Optimal investment with CRRA utility (4/11/93)

Let us return to the situation considered on pp 45–46 of book VIII; we have spot rate process r, stock with $dS_t = (\sigma_t\, dW_t + \mu_t\, dt)\, S_t$ where we assume that r and $\theta \equiv \sigma^{-1}(\mu - r)$ are bounded processes, and the aim is to invest in the bond and stock in such a way as to maximise $E\, U(X_T)$, where X is the wealth process, and

$$U(x) = x^{1-R}/(1-R)$$

is a CRRA utility function $(0<R<1)$. The "value function" for this problem has to be of the form

$$V_t(x) = U(x)\, Y_t ,$$

where $\eta_t \equiv Y_t \exp((1-R)\int_0^t r_u\, du)$ is a supermartingale. Writing

$$d\eta_t = \eta_t \{ \psi_t\, dW_t - \alpha_t\, dt \},$$

we found that the optimal portfolio process φ^* can be expressed in terms of ψ as

$$\boxed{\varphi_t^* = \frac{\theta_t + \psi_t}{R\, \sigma_t S_t}\, X_t^* ,}$$

and that α has to be given as

$$\boxed{\alpha_t = \frac{1-R}{2R}(\theta_t + \psi_t)^2 .}$$

The key point is that we have boundary conditions $Y_T = 1$, $Y_0 = \text{const}$; can we choose ψ to satisfy this?

Well, we have

$$dY_t / Y_t = \psi_t\, dW_t - \alpha_t\, dt - (1-R) r_t\, dt$$

so

$$Y_t = Y_0 \exp\left[\int_0^t \psi_s\, dW_s - \tfrac{1}{2}\int_0^t \psi_s^2\, ds - \frac{1-R}{2R}\int_0^t (\psi_s + \theta_s)^2\, ds - \int_0^t (1-R) r_s\, ds \right]$$

$$= Y_0 \exp\left[\int_0^t \psi_s\, dW_s - \tfrac{1}{2}\int_0^t \frac{\psi_s^2}{R}\, ds - \int_0^t \frac{1-R}{R}\psi_s\theta_s\, ds - \int_0^t R\gamma_s\, ds \right]$$

where for short $R\gamma_s \equiv \frac{1-R}{2R}\theta_s^2 + (1-R) r_s$;

thus

$$\left(Y_t/Y_0\right)^{1/R} = \exp\Big[\int_0^t \frac{\psi_s}{R}\, dW_s - \tfrac{1}{2}\int_0^t \left(\frac{\psi_s}{R}\right)^2 ds - \int_0^t \frac{1-R}{R^2}\, \psi_s \theta_s\, ds - \int_0^t \gamma_s\, ds \Big]$$

$$= \exp\Big[\int_0^t \frac{\psi_s}{R}\, d\bar{W}_s - \tfrac{1}{2}\int_0^t \left(\psi_s/R\right)^2 ds - \int_0^t \gamma_s\, ds \Big]$$

$$\text{where } d\bar{W}_s = dW_s - \frac{1-R}{R}\theta_s\, ds.$$

If we now change measure to $\bar{P}$, where $d\bar{P}/dP \equiv \exp\Big[\int_0^T \frac{1-R}{R}\theta_s\, dW_s - \tfrac{1}{2}\int_0^T \left(\frac{1-R}{R}\right)^2 \theta_s^2\, ds \Big]$,

and set $M_t \equiv \bar{E}\left[\xi \mid \mathcal{F}_t\right]$, where $\xi \equiv \exp\left(\int_0^T \gamma_s\, ds\right)$, then we define ψ by

$$\boxed{dM_t = \frac{\psi_t}{R} M_t\, d\bar{W}_t}$$

so that $$\xi = \bar{E}(\xi) \exp\Big[\int_0^T \frac{\psi_s}{R}\, d\bar{W}_s - \tfrac{1}{2}\int_0^T \left(\frac{\psi_s}{R}\right)^2 ds \Big],$$

and

$$\left(\frac{Y_T}{Y_0}\right)^{1/R} = 1/\bar{E}(\xi) \qquad \therefore \quad Y_0 = \bar{E}(\xi)^R.$$

Re-expressing this in terms of fundamental quantities,

$$\boxed{Y_0 = \left(E \exp\left\{ \int_0^T \frac{1-R}{R}\theta_t\, dW_t + \frac{1-R}{R}\int_0^T \left(r_s + \tfrac{1}{2}\theta_s^2\right) ds \right\} \right)^R.}$$

In the case where r, θ are constant, this reduces to a known result. It is not hard to verify the solution also for the case where r, θ are deterministic – and in fact all that one _actually_ needs is that

$$\tfrac{1}{2}\theta_t^2 + R r_t \;\; \underline{\text{is deterministic}},$$

so even some (very special) random θ, r are covered.

Why and how does a bookie change the odds? (20/11/93)

Suppose that at time t, a bookie is offering to pay out $\alpha_i(t)$ (= stake + winnings) if horse i wins (unit stake), and that the cumulative amount staked on horse i by time t is $C_i(t)$, $\sum_{i=1}^{N} C_i(t) = C(t)$. If all bets have to stop at time T, then the worst the bookie will have to pay out is

$$\max_i \int_0^T \alpha_i(s)\, dC_i(s)$$

and this should be no more than $C(T).(1-\varepsilon)$. So the bookie needs to worry if for some i

$$\dot{C}_i(t).\alpha_i(t) > (1-\varepsilon)\,\dot{C}(t).$$

So he'll adjust α_i to keep the inequality

$$\alpha_i(t) \leq (1-\varepsilon)\,\dot{C}(t)/\dot{C}_i(t),$$

at least for the horse carrying the largest potential loss. If we used the quoted odds to estimate the probabilities, then we form the estimates

$$\hat{p}_i(t) = \frac{1}{\alpha_i(t)} \qquad \left(\text{Expected gain} = (1-p_i) - p_i(\alpha_i - 1) = 0\right)$$

so we would expect to find

$$\sum \hat{p}_i(t) \geq \frac{1}{1-\varepsilon} > 1.$$

at any time. Of course, this need not hold; for the horses for which $\int \alpha\, dc$ is small, the values of α could be (temporarily) quite big.
[Odds of "x to 1" means just that $p = 1/(1+x)$]

The long rate can increase (30/11/93)

If r_t is <u>either</u> 1 for all t <u>or</u> 1 for $t<1$, then 2, each eventuality with equal probability, then for $0<t<1<T$

$$P(t,T) = \tfrac{1}{2}\, e^{-(T-t)} + \tfrac{1}{2}\, e^{-2(T-1)-(1-t)} \qquad \therefore \text{long rate} = 1$$

but for $1 \leq t < T$,

$$P(t,T) = \begin{cases} e^{-(T-t)} & \text{in first case} \\ e^{-2(T-t)} & \text{in second} \end{cases}$$

– so if second occurs, long rate rises.

A simple-minded equilibrium model for the spot-rate process (30/11/93)

(i) Suppose there is one riskless asset, one risky asset, $dS_t = S_t(\sigma_t\, dW_t + \mu_t\, dt)$ and the representative agent is attempting to consume optimally, so as to maximise

$$E\left[\int_0^\infty e^{-\rho t}\, U(C_t)\, dt\right].$$

The wealth process satisfies

$$d\tilde{X}_t = \theta_t\, d\tilde{S}_t - \beta_t C_t\, dt$$

where $\beta_t = \exp\{-\int_0^t r_u\, du\}$, $\tilde{X}_t \equiv \beta_t X_t$, $\tilde{S}_t \equiv \beta_t S_t$, and θ_t is the number of shares held at time t. The market clearing condition

"holding of the bond is always 0"

translates into saying $X_t = \theta_t S_t$, so that $\tilde{X}_t = \theta_t \tilde{S}_t$, and

$$\boxed{S_t\, d\theta_t = -C_t\, dt.}$$

In complete generality, the optimal consumption process $\hat{C}_t$ has the characterisation

$$\hat{C}_t = I(\lambda \zeta_t e^{\rho t}) \qquad (U' \circ I(x) \equiv x)$$

where $\zeta_t \equiv \beta_t Z_t$ is the usual deflator, Z the EMM change-of-measure. The value of λ is fixed by the budget constraint

$$x_0 = E\left[\int_0^\infty \zeta_t \hat{C}_t\, dt\right] = E\int_0^\infty \zeta_t\, I(\lambda e^{\rho t}\zeta_t)\, dt,$$

where x_0 is initial wealth.

Assuming $S_0 = 1$, we must have $\theta_0 = x_0$, therefore

$$\boxed{x_0 = \int_0^\infty \frac{\hat{C}_t\, dt}{S_t} = \int_0^\infty \frac{I(\lambda e^{\rho t}\zeta_t)}{S_t}\, dt}$$

Does this uniquely determine r?

(ii) We have $d\left(\frac{1}{S}\right) = \frac{1}{S}\left\{-\sigma\, dW - \mu\, dt + \sigma^2\, dt\right\}$ $\qquad \left(\tilde{\mu} \equiv \frac{\mu - r}{\sigma}\right)$

$$d\left(\frac{1}{S}\right) = \frac{1}{S}\left\{r\, dt + \tilde{\mu}\, d\tilde{W} + \tilde{\mu}^2\, dt\right\}$$

Hence $d\left(\frac{1}{\zeta S}\right) = \frac{1}{\zeta S}\Big[(\tilde{\mu}-\sigma)\,dW + (-\mu+\sigma^2+r+\tilde{\mu}^2)\,dt + (-\sigma\tilde{\mu})\,dt \Big]$

$$= \frac{1}{\zeta S}\left[(\tilde{\mu}-\sigma)\,dW + (\tilde{\mu}-\sigma)^2\,dt\right]$$

So the condition which must be satisfied can be re expressed as

$$x_0 = \int_0^\infty \zeta_t \hat{C}_t \; e^{\int_0^t \alpha_s dW_s + \frac{1}{2}\int_0^t \alpha_s^2 ds}\, dt \qquad \alpha_s \equiv \tilde{\mu}_s - \sigma_s$$

(iii) In the special case $U(x) = \log x$, this reduces to

$$\lambda = 1/\rho x_0$$

$$x_0 = \int_0^\infty \frac{1}{\lambda e^{\rho t}} \; e^{\int_0^t \alpha_s dW_s + \frac{1}{2}\int_0^t \alpha_s^2 ds}\, dt$$

$$\therefore \quad 1 = \int_0^\infty \rho e^{-\rho t} \exp\left[\int_0^t \alpha_s dW_s + \frac{1}{2}\int_0^t \alpha_s^2 ds\right] dt$$

which can only occur if $\alpha \equiv 0$, that is

$$\boxed{r_t = \mu_t - \sigma_t^2.}$$

Simplifying Schachermayer's example (20/12/93)

(i) Schachermayer ("A counter-example to several problems in mathematical finance") constructs a uniformly bdd cts semimg $(S_t)_{0\le t\le 1}$ defined on some $(\Omega, (\mathcal{F}_t)_{0\le t\le 1}, P)$ with the properties (i) Every $(\mathcal{F}_t)$-martingale is cts

(ii) The minimal martingale measure does not exist

(iii) There is some $P^* \sim P$ st. S is a mg under P^*, and dP^*/dP is bdd

We now build more directly an example satisfying these conditions.

(ii) Take $\Omega_0 \equiv C(\mathbb{R}^+, \mathbb{R}^2)$ with X, Y denoting the canonical processes, and set

$$\sigma \equiv \inf\{t;\ Y_t = 1\}$$

$$\tau \equiv \inf\{t;\ X_t + \tfrac{1}{2}t = -1\}$$

and take

$$\Omega = \{ (X_t, Y_t)_{t\ge 0} \in \Omega_0 : X_u = X_{\tau\wedge\sigma},\ Y_u = Y_{\tau\wedge\sigma}\quad \forall u \ge \tau\wedge\sigma \},$$

with the canonical filtration. Under P, $X_t + t \equiv B_t$ is a Brownian motion, $Y_t + t \equiv \beta_t$ is a Brownian motion, at least up until the stopping time $\zeta \equiv \tau\wedge\sigma$. Under Q, $X_t + t$ will again be a BM, but $Y_t - t \equiv W_t$ will be a Brownian motion.

Thus $$\frac{dQ}{dP} = \exp[\,2\,Y_{\tau\wedge\sigma}\,],$$

which is bounded.

The (attempt at) the minimal mg measure which will turn X into a martingale will be

$$Z_t = \exp\{ B_t - \tfrac{1}{2}t \} \quad \text{stopped at } \zeta$$

$$= \exp\{ X_t + \tfrac{1}{2}t \}$$

Now under Q, Z gets stopped at $\tau\wedge\sigma$, where σ is a.s. finite, and indept of Z. Thus under Q, $(Z_{t\wedge\zeta})$ is a UI martingale, but under P, $\sigma = +\infty$ with positive probability, and $(Z_{t\wedge\zeta})$ is not a UI martingale. So if we take

$$\frac{dP^*}{dP} = \exp(2\,Y_\zeta)\,.\,Z_\zeta$$

What should we do in general?

$$\max\left\{ E[Z_1\, U(X_1)] : X \text{ is marketed wealth pr}, X_0 = 1, \; Z \text{ is non-neg mg} \perp \text{ marketed space}, Z_0 = 1 \right\}?$$

(Couldn't allow *any* Z, because we could then change law in the marketed space with great advantage.)

we get everything except the boundedness of dP^*/dP

[We can put everything on time interval $[0,1]$ by mapping $[0,1] \ni t \to \frac{t}{1-t}$, and we can make bdd S by defining

$$S_0 = 1, \quad dS_t = S_t\,(2 - S_t)\, dX_t. \qquad]$$

(iii) But even the boundedness of dP^*/dP can be achieved, if we stop also at

$$T \equiv \inf\{ u : \ 2Y_u + B_u - \tfrac{1}{2}u > 4 \}.$$

What we must prove is that (under P) $Z_{t \wedge T \wedge \zeta}$ is not a UI martingale. Since Z is continuous, bounded in L^1, it is nts to prove that

$$\lambda P[\ \sup\{Z_u : u \leq T \wedge \zeta\} > \lambda] \not\to 0.$$

Now $P[Z \text{ reaches } \lambda \text{ before time } \zeta]$

$$= \frac{e-1}{\lambda e - 1}$$

and, given that $B_t - \frac{1}{2}t$ does reach $\log \lambda$ before -1, what we see is a BM with drift $+\frac{1}{2}$, until $\log\lambda$ is hit. Let $H_\lambda \equiv \inf\{u : B_u + \frac{1}{2}u = \log\lambda\}$

Now

$$P[\ B_t + \tfrac{1}{2}t \geq -1, \ 2Y_t + (B_t + \tfrac{1}{2}t) \leq 4 \text{ for all } t \leq H_\lambda\ ]$$

$$\geq P[\ B_t + \tfrac{1}{2}t \geq -1 \text{ for all } t, \ B_t + \tfrac{1}{2}t \leq 1 + t \text{ for all } t, \text{ and } Y_t \leq 1 - \tfrac{t}{2} \text{ for all } t\ ]$$

> 0, same for all λ.

Thus $P[Z \text{ reaches } \lambda \text{ before time } T \wedge \zeta] \geq \text{Const.} \dfrac{e-1}{\lambda e - 1}$, and so

Z is not UI if we stop at $T \wedge \zeta$.

Wiener-Hopf reversal with general rates (12/1/94)

The reversal of the standard WH with general rates looks a little different. If $V \equiv \begin{pmatrix} V_+ & \cdot \\ \cdot & -V_- \end{pmatrix}$, $J = \begin{pmatrix} I & \cdot \\ \cdot & -I \end{pmatrix}$ $\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \equiv \begin{pmatrix} V_+ & \cdot \\ \cdot & V_- \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = JV^{-1}Q$,

then we can calculate

$$\hat{\Pi}_+ = \tilde{M}_-^{-1} \Pi_-^T \tilde{M}_+ , \quad \hat{\Pi}_- = \tilde{M}_+^{-1} \Pi_+^T \tilde{M}_-$$

($\tilde{M} \equiv JVM$, the invariant law for chain time-changed with $|V|$.)

$$(\hat{G}_+)^T = \tilde{M}_+ (\tilde{A} + \Pi_- \tilde{C}) \tilde{M}_+^{-1}$$

$$(\hat{G}_-)^T = \tilde{M}_- (\tilde{D} + \Pi_+ \tilde{B}) \tilde{M}_-^{-1}$$

(This is really rather obvious, isn't it?)

We can rework the mean-busy-period calculations of book VIII p 35–36 to obtain that, if $z_i \equiv E^i[\tau_0^-]$ for $i \in E_+$, then

$$z^T = -(1^T M_- \hat{\Pi}_+ + 1^T M_+) \hat{G}_+^{-1} M_+^{-1} V_+^{-1}$$

Some remarks on multi-type branching diffusions (26/1/94)

1) Johnathan Warren is considering diffusions in $(\mathbb{R}^+)^d$ of the form

$$dX_t^i = \sqrt{X_t^i}\, dB_t^i + \sum_j a_{ij} X_t^j \, dt \qquad (i = 1, \dots, d)$$

where the B^i are independent Brownian motions, and we shall assume for now that

$$\boxed{a_{ij} \geq 0 \quad \text{for } i \neq j}$$

which guarantees that X stays in $(\mathbb{R}^+)^d$.

Observe that if X and Y are independent copies of this diffusion, $X_0 = x$, $Y_0 = y$, then $X+Y$ also solves the same SDE, with initial point $x+y$.

Hence for any $\lambda \in (\mathbb{R}^+)^d$, there exists for each $t \geq 0$ a map $\varphi(t, \cdot) : (\mathbb{R}^+)^d \to (\mathbb{R}^+)^d$ such that

$$\boxed{E\left[\exp(-\lambda \cdot X_t) \,\middle|\, X_0 = x\right] = \exp\left[-x \cdot \varphi(t, \lambda)\right]}$$

In particular, for any $T>0$ we have for $t \leq T$

$$M_t \equiv \exp[-X_t \cdot \varphi(T-t,\lambda)] = E[e^{-\lambda \cdot X_T} / \mathcal{F}_t]$$

is a martingale. A couple of lines of Itô calculus give us that φ must solve

$$\boxed{\dot{\varphi} - A^T\varphi + \tfrac{1}{2}\varphi^2 = 0}$$

or more explicitly in coordinates

$$\boxed{\dot{\varphi}^i(t,\lambda) - \sum_j a_{ji}\varphi^j(t,\lambda) + \tfrac{1}{2}\varphi^i(t,\lambda)^2 = 0,} \qquad \varphi(0,\lambda) = \lambda.$$

2) By letting $T\to\infty$, we have, with $\eta \equiv \eta(\lambda) \equiv \lim_{T\to\infty} \varphi(T,\lambda)$, that

$$P^x[X_t \to 0 \text{ as } t\to\infty] = e^{-\eta\cdot x}$$

so, in particular, the limit η is independent of λ and satisfies

$$\boxed{\tfrac{1}{2}\eta_i^2 = \sum_j q_{ji}\eta_j.}$$

[This requires that $\eta_i > 0$ for all i; this isn't necessarily true in general, but if we assume that we get an irreducible chain when we take $q_{ij} \equiv q_{ji}$ $(i\neq j)$ then clearly if $\eta_i = 0$ for some i, then $\eta_j = 0$ for all j. So, making this assumption, the set

$$\{x \geq 0 : \tfrac{1}{2}x_i^2 = \sum a_{ji}x_j \text{ for all } i\}$$

contains 0 and perhaps other solutions, all of which are strictly positive.

Actually, we need also to know that X either tends to infinity, or to zero but can't hang about inbetween. To see this, note that

$$d(1\cdot X_t) = \sqrt{1\cdot X_t}\, d\beta_t + 1^T A X_t\, dt$$

and that the drift $1^T A X_t$ is bounded above by $K 1^T X_t$, for some large enough $K>0$. Then easily

$$\exp(-2K\, 1\cdot X_t) \text{ is a submartingale}$$

and so $1\cdot X_t$ is convergent in $[0,\infty]$. Only possible if $1\cdot X_t \to 0$ or ∞.

From this it follows that if $\lambda \gg 0$, then $\lim_{t\to\infty} \varphi(t,\lambda)$ must exist and be independent of λ.

Since for any $v \gg 0$ satisfying $\frac{1}{2} v_i^2 = \sum_j a_{ji} v_j$ the process $\exp(-v\cdot X_t)$ will be a martingale, it must be that $\exp(-v\cdot x) = P^x($~~tend~~ to 0$)$ and so there is at most one solution to $\frac{1}{2}v_i^2 = \sum_j a_{ji} v_j$ which is $\gg 0$. $\lrcorner$

Remark If we do not make the irreducibility assumption, then we have at least 2 non-interconnecting subpopulations, and if we took λ to be 1 on one of the subpopulations, zero elsewhere, we have $\exp[-\lim \varphi(t,\lambda)\cdot x] = P^x($population 1 gets extinct$) \geq P^x($all populations get extinct$) = \exp[-\lim \varphi(t,1)\cdot x]$.

3) To understand the eigenstructure of A, let's note that for some K large enough $A+KI$ has only non-negative entries, so we may use the Perron-Frobenius theorem; there exist vectors $u, z \geq 0$, not equal to zero, and some $0<\mu<\infty$ such that

$$(A+KI)u = \mu u, \qquad z^T(A+KI) = \mu z^T$$

and μ is the maximal eigenvalue of $A+KI$. It is clear that if A is irreducible, then $u_i\, z_i > 0$ for all i.

PROPOSITION. Assume irreducibility of A. Then the following are equivalent.

(i) There exists a solution $\eta \gg 0$ to

$$\frac{1}{2}\eta_i^2 = \sum_j a_{ji}\eta_j$$

(ii) A has a strictly positive real eigenvalue.

Proof. (i)$\Rightarrow$(ii). Pick $K \equiv \sup\{-a_{ii} : i=1,\dots,d\}$. Then we have

$$\frac{1}{2}\eta_i^2 + K\eta_i = \sum_j (a_{ji} + K\delta_{ij})\eta_j$$

and if $\delta = \inf\{\eta_i : i=1,\dots,d\} > 0$, we conclude that

$$(A^T+KI)\eta \geq (K+\tfrac{1}{2}\delta)\eta .$$

Now by the Perron-Frobenius theorem [see, for example, Appendix C of "Economics for Mathematicians", J.W.S. Cassels, especially Theorem 1 (iii)], we conclude that $K+\frac{1}{2}\delta \leq \mu$, where μ is the top eigenvalue of A^T+KI.

(ii) $\Rightarrow$ (i) Take the non-negative matrix $A^T + KI$, with top eigenvalue $\mu \equiv K+\delta$, $\delta > 0$, and corresponding eigenvector $y \gg 0$. We shall describe a recursive scheme for generating approximations $\eta^{(n)}$ to the η which will solve the given equation. We shall have

$$\tfrac{1}{2}\left(\eta_i^{(n+1)}\right)^2 + K\eta_i^{(n+1)} = \sum_j (a_{ji} + K\delta_{ij})\eta_j^{(n)}$$

where when we solve the quadratic

$$\tfrac{1}{2}x^2 + Kx - c = 0$$

we always take the larger root $\sqrt{K^2+2c} - K > 0$ (when $c>0$, as will always be the case). The starting value $\eta^{(0)}$ will be εy for some small enough $\varepsilon > 0$; it will then turn out that $\eta_i^{(1)} \geq \eta_i^{(0)}$ for all i, and then it follows that $\eta^{(0)} \leq \eta^{(1)} \leq \eta^{(2)} \leq \dots$ Once we prove that this sequence cannot grow unboundedly, it follows that it has a limit, which solves the equation of interest. Taking $\eta^{(0)} = \varepsilon y$, we find

$$\eta_i^{(1)} = (K^2 + 2\varepsilon(K+\delta)y_i)^{\frac{1}{2}} - K \geq \varepsilon y_i$$

$$\text{iff} \qquad K^2 + 2\varepsilon(K+\delta)y_i \geq K^2 + 2K\varepsilon y_i + \varepsilon^2 y_i^2$$

$$\text{iff} \qquad 2\delta \geq \varepsilon y_i$$

so by taking $\varepsilon \equiv \min_i \{2\delta / y_i\}$ we get $\eta_i^{(1)} \geq \eta_i^{(0)}$ for all i

Finally, to check that the sequence $\eta^{(n)}$ cannot grow unboundedly, suppose that for some n, $\max_i \eta_i^{(n)} = M$, very large. Let $b \equiv \max_i \left(\sum_j |a_{ji}| + K\right)$

so that for all i, $\eta_i^{(n+1)} \leq \sqrt{K^2 + 2bM} - K < M$

$$\text{iff} \quad K^2 + 2bM < K^2 + 2KM + M^2$$

$$\text{iff} \quad 2(b-K) < M$$

So if we take $M = 1 + 2\max_i \sum_j |a_{ji}|$, it's impossible to find η such that $\max_i \eta_i^{(n)} \geq M$.

Observations on sequential analysis (27/1/94)

Suppose we observe IID $(X_n)_{n\geq 0}$ which are normal, unknown mean μ and variance σ^2. Want to find some stopping rule to test $H_0: \mu=0$ vs $H_1: \mu\neq 0$.

(i) If we just were to fix the sample size N, then

$$2\log L = -\sum_1^N (X_i-\mu)^2/\sigma^2 - N\log(2\pi\sigma^2)$$

and under H_0 this is maximised by $\sigma^2 = \sum_1^N X_i^2/N$ to value

$$-N - N\log\left(\frac{2\pi}{N}\sum X_i^2\right)$$

and under H_1 it is maximised to

$$-N - N\log\left(\frac{2\pi}{N}\sum_1^N (X_i-\bar{X})^2\right). \qquad (N\geq 2 \text{ assumed}).$$

If we take a test of the form "Stop at the first time N that the LR exceeds c" then this amounts to

$$\log\left(\sum X_i^2\right) - \log\left(\sum X_i^2 - N\bar{X}^2\right) > \frac{c}{N}$$

or again

$$1 - \frac{N\bar{X}^2}{\sum X_i^2} < e^{-c/N}$$

or again, wait until

$$\left(\sum_1^N X_i\right)^2 > N\left(\sum_1^N X_i^2\right)\left(1-e^{-c/N}\right).$$

Even under the null, this will eventually happen; the RHS $\sim cN$, and the LHS will eventually exceed this whatever c (LIL).

(ii) If we want a Bayesian analysis, we could take a prior on $(\mu, v)\equiv(\mu,\sigma^2)$ of the form const $\exp(-\varepsilon\mu^2/2)\, v^{-\gamma}\exp(-a/2v)$ $\quad(\gamma>1)$

and get a posterior proportional to

$$(2\pi v)^{-N/2}\exp\left(-\sum_1^N (X_i-\mu)^2/2v - \varepsilon\mu^2/2 - a/2v\right) v^{-\gamma}.$$

It's now easy to integrate out v, to obtain the posterior for μ:

$$\propto e^{-\varepsilon\mu^2/2}\left[a+\sum_1^N (X_i-\mu)^2\right]^{-\gamma-N/2+1}$$

We could let $\varepsilon\downarrow 0$, and then stop once the probability that $\bar{X}\mu<0$ was smaller than the size required.

Note that $d\tilde{\xi}_t = R_t^T \left(dX_t - \xi_t \frac{dt}{t} \right) - A_t R_t^T \xi_t \, dt$

$$= d\tilde{X}_t - \tilde{\xi}_t \frac{dt}{t} - A_t \tilde{\xi}_t \, dt$$

$$\therefore \quad d\tilde{\xi}_i = d\tilde{X}_i - \tilde{\xi}_i \frac{dt}{t} - \tilde{\xi}_i \sum_{j \neq i} \frac{\tilde{\xi}_j^2}{\lambda_j - \lambda_i} \, dt$$

~~Hence~~ $\boxed{M_i(t) = t \, \tilde{\xi}_i(t) \prod_{j \neq i} \left(\lambda_j(t) - \lambda_i(t) \right)}$ ~~is a martingale for each i~~

~~and $dM_i \, dM_j = 0$ if $i \neq j$~~

Brownian motion in $\mathbb{R}^n$ shifted and rotated (31/1/94)

If X is a $BM(\mathbb{R}^n)$, and $\bar{X}_t \equiv t^{-1}\int_0^t X_u\,du$, then we shall consider

$$\xi_t \equiv X_t - \bar{X}_t .$$

(i) What can we say about ξ? Firstly, ξ solves

$$\boxed{d\xi_t = dX_t - \xi_t \frac{dt}{t},} \qquad \xi_0 = 0,$$

so $d(t\xi_t) = t\,dX_t$. Hence pathwise uniqueness holds for this SDE.

Also, ξ is a Gaussian process with zero mean. It is not hard to verify that for $0 < s \leq t$,

$$\boxed{E\,\xi_s \xi_t^T = \frac{s^2}{3t} I.}$$

[This is easiest to see from $d(t\xi_t) = t\,dX_t$]. Somewhat surprisingly, we have an alternative representation

$$\boxed{t\xi_t = B(t^3/3).}$$

(ii) Suppose now we define

$$V_t \equiv \int_0^t (X_s - \bar{X}_t)(X_s - \bar{X}_t)^T ds$$

$$= \int_0^t (X_s - \bar{X}_s)(X_s - \bar{X}_s)^T ds,$$

as can be easily confirmed. Suppose also that we have a C^1 diagonalisation of V,

$$V_t = R_t \Lambda_t R_t^T$$

with $\dot{R}_t = R_t A_t$, and A_t is an antisymmetric matrix for each t. Then

$$\dot{\Lambda}_t = (R_t^T \xi_t)(\xi_t^T R_t) - A_t \Lambda_t + \Lambda_t A_t$$

so, writing $\tilde{\xi}_t \equiv R_t^T \xi_t$, we shall have (assuming distinct evalues)

$$\boxed{\begin{aligned} \dot{\lambda}_i(t) &= (\tilde{\xi}_t^i)^2 \\ a_{ij}(t) &= \tilde{\xi}_t^i \tilde{\xi}_t^j / (\lambda_j(t) - \lambda_i(t)) \qquad (i \neq j) \end{aligned}}$$

Limiting direction for the multitype branching diffusion (6/2/94)

Back to the earlier situation. Assume wmlog that A has distinct e-values, $\lambda_0, \lambda_1, \dots, \lambda_{d-1}$ with λ_0 the top (Perron-Frobenius) eigenvalue, and let v_j be the corresponding e-vector of A^T.

We know that for each $j=0,\dots,d-1$, $e^{-\lambda_j t}(v_j \cdot X_t)$ is a martingale, and indeed

$$d(e^{-\lambda_j t} v_j \cdot X_t) = e^{-\lambda_j t} \sum_i v_j^i \sqrt{X_t^i}\, dB_t^i$$

so that

$$\begin{aligned} E\, e^{-2\lambda_j t}(v_j \cdot X_t)^2 &= (v_j \cdot X_0)^2 + E\int_0^t e^{-2\lambda_j s}(v_j^i)^2 X_s^i \, ds \\ &\leq (v_j \cdot X_0)^2 + c.\ E\int_0^t e^{-2\lambda_j s}\, v_0 \cdot X_s\, ds \\ &= (v_j \cdot X_0)^2 + c.\left[\{e^{(\lambda_0 - 2\lambda_j)t} - 1\} / (\lambda_0 - 2\lambda_j)\right](v_0 \cdot X_0) \end{aligned}$$

$$\Rightarrow \quad E\,(v_j \cdot X_t)^2 \leq (v_j \cdot X_0)^2 e^{2\lambda_j t} + v_0 \cdot X_0 \frac{c}{\lambda_0 - 2\lambda_j}\left(e^{\lambda_0 t} - e^{2\lambda_j t}\right)$$

$$\Rightarrow \quad E\ e^{-2\lambda_0 t}(v_j \cdot X_t)^2 \longrightarrow 0 \qquad \text{for } j>0.$$

[In fact, this argument is written implicitly assuming λ_j is real, but it works just as well if λ_j is complex. The key point is that, by PF theory, the real part of λ_j is always $< \lambda_0$, $j>0$.]

It is clear that $e^{-\lambda_0 t}(v_j \cdot X_t) \to 0$ as $t \to \infty$ through $\mathbb{N}$, so we shall show that infact $e^{-\lambda_0 t}(v_j \cdot X_t) \to 0$ as $t \to \infty$ unrestrictedly.

Indeed,

$$\begin{aligned} &P\left[\sup_{n \leq t \leq n+1} \left|\int_n^t e^{-\lambda_j s} \sum_i v_j^i \sqrt{X_s^i}\, dB_s^i\right| > a\right] \\ &\leq \frac{4}{a^2} \cdot c \cdot E\left[\int_n^{n+1} e^{-2(\mathrm{Re}\,\lambda_j)s} \sum_i X_s^i \, ds\right] \\ &\leq \frac{c}{a^2}\, e^{(\lambda_0 - 2\mathrm{Re}\,\lambda_j)n} \end{aligned}$$

so if we write $M_t \equiv e^{-\lambda_j t}(v_j \cdot X_t)$ for the (perhaps complex-valued) martingale of interest, we can estimate

$$P\left[\sup_{n\le t\le n+1} \left| e^{\lambda_j t - \lambda_0 t} M_t - e^{\lambda_j n - \lambda_0 n} M_n \right| > a_n \right]$$

$$\le P\left[\left|e^{\lambda_j n - \lambda_0 n} M_n\right| \left|e^{\lambda_j - \lambda_0} - 1\right| > a_n/2 \right]$$

$$+ P\left[\sup_{n\le t\le n+1} |M_t - M_n| \cdot \left|e^{\lambda_j n - \lambda_0 n}\right| > a_n/2 \right]$$

Now for large enough t, we have

$$E\,|M_t| \le (E|M_t|^2)^{\frac{1}{2}} \le C.\ e^{(\frac{1}{2}\lambda_0 - \mathrm{Re}\,\lambda_j)t},$$

so we can estimate the first piece by

$$C.\ e^{(\frac{1}{2}\lambda_0 - \mathrm{Re}\,\lambda_j)n}\ a_n^{-1}\ e^{(\mathrm{Re}\,\lambda_j - \lambda_0)n}$$

$$= C.\ e^{-n\gamma - \frac{1}{2}\lambda_0 n} \qquad \text{if } a_n \equiv e^{\gamma n}, \quad 0 > \gamma > -\lambda_0/2$$

The second term is bounded above by

$$C.\ e^{(\lambda_0 - 2\mathrm{Re}\,\lambda_j)n} \cdot e^{-2n\gamma}\ e^{2(\mathrm{Re}\,\lambda_j - \lambda_0)n}$$

$$= C.\ e^{-\frac{1}{2}\lambda_0 n - n\gamma}.$$

So Borel-Cantelli gives us that a.s. for all large enough n,

$$\sup_{n\le t\le n+1} \left| e^{\lambda_j t - \lambda_0 t} M_t - e^{\lambda_j n - \lambda_0 n} M_n \right| \le e^{\gamma n},$$

and hence $e^{\lambda_j t - \lambda_0 t} M_t \equiv e^{-\lambda_0 t}(v_j \cdot X_t) \to 0$ a.s.

On the non-extinction event, $e^{-\lambda_0 t}(v_0 \cdot X_t) \longrightarrow$ positive limit a.s., so on the non-extinction event, for any $j > 0$,

$$\frac{v_j \cdot X_t}{v_0 \cdot X_t} = \frac{e^{-\lambda_0 t}(v_j \cdot X_t)}{e^{-\lambda_0}(v_0 \cdot X_t)} \longrightarrow 0.$$

Thus $\boxed{\dfrac{X_t}{v_0 \cdot X_t} \xrightarrow{a.s.} f_0}$, the top right e-vector of A.

Joint law of sup + inf of stopped random walk (13/2/94)

(i) Take simple symmetric rw $(S_n)_{n\geq 0}$, $S_0=0$, and stop it at some a.s. finite stopping time T, which only happens when S is in a new location for the first time. What are the possible joint laws of $(\bar{S}_T, -\underline{S}_T)$, where $\bar{S}_k = \max_{r\leq k} S_r$, $\underline{S}_k = \min_{r\leq k} S_r$?

If we take $R_n = \bar{S}_n - \underline{S}_n$, and let (τ_n) be the time-change associated with R ($\tau_n = \inf\{k : R_k = n\}$), then $(\bar{S}(\tau_n), -\underline{S}(\tau_n), \operatorname{sgn}(S_{\tau_n}))_{n\geq 0}$ is a discrete-time transient Markov process on $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \{-1,+1\}$, whose Green f^{n} is easy to describe.

(ii) We start with initial law μ_0 conc on $\{(0,0,1), (0,0,-1)\}$. Then the law μ is the law of $(\bar{S}(\tau_n), -\underline{S}(\tau_n), \sigma_n)$ stopped at some stopping time iff

$$\boxed{\mu_0 G \geq \mu G \quad \text{everywhere,}}$$

where G is the Green kernel (and $\sigma_n \equiv \operatorname{sgn}(S(\tau_n))$, of course.) This is well known, but can it be rephrased in a more ~~transparent~~ (martingale) form?

Fix $a, b > 0$. Then

$$\mu_0 G(b,a,+) = \frac{b}{a+b} \cdot \frac{1}{a+b+1}$$

and

$$\mu G(b,a,+) = P\left[\text{ visit } (b,a,+) \text{ at or after time } \tau\right]$$

$$= P\left[\text{visit } (b,a,+) \text{ after } \tau,\ \tau < H_{-a} \wedge H_b\right]$$
$$+ P\left[\text{visit } (b,a,+) \text{ after } \tau,\ H_{-a} \leq H_b \wedge \tau,\ \tau < H_{-a-1}\right]$$

$$= E\left[\frac{b - S_\tau}{(b+a)(a+b+1)};\ \tau < H_{-a} \wedge H_b\right]$$

$$+ E\left[\frac{S_\tau + a + 1}{a+b+1};\ H_{-a} \leq \tau \leq H_{-a-1} \wedge H_b\right].$$

Cross-multiplying by $(a+b)(a+b+1)$, the inequality characterising embeddable laws says

$$b \geq E\left[b - S_\tau;\ \tau < H_{-a} \wedge H_b\right] + (a+b)\, E\left[S_\tau + a + 1;\ H_{-a} \leq \tau \leq H_{-a-1} \wedge H_b\right].$$

Using the OST at $\tau \wedge H_{-a} \wedge H_b$, the first term in the RHS is

$$b - (a+b) P[H_{-a} \leq \tau \wedge H_b]$$

which transforms the inequality into

$$P[H_{-a} \leq \tau \wedge H_b] \geq E[S_\tau + a + 1 \; ; \; H_{-a} \leq \tau \leq H_{-a-1} \wedge H_b].$$

If now we condition on $H_{-a} \leq \tau \wedge H_b$, and stop at $\tau \wedge H_b \wedge H_{-a-1}$, the OST delivers us

$$1 = E[S_{\tau'} + a + 1 \mid H_{-a} \leq \tau \wedge H_b] \qquad \tau' \equiv \tau \wedge H_b \wedge H_{-a-1}$$

$$= (b+a+1) P[H_b \leq \tau \wedge H_{-a-1} \mid H_{-a} \leq \tau \wedge H_b]$$
$$+ E[S_{\tau'} + a + 1 \; ; \; \tau < H_{-a-1} \wedge H_b \mid H_{-a} \leq \tau \wedge H_b]$$

$$\geq E[S_\tau + a + 1 \; ; \; \tau \leq H_{-a-1} \wedge H_b \mid H_{-a} \leq \tau \wedge H_b],$$

which is a martingale interpretation of the potential-theoretic condition.

Now

$$P[H_{-a} \leq \tau \wedge H_b] = P[H_{-a} \leq H_b] - P[\tau < H_{-a} < H_b]$$

$$= \frac{b}{a+b} - E\left[\frac{b - S_\tau}{a+b} \; ; \; \tau < H_{-a} \wedge H_b \right]$$

$$= \frac{b}{a+b} - \sum_{j=0}^{a-1} \sum_{k=0}^{b-1} \left\{ \mu_{jk+} \frac{b-k}{a+b} + \mu_{jk-} \frac{b+j}{a+b} \right\},$$

so we can in principle work this out from μ, but it looks not a lot better...

A nice portfolio optimisation example (23/2/94)

(i) Here's another example of an optimal portfolio problem where one can work everything out explicitly. Assume $r = 0$, and

$$\frac{dS_t}{S_t} = dB_t + \mu\, dt \equiv dx_t, \text{ say,}$$

where μ is either $+1$ or -1 (with prob's p, q respectively) but is not known and has to be inferred from the observed path of S. We have

$$P[\mu = 1 \mid \mathcal{F}_t] = p e^{x_t} / (p e^{x_t} + q e^{-x_t}) \qquad p/q \equiv e^{-2a}$$

$$= e^{x_t - a} / \{ e^{x_t - a} + e^{a - x_t} \},$$

so $E[\mu \mid \mathcal{G}_t] = \tanh(x_t - a)$.

Thus we have

$$\boxed{dx_t = d\beta_t + \tanh(x_t - a)\, dt} \qquad \text{for some BM } \beta.$$

Thus the change-of-measure martingale will be

$$d\zeta_t / \zeta_t = -\tanh(x_t - a)\, d\beta_t,$$

which can be solved explicitly:

$$\boxed{\zeta_t = e^{t/2} \operatorname{sech}(x_t - a) . \cosh a.}$$

(ii) Now suppose our utility is $U(x) \equiv x^{1-R}/(1-R)$, so the optimal wealth process with horizon T will satisfy

$$X_t^* = E^*[\, I(\lambda_T \zeta_T) \mid \mathcal{F}_t] \qquad (0 \leq t \leq T)$$

where $dP^*/dP = \zeta_T$, and λ_T is chosen to match up the initial wealth. This is not so easy in general, but when $\boxed{R = \tfrac{1}{2},}$ we do get something nice in closed form. Indeed, $I(x) = x^{-2}$, so

$$X_t^* = E^* \left[I(\lambda \zeta_T) \mid \mathcal{F}_t \right] = \lambda^{-2} E^* \left[e^{-T} \cosh^2(x_T - a) \operatorname{sech}^2 a \mid \mathcal{F}_t \right]$$

$$= \lambda^{-2} e^{-T} \operatorname{sech}^2 a \,\{ 1 + e^{2(T-t)} \cosh(2x_t - 2a) \}/2$$

since x is a BM under P^*. Picking the good value of λ gives us

One can also do the analogous calculations with spot-rate $\equiv r \neq 0$.
One obtains

$$X_t^* = X_0 \, e^{-2r x_t + rt} \; \frac{e^{2(T-t)} \cosh 2\left(a + r(T-t) - x_t\right) + 1}{e^{2T} \cosh 2(a + rT) + 1}$$

and hence

$$dX_t^* = X_t^* \left\{ -2r + 2 \frac{\sinh 2\left(x_t - a - r(T-t)\right)}{\cosh 2\left(x_t - a - r(T-t)\right) + e^{-2(T-t)}} \right\} dx_t$$

so as $T \to \infty$, if $r > 0$, we conclude that the optimal portfolio converges to

$$- X_t^* \; 2(r+1) / S_t$$

so hold a proportion $-2r-2$ of wealth in the risky asset!

(See also reverse of p 19) $\rightarrow$

The case $R=1$ is even easier, in fact! We have

$$X_t^* = X_0 / \zeta_t = X_0 / \left\{ e^{r x_t - \frac{1}{2} r^2 t} \, e^{t/2} \, \frac{\cosh a}{\cosh(x_t - a)} \, e^{-rt} \right\}$$

$$= X_0 \, e^{-r x_t} \, \frac{\cosh(x_t - a)}{\cosh a} \, e^{rt - r^2 t/2}$$

So here $\quad dX_t^* = X_t^* \left\{ -r + \tanh(x_t - a) \right\} dx_t + 0$

so $\quad \boxed{\theta_t^* = \tanh(x_t - a) - r}$

This one is of the form $(\mu_t - r_t)/\sigma_t^2 R$, as in the Merton paradigm!

$$\boxed{X_t^* = X_0 \frac{1 + e^{2(T-t)} \cosh(2x_t - 2a)}{1 + e^{2T} \cosh 2a}}$$

When we consider the stochastic integral representing this P^*-martingale, we get

$$dX^* = \frac{2X_0\, e^{2(T-t)}}{1 + e^{2T}\cosh 2a} \sinh(2x_t - 2a)\, dx_t$$

$$\to \quad 2X_0\, e^{-2t} \operatorname{sech} 2a \quad \sinh(2x_t - 2a)\, dx_t$$

which is most certainly <u>not</u> a constant proportion of wealth! The proportion of wealth invested in the risky asset at time t is

$$\frac{2e^{2(T-t)} \sinh(2x_t - 2a)}{1 + e^{2(T-t)}\cosh(2x_t - 2a)}.$$

Observe that if t is large, $T \gg t$, we see roughly $2\tanh(2x_t - 2a)$ which will be about 2 if μ were in fact 1, or about -2 if $\mu = -1$. This is consistent with what we'd expect if μ was not random + was known.

<u>Itô's formula for random C^2 functions (23/2/94)</u>

Suppose we had an adapted continuous function $F(t,x)$ which was C^2 in x, and its x and t derivatives up to order 2, 1 are cts. If Y is a cts semimg, what may one say about $F(t, Y_t)$?

Suppose F is represented as $F(t,x) = \sum_{n=0}^{N} u_n(t) x^n$. Then

$$dF(t, Y_t) = \sum_0^N du_n(t).\, Y_t^n + \sum_0^N u_n(t)\left(n Y_t^{n-1} dY_t + \tfrac{1}{2} n(n-1)\, d\langle Y\rangle_t .\, Y_t^{n-2}\right)$$
$$+ \sum_0^N n Y_t^{n-1}\, dY_t\, du_n(t)$$

$$= F(dt, Y_t) + F'(t, Y_t)\, dY_t + \tfrac{1}{2} F''(t, Y_t)\, d\langle Y\rangle_t$$
$$+ dF'(t,x)\, dY_t \Big|_{x = Y_t}.$$

This must be known somewhere, and generalised.

For the case $R=\frac{1}{2}$, when we maximise, the expected utility achieved is

$$\frac{1}{2}\, e^{r^2 T/2} \left\{ 2X_0\, e^{(r-1)T} \left(e^{2T}\cosh 2(a+rT)+1 \right) \right\}^{\frac{1}{2}} \Big/ \cosh a$$

If we compare with the situation where $\frac{dS}{S} = dB + \mu dt$, μ = constant, we get

$$E(X_T^*)^{1-R} = X_0^{1-R} \exp\left[\frac{1-R}{R} T \left(rR + (\mu-r)^2/2 \right) \right]$$

When $R=\frac{1}{2}$, growth rate reduces to $\frac{1}{2}\left(r + (\mu-r)^2\right)$. Taking the above expression for maximised expected utility and letting $a \to +\infty$ or $-\infty$, we get as growth rate

$$\frac{r^2}{2} + \frac{r-1}{2} + 1 \pm r = \frac{1}{2}\left(r^2 + r + 1 \pm 2r\right)$$

which matches what we'd have for the constant case

Maximising expected utility of terminal wealth (23/2/94)

(i) Let's consider the problem of finding $\max E\, U(\eta + \int_0^T \theta_u \, dX_u)$, where η is some $\mathcal{F}_T$-measurable r.v., and X is a cts. semimartingale, $T > 0$ is fixed. Suppose that we could find

$$\varphi(t,x) \equiv \sup\left\{ E_t\, U(\eta + x + \int_t^T \theta_u \, dX_u) \; ; \; \theta \text{ "admissible" } \right\}.$$

Then we would expect to find (if enough derivatives exist etc) that

$\varphi(t, Y_t)$ is a supermartingale whatever $Y_t \equiv \int_0^t \theta_u \, dX_u$

and is a martingale if the optimal policy is in use.

This requires that we can do Itô's formula on φ. Let us also notice that for fixed x, $\varphi(\cdot, x)$ must be a supermartingale. Well,

$$\begin{aligned} d\varphi(t, Y_t) &= \varphi(dt, Y_t) + \varphi'(t, Y_t)\, dY_t + \tfrac{1}{2}\varphi''(t, Y_t)\, d\langle Y \rangle_t \\ &\qquad + d_t \varphi'(t,x)\, dY_t \Big|_{x = Y_t} \\ &= \varphi(dt, Y_t) + \theta_t \varphi'(t, Y_t)\, dX_t + \tfrac{1}{2}\varphi''(t, Y_t)\, \theta_t^2\, d\langle X \rangle_t \\ &\qquad + d_t \varphi'(t,x)\, dX_t \Big|_{x=Y_t} \cdot \theta_t . \end{aligned}$$

Thus we'd get a quadratic in θ to maximise (it's clear that $\varphi(t,\cdot)$ must always be increasing concave).

Without more explicit assumptions, this is not going any further.

(ii) Take the special example $U(x) = -e^{-x}$, and abbreviate $\varphi(t,0)$ to ζ_t. Then ζ is a supermg, and

$$\boxed{\varphi(t,x) = e^{-x} \zeta_t}.$$

Let's suppose that $\boxed{dX = dM + \alpha\, d\langle M \rangle}$ for some cts loc mg M, and previsible α. Then if $Y_t = \int_0^t \theta_u \, dX_u$, and we analyse $\varphi(t, Y_t)$, we obtain

$$d\,\varphi(t,Y_t) = d\left[\zeta_t\, e^{-Y_t}\right]$$

$$= e^{-Y_t}\left\{ d\zeta_t - \zeta_t\, dY_t + \zeta_t \cdot \tfrac{1}{2}\, d\langle Y\rangle_t + d\zeta_t\,(-dY_t)\right\}$$

$$= e^{-Y_t}\left[d\zeta_t - \zeta_t\theta_t\, dX_t + \tfrac{1}{2}\theta_t^2\zeta_t\, d\langle X\rangle_t - \theta_t\, d\zeta_t\, dX_t\right].$$

Now suppose that

$$\boxed{d\zeta = \zeta\{\rho\, dM + dN + \gamma\, d\langle M\rangle\}}$$

for some previsible processes ρ, γ, and cts loc mg $N \perp M$. We therefore get

$$d\varphi(t,Y_t) \doteq \zeta_t e^{-Y_t}\left[\gamma\, d\langle M\rangle - \theta\alpha\, d\langle M\rangle + \tfrac{1}{2}\theta^2 d\langle M\rangle - \theta\rho\, d\langle M\rangle\right]$$

Maximizing over θ, the best θ is $\theta = (\rho+\alpha)$, and $\gamma = \frac{1}{2}(\rho+\alpha)^2$.

Thus

$$\zeta_t = \zeta_0 \exp\left[N_t + \int_0^t \rho_s\, dM_s + \int_0^t \left(\tfrac{1}{2}(\rho_s+\alpha_s)^2 - \tfrac{1}{2}\rho_s^2\right) d\langle M\rangle_s\right]$$

$$= \zeta_0 \exp\left[N_t + \int_0^t \rho_s\, dX_s + \tfrac{1}{2}\int_0^t \alpha_s^2\, d\langle M\rangle_s\right]$$

Now it's clear that

$$\zeta_T = -e^{-\eta}$$

So, <u>if we can find a representation</u>

$$\boxed{-\eta = c + N_T + \int_0^T \rho_s\, dX_s + \tfrac{1}{2}\int_0^T \alpha_s^2\, d\langle M\rangle_s}$$

for some $\mathcal{F}_0$-measurable c, we have solved the problem; $\theta_t^* = \rho_t + \alpha_t$, and

$$\zeta_0 = -e^{-c}.$$

BM in its own frame; ergodic formulation (24/2/94)

We have (see p 13) that if $\xi_t \equiv X_t - \bar{X}_t$ then $t\,\xi_t = B(t^3/3)$. Then

$$x_t \equiv e^{-t/2}\,\xi(e^t) = e^{-3t/2}\,B(e^{3t}/3)$$

is an OU process, solving

$$\boxed{dx_t = dW_t - \tfrac{3}{2} x_t\,dt.}$$

In terms of this

$$V_t \equiv \int_0^t \xi_u \xi_u^T\,du = \int_{-\infty}^{\log t} e^{2s}\,x_s x_s^T\,ds,$$

so

$$\boxed{e^{-2t} V(e^t) \equiv v_t = \int_{-\infty}^t e^{2s-2t}\,x_s x_s^T\,ds = \int_{-\infty}^0 e^{2u}\,x_{u+t} x_{u+t}^T\,du.}$$

If we diagonalise v to $v = R\Lambda R^T$, we get $(\dot{R} = RA)$

$$\dot{\Lambda} = -2\Lambda + \tilde{x}\tilde{x}^T - A\Lambda + \Lambda A$$

where $\tilde{x} \equiv R^T x$, and, as before, we can conclude

$$\boxed{\begin{aligned} a_{ij} &= \frac{\tilde{x}_i \tilde{x}_j}{\lambda_j - \lambda_i} \\ \dot{\lambda}_i &= -2\lambda_i + \tilde{x}_i^2 \end{aligned}}$$

and $\boxed{d\tilde{x} = d\tilde{W} - \tfrac{3}{2}\tilde{x}\,dt - A\tilde{x}\,dt.}$

We can also find the law of $V_1 = \int_0^1 (X_t - \bar{X}_1)(X_t - \bar{X}_1)^T\,dt$. If we fix some $\theta \in \mathbb{R}^n$, then

$$\begin{aligned} E\exp(-\tfrac{1}{2}\theta^T V_1 \theta) &= E\exp -\tfrac{1}{2}\int_0^1 (\theta^T(X_t - \bar{X}_1))^2\,dt \\ &= E\exp -\tfrac{1}{2}|\theta|^2 \int_0^1 (X_t^1 - \bar{X}_1^1)^2\,dt \qquad \text{by rotational symmetry} \\ &= \left(\frac{|\theta|}{\sinh|\theta|}\right)^{1/2}, \end{aligned}$$

by the time-honoured techniques.

<u>Arc-sin law for drifting BM + related results. (7/3/94)</u>

1) Let $X_t = B_t + ct$ be drifting BM, and $A(t,x) \equiv \int_0^t I_{\{X_u \le x\}}\, du$. What is the law of $A(t,0)$? Assume $c \ge 0$.

Probably quickest to use excursion theory to compute this.

Rate of λ-marked excursions up from 0

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} E^{\varepsilon}(1 - e^{-\lambda H_0})$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \left\{ 1 - e^{-\varepsilon \beta} \right\} \qquad \text{where } \beta = \sqrt{c^2 + 2\lambda} + c$$

$$= \tfrac{1}{2} \left\{ \sqrt{c^2 + 2\lambda} + c \right\}$$

and likewise the rate of λ-marked excursions down from $0 = \frac{1}{2}\{\sqrt{c^2+2\lambda} - c\}$.

Hence

$$E^0\left[e^{-\sigma A(T,0)} \right] = P^0\left[\text{no } \sigma I_{\mathbb{R}^-}\text{-mark by time of first } \lambda\text{-mark} \right] \qquad \left[T \sim \exp(\lambda) \right]$$

$$= \frac{\sqrt{c^2+2\lambda} + c}{\sqrt{c^2+2\lambda} + \sqrt{c^2+2\lambda+2\sigma}} + \frac{\lambda}{\lambda+\sigma} \, \frac{\sqrt{c^2+2\lambda+2\sigma} - c}{\sqrt{c^2+2\lambda} + \sqrt{c^2+2\lambda+2\sigma}}$$

$$= \frac{\sqrt{c^2+2\lambda} + c}{\sqrt{c^2+2\lambda+2\sigma} + c},$$

after a few calculations.

One can similarly calculate

$$E^0 \, e^{-\sigma(T - A(T,0))} = \frac{\sqrt{c^2+2\lambda} - c}{\sqrt{c^2+2\lambda+2\sigma} - c}.$$

Now one may calculate for $\gamma > 0$

$$\left\{ \sqrt{c^2 + 2\gamma} + c \right\}^{-1} = \int_0^\infty e^{-\gamma t} dt \left\{ \frac{e^{-c^2 t/2}}{\sqrt{2\pi t}} - c\, \bar{\Phi}(c\sqrt{t}) \right\}$$

(which is also valid for $c < 0$), so that

$$P^0\left[A(T,0) \in dt \right] = \left\{ \sqrt{c^2+2\lambda} + c \right\} e^{-\lambda t} \left\{ \frac{e^{-c^2 t/2}}{\sqrt{2\pi t}} - c\, \bar{\Phi}(c\sqrt{t}) \right\} dt.$$

[This checks out OK if $c = 0$ too!]

This Laplace transform can also be inverted; we have

$$\int_0^\infty e^{-\lambda s}\, P^0\left[A(s,0)\in dt\right] ds = \frac{2\, dt}{\sqrt{c^2+2\lambda}-c}\cdot e^{-\lambda t}\left\{ \frac{e^{-c^2t/2}}{\sqrt{2\pi t}} - c\,\bar{\Phi}(c\sqrt{t})\right\},$$

and the first term is the LT of $\dfrac{e^{-c^2 s/2}}{\sqrt{2\pi s}} + c\,\bar{\Phi}(-c\sqrt{s})$,

so we finally conclude that

$$\boxed{P^0\left(A(s,0)\in dt\right)/dt = 2.\left\{ \frac{e^{-c^2(s-t)/2}}{\sqrt{2\pi(s-t)}} + c\,\bar{\Phi}\left(-c\sqrt{s-t}\right)\right\}\left\{ \frac{e^{-c^2t/2}}{\sqrt{2\pi t}} - c\,\bar{\Phi}(c\sqrt{t})\right\}}$$

for $0<t<s$

2) Let's now define $g_T \equiv \sup\{u<T : X_u=0\}$, and see whether we can obtain the law of g_T. We have

$$P[g_T>t] = \int_{-\infty}^{\infty} \frac{e^{-(x-ct)^2/2t}}{\sqrt{2\pi t}}\, dx \int_0^{T-t} |x|\, e^{-(x+cs)^2/2s} \frac{ds}{\sqrt{2\pi s^3}}$$

$$= \int_{-\infty}^{\infty} dx \int_0^{T-t} ds\ \exp\left\{-\frac{x^2}{2v} - \tfrac{1}{2}c^2(s+t)\right\} \frac{|x|}{2\pi\sqrt{ts^3}} \qquad \left[\tfrac{1}{v} = \tfrac{1}{s}+\tfrac{1}{t}\right]$$

$$= \int_0^{T-t} ds\ \frac{v}{\pi}\,(ts^3)^{-\frac{1}{2}}\, e^{-c^2(s+t)/2}$$

$$= \int_0^{T-t} \frac{ds}{\pi}\ \frac{e^{-c^2(s+t)/2}}{s+t}\sqrt{\frac{t}{s}}\ .$$

For the special case $c=0$, this agrees with what we know. We can also proceed by excursion theory:

$$E^0 \exp -\alpha g_T = P^0\left[\text{no } \alpha\text{-marked excursion comes before first } \lambda\text{-marked}\right]$$

$$= n\left(1-e^{-\lambda\zeta}\right) \Big/ \left\{ n\left(1-e^{-\lambda\zeta}\right) + n\left(\left(1-e^{-\alpha\zeta}\right)e^{-\lambda\zeta}\right)\right\}$$

$$= \sqrt{\frac{c^2+2\lambda}{c^2+2\lambda+2\alpha}},$$

which is also different from the earlier results.

3) Another observation worth making is that if (S_t) is the sup process of X, and X is BM with drift c, then

$$P\left[S_t \in dx,\ S_t - X_t \in dy\right] = e^{c(x-y) - c^2t/2} \frac{(x+y)\, e^{-(x+y)^2/2t}}{\sqrt{2\pi t^3}}\, dx\, dy$$

from which easily

$$\boxed{\begin{aligned} P\left[S_t \in dx\right]/dx &= e^{2cx}\left\{ \frac{e^{-(x+ct)^2/2t}}{\sqrt{2\pi t}} - c\,\bar{\Phi}\left(\frac{x+ct}{\sqrt{t}}\right)\right\} \\ P\left[S_t - X_t \in dy\right]/dy &= e^{-2cy}\left\{ \frac{e^{-(y-ct)^2/2t}}{\sqrt{2\pi t}} + c\,\bar{\Phi}\left(\frac{y-ct}{\sqrt{t}}\right)\right\} \end{aligned}}$$

Thus we have the pretty result

$$\boxed{P\left[A(t,0) \in ds\right]/ds = 2\, P_c\left[S_s = 0\right]\, P_{-c}\left[S_{t-s} = 0\right].}$$

It then follows that for $x \geq 0$

$$\boxed{P_c\left[A(t,x) \in ds\right]/ds = 2\, P_c\left[S_s = x\right]\, P_c\left[I_{t-s} = 0\right]} \qquad (I_t \equiv \inf_{u \le t} X_u)$$

$$\boxed{\begin{aligned} P_c\left[A(t,-x) \in ds\right]/ds &= 2\, P_{-c}\left[S_{t-s} = x\right]\, P_{-c}\left[I_s = 0\right] \\ &= 2\, P_c\left[S_s = 0\right]\, P_c\left[I_{t-s} = -x\right] \end{aligned}}$$

4) Look again; if $v_T = \sup\{u < T : S_u - X_u = 0\}$, then

$$\begin{aligned} E\, e^{-\alpha(T - v_T)} &= P\left[\text{no } \alpha\text{-killing before the } \lambda\text{-killing on the final excursion down from max}\right] \\ &= n\left(\int_0^\zeta \lambda e^{-\lambda s - \alpha s}\, ds\right) \Big/ n\left(\int_0^\zeta \lambda e^{-\lambda s}\, ds\right) \\ &= \frac{\lambda}{\lambda+\alpha}\, \frac{\sqrt{c^2+2\lambda+2\alpha} - c}{\sqrt{c^2+2\lambda} - c} \\ &= \frac{\sqrt{c^2+2\lambda} + c}{\sqrt{c^2+2\lambda+2\alpha} + c} \\ &= E\, e^{-\alpha A(T,0)} \end{aligned}$$

Another simple excursion argument yields

$$E e^{-\alpha v_T} = \frac{\sqrt{c^2+2\lambda}-c}{\sqrt{c^2+2\lambda+2\alpha}-c} = E\, e^{-\alpha(T-A(T,0))}$$

We therefore conclude that if $X_t = B_t + ct$, then, with $S_t \equiv \sup_{u\le t} X_u$, we have the identities in law

$$\boxed{\begin{aligned} v_t &\equiv \sup\{u\le t:\ S_u - X_u = 0\} \overset{\mathcal{D}}{=} t - A(t,0) \\ t - v_t &\overset{\mathcal{D}}{=} A(t,0). \end{aligned}}$$

$\left[A(t,x) \equiv \int_0^t I_{\{X_u \le x\}}\, du\right]$

5) Using these identities in law, we can give a simple proof of a result of Dassios. This result says that if we fix $0\le\alpha\le t$, and set

$$Y \equiv \sup_{0\le u\le\alpha} X'_u, \qquad Z \equiv -\inf_{0\le u\le t-\alpha} X''_u$$

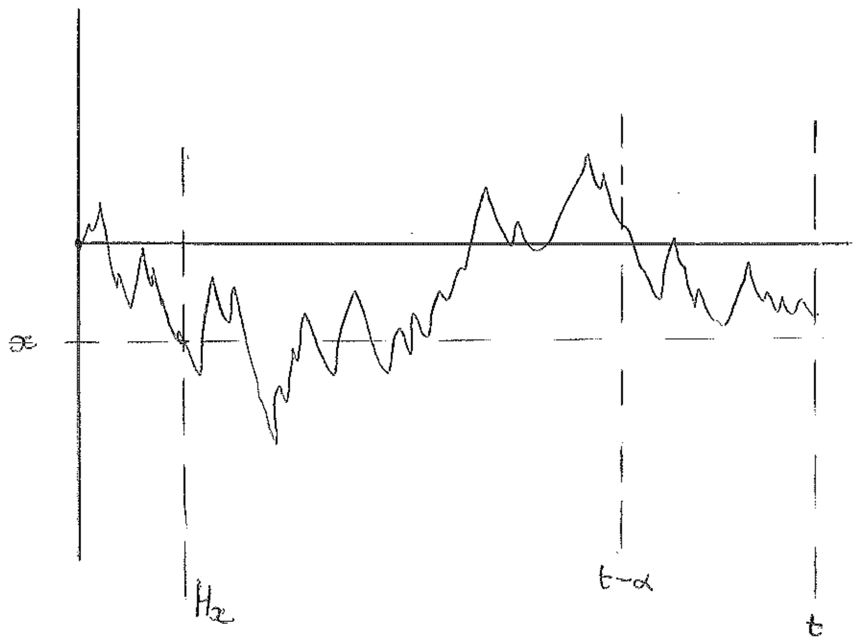
(where X', X'' are independent copies of X), and let $C_t(\alpha) \equiv \inf\{x:\ A(t,x) > \alpha\}$, then

$$\boxed{C_t(\alpha) \overset{\mathcal{D}}{=} Y - Z.}$$

We now develop for $x\le 0$ (the case of $x\ge 0$ follows from this)

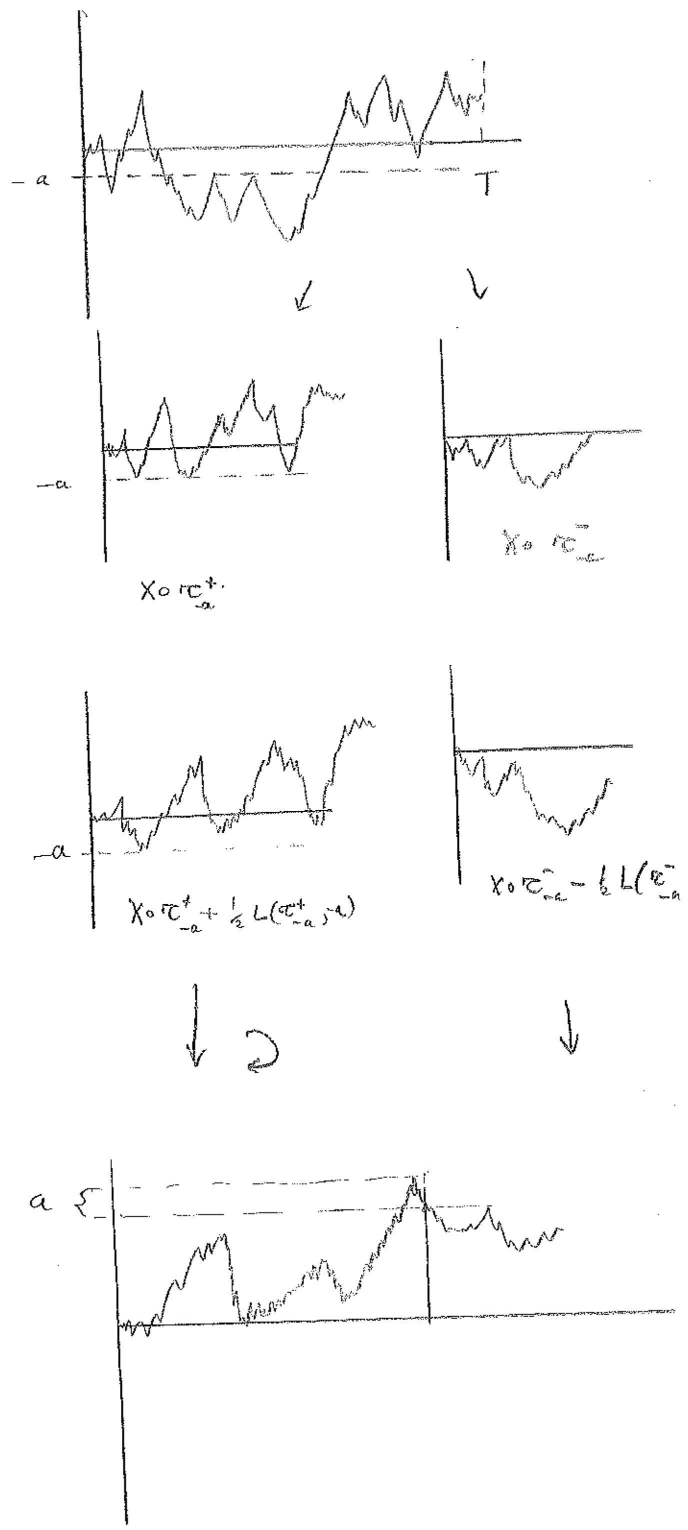
$$P\left[C_t(\alpha) < x\right] = P\left[A(t,x) > \alpha\right] = \int_0^{t-\alpha} P[H_x\in du]\, P\left[A(t,x) - A(u,x) > \alpha \,\middle|\, H_x = u\right]$$

$$= \int_0^{t-\alpha} P[H_x \in du]\, P\left[\sup_{u\le s\le t-\alpha} X_u \ge \sup_{t-\alpha\le s\le t} X_s \,\middle|\, H_x\in du\right].$$

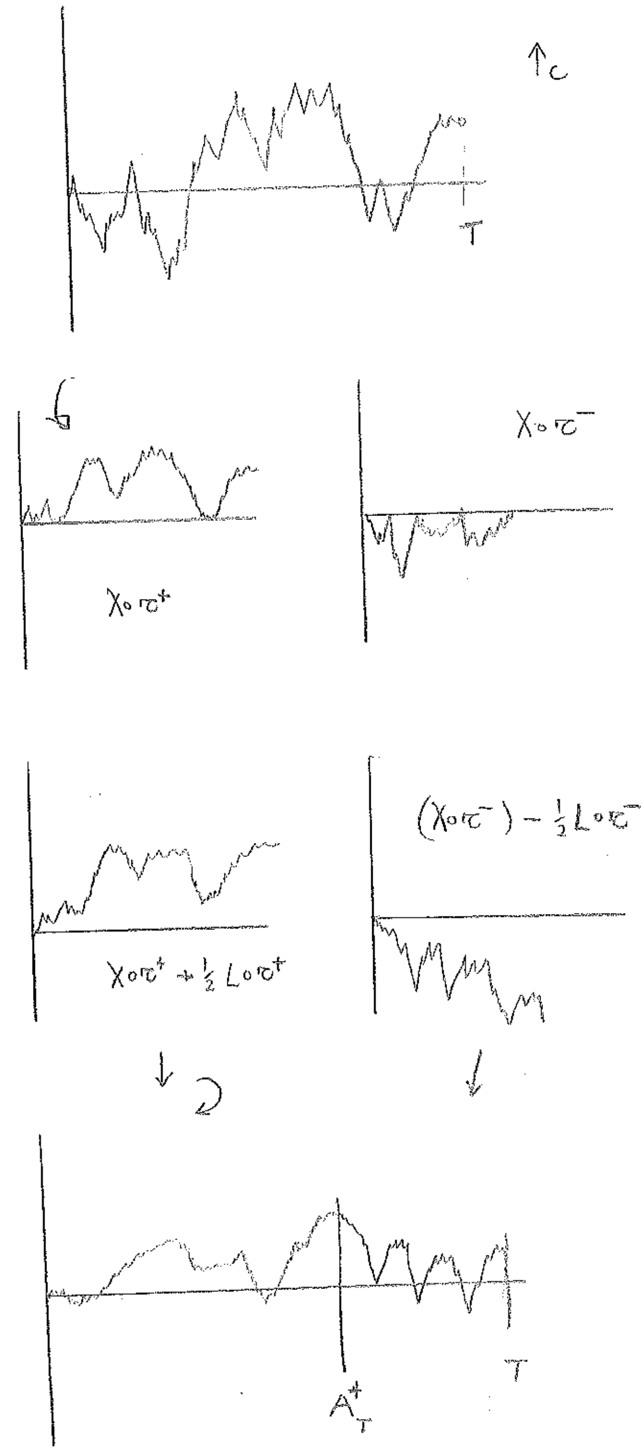


Path decomposition

More generally:



Case a=0



Given that $H_x = u$, the path fragment $\{ X_{u+v} - x : 0 \le v \le t-u \}$ is just a standard Brownian motion with drift c, so we shall have that

$$X'_s \equiv X_{t-\alpha+s} - X_{t-\alpha} \qquad (0 \le s \le \alpha)$$

$$X''_s \equiv -X_{t-\alpha-s} + X_{t-\alpha} \qquad (0 \le s \le t-\alpha-u)$$

are independent path fragments, each of which is a BM with drift c. Hence

$$P[C_t(\alpha) < x] = \int_0^{t-\alpha} P[H_x \in du] . P[\, Y \le - \inf_{0 \le s \le t-\alpha-u} X''_s \,]$$

$$= \int_0^{t-\alpha} P[H_x \in du] \, P[\, Y - x \le -(x + \inf_{0 \le s \le t-\alpha-u} X''_s) \,]$$

$$= P(Y - x \le Z).$$

6) Is there a path-decomposition to explain this? I believe so. If T is independent of $X_t \equiv B_t + ct$, $T \sim \exp(\lambda)$, then we'll decompose $(X_t)_{0 \le t \le T}$ as follows.
Let $A_t^{\pm} \equiv \int_0^t I_{\{\pm X_s \ge 0\}} ds$, with inverses $\tau_t^{\pm}$. Now define

$$Y_t^+ \equiv X_T^+ - X(\tau^+(A_T^+ - t)) + \tfrac{1}{2} \{ L(T,0) - L(\tau^+(A_T^+ - t), 0) \} \qquad (0 \le t \le A_T^+),$$

$$Y_t^- \equiv X(\tau^-(A_T^- - t)) + \tfrac{1}{2} \{ L(T,0) - L(\tau^-(A_T^- - t), 0) \} \qquad (0 \le t \le A_T^-).$$

I claim that Y^+, Y^- are both pieces of upward-drifting BM (drift $+c$) and that

$$\tilde{X}_t \equiv \begin{cases} Y_t^+ & 0 \le t \le A_T^+ \\ Y^-(T-t) + X_T^+ & A_T^+ \le t \le T \end{cases}$$

is identical in law to $(X_t)_{0 \le t \le T}$. We observe that $\tilde{X}_T = X_T$, and $A_T^+ = \sup\{ u \le T : \tilde{S}_u = \tilde{X}_u \}$, so the identity in law of the time below zero, and the time from max to T, would be immediate.
Proof? See Bertoin, Sem de Prob $\underline{\text{XXV}}$, 330 – 344!!

Oseledec theorem: a simple example. (9/3/94)

Let's take a very simple linear system, which will be running for all $t \in \mathbb{R}$ according to

$$\begin{cases} dX_1 = X_1\, dW_1 - \alpha X_1\, dt - \gamma X_2\, dt \\ dX_2 = X_2\, dW_2 - \beta X_2\, dt. \end{cases} \qquad (0 < \alpha < \beta)$$

So if we set

$$Z_2(s,t) \equiv \exp\{ W_2(t) - W_2(s) - (\beta + \tfrac{1}{2})(t-s) \} \qquad (-\infty < s \le t < \infty)$$

$$Z_1(s,t) \equiv \exp\{ W_1(t) - W_1(s) - (\alpha + \tfrac{1}{2})(t-s) \} \qquad ''$$

we see that

$$\boxed{\begin{aligned} X_2(t) &= X_2(s).\, Z_2(s,t) \\ X_1(t) &= X_1(s).\, Z_1(s,t) - X_2(s)\; Z_1(s,t) \int_s^t \gamma Z_1(s,u)^{-1} Z_2(s,u)\, du. \end{aligned}}$$

If we suppose that we want to see the more rapid decay as $t \to \infty$, we have to ensure that

$$X_1(s) - X_2(s) \int_s^t \gamma\; Z_1(s,u)^{-1} Z_2(s,u)\, du \longrightarrow 0$$

so that

$$\boxed{X_1(s) = X_2(s) \int_s^\infty \gamma\; Z_1(s,u)^{-1}\, Z_2(s,u)\, du}$$

This identifies the critical subspace in the forward direction. Let's abbreviate

$$\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \equiv \begin{pmatrix} Z_1(s,t) & Z_{12}(s,t) \\ 0 & Z_2(s,t) \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} \qquad (-\infty < s \le t < \infty).$$

Hence

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{Z_1 Z_2(s,t)} \begin{pmatrix} Z_2(s,t) & -Z_{12}(s,t) \\ 0 & Z_1(s,t) \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$$

In what subspace must $\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$ lie in order that we see the slower growth rate $e^{-\alpha s}$ as $s \to -\infty$? Clearly it's the subspace $\{ \binom{x}{0} : x \in \mathbb{R} \}$!

[Peter Imkeller is interested in such questions.]

Infinite-variance critical branching processes (10/3/94)

Talks by Götz Kersting + Alison Etheridge make me interested in the behaviour of a critical branching process with offspring PGF $\varphi(s) = s + (1-s)^{\beta}/\beta$ where β is a parameter in $(1,2]$. The case $\beta = 2$ is the classical binary branching, but for $\beta \in (1,2)$ the offspring distⁿ has infinite variance.

1) Suppose we look at the continuous-time br. pr., where splitting occurs at rate λ. Then if $f(t,\alpha) \equiv E[\exp(-\alpha Z_t) \mid Z_0 = 1]$, we have

$$f(t,\alpha) = e^{-\alpha} \cdot e^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \varphi(f(t-s,\alpha))\, ds$$

implying that

$$\dot{f} = \lambda(1-f)^{\beta}/\beta \quad , \qquad f(0,\alpha) = e^{-\alpha}.$$

The explicit solution is

$$f(t,\alpha) = 1 - \left[(1-e^{-\alpha})^{-\varepsilon} + \varepsilon \lambda t/\beta \right]^{-1/\varepsilon} \qquad (\varepsilon \equiv \beta - 1).$$

Hence

$$E\left[\exp\left(-\tfrac{\alpha}{n} Z(n^{\varepsilon} t)\right) \;\middle|\; Z_0 = n \right]$$

$$= f\left(n^{\varepsilon} t, \tfrac{\alpha}{n}\right)^n$$

$$= \left[1 - \frac{1 - e^{-\alpha/n}}{\{1 + \beta^{-1} \varepsilon \lambda n^{\varepsilon} t (1 - e^{-\alpha/n})^{\varepsilon}\}^{1/\varepsilon}} \right]^n$$

$$\sim \left[1 - \frac{\alpha/n}{\{1 + \beta^{-1} \varepsilon \lambda t \alpha^{\varepsilon}\}^{1/\varepsilon}} \right]^n$$

$$\to \exp\left[- \frac{\alpha}{(1 + \beta^{-1} \varepsilon \lambda t \alpha^{\varepsilon})^{1/\varepsilon}} \right] \qquad (n \to \infty)$$

This identifies the correct scaling and the form of the limit process.
The discrete-time limit theorem must also be known somewhere:

2) Let's give another description of the limit process, which illuminates the sample path behaviour.

Take a Lévy process $(Y_t)_{t\geq 0}$, $Y_0 \equiv 1$, with characteristic exponent

$$\psi(it) = \int_0^\infty (e^{itx} - 1 - itx) \frac{dx}{x^{1+\beta}} \qquad (1<\beta<2)$$

$$= -\frac{t^2}{2} \qquad (\beta = 2)$$

Now form $A_t \equiv \int_0^t Y_u^{-1}\, du$, $\tau_t \equiv \inf\{s : A_s > t\}$, and finally write

$$\boxed{\zeta_t = Y(\tau_t).}$$

Then (to within trivial rescaling of time) ζ is the limit of the branching processes discussed above. How do we see this?

Well, the generator of Y is

$$\mathcal{G}f(x) = \int_0^\infty \frac{dy}{y^{1+\beta}} \left(f(x+y) - f(x) - y f'(x) \right)$$

so that the generator of $\tilde{Y} \equiv Y_0 \tau \equiv \zeta$ is

$$\tilde{\mathcal{G}} f(x) = x\, \mathcal{G}f(x).$$

Thus if we consider

$$E\left[\exp(-\alpha \zeta_t) \,\middle|\, \zeta_0 = x \right] = \exp\left[-x\, g(t,\alpha) \right]$$

(conjecturing the form given by the RHS for now), we would have to have

$M_t \equiv \exp[-\zeta_t\, g(T-t, \alpha)]$ is a martingale.

The equation for g which results is (at least in the case $\beta \in (1,2)$)

$$\dot{g} + \int_0^\infty \frac{dy}{y^{1+\beta}} \left(e^{-gy} - 1 + gy \right) = 0 , \qquad g(0,\alpha) = \alpha$$

$$= \dot{g} + \frac{\Gamma(2-\beta)}{\beta(\beta-1)}\, g^\beta.$$

Abbreviating $\Gamma(2-\beta)/\beta(\beta-1) \equiv c$, we have an explicit solution:

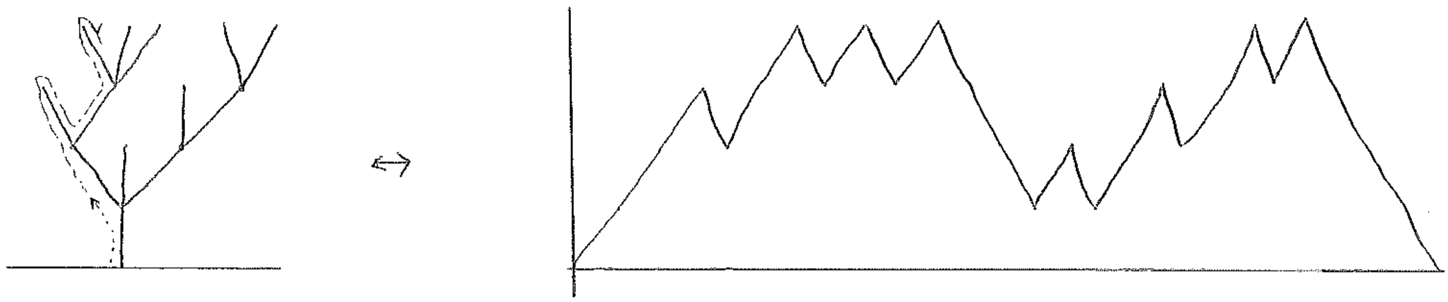
$$\boxed{g(t,\alpha) = \frac{\alpha}{(1+\varepsilon c t\,\alpha^{\varepsilon})^{1/\varepsilon}}} \,.$$

Comparing with the limiting form of the branching process calculated in (1) above, we see that they are the same apart from trivial rescaling of time.

3) Another way to present ζ is to take ζ to solve

$$\boxed{d\zeta_t = \zeta_{t-}^{1/\beta}\, dY_t \,.}$$

4) More interesting is the study of the contour process:



If we take $C_n(\cdot)$ to be the contour process for $Z^{(n)}$, is there a limit result for C_n? What should be the correct scaling? We must take

$$\boxed{n^{-\varepsilon}\, C_n(n^{\theta} t) \qquad \text{where } \theta = \beta.}$$

Why? The time when $C_n(\cdot)$ finally hits 0 is twice the total number of individuals who have ever lived (the slope of C_n is ± 1 everywhere). If $h(s) = E[s^N \mid Z_0 = 1]$, then easily $h(z) = z\,\varphi(h(z))$. Writing $h(1-t) = 1-\delta(t)$, we get

$$1-\delta = (1-t)\,\varphi(1-\delta) = (1-t)\left(1-\delta+\delta^{\beta}/\beta\right) \quad \therefore \quad \frac{\delta^{\beta}}{\beta} = t\,\frac{1-\delta}{1-t},$$

implying $\boxed{\delta(t) \sim (\beta t)^{1/\beta}}$ $(t \downarrow 0)$

From this $E\left[e^{-\lambda n^{-\beta} N} \mid Z_0 = n\right] = h\left(e^{-\lambda n^{-\beta}}\right)^n \longrightarrow e^{-(\lambda\beta)^{1/\beta}}$.

But can we assert any weak convergence of $n^{-\varepsilon}\, C_n(n^{\beta}\cdot)$?? If so, what can be said about the limit process?

Limits of AR processes (22/3/94)

(i) Consider an AR(k) process given by

$$Y_{n+1} = c_0 Y_n + \cdots + c_k Y_{n-k} + \varepsilon_{n+1}$$

where the ε's are IID $N(0,\sigma^2)$, Martin Jacobsen asks about possible limits of such things, so if we have now a sequence $Y^{(n)}$ of AR(k) processes, and we form the process $X_n(t) = Y^{(n)}_{[nt]}$, what limit processes may arise? Really only interested in continuous limits.

(ii) To answer this, we consider just the original AR(k) process, and form

$$\eta_n \equiv (Y_n, Y_{n-1}, \ldots, Y_{n-k})^T,$$

so that $\eta_{n+1} = A\eta_n + \begin{pmatrix} \varepsilon_{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \equiv A\eta_n + \nu_{n+1}$, say, where A is the matrix

$$A \equiv \begin{pmatrix} c_0 & c_1 & \cdots & c_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and $E\,\nu_n \nu_n^T = \sigma^2 e_0 e_0^T$.

We may now represent

$$\eta_n = \sum_{r=0}^{n-1} A^r \nu_{n-r} + A^n \eta_0 .$$

Supposing that η_0 is deterministic, we find that

$$\operatorname{Cov}(\eta_m, \eta_\ell) = \sum_{r=1}^{m\wedge\ell} A^{m-r} e_0 e_0^T A^{\ell-r} \cdot \sigma^2$$

Hence

$$\boxed{\operatorname{cov}(Y_m, Y_\ell) = \sum_{r=1}^{m\wedge\ell} e_0^T A^{m-r} e_0 \cdot e_0^T A^{\ell-r} e_0 \cdot \sigma^2}$$

So if we set

$$a_r \equiv \begin{cases} e_0^T A^r e_0 & (r \geq 0) \\ 0 & (r < 0) \end{cases}$$

we have

$$\boxed{\operatorname{Cov}(Y_m, Y_\ell) = \sigma^2 \sum_{r=1}^{\infty} a_{m-r}\, a_{\ell-r}}$$

This makes it important to understand $a_m \equiv e_0^T A^m e_0$, which is to say, the eigenstructure of A.

(iii) If $Ax = \lambda x$, then we see that $x_{i-1} = \lambda x_i$ $(i = 1, \dots, k)$ and so $x_i = \lambda^{-i}$ $(i = 0, \dots, k)$ to within an irrelevant multiplicative constant. This means that λ must solve

$$\sum_{i=0}^{k} \lambda^{-i} c_i = \lambda$$

or, again, $$P(\lambda^{-1}) \equiv 1 - \sum_{i=0}^{k} \lambda^{-i-1} c_i = 0.$$

Let's assume that there are $k+1$ distinct eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_k$ (since we're only interested in limit behaviour, we lose nothing by assuming this) and so

$$A = S \Lambda S^{-1}$$

where $\Lambda \equiv \mathrm{diag}(\lambda_0, \lambda_1, \dots, \lambda_k)$ and

$$S \equiv \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0^{-1} & \lambda_1^{-1} & \cdots & \lambda_k^{-1} \\ \lambda_0^{-2} & \lambda_1^{-2} & \cdots & \lambda_k^{-2} \\ \vdots & \vdots & & \vdots \\ \lambda_0^{-k} & \lambda_1^{-k} & \cdots & \lambda_k^{-k} \end{pmatrix}.$$

What is the inverse of S? If $(S^{-1})_{ij} \equiv \sigma_{ij}$, we must have

$$\sum_{j=0}^{k} \sigma_{ij} \lambda_r^{-j} \begin{array}{ll} = 0 & \text{if } r \neq i \\ = 1 & \text{if } r = i \end{array}$$

Thus the polynomial $P_i(t) = \sum_{j=0}^{k} \sigma_{ij} t^j$ must vanish at λ_ℓ^{-1}, $\ell \neq i$, and be equal to 1 at λ_i^{-1}.

Hence
$$\boxed{P_i(t) = \prod_{\substack{r=0 \\ r \neq i}}^{k} \frac{t - \lambda_r^{-1}}{\lambda_i^{-1} - \lambda_r^{-1}}}$$

This specifies S^{-1} completely. But we are only concerned with $a_m \equiv e_0^T A^m e_0 = e_0^T S \Lambda^m S^{-1} e_0$, and

$$(S^{-1} e_0)_i = \prod_{\substack{r=0 \\ r \neq i}}^{k} \frac{-\lambda_r^{-1}}{\lambda_i^{-1} - \lambda_r^{-1}} = \prod_{\substack{r=0 \\ r \neq i}}^{k} \frac{\lambda_i}{\lambda_i - \lambda_r}.$$

So
$$\boxed{a_m \equiv e_0^T A^m e_0 = \sum_{r=0}^{k} \lambda_r^m \prod_{\substack{j=0\\ j\neq r}}^{k} \frac{\lambda_r}{\lambda_r - \lambda_j}} \qquad (m \geq 0).$$

We may abbreviate
$$\prod_{\substack{j=0\\ j\neq r}}^{k} \frac{\lambda_r}{\lambda_r - \lambda_j} \equiv \sigma_{r0}$$

Hence
$$\sum_{m\geq 1} \sum_{\ell \geq 1} e^{-\alpha m - \beta \ell} \operatorname{cov}(Y_m, Y_\ell)$$
$$= \sum_{m\geq 1} \sum_{\ell\geq 1} e^{-\alpha m - \beta \ell} \sum_{r=1}^{\infty} a_{m-r}\, a_{\ell - r} \cdot \sigma^2$$
$$= \sum_{r=1}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \exp\left[-\alpha(r+i) - \beta(r+j)\right] a_i a_j \cdot \sigma^2$$
$$= \left(\sum_{i=0}^{\infty} a_i e^{-\alpha i}\right)\left(\sum_{j=0}^{\infty} a_j e^{-\beta j}\right) \frac{e^{-\alpha-\beta}}{1 - e^{-\alpha-\beta}} \cdot \sigma^2.$$

So if we set
$$F(s) \equiv \sum_{i\geq 0} a_i s^i = \sum_{j\geq 0} \sum_{r=0} \lambda_r^j s^j \sigma_{r0} = \sum_{r=0}^{k} \frac{\sigma_{r0}}{1 - \lambda_r s},$$

we have
$$\boxed{\sum_{m\geq 1} \sum_{\ell \geq 1} e^{-\alpha m - \beta \ell} \operatorname{cov}(Y_m, Y_\ell) = \sigma^2 \frac{e^{-\alpha-\beta}}{1 - e^{-\alpha-\beta}} F(e^{-\alpha}) F(e^{-\beta})}$$

(iv) The point of this is now that, in terms of the approximations introduced in (i) above,
$$\int_0^\infty ds \int_0^\infty dt\; e^{-\mu s - \nu t} \operatorname{cov}(X_n(s), X_n(t))$$
$$= \sum_{m\geq 0} \sum_{\ell\geq 0} \operatorname{cov}(Y_m^{(n)}, Y_\ell^{(n)})\, e^{-\mu m/n - \nu \ell / n}\; \frac{1 - e^{-\mu/n}}{\mu} \cdot \frac{1 - e^{-\nu/n}}{\nu}$$
$$= \frac{1 - e^{-\mu/n}}{\mu} \cdot \frac{1 - e^{-\nu/n}}{\nu} \cdot F_n(e^{-\mu/n})\, F_n(e^{-\nu/n}) \cdot \frac{e^{-\mu/n - \nu/n}}{1 - e^{-(\mu+\nu)/n}}\; \sigma_n^2,$$

and if the processes X_n were converging to something, we would have to have

$$\boxed{\int_0^\infty ds \int_0^\infty dt \quad e^{-\mu s - \nu t} \operatorname{cov}(X_s, X_t) = \lim_{n\to\infty} \frac{\sigma_n^2}{n(\mu+\nu)} \quad F_n(e^{-\mu/n})\, F_n(e^{-\nu/n}).}$$

Taking $\mu = \nu$, it follows that we must have for each $\mu > 0$

$$\boxed{\lim_{n\to\infty} \quad \frac{\sigma_n}{\sqrt{n}}\, F_n(e^{-\mu/n}) \quad \text{exists.}}$$

(v) However, some considerable simplification is possible.

PROPOSITION

$$\boxed{F(s) = \prod_{j=0}^{k} (1-\lambda_j s)^{-1}.}$$

Proof. We have $F(s) \equiv \sum_{r=0}^{k} \sigma_{r0}/(1-\lambda_r s) = \dfrac{Q(s)}{\prod_{r=0}^{k}(1-\lambda_r s)}$ for some polynomial Q of degree at most k. Now observe

$$\lim_{s\to\lambda_m^{-1}} (1-\lambda_m s)\,F(s) = \sigma_{m0} = Q(\lambda_m^{-1}) \Big/ \prod_{\substack{r=0\\ r\neq m}}^{k} (1-\lambda_r/\lambda_m)$$

$$\therefore \quad Q(\lambda_m^{-1}) = 1 \quad \text{for all } m=0,\dots,k \qquad \therefore \quad Q \equiv 1.$$

The convergence issue is now concerning $n^{-\frac{1}{2}}\sigma_n \prod_{j=0}^{k} (1-\lambda_j^{(n)} e^{-\mu/n})^{-1}$. If some of the $(\lambda_j^{(n)})$ sequences remain bounded away from 1, then they contribute nothing in the limit, apart from a constant. So wmlog we suppose that $\lambda_j^{(n)} \equiv 1-\delta_j^{(n)} \to 1$ $(n\to\infty)$ for each j. Then

$$1-\lambda_j^{(n)} e^{-\mu/n} = 1-(1-\delta_j^{(n)})(1-\tfrac{\mu}{n}+\theta_n) \qquad \text{where } \theta_n = O(\tfrac{1}{n^2})$$

$$= \delta_j^{(n)} - \delta_j^{(n)}(\tfrac{\mu}{n}-\theta_n) + \tfrac{\mu}{n} - \theta_n$$

$$= \delta_j^{(n)} + \tfrac{\mu}{n} - \delta_j^{(n)}(\tfrac{\mu}{n}-\theta_n) - \theta_n$$

$$\sim \tfrac{\mu}{n} + \delta_j^{(n)}$$

For convergence, we must have $n\delta_j^{(n)} \to \theta_j$ for each j (note that θ_j could in general be any complex number), and we need to have at the same time that $n^{-\frac{1}{2}}\sigma_n \sim n^{-k-1}$.

The conclusion is that

$$\boxed{\int_0^\infty ds \int_0^\infty dt \quad e^{-\mu s - \nu t} \operatorname{Cov}(X_s, X_t) = \frac{c}{\mu+\nu} \prod_{j=0}^{k} \frac{1}{(\theta_j+\mu)(\theta_j+\nu)}}$$

We can undo this LT fairly simply; if f is the function on $\mathbb{R}^+$ whose LT is $\prod_0^k (\lambda+\theta_j)^{-1}$, we shall have for $0 \leq s \leq t$

$$\boxed{\operatorname{Cov}(X_s, X_t) = \int_0^s f(s-u)\, f(t-u)\, du}$$

(because $(\mu,\nu) \mapsto (\mu+\nu)^{-1}$ is the joint LT of Lebesgue measure on the diagonal.)

(vi) <u>Can we identify the X-process which arises in the limit?</u> If

$$dY_t = C Y_t\, dt + \sigma\, dW_t \qquad Y \text{ is } (k+1)\text{-dimensional}, \quad \sigma = e_0,$$

and $C = \begin{pmatrix} a_0 & a_1 & \cdots & a_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ has eigenvalues $-\theta_0, -\theta_1, \dots, -\theta_k$, and if

$X_t = e_k^T Y_t$ is the last component of Y, then $\operatorname{Cov}(X_s, X_t) = \int_0^s f(s-u) f(t-u)\, du$, where $f(t) = e_k^T e^{tC} e_0 = e_k^T S e^{t\Lambda} S^{-1} e_0$, where Λ is the diagonal matrix of eigenvalues, $\Lambda = \operatorname{diag}(-\theta_0, -\theta_1, \dots, -\theta_k)$, then the LT of f is

$$\sum_{r=0}^{k} (-\theta_r)^{-k} \frac{1}{\lambda+\theta_r} \prod_{\substack{j=0\\ j\neq r}}^{k} \frac{\theta_j^{-1}}{\theta_j^{-1}-\theta_r^{-1}}$$

$$= \sum_{r=0}^{k} \frac{1}{\lambda+\theta_r} \left(\prod_{\substack{j=0\\ j\neq r}}^{k} \frac{1}{\theta_j-\theta_r} \right)$$

$$= \prod_{j=0}^{k} \frac{1}{\lambda+\theta_j},$$

so this is the SDE specification of the limit process. We find the a_i by noticing that

$$t^{k+1} - \sum_{i=0}^{k} a_i t^{k-i} = \prod_{j=0}^{k} (t+\theta_j).$$

Further estimations for the Asian option (23/3/94)

Take standard BM $(B_t)_{t\geq 0}$ and let $Z \equiv \int_0^T B_t \, \nu(dt)$ for some measure ν. The (fixed-strike) Asian option is concerned with the r.v.

$$Y \equiv \int_0^T e^{\sigma B_t} e^{(r-\sigma^2/2)t} \mu(dt) - K$$

Abbreviate $r - \sigma^2/2 \equiv c$. Let's also consider the random variable

$$\tilde{Y} \equiv \int_0^T (1+\sigma B_t) e^{ct} \mu(dt) - K \leq Y.$$

(i) The (very sharp) lower bound is based on the estimate

$$0 \leq E(Y^+|Z) - E(Y|Z)^+.$$

But now let's observe that

$$\begin{aligned} 0 &\leq E(Y^+|Z) - E(Y|Z)^+ \\ &\leq E\{\tilde{Y}^+ + Y^+ - \tilde{Y}^+ | Z\} - E(\tilde{Y}|Z)^+ \\ &\leq E[Y - \tilde{Y}|Z] + E(\tilde{Y}^+|Z) - E(\tilde{Y}|Z)^+, \end{aligned}$$

so we may also squeeze the error on the other side.

(ii) We have for $0 \leq s \leq t \leq T$

$$E[B_t|Z] = a_t Z, \quad \text{where } a_t \equiv E(B_t Z)/EZ^2;$$

$$\mathrm{cov}\begin{pmatrix} B_s \\ B_t \end{pmatrix} Z = \begin{pmatrix} \rho_{ss} & \rho_{st} \\ \rho_{ts} & \rho_{tt} \end{pmatrix} \equiv \begin{pmatrix} s - a_s^2 v & s - a_s a_t v \\ s - a_s a_t v & t - a_t^2 v \end{pmatrix},$$

with $v \equiv EZ^2$.

Conditional on Z, $\tilde{Y}$ is a Gaussian variable. How will this compare with what we had earlier? Probably not very well!

(iii) We should be able to improve the bound if we proceed as follows. Instead of $\tilde{Y}$, let's consider

$$Y_z \equiv \int_0^T \{1 + \sigma(B_t - z a_t)\} e^{\sigma z a_t + ct} \mu(dt) - K \leq Y.$$

Given $Z = z$, Y_z is a Gaussian variable, with mean

$$\int_0^T e^{\sigma z a_t + ct} \mu(dt) - K \equiv m(z) \text{ , say,}$$

and variance

$$v(z) \equiv \int_0^T \mu(ds) \int_0^T \mu(dt)\, \sigma^2 \rho_{st} \exp\{ \sigma z (a_s + a_t) + c(s+t) \}.$$

Then the inequalities

$$0 \le E[Y^+ | Z=z] - E[Y|Z=z]^+$$

$$\le E[Y^+ - Y_z^+ | Z=z] + E[Y_z^+ | Z=z] - E[Y_z | Z=z]^+$$

$$\le E[Y - Y_z | Z=z] + E[Y_z^+ | Z=z] - E[Y_z | Z=z]^+,$$

will give an upper bound on the error.

If $X \sim N(m,v)$, then it's easy to compute

$$E\, X^+ - (EX)^+ = \sqrt{\frac{v}{2\pi}}\, e^{-m^2/2v} - |m|\, \bar{\Phi}\left(\frac{|m|}{\sqrt{v}}\right).$$

Also,

$$E[Y - Y_z | Z=z] = \int_0^T \{ e^{\frac{1}{2}\sigma^2 \rho_{tt}} - 1 \} e^{\sigma z a_t + ct} \mu(dt)$$

(iv) In the special case $\nu(dt) = e^{\alpha t} dt$, we have

$$\boxed{\begin{aligned} E\, Z^2 &= \frac{T e^{2\alpha T}}{\alpha^2} - \frac{1}{2\alpha^3}\left\{ 1 + 3e^{2\alpha T} - 4e^{\alpha T} \right\}, \\ E\, B_t Z &= \alpha^{-1} t e^{\alpha T} - \alpha^{-2}(e^{\alpha t} - 1) \end{aligned}}$$

so that

$$a_t = \{ \alpha^{-1} t e^{\alpha T} - \alpha^{-2}(e^{\alpha t} - 1) \} / V, \quad V \equiv \frac{T e^{2\alpha T}}{\alpha^2} - \frac{1 + 3e^{2\alpha T} - 4e^{\alpha T}}{2\alpha^3}$$

$$\rho_{st} = s \wedge t - a_s a_t V.$$

Letting $\alpha \downarrow 0$, we recover $V = T^3/3$, $a_t = 3t(2T - t)/2T^3$, as previously.

(v) Another (very crude) error bound would be to estimate

$$0 \le E(Y^+|Z) - E(Y|Z)^+$$

$$= \tfrac{1}{2}\{E_Z(|Y|) - |E_Z Y|\}$$

$$= \tfrac{1}{2} E_Z(|Y| - |E_Z Y|)$$

$$\le \tfrac{1}{2} E_Z |Y - E_Z Y| \le \tfrac{1}{2} E_Z\, S_0 \int_0^T \left| e^{\sigma B_t} - E_Z e^{\sigma B_t}\right| e^{(r-\sigma^2/2)t}\,\mu(dt)$$

$$= \tfrac{1}{2} S_0 \int_0^T E_Z\left| e^{\sigma(B_t - Za_t)} - e^{\frac{1}{2}\sigma^2\rho_{tt}}\right| \exp\left(Z\sigma a_t + (r-\tfrac{\sigma^2}{2})t\right)\,\mu(dt).$$

$$= S_0 \int_0^T E_Z\left(e^{\sigma(B_t - Za_t)} - e^{\frac{1}{2}\sigma^2\rho_{tt}}\right)^+ \exp\left\{Z\sigma a_t + (r-\tfrac{\sigma^2}{2})t\right\}\,\mu(dt)$$

$$= S_0 \int_0^T \left\{\bar{\Phi}\left(-\tfrac{1}{2}\sigma\sqrt{\rho_{tt}}\right) - \bar{\Phi}\left(\tfrac{1}{2}\sigma\sqrt{\rho_{tt}}\right)\right\} \exp\left\{Z\sigma a_t + (r-\tfrac{\sigma^2}{2})t\right\}\,\mu(dt)\, e^{\sigma^2\rho_{tt}/2}.$$

How would this compare with other bounds? Numerical work shows that this is lousy in comparison with the L^2 estimate obtained from the conditional variance of Y given $\int_0^T B_s\,ds$. Also, the estimate of the conditional variance by replacing $e^{\sigma^2\rho_{st}}-1$ by its modulus is lousy – the cancellations really matter.

<u>Limits of multivariate AR processes: an example (30/3/94)</u>

This was just a simple example of a 2-dimensional AR(2) process, chosen so that one might calculate the whole story and see what comes out. We consider

$$X_{n+1} = \begin{pmatrix}1&1\\0&a\end{pmatrix}X_n + \begin{pmatrix}b&0\\0&b\end{pmatrix}X_{n-1} + \varepsilon_{n+1},$$

so that

$$A = \left(\begin{array}{cc|cc} 1 & 1 & b & 0\\ 0 & a & 0 & 0\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right) \qquad \det A - \lambda I = \lambda(\lambda-a)(\lambda^2-\lambda-b)$$

so if we write $\alpha < 0 < \beta$ for the roots of the quadratic $\lambda^2 - \lambda - b$, we can work out for the e-values $0, a, \alpha, \beta$ the corresponding e-vectors, which are the columns of

$$S \equiv \begin{pmatrix} 0 & -a^2 & \alpha & \beta\\ 0 & a(a+b) & 0 & 0\\ 0 & -a & 1 & 1\\ 1 & a+b & 0 & 0\end{pmatrix}.$$

Inverting gives
$$S^{-1} = \begin{pmatrix} 0 & -1/a & 0 & 1 \\ 0 & \frac{1}{a(a+b)} & 0 & 0 \\ \frac{-1}{\beta-\alpha} & \frac{\beta-a}{(\beta-\alpha)(a+b)} & \frac{\beta}{\beta-\alpha} & 0 \\ \frac{1}{\beta-\alpha} & \frac{a-\alpha}{(\beta-\alpha)(a+b)} & \frac{-\alpha}{\beta-\alpha} & 0 \end{pmatrix}$$

If we form $\sum\limits_{m\geq 0} s^m A^m$ and look only at the part corresponding to the first two components, we get

$$\begin{pmatrix} 0 & -a^2 & \alpha & \beta \\ 0 & a(a+b) & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{1-as} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{1-\alpha s} & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{1-\beta s} \end{pmatrix} \begin{pmatrix} 0 & -1/a \\ 0 & \frac{1}{a(a+b)} \\ \frac{-1}{\beta-\alpha} & \frac{\beta-a}{(\beta-\alpha)(a+b)} \\ \frac{1}{\beta-\alpha} & \frac{a-\alpha}{(\beta-\alpha)(a+b)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{(1-\alpha s)(1-\beta s)} & \frac{(\beta-a)(a-\alpha)\,s}{(a+b)(1-as)(1-\alpha s)(1-\beta s)} \\ 0 & \frac{1}{1-as} \end{pmatrix}$$

What is the general form of this??

Image analysis: Some possible approaches (30/3/94)

We observe a value Y_i at vertex i of some graph, which is a corrupted version of a "true" value X_i, $X_i, Y_i \in \mathbb{R}$. Aim to estimate X by finding x to

$$\min_x \left\{ \sum_i (Y_i - x_i)^2 + \sum_i \sum_j \beta_{ij} L(x_i - x_j) \right\}$$

where $L : \mathbb{R} \to \mathbb{R}^+$ is symmetric, increasing in $\mathbb{R}^+$, and $\beta_{ij} = \beta_{ji}$. Also, $\beta_{ij} = 0$ if i and j are not neighbours. How to find x to minimise?

(i) If $L(x) = \lambda^{-1} x^2/2$, we can differentiate wrto x_p, and set the derivative to 0;

$$0 = 2(\hat{x}_p - Y_p) + 2\lambda^{-1} \sum_j \beta_{pj} (\hat{x}_p - \hat{x}_j) \quad \therefore \quad \lambda Y = \lambda \hat{x} - Q\hat{x}$$

where Q is the Q matrix of the Markov process which jumps $i \to j$ rate β_{ij}.
Thus $\boxed{\hat{x}_i = (\lambda R_\lambda Y)_i \equiv E\left[Y_{X(T)} \mid X_0 = i \right]}$

where $T \sim \exp(\lambda)$ independent of the MC X with Q-matrix Q. In the case of a square grid of pixels, $\beta_{ij} = 1$ if i, j neighbours; $= 0$ if not, the Markov chain is just continuous-time r.w., and could be approximated by Brownian motion, whose resolvent is known explicitly.

(ii) More generally, we get

$$(*) \qquad 0 = 2(\hat{x}_p - Y_p) + 2 \sum_j \beta_{pj} L'(\hat{x}_p - \hat{x}_j).$$

To solve? Assume L' is Lipschitz continuous, $|L'(x) - L'(y)| \le K|x-y|$, and that $\sum_j \beta_{pj} K < 1$ for each p. Then we can generate successive approximations

$$\boxed{x_p^{(n+1)} = Y_p - \sum_j \beta_{pj} L'(x_p^{(n)} - x_j^{(n)}),} \qquad x_p^{(0)} = Y_p \text{ (say)},$$

which converge geometrically-rapidly to the limit which solves $(*)$. This could also be computed pretty rapidly.

Remarks: (i) If $(X_n)_{n\in\mathbb{Z}}$ is a Gaussian process, $X_n = 0$ $\forall n < 0$, and

$$\Phi_X(s,t) = E\Big(\sum_{n\geq 0} s^n X_n\Big)\Big(\sum_{m\geq 0} t^m X_m^T\Big)$$

and $Y_n \equiv \sum_{j=0}^{K} C_j X_{n-j}$, then

$$\Phi_Y(s,t) = E\Big[\sum_{n\geq 0} s^n Y_n \sum_{m\geq 0} t^m Y_m^T\Big]$$

$$C(s) \equiv \sum_{j=0}^{K} s^j C_j$$

$$= C(s)\,\Phi_X(s,t)\,C(t)^T$$

(ii) Also, if $dX_t = \sigma\, dW_t + K X_t\, dt$, $X_0 = 0$, then

$$\int_0^\infty ds \int_0^\infty dt\; E\big[X_s X_t^T\big]\, e^{-\mu s - \nu t} = \frac{1}{\mu+\nu}\,(\mu - K)^{-1}\sigma\sigma^T(\nu - K^T)^{-1}.$$

(iii) If we had an AR process X_n, $X_n = 0$ $\forall n < 0$,

$$X_n = A X_{n-1} + \varepsilon_n \qquad (n \geq 0)$$

(ε_n are IID $N(0,\Gamma)$)

then
$$\varepsilon_n = \begin{cases} X_n - A X_{n-1} & (n \geq 0) \\ 0 & (n < 0)\end{cases}$$

and
$$\Phi_\varepsilon(s,t) = (I - sA)\,\Phi_X(s,t)\,(I - tA^T) = \frac{1}{1-st}\,\Gamma$$

$$\Rightarrow \boxed{\Phi_X(s,t) = (I - sA)^{-1}\,\Gamma\,(I - tA^T)^{-1}\cdot\frac{1}{1-st}}$$

Limits of AR processes in higher dimensions (31/3/94)

Let's have a d-vector AR(k) process (slightly altering notation from pp32-36)

$$Y_n = \sum_{r=1}^{k} C_r Y_{n-r} + \mathcal{E}_n$$

which as before we stack: $\eta_n^T = (Y_n^T, Y_{n-1}^T, \ldots, Y_{n-k+1}^T)$ and form

$$\eta_n = A\,\eta_n + \gamma_n \quad , \quad \gamma_n^T \equiv (\mathcal{E}_n^T, 0, \ldots, 0)$$

and

$$A \equiv \begin{pmatrix} C_1 & C_2 \cdots & C_{k-1} & C_k \\ I & 0 \cdots & 0 & 0 \\ 0 & I \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \cdots & I & 0 \end{pmatrix}.$$

[Useful discussion resumes p44]

(i) What can we say about the diagonalisation of A? As before, the matrix whose columns are e-vectors of A looks like

$$S = \begin{pmatrix} y_1 & y_2 & \cdots & y_M \\ \lambda_1^{-1} y_1 & \lambda_2^{-1} y_2 & \cdots & \lambda_M^{-1} y_M \\ \lambda_1^{-2} y_1 & \lambda_2^{-2} y_2 & \cdots & \lambda_M^{-2} y_M \\ \vdots & \vdots & & \vdots \\ \lambda_1^{-k+1} y_1 & \lambda_2^{-k+1} y_2 & \cdots & \lambda_M^{-k+1} y_M \end{pmatrix} \qquad (M \equiv kd)$$

where the λ_j are e-values, solving $\det\left(\lambda^k I - \sum_{r=1}^{k} \lambda^{k-r} C_r\right) = 0$. We lose little generality by assuming that the λ_j are distinct.

It makes sense to write S^{-1} as

$$S^{-1} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1k} \\ u_{21} & u_{22} & \cdots & u_{2k} \\ \vdots & \vdots & & \vdots \\ u_{M1} & u_{M2} & \cdots & u_{Mk} \end{pmatrix}$$

where the u_{ij} are row d-vectors. Each row of S^{-1} is a left e-vector of A. We have for each j

$$\boxed{u_{j,r+1} = \lambda_j^{r} u_{j1} - \sum_{p=1}^{r} \lambda_j^{r-p} u_{j1} C_p} \qquad (r = 1, \ldots, k-1)$$

as well as $\quad u_{j1}\left(\lambda_j^k - \sum_{p=1}^{k}\lambda_j^{k-p}c_p\right) = 0.$

We may also express S^{-1} with more regard to its structure as

$$\begin{pmatrix} u_{11} & \lambda_1 u_{11} & \lambda_1^2 u_{11} & \cdots & \lambda_1^{k-1}u_{11} \\ u_{21} & \lambda_2 u_{21} & \lambda_2^2 u_{21} & \cdots & \lambda_2^{k-1}u_{21} \\ \vdots & \vdots & \vdots & & \vdots \\ u_{M1} & \lambda_M u_{M1} & \lambda_M^2 u_{M1} & \cdots & \lambda_M^{k-1}u_{M1} \end{pmatrix}\begin{pmatrix} I & -c_1 & -c_2 & \cdots & -c_{k-1} \\ 0 & I & -c_1 & \cdots & -c_{k-2} \\ 0 & 0 & I & \cdots & -c_{k-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \end{pmatrix}.$$

It's clear that an abbreviation $v_p \equiv u_{p1}$ will be helpful:

$$\boxed{S^{-1} = \begin{pmatrix} v_1 & \lambda_1 v_1 & \cdots & \lambda_1^{k-1}v_1 \\ v_2 & \lambda_2 v_2 & \cdots & \lambda_2^{k-1}v_2 \\ \vdots & \vdots & & \vdots \\ v_M & \lambda_M v_M & \cdots & \lambda_M^{k-1}v_M \end{pmatrix}\begin{pmatrix} I & -c_1 & -c_2 & \cdots & -c_{k-1} \\ 0 & I & -c_1 & \cdots & -c_{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \end{pmatrix}}$$

$$\equiv \begin{pmatrix} v_1 & \lambda_1 v_1 & \cdots & \lambda_1^{k-1}v_1 \\ \vdots & \vdots & & \vdots \\ v_M & \lambda_M v_M & \cdots & \lambda_M^{k-1}v_M \end{pmatrix} \mathcal{U} \quad \text{, say,}$$

$$\equiv V\,\mathcal{U}.$$

Now $\mathcal{U}^{-1}$ is another upper-triangular matrix, and SV has a $k\times k$ structure of $d\times d$ blocks, the $(p\,q)^{th}$ of which will be

$$\sum_{j=1}^{M}\lambda_j^{-p+1}\,y_j\,v_j\,\lambda_j^{q-1} = \sum_{j=1}^{M}\lambda_j^{q-p}\,y_j\,v_j$$

Thus SV has a banded structure; also, $SV = \mathcal{U}^{-1}$, and therefore

$$\boxed{\begin{aligned} \sum_{j=1}^{M}\lambda_j^{q-p}\,y_j\,v_j &= 0 \qquad \text{for } q<p \\ \sum_{j=1}^{M} y_j\,v_j &= I \end{aligned}}$$

(ii) If we want to know about
$$\Phi_\eta(s,t) = \sum_{n\geq 0}\sum_{m\geq 0} s^n t^m E[\eta_n \eta_m^T] \qquad (\eta_j = 0 \ \forall j<0)$$
then using the remark on the reverse of p 41 we have
$$\Phi_\eta(s,t) = (I-sA)^{-1}\ \Gamma\ (I-tA^T)^{-1}\ \frac{1}{1-st}$$
where Γ is the covariance matrix of the noise process γ' (which is only non-zero in the (1,1) block.) But we are really only concerned with what the Y process (= top d components of η) is doing, so we want to know about
$$(I\ 0\ \dots\ 0)(I-sA)^{-1} \equiv (B_1\ B_2\ \dots\ B_k),$$

Say. To solve this,
$$(I\ 0\ 0 \dots 0) = B\begin{pmatrix} I-sC_1 & -sC_2 & -sC_3 & \dots & -sC_{k-1} & -sC_k \\ -sI & I & 0 & \dots & 0 & 0 \\ 0 & -sI & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -sI & I \end{pmatrix}$$
so that
$$\begin{cases} 0 = -sB_1C_k + B_k \\ 0 = -sB_1C_m + B_m - sB_{m+1} & (m=2,\dots,k-1) \\ I = B_1(I-sC_1) - sB_2 \end{cases}$$
whence
$$B_m = \sum_{r=m}^{k} s^{r-m+1} B_1 C_r, \qquad \text{for } m=2,\dots,k,$$
$$\boxed{B_1 = \left(I - \sum_{r=1}^{k} s^r C_r\right)^{-1}}$$

We may equally well use this same method to bound the price of the floating-strike Asian option, taking

$$Y \equiv \int_0^1 \mu(dt)\, e^{\sigma B_t - \sigma^2 t/2 + rt}, \qquad \mu(dt) \equiv dt - \delta_1(dt).$$

and $Z = \int_0^1 B_t\, \mu(dt)$. We have $Z = -\int_0^1 t\, dB_t$, so $EZ^2 = \frac{1}{3}$, and

$$E(B_t|Z) = -\tfrac{3}{2} t^2 Z \equiv m_t Z, \quad \text{cov}(B_s, B_t | Z) = s\wedge t - m_s m_t/3 \equiv v_{st}.$$

Then

$$\text{var}(Y|Z=z) = \int_0^1 \mu(ds) \int_0^1 \mu(dt)\, \exp\left[r(s+t) + \sigma z(m_s+m_t) - \frac{\sigma^2}{6}(m_s^2+m_t^2)\right] \left\{ e^{\sigma^2 v_{st}} - 1 \right\}$$

$$= \int_0^1 \mu(ds) \int_0^1 \mu(dt) \left[e^{r(s+t) + \sigma z(m_s+m_t) - \frac{\sigma^2}{6}(m_s^2+m_t^2)} \left(e^{\sigma^2 v_{st}} - 1\right) - \left\{1 + r(s+t) + \sigma z(m_s+m_t) - \frac{\sigma^2}{6}(m_s^2+m_t^2)\right\} . \sigma^2 v_{st} \right]$$

as for fixed-strike!

Now bound $\left| e^x (e^y - 1) - (1+x) y \right|$

$$\le |e^x - 1 - x|\, |e^y - 1| + |1+x|\, |e^y - 1 - y|$$

$$\le \tfrac{1}{2} x^2 y\, e^{a+b} + \tfrac{1}{2} |1+x|\, y^2\, e^b \qquad \text{if } |x| \le a, \ |y| \le b.$$

Easily $0 \le v_{st} \le 1$ $\forall s,t \in [0,1]$, and $|Z| \le 2$ with very high probability. Once again, we get a bound which is $O(\sigma^4)$.

Asian options: the reason why the lower bound is so good. (7/4/94)

Returning to the estimate

$$0 \le E(Y^+|Z) - E(Y|Z)^+ \le \tfrac{1}{2}\{E(|Y|\,|Z) - |E(Y|Z)|\}$$

$$\le \tfrac{1}{2}\; E(|Y - E(Y|Z)|\;|Z)$$

$$\le \tfrac{1}{2}\sqrt{\mathrm{var}\,(Y|Z)},$$

we have

$$\mathrm{var}\,(Y|Z) = \int_0^1 ds \int_0^1 dt\,\Big\{E_Z\big(e^{\sigma B_s + (r-\sigma^2/2)s + \sigma B_t + (r-\sigma^2/2)t}\big) - E_Z\, e^{\sigma B_s + (r-\sigma^2/2)s} \cdot E_Z\, e^{\sigma B_t + (r-\sigma^2/2)t}\Big\}$$

$$= \int_0^1 ds \int_0^1 dt\; \exp\{r(s+t) + \sigma z(m_s+m_t) + \tfrac{\sigma^2}{2}(v_{ss} - s + v_{tt} - t)\}\,.\,\{\exp(\sigma^2 v_{st}) - 1\}$$

where $E(B_s|Z) = m_s\,.\,Z$, $m_s \equiv 3s(2-s)/2$, $v_{st} \equiv \mathrm{cov}\,(B_s, B_t|Z) = s\wedge t - m_s m_t/3$.

If now we write for short $\varphi(t,z) = rt + \sigma z m_t + \frac{\sigma^2}{2}(v_{tt} - t)$, we have

$$\mathrm{var}\,(Y|Z=z) = \int_0^1 ds \int_0^1 dt\; \exp\{\varphi(s,z) + \varphi(t,z)\}\,\{e^{\sigma^2 v_{st}} - 1\}.$$

$$= \int_0^1 ds \int_0^1 dt\,\Big[e^{\varphi(s,z)+\varphi(t,z)}\{e^{\sigma^2 v_{st}} - 1\} - (1 + \varphi(s,z) + \varphi(t,z))\,\sigma^2 v_{st}\Big]$$

since we have $\boxed{\int_0^1 v_{st}\, ds = \mathrm{cov}\,(\int_0^1 B_s ds,\, B_t\,|\,Z) = 0,}$ any t. Thus there's a simple bound

$$\mathrm{var}\,(Y|Z=z) \le \int_0^1 ds \int_0^1 dt\,\Big\{\,|e^{\varphi(s,z)+\varphi(t,z)} - 1 - \varphi(s,z) - \varphi(t,z)|\,.\,|e^{\sigma^2 v_{st}} - 1|$$

$$+\; |e^{\sigma^2 v_{st}} - 1 - \sigma^2 v_{st}|\,.\,|1 + \varphi(s,z) + \varphi(t,z)|\,\Big\}$$

The inequalities $-\frac{1}{12} \le v_{st} \le \frac{1}{4}$, $0 \le e^x - 1 - x \le \frac{1}{2}x^2 e^x$ $(x \ge 0)$, $\le -x$ $(x \le 0)$, $|e^x - 1| \le |x|\, e^{x^+}$

[Also: for $|x| \le a$, $0 \le e^x - 1 - x \le \frac{x^2}{2} e^a$.]

can be applied here. But let's consider some typical values. We have $\sigma = 0.1 - 0.3$ as typical, and z would be unlikely to lie outside $(-1,1)$ – it's the value of a $N(0,1/3)$ r.v.. So provided $|\varphi(t,z)| \leq 1$, we can bound

$$\left| e^{\varphi(t,z)+\varphi(s,z)} - 1 - \varphi(t,z) - \varphi(s,z) \right| \leq \{\varphi(t,z)+\varphi(s,z)\}^2 . 4$$

But here we have $|\varphi(t,z)| \leq 1 \Leftrightarrow$

$$\left(-1 - rt + \frac{\sigma^2}{6} m_t^2\right)/\sigma m_t \leq z \leq \left\{1 - rt + \frac{\sigma^2}{6} m_t^2\right\}/\sigma m_t$$

Now the interval for z is of length $\dfrac{2}{\sigma m_t} = \dfrac{4}{3\sigma\, t(2-t)} \geq \dfrac{4}{3\sigma} \sim \dfrac{40}{9}$, which is big as far as z is concerned, and the interval is centred at

$$-\frac{rt}{\sigma m_t} + \frac{\sigma m_t}{6} = \frac{\sigma}{4} t(2-t) - \frac{2r}{3\sigma(2-t)},$$

which is pretty small. Thus we get an upper bound for the variance of

$$4 \int_0^1 ds \int_0^1 dt \, \{\varphi(t,z)+\varphi(s,z)\}^2 . \frac{\sigma^2}{4} e^{\sigma^2/4}$$

$$+ 4 . (\sigma^2/4)^2 \; 3$$

$$\leq 4\sigma^2 e^{\sigma^2/4} + \frac{3}{4}\sigma^4 .$$

But this is too crude; we shall typically have $r = O(1/10)$, σ similarly, so, when z is about 1, $\varphi(t,z)$ would be at worst of the order $1/10$. So our upper bound looks somewhat like

$$\sigma^4 e^{\sigma^2/4} + \frac{3}{4}\sigma^4$$

and so $\boxed{0 \leq E_Z Y^+ - (E_Z Y)^+ \sim \sigma^2}$, very small

To set things up really ready to use a limit result, we ought to pick our coordinate system cunningly... It looks like we need to take

$$\zeta_n \equiv \begin{pmatrix} (D^{p-1}X)_n \\ (D^{p-2}X)_{n-1} \\ \vdots \\ X_{n-p+1} \end{pmatrix} \equiv \begin{pmatrix} 1 & -\binom{p-1}{1} & \binom{p-1}{2} & -\binom{p-1}{3} & \cdots & (-1)^{p-1} \\ 0 & 1 & -\binom{p-2}{1} & \binom{p-2}{2} & \cdots & (-1)^{p-2} \\ 0 & 0 & 1 & -\binom{p-3}{1} & \cdots & (-1)^{p-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_{n-p+1} \end{pmatrix}$$

$$\equiv U \xi_n \qquad \xi_n^T = (X_n, X_{n-1}, \ldots, X_{n-p+1})$$

It's not hard to check that the inverse of U is the upper-triangular matrix

$$U^{ij} = \binom{p-1-i}{j-i} I_{\{i \le j\}} .$$

Now $\xi_n = K\xi_{n-1} + \begin{pmatrix} \varepsilon_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ where $K = \begin{pmatrix} c_1 & c_2 & \cdots & c_p \\ I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

so $\zeta_n = UKU^{-1}\zeta_{n-1} + \begin{pmatrix} \varepsilon_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Clearly $(D^{p-r-1}X)_{n-r} = (D^{p-r-1}X)_{n-r-1} + (D^{p-r}X)_{n-r}$ $\quad (r = 1, \ldots, p-1)$

$$\boxed{\therefore\ \zeta_n^r = \zeta_{n-1}^r + \zeta_{n-1}^{r-1}} \qquad (r = 1, \ldots, p-1)$$

and

$$\boxed{\zeta_n^0 = \sum_{k=0}^{p-1} \sum_{i=0}^{k} c_{i+1} \binom{p-i-1}{k-i} \zeta_{n-1}^k - \sum_{k=0}^{p-1} \binom{p}{k+1} \zeta_{n-1}^k + \hat{\varepsilon}_n + \zeta_{n-1}^0}$$

[? - see pp 55, 56]

Limits of ARMA processes: some examples (10/4/94)

Suppose that X is a (d-vector) ARMA(p,q) process:

$$X_n = \sum_{r=1}^{p} C_r X_{n-r} + \nu_n, \quad \nu_n \equiv \sum_{j=0}^{q-1} B_j \xi_j$$, where the ξ_j are

IID $N(0,\Gamma)$, and where wlog $B_0 = I$. Then

$$\Phi_X(s,t) \equiv \sum_{n\geq 0}\sum_{m\geq 0} s^m t^n E\left[X_m X_n^T\right] \qquad \left(B(s) \equiv \sum_{r=0}^{q-1} s^r B_r\right)$$

$$= (I - C(s))^{-1} B(s)\,\Gamma\, B(t)^T (I - C(t)^T)^{-1} \frac{1}{1-st},$$

as we've seen. Let's consider the case $d=1$, $p=1$, $q=3$, so we get

$$\Phi_X(s,t) = \frac{1}{1-st}\,\frac{b_0 + b_1 s + b_2 s^2}{1 - cs}\cdot\frac{b_0 + b_1 t + b_2 t^2}{1-ct}\cdot\sigma^2.$$

In the limit,

$$\lim_N \int_0^\infty e^{-\alpha s}\,ds \int_0^\infty e^{-\beta t}\,dt\; E X_s^{(N)} X_t^{(N)}$$

$$= \lim_N \frac{1}{N^2}\sum_{j\geq 0}\sum_{k\geq 0} e^{-\alpha j/N - \beta k/N}\, E X_j^{(N)} X_k^{(N)}$$

$$= \lim \frac{1}{N^2}\,\Phi_{X^{(N)}}\left(e^{-\alpha/N}, e^{-\beta/N}\right)$$

$$= \lim \frac{1}{\alpha+\beta}\cdot\frac{\sigma_N^2}{N}\,\frac{B_N(s)}{1 - c_N s}\cdot\frac{B_N(t)}{1-c_N t} \qquad s = e^{-\alpha/N},\ t = e^{-\beta/N}.$$

Now we'll suppose that $c_N = e^{-\theta/N}$ for some fixed $\theta > 0$, and take $B_N(s) = B(s)$, same for all N, giving

$$\frac{1}{\alpha+\beta}\,\frac{1}{\alpha+\theta}\,\frac{1}{\beta+\theta}\,\lim_{N\to\infty} N\sigma_N^2\, B_N(e^{-\alpha/N})\, B_N(e^{-\beta/N}).$$

<u>What sorts of limits can arise?</u> This depends on the number of unit roots in B.

<u>Case 1</u>: B has no unit roots; here, then, we choose $\sigma_N^2 = \sigma^2/N$, and suppose that $B(1) = 1$, wlog, yielding the limiting form of the covariance to be

$$\frac{1}{\alpha+\beta}\cdot\frac{\sigma^2}{(\alpha+\theta)(\beta+\theta)},$$

which corresponds to OU process in the limit

Case 2: B has 1 unit root. This time, $\sigma_N^2 = \sigma^2 N$ and in the limit we obtain

$$\frac{1}{\alpha+\beta} \frac{\sigma^2 \alpha\beta}{(\alpha+\theta)(\beta+\theta)}.$$

Now $\alpha\beta(\alpha+\theta)^{-1}(\beta+\theta)^{-1}$ is the LT of the measure $(\delta_0(dx)-\theta e^{-\theta x}dx)(\delta_0(dy)-\theta e^{-\theta y}dy)$ which makes it highly unlikely that there is some nice limit process. To see conclusively that there can be no continuous Gaussian process $(X_t)_{t\geq 0}$ for which $\mathrm{var}(X_t)$ is loc. bdd and

$$\int_0^\infty e^{-\alpha s}ds\int_0^\infty e^{-\beta t}dt\; EX_sX_t = \sigma^2\alpha\beta/(\alpha+\beta)(\alpha+\theta)(\beta+\theta)$$

note that $\displaystyle\lim_{\beta\to\infty}\int_0^\infty e^{-\alpha s}ds\int_0^\infty \beta e^{-\beta t}dt\; E(X_sX_t) = 0,$

since $X_t\to 0$ in L^2 as $t\to 0$ if $\mathrm{var}(X_t)$ is loc. bdd and if X is cts, $X_0=0$. But the limit is $\sigma^2\alpha(\alpha+\theta)^{-1}\neq 0$.

Case 3: B has 2 unit roots. Similarly to case 2.

The final statement in the real-valued case is the following.

THEOREM. Suppose $(X_t)_{t\geq 0}$ is a (weak) limit of $(X^{(N)}_{[Nt]})_{t\geq 0}$, where each $X^{(N)}$ is an ARMA(p,q) process with p,q fixed. Suppose also that X is continuous in L^2. Then the only possible limits which can arise have the form

$$E\left[\int_0^\infty e^{-\alpha s}ds\int_0^\infty e^{-\beta t}dt\; X_sX_t\right] = \frac{1}{\alpha+\beta}\cdot\frac{Q(\alpha)}{P(\alpha)}\cdot\frac{Q(\beta)}{P(\beta)},$$

where Q,P are polynomials, $0\leq \deg(Q)<\deg(P)\leq p$.

Moreover, every such process may arise, and can be represented in the form $X_t = v\cdot Z_t$, where Z_t is an $\mathbb{R}^p$-valued linear Gaussian process

$$dZ_t = KZ_t\,dt + \sigma\, dW_t$$

with

$$K = \begin{pmatrix} k_1 & k_2 & \cdots & k_{p-1} & k_p\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & & 1 & 0\end{pmatrix},\quad \sigma = \begin{pmatrix}\sigma_1\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

Proof. Firstly, note that

$$\left|E\int_0^\infty \alpha e^{-\alpha s}ds\int_0^\infty \beta e^{-\beta t}dt\; X_sX_t\right| \leq \int_0^\infty \alpha e^{-\alpha s}\sqrt{EX_s^2}\,ds\int_0^\infty \beta e^{-\beta t}\sqrt{EX_t^2}\,dt$$

$$\leq \left(\int_0^\infty \alpha e^{-\alpha s} EX_s^2\, ds\right)^{1/2} \left(\int_0^\infty \beta e^{-\beta t} EX_t^2\, dt\right)^{1/2} \to 0 \qquad (\alpha \to \infty)$$

provided the integral is finite for some α. This forces $\deg Q < \deg P$, if we can establish such a representation.

But, as we saw on p 47, we shall have

$$\Phi_{X^{(N)}}(e^{-\alpha/N}, e^{-\beta/N}) = \frac{1}{1-e^{-(\alpha+\beta)/N}} \cdot \Gamma_N \cdot \frac{B_N(e^{-\alpha/N})}{1-C_N(e^{-\alpha/N})} \cdot \frac{B_N(e^{-\beta/N})}{1-C_N(e^{-\alpha/N})}$$

which, if it converges, must converge to something of the form given. Now to understand how all such things may arise; we could choose the ARMA coefficients exactly so as to give what we require!!

Representing them as solutions of linear SDEs is also quite easy. The Z process can be written as

$$Z_t = e^{tK} \int_0^t e^{-uK} \sigma\, dW_u$$

so that for $0 \leq s \leq t$ we get

$$E Z_s Z_t^T = \int_0^s e^{(s-u)K} \sigma\sigma^T e^{(t-u)K^T}\, du.$$

Our goal now is to choose v, K in such a way that

$$\int_0^\infty v^T e^{tK} \sigma\, e^{-\alpha t}\, dt = Q(\alpha)/P(\alpha)$$

$$= v^T (\alpha - K)^{-1} \sigma.$$

But let's notice that if $(\alpha - K)x = e_1$, then $\alpha x_1 - \sum_1^p k_r x_r = 1$ and $\alpha x_r = x_{r-1}$ $(r > 1)$, so $x_r = \xi \cdot \alpha^{-r}$, with $\xi = \left(1 - \sum_{r=1}^p \alpha^{-r} k_r\right)^{-1}$.

Hence

$$(\alpha - K)^{-1}\sigma = \begin{pmatrix} \alpha^{p-1} \\ \alpha^{p-2} \\ \vdots \\ \alpha \\ 1 \end{pmatrix} \cdot \frac{1}{\alpha^p - \sum_{r=1}^p k_r \alpha^{p-r}}$$

Now it's obvious how we're going to do it!!

In the d-vector case, the scalars k_r become $d \times d$ matrices K_r, and $(\alpha - K)x = y = \begin{pmatrix} y_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ gets solved as above: $(\alpha - K)^{-1} y = (\alpha^{p-1}, \alpha^{p-2}, \dots, \alpha, 1)^T \left(\alpha^p I - \sum_{r=1}^p \alpha^{p-r} K_r\right)^{-1} y_1$.

A useful result (extending Black-Scholes) (6/5/94)

Suppose we have that (X, Y) are bivariate normal with mean zero and covariance matrix $V \equiv \begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix}$ and A, B are two positive reals.

Then

$$E\left(Ae^{X+Y} - Be^{Y}\right)^{+} = \exp\left\{\frac{v_{xx}v_{yy} - v_{xy}^2}{2v_{xx}}\right\}\left[A e^{(v_{xx}+v_{xy})^2/2v_{xx}}\,\Phi\left(\frac{v_{xx}+v_{xy}-\gamma}{\sqrt{v_{xx}}}\right) - Be^{v_{xy}^2/2v_{xx}}\,\Phi\left(\frac{v_{xy}-\gamma}{\sqrt{v_{xx}}}\right)\right] \qquad (\gamma \equiv \log(B/A))$$

Proof Given X, Y has mean θX, where $\theta \equiv v_{xy}/v_{xx}$, and variance $v = v_{yy} - v_{xy}^2/v_{xx}$. Thus

$$E\left(Ae^{X+Y} - Be^{Y}\right)^{+} = E\, e^{Y}\left(Ae^{X} - B\right)^{+}$$

$$= \exp(\tfrac{1}{2}v)\, E\, e^{\theta X}\left(Ae^{X} - B\right)^{+}$$

$$= e^{v/2}\int_{\gamma}^{\infty}\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, e^{\theta x}\left(Ae^{x} - B\right)dx \qquad \begin{array}{l}\sigma^2 \equiv v_{xx} \\ \gamma \equiv \log(B/A)\end{array}$$

$$= e^{v/2}\int_{\gamma/\sigma}^{\infty}\exp(-\tfrac{1}{2}z^2)\, e^{\theta\sigma z}\left(Ae^{\sigma z} - B\right)\frac{dz}{\sqrt{2\pi}}$$

$$= e^{v/2}\left[Ae^{\sigma^2(1+\theta)^2/2}\,\bar{\Phi}\left(\frac{\gamma}{\sigma} - \sigma(1+\theta)\right) - Be^{\sigma^2\theta^2/2}\,\bar{\Phi}\left(\frac{\gamma}{\sigma} - \sigma\theta\right)\right]$$

$$= e^{v/2}\left[Ae^{(v_{xx}+v_{xy})^2/2v_{xx}}\,\bar{\Phi}\left(\frac{\gamma - v_{xx} - v_{xy}}{\sqrt{v_{xx}}}\right) - Be^{v_{xy}^2/2v_{xx}}\,\bar{\Phi}\left(\frac{\gamma - v_{xy}}{\sqrt{v_{xx}}}\right)\right]$$

Budget equation with dividends and discontinuous processes (9/5/94)

(i) How should the budget equations look when each share pays a stream of dividends, but the price and dividend processes are discontinuous semimartingales?

Let's suppose we have a $(d+1)$-vector semimg price process

$$\bar{S}_t^T \equiv (S_t^0, S_t^T)$$

and that there is an FV process $\bar{D}_t^T \equiv (0, D_t^T)$ of dividends paid out on the assets [NB: we're supposing that asset 0 pays no dividends – this is consistent with regarding asset 0 as cash, because a dividend on a share is like an enforced conversion of some of its value into cash. So a dividend on cash would be tautologous!]

Now we are going to choose security 0 as a numeraire, and define

$$\tilde{S}_t \equiv S_t / S_t^0$$

with an appropriate definition of $\tilde{D}$, the dividend process in this reference frame. The appropriate definition will become apparent.

(ii) Let's now think how things look referred to this numeraire. Using $\sim$ to denote quantities in this frame, $\tilde{S}^0 = 1$, and if $\tilde{X}$ is the discounted wealth process if we follow portfolio θ in the assets $1, \dots, d$, we must have

$$\boxed{\tilde{X}_t = \theta_t^0 + \theta_t \cdot S_t = \tilde{X}_0 + \int_{(0,t]} \theta_u \left(d\tilde{S}_u + d\tilde{D}_u\right).}$$

Here, the process θ^0 is the holding of the 0^{th} asset, cash; the second equation can be regarded as the definition of θ^0. In general, θ^0 will be adapted, but will not have any sample-path regularity.

So now we develop $X_t \equiv \tilde{X}_t S_t^0$ using Itô's formula:

$$dX_t = S_{t-}^0 \, d\tilde{X}_t + \tilde{X}_{t-} \, dS_t^0 + d[S^0, \tilde{X}]_t$$

$$= S_{t-}^0 \left(\theta_t \, d\tilde{S}_t + \theta_t \, d\tilde{D}_t\right) + \tilde{X}_{t-} \, dS_t^0 + d[S^0, \tilde{X}]_t$$

Note that if $D \equiv 0$, we have θ^0 must be continuous!

Also, it is worth remarking that if X is any self-financing wealth process

$$X_t = \theta_t \cdot S_t = X_0 + \int_{[0,t]} \theta_u \cdot dS_u$$

and φ is any (positive) semimartingale, then if $\tilde{X} \equiv X\varphi$, $\tilde{S} \equiv S\varphi$, we have

$$d\tilde{X} = \theta \, d\tilde{S}, \quad \tilde{X}_t = \theta_t \cdot \tilde{S}_t .$$

$$= \theta_t \left\{ S^0_{t-} d\tilde{S}_t + \tilde{S}_{t-} dS^0_t + d[S^0, \tilde{S}]_t \right\} - \theta_t \left(\tilde{S}_{t-} dS^0_t + d[\tilde{S}, S^0]_t \right)$$

$$+ S^0_{t-} \theta_t d\tilde{D}_t + \tilde{X}_{t-} dS^0_t + d[\tilde{X}, S^0]_t$$

$$= \theta_t dS_t + S^0_{t-} \theta_t d\tilde{D}_t + \left\{ (\tilde{X}_{t-} - \theta_t \tilde{S}_{t-}) dS^0_t + \theta_t d[\tilde{D}, S^0]_t \right\}.$$

Now

(*) $\quad \theta^0_t = \tilde{X}_t - \theta_t \tilde{S}_t = \tilde{X}_{t-} - \theta_t \tilde{S}_{t-} + \theta_t \Delta\tilde{D}_t,$

so the expression for dX_t becomes

$$\boxed{dX_t = \theta_t dS_t + S^0_{t-} \theta_t d\tilde{D}_t + \left\{ (\theta^0_t - \theta_t \Delta\tilde{D}_t) dS^0_t + \theta_t d[\tilde{D}, S^0]_t \right\}}$$

(iii) Special case: S^0 is an FV process. In this case, the equation collapses to

$$\boxed{dX_t = \theta_t dS_t + \theta^0_t dS^0_t + \theta_t S^0_{t-} d\tilde{D}_t}$$

from which we conclude that the relationship between D and $\tilde{D}$ must be

$$\boxed{S^0_{t-} d\tilde{D}_t = dD_t.}$$

(why??)

(iv) Special case: $\tilde{D}$ is previsible. This time, we see that θ^0 must be previsible, from (*), and thus

$$\boxed{dX_t = \theta^0_t dS^0_t + \theta_t dS_t + \theta_t \left\{ S^0_{t-} d\tilde{D}_t + d\langle S^{0c}, \tilde{D}^c \rangle_t \right\}}$$

The relationship between $\tilde{D}$ and D this time is

$$\boxed{S^0_{t-} d\tilde{D}_t + d\langle S^{0c}, \tilde{D}^c \rangle_t = dD_t.}$$

[NB: under the natural assumption that D is FV, this reduces to the same thing as above]

This is easily reworked to

$$\boxed{S^0_t \tilde{D}_t = D_t + \int_{(0,t]} \tilde{D}_u S^0_u \frac{dS^0_u}{S^0_u}.}$$

A question of Simon Babbs (10/5/94)

(i) Suppose that r solves the SDE

$$dr_t = \sigma \sqrt{r_t}\, dW_t + (\alpha - \beta r_t)\, dt$$

where $\sigma, \alpha > 0$, $\beta \in \mathbb{R}$, and that now we consider the exponential SDE

$$dZ_t = \varepsilon \sigma \sqrt{r_t}\, Z_t\, dW_t, \quad Z_0 = 1.$$

When is this exponential local martingale Z actually a true martingale?

(ii) The answer is, "Always". Let's see why. We may write

$$Z_t = \exp\left[\varepsilon \int_0^t \sigma \sqrt{r_s}\, dW_s - \tfrac{1}{2} \int_0^t \sigma^2 \varepsilon^2 r_s\, ds\right]$$

$$= \exp\left[\varepsilon (r_t - r_0) - (\tfrac{1}{2}\sigma^2\varepsilon^2 - \beta\varepsilon) \int_0^t r_s\, ds - \alpha\varepsilon t\right],$$

so that Z is a true martingale iff for all $t \geq 0$

$$(*) \quad \boxed{E \exp\left\{\varepsilon r_t - (\tfrac{1}{2}\sigma^2\varepsilon^2 - \beta\varepsilon) \int_0^t r_s\, ds\right\} = \exp(\varepsilon r_0 + \alpha\varepsilon t).}$$

Suppose we could prove that (with $\varepsilon \in \mathbb{R}$ fixed) there exists $\delta > 0$ so small that $(*)$ holds for all $t \leq \delta$, for all r_0; then is easy to see that $(*)$ must hold for all r_0, for all t.

(iii) The task now is to estimate r and $\int r_s\, ds$. Let's abbreviate $|\tfrac{1}{2}\sigma^2\varepsilon^2 - \beta\varepsilon| \equiv c$. If we could prove that there is $\delta > 0$ such that for $t \leq \delta$

$$E \exp\left\{4\varepsilon \sup_{u \leq t} r_u\right\} < \infty,$$

$$E \exp\left\{4c \int_0^t r_u\, du\right\} < \infty$$

whatever r_0, then the local martingale Z is L^2-bounded on $[0, \delta]$, and hence is a true martingale.

To obtain these estimates, we do a little stochastic calculus. Firstly, if we set

$$\tilde{r}_t \equiv e^{+\beta t} r_t,$$

We find that

$$d\tilde{r}_t = \sigma e^{\beta t/2} \sqrt{\tilde{r}_t}\, dW_t + \alpha e^{\beta t} dt.$$

Accordingly, if $A_t \equiv \frac{\sigma^2}{4} \int_0^t e^{\beta s} ds$, $\tau_t = \inf\{u : A_u > t\}$, we get that $\rho_t \equiv \tilde{r}(\tau_t)$ solves

$$d\rho_t = 2\sqrt{\rho_s}\, dW_t + \frac{4\alpha}{\sigma^2} dt$$

so is genuinely a BESQ process. If we replace $4\alpha/\sigma^2$ by an integer $n > \frac{4\alpha}{\sigma^2}$, we make the solution stochastically larger, so we may wlog assume that $4\alpha/\sigma^2 = n$ is integer. If B denotes a BM in $\mathbb{R}^n$, $B_0 = 0$, and W is $BM(\mathbb{R})$ we have

$$P\left[|B_u|^2 > a \text{ for some } u \le \delta\right] \le n\, P\left[W_u^2 > a/n \text{ for some } u \le \delta\right]$$

$$\le \frac{c_n}{\sqrt{\delta}} \exp\left(-a/2n\delta\right).$$

Hence

$$\sup_{0 \le u \le \delta} |x + B_u|^2 \equiv Y_\delta, \text{ say},$$

will satisfy $E \exp 4\varepsilon Y_\delta < \infty$ for δ small enough. Only tidying up remains.

(iv) The basic idea that it's enough to get exponential moments for a small time interval is not new; Wolfgang Stummer's article in PTRF 97 515–542 contains several references to these tricks, for example

ARMA processes expressed in terms of differences (13/5/94)

(i) We take the basic $X_n = \sum_{r=1}^{p} c_r X_{n-r} + \nu_n$, for some noise process ν, and rewrite in terms of the state vector

$$\zeta(n) \equiv \begin{pmatrix} (\Delta^{p-1}X)_n \\ (\Delta^{p-2}X)_{n-1} \\ \vdots \\ (\Delta X)_{n-p+2} \\ X_{n-p+1} \end{pmatrix} \equiv \begin{pmatrix} \zeta_0(n) \\ \zeta_1(n) \\ \vdots \\ \zeta_{p-2}(n) \\ \zeta_{p-1}(n) \end{pmatrix}.$$

Then clearly for each $j = 1, \dots, p-1$,

$$\boxed{\zeta_j(n) = \zeta_j(n-1) + \zeta_{j-1}(n-1),}$$

so we can express

$$\zeta(n) = \begin{pmatrix} V_0 & V_1 & \cdots & V_{p-1} \\ I & I & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{pmatrix} \zeta(n-1) + \begin{pmatrix} \nu_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let's observe that

$$X_{n-p+r} = \sum_{j=0}^{r} \binom{r}{j} \zeta_{p-1-j}(n-1) \qquad (r = 0, \dots, p-1),$$

so

$$\zeta_0(n) = \sum_{l=0}^{p-1} (-1)^l \binom{p-1}{l} X_{n-l}$$

$$= \sum_{r=1}^{p-1} (-1)^{p-r} \binom{p-1}{p-r} \sum_{j=0}^{r} \binom{r}{j} \zeta_{p-1-j}(n-1) + X_n$$

$$= \nu_n + \sum_{r=1}^{p} c_r X_{n-r} + \sum_{j=1}^{p-1} \sum_{r=j}^{p-1} (-1)^{p-r} \binom{p-1}{p-r}\binom{r}{j} \zeta_{p-1-j}(n-1)$$

$$- \zeta_{p-1}(n-1)$$

$$= \nu_n + \sum_{r=0}^{p-1} c_{p-r} \sum_{j=0}^{r} \binom{r}{j} \zeta_{p-1-j}(n-1) + \sum_{j=1}^{p-2} \sum_{r=j}^{p-1} (-1)^{p-r} \binom{p-1}{p-r}\binom{r}{j} \zeta_{p-1-j}(n-1)$$

$$- (p-1)\zeta_0(n-1) - \zeta_{p-1}(n-1)$$

If we go back to the formulation on pp 47–48, we find that we need limit for

$$C^{(N)}(e^{-\alpha/N}) \doteq I - \sum_{r=1}^{p} \left(1-\frac{\alpha}{N}\right)^{r} C_r^{(N)}$$

$$= \left(I - \sum_{r=1}^{p} C_r^{(N)}\right) - \sum_{j=1}^{p} \sum_{r=j}^{p} \left(-\frac{\alpha}{N}\right)^{j} \binom{r}{j} C_r^{(N)}$$

so the existence of a limit comes down to

$$\begin{cases} I - \sum_{r=1}^{p} C_r^{(N)} \sim N^{\gamma} A'_{p-1} & \\ \sum_{r=j}^{p} \binom{r}{j} C_r^{(N)} \sim N^{j+\gamma} A'_{p-j-1} & (j = 1, \ldots, p) \end{cases}$$

$$= \gamma_n + \sum_{j=0}^{p-1} \left(\sum_{r=j}^{p-1} \binom{r}{j} C_{p-r} \right) \zeta_{p-1-j}(n-1) - \sum_{j=1}^{p-2} \binom{p}{j} \zeta_{p-1-j}(n-1)$$

$$- (p-1)\zeta_0(n-1) - \zeta_{p-1}(n-1)$$

$$= \gamma_n + \left(C_1 - (p-1)I\right)\zeta_0(n-1) + \sum_{j=1}^{p-2} \left\{ \sum_{r=j}^{p-1} \binom{r}{j} C_{p-r} - \binom{p}{j} I \right\} \zeta_{p-1-j}(n-1)$$

$$+ \left(\sum_{r=1}^{p} C_r - I \right) \zeta_{p-1}(n-1).$$

Hence we have

$$\boxed{\begin{aligned} V_0 &= C_1 - (p-1)I \\ V_k &= \sum_{i=0}^{k} \binom{p-1-i}{k-i} C_{i+1} - \binom{p}{k+1} \\ V_{p-1} &= \sum_{r=1}^{p} C_r - I \end{aligned}} \qquad (k = 1, \ldots, p-2)$$

(ii) Now we consider a sequence of these things, denoted by $^{(N)}$, and we'll set

$$Z_j^{(N)}(n) \equiv N^{p-1-j} \zeta_j^{(N)}(n),$$

so that

$$\boxed{Z_j^{(N)}(n) - Z_j^{(N)}(n-1) = N^{-1} Z_{j-1}^{(N)}(n-1)} \qquad (j = 1, \ldots, p-1)$$

and

$$\boxed{\begin{aligned} Z_0^{(N)}(n) = {} & N^{p-1}\gamma_n + \left(C_1^{(N)} - p + 1\right) Z_0^{(N)}(n-1) \\ & + \sum_{k=1}^{p-2} N^k \left\{ \sum_{i=0}^{k} \binom{p-1-i}{k-i} C_{i+1}^{(N)} - \binom{p}{k+1} \right\} Z_k^{(N)}(n) \\ & + N^{p-1} \left(\sum_{r=1}^{p} C_r^{(N)} - I \right) Z_{p-1}^{(N)}(n-1) \quad . \end{aligned}}$$

For nice limit, we shall need to get

$$\boxed{\begin{aligned} & N\left(C_1^{(N)} - pI\right) \to A_0 \\ & N^{k+1} \left\{ \sum_{i=0}^{k} \binom{p-1-i}{k-i} C_{i+1}^{(N)} - \binom{p}{k+1} \right\} \to A_k \qquad k = 1, \ldots, p-2 \\ & N^p \left(\sum_{r=1}^{p} C_r^{(N)} - I \right) \to A_{p-1} \end{aligned}}$$

(in fact, the middle statement holds for $k = 0, \ldots, p-1$)

Correlation of jump processes with variable intensities (13/5/94)

(i) We're going to consider random measures ν_i on $\mathbb{R}^+ \times \mathbb{R}$ which are integer-valued, and have intensities $\lambda_i(t)$, so that

$$\tilde{N}_i(t) \equiv \iint_{(0,t]\times\mathbb{R}} \nu_i(ds, dy) - \int_0^t \lambda_i(s)\,ds$$

is a martingale for each i, and there are no jumps in common for any $i \neq j$. Write the compensator of ν_i in the form

$$\tilde{\nu}_i(ds, dy) = \lambda_i(s)\, F_i(s, dy)\, ds$$

and set

$$\mu_i(s) \equiv \int F_i(s, dy)\, y.$$

We shall define

$$Y_i(t) \equiv \iint_{(0,t]\times\mathbb{R}} y\, \nu_i(ds, dy)$$

$$M_i(t) \equiv Y_i(t) - \int_0^t \mu_i(s)\lambda_i(s)\,ds \quad \text{, a martingale}$$

Then we get for $i \neq j$

$$\boxed{\begin{aligned} \operatorname{cov}\left[Y_i(t), Y_j(t)\right] &= \operatorname{cov}\left(\int_0^t \lambda_i(s)\mu_i(s)\,ds,\ \int_0^t \lambda_j(s)\mu_j(s)\,ds\right) \\ &\quad + E\int_0^t \left\{M_i(s)\lambda_j\mu_j(s) + M_j(s)\lambda_i\mu_i(s)\right\} ds \end{aligned}}$$

$$\boxed{\begin{aligned} \operatorname{var}\left(Y_i(t)\right) &= \operatorname{var}\left(\int_0^t \lambda_i\mu_i(s)\,ds\right) + E\int_0^t \lambda_i(s)\,ds \int y^2 F_i(s, dy) \\ &\quad + 2E\int_0^t M_i(s)\lambda_i\mu_i(s)\,ds \end{aligned}}$$

(ii) If we make some simplifying assumptions, that the μ_i are <u>constant</u> processes, we don't in general get rid of the ugly final terms in these expressions.

For example, if $\tilde{N}_1(t) = N(\xi t) - \xi t$, where ξ is indept of the standard Poisson pr. N, and $1 \le \xi \le 2$, and $\lambda_2(t) = f(N_t)$ then

$$E\left[\tilde{N}_1(t)\,\lambda_2(t)\right] = E\left[f(N_t)(N_t - t)\right] \neq 0 \text{ in general.}$$

One case where a little progress can be made is where

$$d\lambda_i = \sigma_i \sqrt{\lambda_i}\, dW^i + (\alpha^i - \beta^i \lambda_i)\, dt \qquad , \quad dW^i dW^j = \rho_{ij}\, dt.$$

Here,

$$E\left[\lambda_i(t)\, M_j(t)\right] = E \int_0^t M_j(s)\, (\alpha^i - \beta^i \lambda_i(s))\, ds$$

$$= -\beta^i \int_0^t E\left(M_j(s)\lambda_i(s)\right) ds$$

which must therefore be zero. The problem comes with cov $(\Lambda^i_t, \Lambda^j_t)$ when $\rho_{ij} \neq 0$ for $i \neq j$, $\Lambda^i_t \equiv \int_0^t \lambda_i(s)\, ds$.

(iii) If we assume that

$$Y_i(t) = \eta_i(\Lambda_i(t))$$

where the $\Lambda_i(\cdot)$ are continuous increasing processes, and the η_i are compound Poisson processes, jumping at rate 1, jump-size dist^{rs} F_i, all independent of each other and the Λ_i, then

$$E\, Y_i(t) = \mu_i\, E\left[\Lambda_i(t)\right] \qquad \left(\mu_i \equiv \int x F_i(dx),\ \sigma_i^2 \equiv \int (x-\mu_i)^2 F_i(dx)\right)$$

$$E\left[Y_i(t)Y_j(t)\right] = \mu_i \mu_j\, E\left[\Lambda_i(t)\Lambda_j(t)\right] \qquad (i \neq j)$$

$$E\left[Y_i(t)^2\right] = (\mu_i^2 + \sigma_i^2)\, E\,\Lambda_i(t) + \mu_i^2\, E\left[\Lambda_i(t)^2\right]$$

from which

$$\boxed{\mathrm{cov}\left(Y_i(t), Y_j(t)\right) = \mu_i \mu_j\, \mathrm{cov}\left(\Lambda_i(t), \Lambda_j(t)\right) + \delta_{ij}\, (\mu_i^2 + \sigma_i^2)\, E\,\Lambda_i(t)}$$