Hybrid Derivatives Pricing under the Potential Approach*

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Abstract

We present a general framework to price contingent claims whose payoffs involve equity, credit and interest rate components. The common cross-market dynamics are modeled via a Markov-chain ξ. The model is dynamically consistent and allows for a high degree of flexibility. Prices of various vanilla and more complex derivative products can be derived analytically or resorting to integral transform techniques.

1 Introduction

In recent years the financial industry has witnessed the launch and growth of a new class of derivatives products, so-called *hybrid* because they combine features

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and risks from different markets such as interest rates, equity and credit. From a business point of view the purpose of these new financial instruments is to capture and trade the correlation between different markets. Modeling the common dynamics of different markets in a tractable and realistic way is thus of extreme importance.

In this paper we apply the potential approach of Rogers [R1] (see also [R2] and [R3]). Although the original presentation of the methodology did not discuss default risk, it is not hard to see how to incorporate this into the modelling; since the basic probabilistic object in the potential approach is some fairly abstract Markov process, it is natural to try to explain the default risk in terms of this Markov process. Given that this is what we are going to do, we propose to let the default intensity be simply a function of the underlying Markov process - the *doubly stochastic* modelling idea used by Duffie, Saita & Wang [DSW], Frey & Backhaus [FB], Di Graziano & Rogers [DR3], and others. Where our approach differs is in the choice of a finite-state Markov chain as the underlying Markovian driver. We make no attempt to interpret this chain as any sort of observable; its sole purpose is to drive the stochastic processes of interest, just as the driving Brownian motion in a SDE has no interpretation, though the solution generally does.

This modelling approach is well suited to pricing hybrid derivatives. In particular, we will concentrate on products involving credit, equity and interest rate risk. The results presented can be easily generalized to include foreign exchange risk.

In more detail, we let the Markov chain be denoted by ξ a continuous-time Markov chain with finite state space. We then model the stochastic hazard process $(\lambda_t^i)_{t\geq 0}$ for a given company i as a function of ξ, say $\lambda_t^i \equiv \lambda^i(\xi_t)$ for $t \geq 0$. The stock price of a given company is viewed as the expected sum of all future (stochastic) dividends and, given our modelling assumption, turns out to be a Markov-modulated diffusion with jumps. The short rate r_t is also assumed to be a deterministic function of ξ_t. Conditional on ξ, everything (e.g. default time of any two companies, or equity and credit risk) is independent, which in turn allows us to express many pricing formulae in analytic or semi-analytic form. How does the potential approach fit in this framework?

According to standard asset pricing theory, if we assume no arbitrage than there exists some equivalent martingale measure under which discounted asset prices are martingales. Equivalently, one can work under some *reference* probability and discount assets by a state price density ζ obtaining again a martingale. The potential approach offers us a recipe to choose the state price density in an efficient, flexible and consistent way. In particular, in this paper we shall assume that ζ is again some functional of ξ.

Note that one of the most appealing features of the approach presented in the paper is that correlation between assets and markets is modelled *endogenously* via the process ξ without having to resort to exogenous structures like in the copula approach. Moreover correlation is modeled in a dynamically consistent way. Another strength of the approach is that passing from single-name derivatives to multi-name derivatives requires *no change of the model!*

We are currently investigating the ability of the model to fit quotes of liquid instruments in multiple markets, in particular CDS and option prices.

2 Model setup

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ and assume there exists a pricing operator π_{tT} such that if Y_T is a $\mathcal{F}_T$ measurable random variable (such as the payoff of an option maturing at time T), then its price Y_t at time t is given by

$$Y_t = \pi_{tT}(Y_T). \tag{1}$$

Taking a simple axiomatic approach, Rogers [R3] proves that such an operator can be represented in terms of a positive supermartingale $(\zeta_t)_{t\geq 0}$, the so-called *state price density* process:

$$\pi_{tT}(Y_T) = \frac{E_t[\zeta_T Y_T]}{\zeta_t}, \qquad t \leq T. \tag{2}$$

Note that asset prices are martingales under the *reference* probability measure, when discounted by ζ. All we need to do in order to be able to price contingent claims then, is to specify a form for the state price density process ζ. Rogers [R1] gives various examples of pricing kernels based on Markov processes. In this paper, we will restrict our attention to a very simple but convenient form for ζ,

$$\zeta_t = \exp\left(-\int_0^t \alpha(\xi_u)du\right) f(\xi_t), \tag{3}$$

where $\xi \in \{1, \ldots, N\}$ is a continuous time, irreducible, N-state Markov chain, with infinitesimal generator Q. Note that the matrix Q fully characterizes the transition semigroup of the chain. In order for ζ to be a supermartingale we shall require that

$$Qf - \alpha f \leq 0, \tag{4}$$

where in this equation α denotes the point-wise multiplication operator, $(\alpha f)_i = \alpha_i f_i$.

Remarks (i) Since the process ξ_t can only take a finite number of values, any function of the chain $g(\xi)$ can be thought of as a N-dimensional vector whose i^{th}

component is given by $g_i \equiv g(\xi) \mid_{\xi=i}$. In this paper, we shall use the notation $g(\xi)$ or g_ξ to indicate component ξ of the vector and g without subscripts to denote the whole vector.

(ii) The functions $f(\cdot)$, $\alpha(\cdot)$, the infinitesimal generator Q and the other functions of ξ which shall use later in the paper will be specified by calibrating to market data. The vectors f, α, the matrix Q, etc should be seen as parameters of the problem and calibrated to volatility surfaces, CDS quotes and risk-free bonds. Depending on the availability of data it may be necessary to adjust the number of states of the chain and/or put some structure on the parameters.

(iii) The state-price density process ζ can be viewed as the product of the discount factor and the change-of-measure martingale which transforms from the reference measure to the pricing measure. In the simple modelling set-up which we adopt here, the riskless rate is a function of the chain, and is

$$r_t \;=\; \frac{(\alpha - Q)f(\xi_t)}{f(\xi_t)} \tag{5}$$

consult [R3] again.

(iv) We shall adopt the following notational convention: if A is a matrix and b is a vector, we will write $A - b$ to indicate $A - \text{diag}b$. Similarly, by $Ab(i)$ we mean $(Ab)_i$.

2.1 Stock price derivation

The goal of this section is to derive the price of a stock, when we allow the company to default. In our set up, the stock price today is given by the expected sum of all future dividends, up to default, appropriately discounted.

Assume, for the time being, that the market consists of a risky, defaultable asset (the *stock*) paying a continuous stochastic dividend δ which obeys

$$\frac{d\delta_t}{\delta_t} = \mu(\xi_t)dt + \sigma(\xi_t)dW_t. \tag{6}$$

Here, μ and σ are deterministic functions of ξ, and W is a one dimensional standard Brownian motion independent of the Markov chain ξ. Following the approach of Di Graziano and Rogers [DR3], we shall define the probability of survival of a given firm, conditional on the path of the chain up to time t as

$$q_t \equiv P\left(\tau > t \mid F_t^\xi\right) = \exp\left(-\int_0^t \lambda(\xi_u)du\right), \tag{7}$$

where τ is the default time and $F_t^\xi = \sigma(\xi_s : s \leq t)$. Although in this section we will be dealing with one defaultable stock only, it is straightforward to extend the analysis to incorporate multiple defaultable companies. We will come back to the more general case in Section 3.

The value of the stock can be viewed as the expected sum of all future dividends up until default appropriately discounted, more precisely

$$S_t = \zeta_t^{-1} E_t \left[\int_t^\tau \zeta_u \delta_u du \right]. \tag{8}$$

Using specification (3) for the state-price density, it is possible to derive explicitly the stock price S_t at time t in terms of the dividend process and a function of the Markov chain. Taking $t = 0$, with no loss of generality and defining the martingale

$$M_t \equiv \exp\left(-\frac{1}{2} \int_0^t \sigma^2(\xi_u) du + \int_0^t \sigma(\xi_u) dW_u \right), \tag{9}$$

and setting,

$$\bar{\alpha} \equiv \alpha + \lambda - \mu \tag{10}$$

we have,

$$\begin{aligned}
S_0 &= \zeta_0^{-1} E_0 \left[\int_0^\tau \zeta_t \delta_t dt \right] \\
&= \frac{\delta_0}{f(\xi_0)} E_0 \left[\int_0^\infty M_t 1_{\{\tau > t\}} \exp\left(\int_0^t (\mu - \alpha)(\xi_u) du \right) f(\xi_t) dt \right] \\
&= \frac{\delta_0}{f(\xi_0)} E_0 \left[\int_0^\infty E[M_t 1_{\{\tau > t\}} \mid F_t^\xi] \exp\left(\int_0^t (\mu - \alpha)(\xi_u) du \right) f(\xi_t) dt \right] \\
&= \frac{\delta_0}{f(\xi_0)} E_0 \left[\int_0^\infty \exp\left(- \int_0^t \bar{\alpha}(\xi_u) du \right) f(\xi_t) dt \right] \\
&= \frac{\delta_0}{f(\xi_0)} (\bar{\alpha} - Q)^{-1} f(\xi_0).
\end{aligned} \tag{11}$$

A proof of the last equality (11) can be found in appendix A. Setting

$$v(\xi) \equiv \frac{(\bar{\alpha} - Q)^{-1} f(\xi)}{f(\xi)}, \tag{12}$$

we shall have more generally that

$$S_t = \delta_t v(\xi_t). \tag{13}$$

Note that the stock price at any time depends on the current state of the chain ξ_t, which we assume to be unobservable. We will estimate the distribution of the current chain state from market prices.

Remarks. (i) Between jumps, the stock price evolves as a geometric Brownian motion with constant drift and volatility. As the chain jumps to state j, say, the stock price jumps to a new level and its dynamics will be characterized by a new drift $\mu(j)$ and volatility $\sigma(j)$ until the next jump.

(ii) The use of Markov-modulated log-Brownian dynamics is gaining in popularity; see, for example, [BE], [C], [DR1], [GZ], [JR], [YZ]. In our notation, the effect of this is to take $S_t \equiv \delta_t$, satisfying the dynamics

$$dS_t = S_t\big\{\, \sigma(\xi_t)dW_t + \mu(\xi_t)dt \,\big\}.$$

Though such a model lacks a clear equilibrium derivation, it may be preferred for some purposes.

The ultimate goal of the paper is to derive a model capable of pricing derivatives whose payoff depends on the behavior of the underlying stock as well as the time of default. A viable model thus has to be able to fit liquid instruments in the credit, volatility and bond market. In the following few sections we will derive analytic and semi-analytic formulas to price bonds, CDS and vanilla options (put and call), which will be used in the calibration.

2.2 Pricing bonds

The simple techniques introduced in the previous sections can be applied in a straightforward manner to price bonds. As an example, consider a security which pays 1 at maturity if default has not occurred and zero otherwise. For notational convenience set,

$$\tilde{\alpha} \equiv \alpha + \lambda, \tag{14}$$

then the price of the zero recovery bond is given by

$$\begin{aligned}
\bar{P}(0,T) &\equiv \frac{1}{\zeta_0} E_0\left[1_{\{\tau > T\}}\zeta_T\right] \\
&= \frac{1}{f(\xi_0)} E_0\left[\exp\left(-\int_0^T \tilde{\alpha}(\xi_u)du\right) f(\xi_T)\right] \\
&= \frac{1}{f(\xi_0)} \exp\big(T(Q-\tilde{\alpha})\big) f(\xi_0) \qquad (15)
\end{aligned}$$

The price of a non defaultable bond $P(0,T)$ can be obtained from (15) simply by setting $\lambda(\xi) = 0$ for all ξ.

2.3 CDS pricing

The most traded (vanilla) instruments in the default market are credit default swaps (CDS). In a standard CDS contract, the protection seller agrees to pay par minus recovery in case of default of the reference entity (default leg). In order to be compensated for his default exposure, the protection seller periodically receives a constant spread s until maturity or default, whichever happens first (premium leg). The initial spread s is chosen in such a way that the two legs of the swap have equal value. In the simple framework introduced in the previous sections, it is possible to derive the CDS spread explicitly.

Assume for simplicity that the recovery rate R at default is independent of ξ. The default leg of the CDS maturing at T is then equal to

$$\begin{aligned}\frac{DL_T}{1-E[R]} &\equiv \frac{1}{\zeta_0}E_0\left[\zeta_\tau 1_{\{\tau\le T\}}\right]\\ &= \frac{1}{f(\xi_0)}E_0\left[\int_0^T \exp\left(-\int_0^t \tilde{\alpha}(\xi_u)du\right)\lambda(\xi_t)f(\xi_t)dt\right]\\ &= \frac{1}{f(\xi_0)}\tilde{Q}^{-1}(e^{\tilde{Q}T}-I)\bar{f}(\xi_0),\end{aligned}$$

where $\tilde{Q}\equiv Q-\tilde{\alpha}$, $\bar{f}=f\lambda$.

The premium leg can be calculated similarly. As a first approximation, assume that no accrued is paid at default. Let $0=T_1<\ldots<T_n=T$ be the payment dates of the CDS and call Δ_j the daycount fraction for the interval $(T_{j-1},T_j]$. The ex-accrued premium leg (PL) is then simply,

$$\begin{aligned}PL_T &\equiv \frac{s}{\zeta_0}E\left[\sum_{j=1}^n \Delta_j 1_{\{\tau>T_j\}}\zeta_{T_j}\right]\\ &= s\sum_{j=1}^n \Delta_j\bar{P}(0,T_j)\end{aligned}$$

where $\bar{P}(0,t)$ is the value of a zero recovery risky bond as given by (15).

If the reference entity defaults between two payment dates, the protection buyer has to pay his counterpart the amount of the protection fee matured up the default date, that is, the accrued spread. Let D be the length of the reference period defined in the day-count convention (e.g. 360 or 365 days). The value of the accrued leg is equal to

$$
\begin{aligned}
A_T &\equiv \frac{s}{\zeta_0 D} E\left[\sum_{j=1}^{n} 1_{\{T_{j-1}<\tau\le T_j\}}(\tau - T_{j-1})\zeta_\tau\right] \\
&= \frac{s}{f(\xi_0)D}\sum_{j=1}^{n} E\left[\int_{T_{j-1}}^{T_j}(u-T_{j-1})\exp\left(-\int_0^u \tilde{\alpha}(\xi_s)ds\right)\bar{f}(\xi_u)du\right] \\
&= \frac{s}{f(\xi_0)D}\left\{\tilde{Q}^{-2}(I-\exp(T\tilde{Q}))\bar{f}(\xi_0) \;+\; \sum_{j=1}^{n}(T_j - T_{j-1})\tilde{Q}^{-1}e^{\tilde{Q}T_j}\bar{f}(\xi_0)\right\}.
\end{aligned}
$$

The initial spread can be recovered as usual as the ratio between the default leg and the premium leg. The simple results of this section can be used to calibrate the parameters of the model to quoted CDS prices.

2.4 Hybrid option pricing

In this section we shall show how to use integral transforms to price vanilla options involving equity and credit. Note that in our set-up, the Markov chain connects the equity and the credit market; in particular, ξ determines the interaction between equity and credit. Our framework can be easily extended to incorporate the interest-rate and FX markets as well.

As a first example we shall show how to price a simple hybrid put option which pays $(K - S_T)^+$ if the company does not default before the maturity T and zero otherwise. We shall derive the Laplace transform of the price explicitly and then invert it numerically using the Hosono-Abate-Whitt method [Ho], [AW].

The price of a no-default put option with maturity T is given by

$$P_T(k) \equiv \frac{1}{\zeta_0} E\left[\zeta_T\left(e^k - e^s\right)^+; \tau \ge T\right], \tag{16}$$

where s and k are the log stock and the log strike respectively. The following calculations are very similar to the ones appearing in [DR1] to which we refer to for more details. Given a complex number η such that $\mathrm{Re}(\eta) > 1$ we define the Laplace transform of the put price as

$$\widehat{P}_T(\eta) \equiv \int_{-\infty}^{\infty} e^{-\eta k} P_T(k)dk. \tag{17}$$

In order to ease notation set

$$z_\eta \equiv (1-\eta)(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}(1-\eta)^2\sigma^2 - \alpha - \lambda, \tag{18}$$

and

$$M_t^\eta \equiv \exp\left((1-\eta)\int_0^t (\mu - \frac{1}{2}\sigma^2)(\xi_s)ds + (1-\eta)\int_0^t \sigma(\xi_s)dW_s \right). \qquad (19)$$

Applying Fubini's theorem to (17), and performing some standard calculation, it follows that

$$\begin{aligned}
\widehat{P}_T(\eta) &= \frac{1}{\zeta_0}\int_{-\infty}^{\infty} e^{-\eta k} E\left[\zeta_T \left(e^k - e^s\right)^+ ; \tau \geq T\right] dk \\
&= \frac{1}{\eta(\eta-1)\zeta_0} E\left[\zeta_T \exp(-(\eta-1)s); \tau \geq T\right] \\
&= \frac{1}{\eta(\eta-1)f(\xi_0)} E\left[\exp\left(-\int_0^T \alpha(\xi_u)du\right) f(\xi_T) S_T^{1-\eta}; \tau \geq T\right] \\
&= \frac{\delta_0^{1-\eta}}{\eta(\eta-1)f(\xi_0)} E\left[\exp\left(-\int_0^T (\alpha+\lambda)(\xi_u)du\right) M_T^\eta f(\xi_T) v(\xi_T)^{1-\eta}\right] \\
&= \frac{\delta_0^{1-\eta}}{\eta(\eta-1)f(\xi_0)} E\left[\exp\left(\int_0^T z_\eta(\xi_u)du\right) f(\xi_T) v(\xi_T)^{1-\eta}\right] \\
&= \frac{\delta_0^{1-\eta}}{\eta(\eta-1)f(\xi_0)} \left(e^{(Q+z_\eta)T}\hat{f}\right)(\xi_0)
\end{aligned}$$

where $\hat{f}(\xi) \equiv f(\xi)v(\xi)^{1-\eta}$ for all $\xi \in \{1, \ldots, N\}$.

Remark: The transform of the classical vanilla put can be recovered from the previous calculation simply by setting the vector $\lambda = 0$.

Defaultable vanilla options with barriers can be also priced in our framework thanks to the techniques developed by Di Graziano and Rogers [DR2].

So far we have only dealt with single name options. Extending the model to handle options involving more than one reference entity is straightforward. The following section gives some basic examples of hybrid options involving stocks and portfolios of defaultable entities. Some of the results are related to the model of Di Graziano and Rogers [DR3], which however did not involve any equity component.

3 Hybrid multiname structures

Consider a market consisting of M defaultable securities, say CDS, and let $\{S_t^i\}_{t\geq 0}$ and $\{\lambda^i(\xi_t)\}_{t\geq 0}$ be the the stock price process and the default intensity process of company i respectively. Let l_i be the (possibly random) individual losses at default and $L_t \equiv \sum_{i=1}^{N} l_i 1_{\{\tau^i \leq t\}}$ be the cumulative losses of the CDS portfolio at time t. Note that conditional on the filtration generated by the chain $\mathcal{F}_t^\xi$, defaults are independent.

As an example, suppose we want to price a European call on asset i which gets knocked out if the total losses of some the basket exceeds some threshold; the final payoff is

$$C_T \equiv (S_T^i - K)^+ 1_{\{L_T \leq x\}}. \tag{20}$$

The contingent claim considered above is a call option on stock i, which pays nothing if losses in the credit portfolio exceeds the threshold x. The price at $t = 0$ of such a claim will be given by

$$C_0 = \frac{1}{\zeta_0} E\left[\zeta_T (S_T^i - K)^+ 1_{\{L_T \leq x\}}\right], \tag{21}$$

which requires the knowledge of the joint distribution of S_T^i and L_T. The key thing to notice here is that conditional on the chain, S and L are independent. Using standard techniques it is easy to prove that,

$$C_T^\xi \equiv E\left[(S_T^i - K)^+ \mid F_T^\xi\right] = A\bar{\Phi}(a - \Sigma_T) - K\bar{\Phi}(a), \tag{22}$$

where $\bar{\Phi}$ is the tail of the standard $N(0,1)$ law, $A \equiv \delta_i(0) v_i(\xi_T) \exp(\int_0^T \mu_i(\xi_s)\, ds)$, $\Sigma_T^2 \equiv \int_0^T \sigma_i(\xi_s)^2\, ds$ and

$$a = \frac{\log(K/A) + \frac{1}{2}\Sigma_T^2}{\Sigma_T}$$

The joint density can be derived by inverting the Laplace transform of the *modified* loss process,

$$\begin{aligned}
\hat{C}_T(\theta) &\equiv \frac{1}{\zeta_0} E\left[\zeta_T (S_T^i - K)^+ e^{-\theta L_T}\right] && (23)\\
&= \frac{1}{\zeta_0} E\left[\zeta_T E\left[(S_T^i - K)^+ \mid F_T^\xi\right] E\left[e^{-\theta L_T} \mid F_T^\xi\right]\right] && (24)\\
&= \frac{1}{f(\xi_0)} E\left[\exp\left(-\int_0^t \alpha(\xi_s) ds\right) f(\xi_T) C_T^\xi \Pi_T^\xi(\theta)\right], && (25)
\end{aligned}$$

where $\Pi_T^\xi(\theta)$ is the Laplace transform of the cumulative loss L_T, conditional on F_T^ξ,

$$\Pi_T^\xi(\theta) \equiv \prod_{j=1}^{N} \left(q_t^j + (1 - q_t^j) E\left[e^{-\ell_i \theta}\right]\right) \tag{26}$$

and q_t^j is the conditional survival probability for company j,

$$q_t^j \equiv \exp\left(-\int_0^t \lambda^j(\xi_u)du\right). \tag{27}$$

Note that the expectation on the right most hand-side of (25) only involves functionals of the chain. The easiest way to calculate $\hat{C}_T(\theta)$ is thus to simulate the path of the chain up to time T. This can be done efficiently using the algorithm presented in appendix B. Once the modified Laplace transform of the loss process has been computed, we can used the Hosono-Abate-Whitt inversion method to recover the desired density.

4 Conclusions

We presented a framework for the pricing of hybrid derivatives, in particular we considered contingent claims whose pay-off depends on default, equity and interest rate risk. By modelling the common dynamics of different company/markets via a continuous time-finite state Markov chain ξ we obtain tractable solutions for a range of hybrid structures. The modelling approach is also dynamically consistent and allows for enough flexibility to calibrate (at least in theory) to market instruments. We are currently investigating how well the model does in simultaneously fitting CDS prices, volatility surfaces and bond prices. The foreign exchange market has not been considered explicitly in the paper, but the potential approach allows us to extend the analysis to incorporate FX risk in a natural fashion (see Rogers [R1] for some example).

A Exponential functionals of continuous time Markov Chains

This section contains some basic results about continuous time Markov chains. This material is well known in some circles, but perhaps not amongst practitioners. The reader is referred to Rogers and Williams [RW] and Bielecki and Rutkowski [BR] for a more detailed and rigorous treatment.

In order to calculate individual survival probabilities, stock and other security prices, we had to evaluate expressions of the form

$$g(0,\xi_0) \equiv E\left[\exp\left(\int_0^T \phi(\xi_u)du\right)\eta(\xi_T)\right], \tag{28}$$

where ξ is a continuous-time, irreducible, N-state Markov-chain with infinitesimal generator Q and $\phi(\cdot)$ is a deterministic function of ξ. Since the chain takes values in the set $\{1, 2, \ldots, N\}$, we can view g as a N-dimensional column vector whose i^{th} component is given by $g_i \equiv g(\xi) \mid_{\xi=i}$. Define

$$g(t, \xi_t) \equiv E_t \left[\exp\left(\int_t^T \phi(\xi_s) ds \right) \eta(\xi_T) \right]. \tag{29}$$

By Dynkin's formula, the process

$$M_t \equiv g(t, \xi_t) - g(0, \xi_0) - \int_0^t (\frac{\partial g}{\partial t} + Qg)(u, \xi_u) du \tag{30}$$

is a martingale (and not merely a local martingale). Of course,

$$(Qg)_i = \sum_{j=1}^{N} Q_{ij} g(t, j). \tag{31}$$

By applying Itô's formula to the *martingale*

$$Y_t \equiv g(t, \xi_t) \exp\left(\int_0^t \phi(\xi_u) du \right) = E \left[\exp\left(\int_0^T \phi(\xi_u) du \right) \eta(\xi_T) \,\middle|\, \mathcal{F}_t \right]$$

we obtain

$$dY_t \stackrel{\circ}{=} \exp\left(\int_0^t \phi(\xi_s) ds \right) \left[\frac{\partial g\,(t, \xi_t)}{\partial t} + (Qg)\,(t, \xi_t) + \phi(\xi_t) g(t, \xi_t) \right] dt, \tag{32}$$

where the symbol $\stackrel{\circ}{=}$ indicates that the two sides of (32) differ by a martingale. This implies that

$$\frac{dg}{dt} + Qg + \phi g = 0, \tag{33}$$

where $w(t)$ is a vector function whose i^{th} component is $w(t, i)$. The system (33) of ordinary differential equations in t, with boundary $w(T, \xi) = \eta(\xi)$, has solution

$$g(t, \xi) = \exp((T - t)(Q + \phi)\,)\eta(\xi).$$

B Monte Carlo simulation for continuous time Markov-chains

One of the great advantages of using Markov-chains as opposed to other continuous time Markov processes is that the simulation of their sample path is easy, accurate and fast. In this section, we review some standard techniques used to simulate continuous time, discrete space Markov-chains.

The following algorithm assumes the knowledge of the Q-matrix Q (in what follows we will let $Q_{ij} \equiv q_{ij}$) and the initial state of the chain, say $\xi \in \{1, \dots, M\}$, where is the number of states of the chain. Let $\xi_0 = i$ and note that $q_i = -q_{ii}$ is the is the rate at which the chain jumps out of state i.

1. Let i be the current state of the chain. Generate an exponential(1) random variable z,
2. let τ denote the time elapsed from the last jump: set $\tau = z/q_i$,
3. if $\tau \geq T$, stop otherwise go to step 4,.
4. sample $\xi(\tau)$ according to probabilities (q_{ij}/q_i), where $j \neq i$ and $j \leq M$,
5. go to step 1, and set $i = \xi(\tau)$.

Remark. Note that a new simulation of the path of the chain is only needed if we change its transition matrix Q. In order words, we can simply simulate a number of paths once and then re-use the same paths to price a variety of contingent claims. Even changing the functions μ, σ, λ^i etc, does not require a new simulation, with obvious computational advantages.

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