\pm(a) &= (e^{-aG_\mp} K_\mp)^*\, K_\mp(a)^*\, (K_\mp^{-1} e^{aG_\mp})^* \\ (K_\pm(a))^* &= (\hat K_\mp^{-1} e^{a\hat G_\mp})\, \hat K_\mp(a)\, (e^{-a\hat G_\mp} \hat K_\mp) \\ p_- &= m_-\, \hat K_+(a)^*\, e^{a\hat G_+^*}\, (\hat K_+^{-1})^* \end{aligned}}$$

An observation on convex dual functions (13/4/93)

Varadhan's book on large deviations quotes the "minimax theorem" which says that for compact convex C, and a convex function f one has

$$\inf_{y\in C} \sup_\theta \{ y\cdot\theta - f(\theta) \} = \sup_\theta \inf_{y\in C} \{ y\cdot\theta - f(\theta) \}.$$

The inequality $\geq$ is trivial, and does not need convexity. If we set $g(x) = 0$ for $x \in C$; $= +\infty$ for $x \notin C$, then g is convex, and the statement above says

$$\inf_y \sup_\theta \{ y\cdot\theta + g(y) - f(\theta) \} = \sup_\theta \inf_y \{ y\cdot\theta + g(y) - f(\theta) \}$$

or, again

$$\inf_y \{ g(y) + f^*(y) \} = \sup_\theta \{ -g^*(-\theta) - f(\theta) \}$$

where $f^*(y) \equiv \sup(y\cdot\theta - f(\theta))$ is the convex dual function.

But recall that if f_1, f_2 are convex, and

$$f_{12}(x) \equiv \inf_a \{ f_1(a) + f_2(x-a) \}$$

then

f_{12} is convex, and $f_{12}^* = f_1^* + f_2^*$

Apply this to $f_1(x) \equiv g(-x)$, $f_2(x) \equiv f^*(x)$. We get

$$f_{12}(0) = \inf_a \{ g(a) + f^*(a) \} = \sup_\theta \{ -(f_1^*(\theta) + f_2^*(\theta)) \}$$

$$= \sup_\theta \{ -g^*(-\theta) - f(\theta) \}$$

as stated.

A proposition on entropy (27/4/93)

I just record here a result that Nina made me aware of: if $H(P|P_n) \equiv \int \log(dP/dP_n)\, dP \to 0$, then $\|P - P_n\|_v \to 0$.

Proof. Set $f_n \equiv dP/dP_n$. Then $H(P|P_n) \Rightarrow P_n(|f_n - 1| > \varepsilon) \to 0$ for each $\varepsilon > 0$.

Now $\|P - P_n\| = \int |f_n - 1|\, dP_n$, and $\int_{\{f_n > 1+\varepsilon\}} (f_n - 1)\, dP_n \leq \int_{\{f_n > 1+\varepsilon\}} f_n \log f_n\, dP_n$

tends to zero, and $\int_{\{f_n < 1-\varepsilon\}} (1 - f_n)\, dP_n \leq P_n(f_n < 1-\varepsilon) \to 0$

and that does it. I reckon the same proof (essentially) proves that $H(P_n|P) \to 0 \Rightarrow \|P_n - P\|_v \to 0$.

The OU bridge (22/4/93)

Worth recording this calculation. If $\lambda > 0$, and

$$dX_t = dW_t - \tfrac{1}{2}\lambda X_t\, dt,$$

then for $0 < t < T$ we have

$$\boxed{E\left[X_t \mid X_0 = x,\ X_T = y\right] = \frac{\sinh \lambda(T-t)/2}{\sinh \lambda T/2}\, x + \frac{\sinh \lambda t/2}{\sinh \lambda T/2}\, y}$$

and

$$\boxed{\operatorname{var}(X_t \mid X_0, X_T) = \frac{\sinh \lambda T - \sinh \lambda t - \sinh \lambda(T-t)}{2\lambda \sinh^2(\lambda T/2)}}.$$

The limit as $\lambda \downarrow 0$ does the correct thing.

Conditioning on $X_T = y$ converts the SDE to

$$\boxed{dX_t = dW_t - \tfrac{1}{2}\lambda X_t\, dt - \frac{\lambda\left(X_t - e^{\lambda(T-t)/2} y\right)}{e^{\lambda(T-t)} - 1}\, dt}$$

We can take an OU bridge from $(0,0)$ to $(0,T)$ and add to it the function

$$\mu_t \equiv \frac{x \sinh \lambda(T-t)/2 + y \sinh \lambda t/2}{\sinh(\lambda T/2)}$$

and this converts it into an OU bridge from $(x,0)$ to (y,T).

The LD view of conditioning to stay out of a set (11/5/93)

1) If we take a finite irreducible continuous-time Markov chain, and if $Q_{x,t}$ is the law of the normalised occupation measure at time t if the chain starts at x, then the result of Donsker + Varadhan (CPAM 28, 1–47, 1975) Theorem 3 is that

$$\overline{\lim}\ \frac{1}{t} \log Q_{x,t}(C) \leq -\inf_{\mu \in C} I(\mu) \qquad (C \text{ closed, } G \text{ open})$$

$$\underline{\lim}\ \frac{1}{t} \log Q_{x,t}(G) \geq -\inf_{\mu \in G} I(\mu),$$

where $I(\mu) = -\inf\left\{ \int \frac{\mathcal{G}u}{u}\, d\mu : u > 0 \right\}$.

2) Suppose now we partition the statespace I into $I = I_0 \cup I_1$, and partition the Q-matrix as

$$Q = \begin{array}{cc} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \begin{array}{c} I_0 \\ I_1 \end{array} \end{array}$$

What is the chain "conditioned not to enter I_0"? For $i \in I_1$, $\tau \equiv \inf\{t : X_t \in I_0\}$,

$$P^i(\tau > t) = (\exp(tD)\mathbb{1})_i$$

and if we make an e-vector expansion $D = \sum \lambda_j u_j v_j^T$, where the e-values satisfy $0 < -\lambda_1 < \operatorname{Re}\lambda_2 \leq \cdots$ (Perron-Frobenius), then

$$P^i[\tau > t]\, e^{-\lambda_1 t} \to u_1(i)\, v_1^T \mathbb{1}$$

and so the chain conditioned not to enter I_0 before T converges to the chain with Q-matrix

$$\boxed{\begin{array}{ll} \tilde{q}_{ij} = q_{ij}\, u_1(j)/u_1(i) & (i \neq j) \\ \text{and} \quad \tilde{q}_{jj} = -q_j - \lambda_1 . & \end{array}}$$

3) What happens if we take $F = \{\mu \in \mathcal{P}(I) : \mu(I_0) = 0\}$, and compute

$$\inf\{I(\mu) : \mu \in F\}?$$

The question is the same as computing

$$\sup_{\mu \in F} \inf_{u > 0} \sum_i \sum_j \mu_i q_{ij} u_j / u_i .$$

Note that if $\mu \in F$, we can see directly that the inf over $u > 0$ will be among functions u for which $u = 0$ on I_0, so we restrict the sums to $i, j \in I_1$.

Writing $\theta_i = \mu_i / u_i$, the problem is

$$\sup_{\theta > 0} \inf_{u > 0} \sum_i \sum_j \theta_i q_{ij} u_j \qquad \text{subject to } \sum \theta_i u_i = 1.$$

I claim that the solution to this problem is λ_1.

Why? Because if we consider

$$(*) \qquad \sup_{\theta>0} \inf_{u>0} \left\{ \sum\sum \theta_i (q_{ij} - \lambda_1 \delta_{ij}) u_j + \lambda_1 \right\}$$

$$\geq \sup_{\theta>0} \inf_{u>0} \left\{ \sum\sum \theta_i q_{ij} u_j \quad : \quad \sum \theta_i u_i = 1 \right\}$$

and if the solution to $(*)$ satisfies the constraint $\theta \cdot u = 0$, then it's optimal for the original problem.

Now if $\theta^T (Q - \lambda_1)_j < 0$ for some j, then $\inf_{u>0} \theta^T (Q - \lambda_1) u = -\infty$.

The only hope then is that we have

$$\theta^T (Q - \lambda_1)_j \geq 0 \qquad \forall j.$$

How can this happen? If it does, we have for all $t \geq 0$

$$\theta^T \exp t(Q - \lambda_1 I) \geq \theta^T$$

and so, letting $t \to \infty$,

$$\theta^T u_1 v_1^T \geq \theta^T$$

But $\langle u_1, v_1 \rangle = 1$, so right-multiplying by the strictly positive vector u_1 yields the conclusion that we must have had equality throughout, θ is a multiple of v_1, and the sup in $(*)$ can only happen with θ a multiple of v_1, in which case the sup is λ_1, and achieved by, for example,

$$\theta = v_1, \quad u = u_1$$

which together fit the constraint.

So the LD method picks out the conditioned law correctly!

5) Ofer Zeitouni says that the following conjecture is known to be true in some special cases, but a general proof is not known. If A, C are convex symmetric subsets of $\mathbb{R}^n$, then

$$P(B_1 \in A \cap C) \geq P(B_1 \in A)\, P(B_1 \in C).$$

[Loren Pitt did the two-dimensional case.]

6) Take $X_t \equiv B_t + (1+t)^{\gamma} - 1$, and consider (where $0 \leq \gamma \leq 1$ is fixed)

$P[X \text{ reaches } -N \text{ before } 1]$ — how does this behave if $N \to \infty$?

7) A paper which Wilfrid asked me to comment on had the following nice question. Consider a random graph with vertex set $\mathbb{N}$. This random graph G has the property that if one restricts G to the finite ordered vertex set V, then the law of $G|_{V'}$ is the same for any finite ordered vertex set V' with $|V'| = |V|$. Clearly if each edge was present or absent with prob'y p independently of all others one gets such a graph, with law $\mathbb{P}_p$ say. The result (Michael Albert?) is that the only such measures are of the form $\int \nu(dp)\, \mathbb{P}_p$.

8) Don Burkholder asked about getting joint-law statements for $\left(B_\tau, \int_0^\tau \theta_u \, dB_u\right)$.

9) I guessed/conjectured that if we take a finite Mkv chain and do WH on it in the balanced case, then the invariant law of G_+ might be the same as the invariant law of $\pi_- \Pi_+$. However, the example

$$Q = \left(\begin{array}{cc|c} -a & a & 0 \\ 0 & -b & b \\ \hline c & 0 & -c \end{array}\right) \begin{array}{l} E_+ \\ \\ E_- \end{array} \qquad \text{with} \quad \begin{array}{l} c = 1 \\ a = 1/\theta \\ b = \frac{1}{1-\theta} \end{array}$$

with $\theta \in (0,1)$ some parameter is always balanced, and one can compute

$\pi_+ = (\theta, 1-\theta)$ $\qquad G_+ = \begin{pmatrix} -1/\theta & 1/\theta \\ \frac{\theta}{1-\theta} & -\frac{\theta}{1-\theta} \end{pmatrix}$

so the invariant law of G_+ is $\propto (\theta, (1-\theta)/\theta)$, of $\pi_- \Pi_+$ is $\propto (\theta, 1-\theta)$ so

Nice questions.

1) Thomas Bruss + Tom Ferguson have invested a lot of time on the optimal choice problem where you have N candidates, each of a $U[0,1]$ quality, and you win the rank of the chosen candidate. The true policy has to carry around lots of history, but is there a simpler asymptotic form as $N \to \infty$? Herb Robbins also thinks this is a good question.

2) Claudia + Balkema have been considering asymptotics of tilted distⁿˢ in $\mathbb{R}^n$. If X is $\mathbb{R}^n$-valued, and

$$E\, e^{\theta\cdot X} \equiv \exp \psi(\theta)$$

exists for all $\theta \in \mathbb{R}^n$, then one defines the tilted distⁿ by $dP_\theta = e^{\theta\cdot x - \psi(\theta)}\, dP$. Under P_θ, X has mean $\nabla\psi(\theta)$, and variance $D^2\psi(\theta)$, so if we consider the P_θ-law of $Y_\theta \equiv D^2\psi(\theta)^{-\frac12}(X - \nabla\psi(\theta))$, then

$$E_\theta \exp \alpha\cdot Y_\theta = \exp\left[\psi(\theta + D^2\psi(\theta)^{-\frac12}\alpha) - \psi(\theta) - \nabla\psi(\theta)\cdot D^2\psi(\theta)^{-\frac12}\alpha\right]$$

and we'd like to be able to say (under suitable conditions) that for each fixed $\alpha \in \mathbb{R}^n$

$$\psi(\theta + D^2\psi(\theta)^{-\frac12}\alpha) - \psi(\theta) - \nabla\psi(\theta)\, D^2\psi(\theta)^{-\frac12}\alpha \longrightarrow -\tfrac12|\alpha|^2 \qquad (\theta\to\infty) \ldots$$

Any neat characterisation / sufficient conditions?

3) Jean-Dominique + Ofer are keen to know the large n (small ε?) asymptotics of

$$P(X_i \geq \varepsilon \text{ for } 1 \leq i \leq n),$$

where X is zero-mean Gaussian process, and $\mathrm{cov}(X_i, X_j) \sim |i-j|^{-\alpha}$ for some $0<\alpha<1$.

4) If you have N agents, each of whom observes B_t with a noise ($\sigma_j W_j(t)$ for jth agent), then at times of Poisson p, two agents at random trade at the average of their conditional expectations of B, and this trade is made public, what happens?