The correlation of the maxima of correlated Brownian motions

L. C. G. Rogers and Larry Shepp
University of Cambridge and Rutgers University

February 22, 2006

Abstract. We obtain an expression for the correlation of the maxima of two correlated Brownian motions.

Keywords: Brownian motion, correlation, Kantorovich-Lebedev transform.

AMS 2000 Subject Classifications: 60J65.

1 Introduction

Finding the expectation (or the law) of a functional of a Brownian path is usually either quite straightforward, or quite impossible[1], and it is usually not too hard to guess into which category a particular question falls. However, the question which we deal with in this short note is innocent to state, surprisingly tricky to deal with, and falls somewhere between the two types of problem. The question arose naturally in an application to the estimation of the correlation between two stocks (see (Rogers and Zhou, 2006) for a full account), and can be simply stated as follows. Let $(W_i(t))_{t\geq 0}$, $i = 1, 2$, be two standard Brownian motions with constant correlation ρ:

$$E[W_1(s)W_2(t)] = \rho \min\{s, t\} \quad \forall s, t \geq 0$$

and let $S_i(t) \equiv \sup_{s\leq t} W_i(s)$; what is

$$E[\ S_1(t)S_2(t)\ ]\ ? \tag{1}$$

[1](Borodin and Salminen, 2002) is an encyclopedia of results of the first kind.

Brownian scaling tells us that there must be some positive function $c : [-1,1] \to (0,\infty)$ such that

$$E[\ S_1(t)S_2(t)\ ] = c(\rho)t,$$

so all we have to do is to find c. The values $c(1) = 1$, $c(0) = 2/\pi = 0.6366198$, and $c(-1) = 2\log(2) - 1 = 0.3862994$ are known (for the last, see, for example, (Garman and Klass, 1980)); they reduce to calculations for a single Brownian motion. The three values are not of course collinear, so the functional form of c is not obviously trivial, but the departure from collinearity is not great:

$$\frac{1}{2}\{\ c(-1) + c(1)\ \} - c(0) = 0.0565274.$$

In this note, we shall derive the explicit form

$$c(\rho) = \cos\alpha \int_0^\infty d\nu\ \frac{\cosh\nu\alpha}{\sinh\nu\pi/2}\ \tanh\nu\gamma \tag{2}$$

for the function c, where $\rho = \sin\alpha$, $\alpha \in (-\pi/2, \pi/2)$, and $2\gamma = \alpha + \pi/2$.

2 Calculating c.

We begin with some notation. We write

$$\mathcal{G} \equiv \frac{1}{2}\{\ D_1^2 + 2\rho D_1 D_2 + D_2^2\ \}$$

for the infinitesimal generator of W, where $D_i \equiv \partial/\partial x_i$. We shall write

$$X_i(t) \equiv S_i(t) - W_i(t), \quad i = 1, 2,$$

for the process of the heights below the maxima, which is a correlated two-dimensional Brownian motion in R_+^2 with normal reflection on the axes. We shall write T for an exponential(λ) variable independent of W, and we set $\theta \equiv \sqrt{2\lambda}$.

We break the calculation into a sequence of goals, each a consequence of the next, until we finally arrive at a goal we can attain.

GOAL 1: *Calculate*

$$\begin{aligned} f(x_1,x_2) &\equiv P\big[\,x_1 \le S_1(T), x_2 \le S_2(T)\,\big] \\ &= \int_0^\infty \lambda e^{-\lambda t} P[x_1 \le S_1(t), x_2 \le S_2(t)]dt. \end{aligned}$$

This is as good as solving the problem, because then we shall obtain

$$\lambda^{-1} c(\rho) = \int_0^\infty \int_0^\infty f(x_1, x_2)\, dx_1\, dx_2.$$

To achieve Goal 1, we aim for:

GOAL 2: *Calculate*

$$\begin{aligned} \tilde{f}(x_1, x_2) &\equiv P\big[\, S_1(T) \le x_1, S_2(T) \le x_2 \,\big] \\ &= P[\, \tau > T \mid X_1(0) = x_1, X_2(0) = x_2 \,], \end{aligned}$$

where $\tau = \inf\{t : X_1(t)X_2(t) = 0\}$. This will give us Goal 1, because

$$\begin{aligned} 1 - f(x_1, x_2) &= P[S_1(T) \le x_1] + P[S_2(T) \le x_2] - \tilde{f}(x_1, x_2) \\ &= 1 - e^{-\theta x_1} + 1 - e^{-\theta x_2} - \tilde{f}(x_1, x_2). \end{aligned} \quad (3)$$

Now

$$\begin{aligned} \hat{f} \equiv 1 - \tilde{f}(x_1, x_2) &= P[\, \tau < T \mid X_1(0) = x_1, X_2(0) = x_2 \,] \\ &= E[\, e^{-\lambda\tau} \mid X_1(0) = x_1, X_2(0) = x_2 \,] \end{aligned}$$

will clearly satisfy

$$(\lambda - \mathcal{G})\hat{f} = 0$$

with boundary conditions $\hat{f} = 1$ on the axes. Using this, we see from (3) that

$$f(x_1, x_2) = e^{-\theta x_1} + e^{-\theta x_2} - \hat{f}(x_1, x_2)$$

must solve

$$(\lambda - \mathcal{G})f = 0 \quad (4)$$

with boundary conditions

$$f(x_1, 0) = e^{-\theta x_1}, \quad (5)$$

$$f(0, x_2) = e^{-\theta x_2}. \quad (6)$$

Our next goal therefore is:

GOAL 3: *Solve the PDE (4), (5), (6).* For this, we transform the state variables:

$$X_t \equiv X_1(t) \sec\alpha - X_2(t) \tan\alpha, \quad Y_t = X_2(t),$$

where $\rho = \sin\alpha$. As is easily confirmed, the process $Z_t \equiv X_t + iY_t$ is now a complex Brownian motion in the wedge

$$\Omega_\rho \equiv \{re^{i\varphi} : r \ge 0,\ 0 \le \varphi \le 2\gamma\},$$

where we write

$$2\gamma = \alpha + \frac{\pi}{2}.$$

The Brownian motion Z experiences skew reflection on the boundary of Ω_ρ, in the direction $(-\sin\alpha, \cos\alpha)$ on R_+ and in direction $(1, 0)$ on the other side of the wedge.

Remark. Brownian motion in the wedge with skew reflection was studied by (Varadhan and Williams, 1985), who gave criteria for the corner of the wedge to be visited, and for there to be possible escape from the corner; see also (Rogers, 1989) for a brisk summary of the results. The criterion of Varadhan & Williams leads to the (initially surprising) conclusion that if $\rho > 0$ then the corner of the wedge *will* be visited; in terms of W, this says that there will be times t such that

$$W_1(t) = S_1(t) \quad \textit{and} \quad W_2(t) = S_2(t),$$

a property that would certainly not be satisfied if the Brownian motions were independent.

Writing $h(x + iy) = f(x_1, x_2)$, we therefore have that h satisfies the PDE

$$(\lambda - \frac{1}{2}\Delta)h = 0$$

with the boundary condition that

$$h(re^{i\varphi}) = \exp(-\theta r \cos\alpha) \tag{7}$$

for $\varphi = 0, 2\gamma$. Writing the Laplacian in polar coordinates, we obtain the PDE for h:

$$(\theta^2 - D_{rr} - \frac{1}{r}D_r - \frac{1}{r^2}D_{\varphi\varphi})h = 0 \tag{8}$$

where (for example) $D_{rr} \equiv \partial^2/\partial r^2$. Now the PDE (8) has separable solutions of the form

$$h_\nu(re^{i\varphi}) = K_{i\nu}(\theta r)\cosh(\nu(\varphi - \gamma))$$

for $\nu > 0$, in terms of the usual Bessel functions K_β, and the key is to combine these using Kantorovich-Lebedev transforms, a technique we learned from Henry McKean. We claim that the integral combination

$$h(re^{i\varphi}) = \frac{2}{\pi}\int_0^\infty \cosh(\nu\alpha)\,\frac{\cosh\nu(\varphi - \gamma)}{\cosh\nu\gamma}\,K_{i\nu}(\theta r)\,d\nu \tag{9}$$

is the solution to the PDE (8) with the required boundary conditions (7). The fact that this solves the PDE follows from that fact that it is a linear combination of separable solutions, and to confirm the boundary behaviour, we quote the identity

$$\frac{2}{\pi}\int_0^\infty K_{i\nu}(r)\cosh\nu\alpha\,d\nu = \exp(-r\cos\alpha) \tag{10}$$

for $0 \leq \alpha \leq \pi/2$; see (Oberhettinger, 1972) page 244.

The expression (9) achieves Goal 3, hence Goal 2 and finally Goal 1. To obtain the constant $c(\rho)$ we just have to integrate the solution h over the domain Ω_ρ, not forgetting the (constant) Jacobian: we find that

$$\begin{aligned}
\lambda^{-1} c(\rho) &= \cos\alpha \int_0^\infty r\,dr \int_0^{2\gamma} d\varphi \int_0^\infty d\nu\, \frac{2}{\pi} \cosh(\nu\alpha)\, \frac{\cosh\nu(\varphi-\gamma)}{\cosh\nu\gamma}\, K_{i\nu}(\theta r) \\
&= \frac{\cos\alpha}{\theta^2} \int_0^{2\gamma} d\varphi \int_0^\infty d\nu\, \frac{\cosh\nu\alpha}{\sinh\nu\pi/2}\, \frac{\cosh\nu(\varphi-\gamma)}{\cosh\nu\gamma} \\
&= \frac{\cos\alpha}{\theta^2} \int_0^\infty d\nu\, \frac{\cosh\nu\alpha}{\sinh\nu\pi/2}\, 2\tanh\nu\gamma.
\end{aligned}$$

Finally, we have explicitly

$$c(\rho) = \cos\alpha \int_0^\infty d\nu\, \frac{\cosh\nu\alpha}{\sinh\nu\pi/2}\, \tanh\nu\gamma,$$

as stated at (2).

References

Borodin, A. N. and Salminen, P. (2002). *Handbook of Brownian Motion*. Birkhauser.

Garman, M. B. and Klass, M. J. (1980). On the estimation of security price volatilities from historical data. *Journal of Business*, 53:67–78.

Oberhettinger, F. (1972). *Tables of Bessel Transforms*. Springer.

Rogers, L. C. G. (1989). A guided tour through excursions. *Bulletin of the London Mathematical Society*, 21:305–341.

Rogers, L. C. G. and Zhou, F. (2006). Estimating correlation from high, low, opening and closing prices. Technical report, Statistical Laboratory, University of Cambridge.

Varadhan, S. R. S. and Williams, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Communications in Pure and Applied Mathematics*, 38:405–443.