Fitting Potential Models to Interest Rate and Foreign Exchange Data

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1 Introduction

Most term structure models to date fall into one of two classes: either one models the spot rate process $(r_t, t \geq 0)$, as in Vasicek [Vas77], Cox, Ingersoll and Ross [CIR85], Longstaff and Schwartz [LS91]; or alternatively one models the process of forward rates, as in Ho and Lee [HL86], Heath, Jarrow and Morton [HJM92], Babbs [Bab90]. However, as a recent paper of Rogers [Rog95] shows, there are considerable advantages to a third approach, called the *potential approach*. The idea here is that the fundamental is the state-price density $(\zeta_t, t \geq 0)$, a positive supermartingale, in terms of which the bond prices have the simple expression

$$P(t,T) = \tilde{E}_t(\zeta_T)/\zeta_t, \tag{1.1}$$

where $0 \leq t \leq T$, and $P(t,T)$ denotes the price at time t of a zero-coupon bond which pays out 1 at time T, and $\tilde{P}$ is a reference measure. This approach was also advocated by Constantinides [Con92], though this paper did not develop one of the most exciting consequences, namely the simplicity with which exchange rates can be modelled. As Rogers [Rog96] shows, if at time t one unit of country j's currency is worth Y_t^{ij} units of country i's currency, then under certain assumptions (satisfied in the complete markets case)

$$Y_t^{ij} = Y_0^{ij} \zeta_t^j / \zeta_t^i; \tag{1.2}$$

this important observation was also made by Saa-Requejo [SR93]. To obtain a wide family of models, then, one needs to have a way of generating positive supermartingales, and [Rog96] showed how one could make use of classical Markov process theory to generate such examples. By taking a Markov process $(X_t, t \geq 0)$ with resolvent $(R_\lambda)_{\lambda > 0}$, the recipe

$$\zeta_t \equiv e^{-\alpha t} R_\alpha g(X_t) \tag{1.3}$$

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defines a positive supermartingale whenever the function g is positive and suitably integrable. Different choices of g and α give a wide range of possible potentials, even within the context of a fixed Markov process. Using this framework, the bond prices can be compactly expressed as

$$P(0,t) = \tilde{E}^x(e^{-\alpha t}R_\alpha g(X_t))/R_\alpha g(X_0), \tag{1.4}$$

where $\tilde{P}^x$ denotes the law of the process started at x. Further simple formulae for other derivative prices can be obtained. One even has that the spot rate process can be given as

$$r_x = \frac{g(X_t)}{R_\alpha g(X_t)}. \tag{1.5}$$

The objective of this paper is to examine the fit of a few such models to data. Throughout, we shall take the underlying Markov process to be a two-dimensional (Gaussian) diffusion X solving

$$dX_t = dW_t - BX_t dt, \tag{1.6}$$

where B is a 2×2 matrix, and W is a Brownian motion in two dimensions. It should be noted that what we are attempting here, namely to explain *simultaneously* the yield curves in two countries and the exchange rates between them, using a model with *only two underlying sources of noise*, is ambitious; other models would introduce at least one source of noise for each country and one for each exchange rate.

The data we used to fit the model was daily yield curve data for USD and GBP, along with daily data for the exchange rate between the two currencies. We give more details on the data in Section 2.

The reason to restrict attention to this particular Markov process is that the bond price (and the prices of other derivatives more generally) is expressed as an expectation of a function of the process X at some later time, and such expectations are comparatively simple in this case. Indeed, we have that under $\tilde{P}^x$

$$X_t \sim N(e^{-tB}x, V_t), \quad V_t \equiv \int_0^t e^{-sB}(e^{-sB})^T ds, \tag{1.7}$$

as is easily shown. Thus the calculation of prices can commonly be reduced to an integration with respect to a Gaussian density; often, the bond prices can be given in closed form, as examples in [Rog95] demonstrate. Nevertheless, it is important to emphasise that the method is not restricted to this one diffusion, nor even to *any* diffusion; it may well turn out in practice that we should take as the underlying process a finite Markov chain, in which case the computations would inevitably be numerical.

Having chosen the underlying Markov process, we now have to pick the function $g \geq 0$, or, more conveniently, $f \equiv R_\alpha g$. Familiar properties of the resolvent then allow us to recover g from f by $g = (\alpha - G)f$, and [Rog95] gave numerous examples. In this paper, we shall fit the models where for some 2-vector c, some 2×2 positive-definite symmetric matrix Q and real positive γ chosen suitably, f takes one of the forms:

$$(A) \qquad f(x) = \exp(\frac{1}{2}(x-c) \cdot Q(x-c));$$

$$(B) \qquad f(x) = \gamma + \frac{1}{2}(x-c) \cdot Q(x-c).$$

Before getting into the details of these models, which we shall attend to in Section 3, we describe the general methodology used in the fitting procedure. In all cases, we are dealing with a family of models parametrised by some vector θ. Some of the components of θ will be parameters relating to the movement of the underlying Markov process X (so in our example, the entries of the matrix B, or, more usefully, the eigenvalues and unit eigenvectors of B), the remainder will be components relating to the functions used in the potential description. On each day, we will have the market values of some K observables; let us denote the market value of the jth observable on day n by y_n^j, and let us denote the model price of the jth asset by $y^j(x, \theta)$, a function of the state x of the Markov process, and the parameter vector θ. If we simply want to fit day-by-day, allowing different values of the parameters each day, then on day n we can minimise $F_n(\theta, X)$ defined by

$$F_n(\theta, X) \equiv \frac{1}{2} \sum_{j=1}^{K} \varepsilon_j (y_n^j - y^j(x, \theta))^2,$$

where the ε_j are positive weights which are at our disposal. The observables y_n^j do not have to be prices; they could be implied volatilities, log-prices, historical volatilities, or any other observable whose value can be computed within the model, and which we care to use for the calibration. The results of the day-by-day fits to the data are discussed in Section 4.

Allowing the parameters to change each day is a violation of the model assumptions; we should insist that they are the same for all time. Nevertheless, it is not reasonable to imagine that the parameters in a model for interest rates remain absolutely unchanged over very long periods, so for a more sophisticated analysis we will allow the parameters to shift gradually with time; as is to be expected, the more we force the parameters to remain stable, the poorer the fit to the data. If we abbreviate $(x^T, \theta^T) \equiv z^T$, then what we shall do is to minimise each day the function $\tilde{F}_n(z)$ defined by

$$\tilde{F}_n(z) \equiv \frac{1}{2} \sum_{j=1}^{K} \varepsilon_j (y_n^j - y^j(z))^2 + \frac{1}{2} (z - \tilde{\mu}_n)^T \tilde{V}_n^{-1} (z - \tilde{\mu}_n),$$

where the vectors $\tilde{\mu}_n$ and the matrices $\tilde{V}_n$ are defined recursively, as is explained in detail in Section 5. Section 6 discusses the results of this fitting procedure.

2 The Data

The US yield curve data (figure **??**) and the GBP/USD exchange rate(figure **??**) were obtained from the World Wide Web site of the Federal Reserve Bank of Chicago. [2]. The data that has been used covers the period January 1991 to November 1991. For each day we have 9 values corresponding to different maturities for the bonds. The maturities are 3, 6 months, 1, 2, 3, 5, 7, 10, 30 years. The UK yield curve data (figure **??**) was obtained from S. Babbs and it covers the same period of time and has the same maturities.[3]
Let us define some notation for the data that will be used later on. Let M=[.25 .5 1 2 3 5 7 10 30] be the vector that represents the maturities of the bonds under consideration; let y_{ni}^{US} (resp. y_{ni}^{UK}) be the value of the yield on the n-th day for the US (resp. UK) bond with maturity $M(i)$, and y_n^{FX} the logarithm of the foreign exchange rate on the n-th day divided by the foreign exchange rate on the $(n-1)$-th day.

3 The Models

3.1 Quadratic

Let us consider the case (see [Rog95]) in which the function $f : R^d \to R$ is given by

$$f(x) = \gamma + \frac{1}{2}(x - c)^T Q(x - c) \tag{3.8}$$

and the d-dimensional diffusion $(X_t, t \geq 0)$ is the solution of the linear stochastic differential equation

$$dX_t = -B\, X_t\, dt \,+\, dW_t \qquad X_0 = x_0 \tag{3.9}$$

[2]http://gopher.great-lakes.net:2200/1/partners/ChicagoFed/finance
[3]For simplicity the dates not in common have been eliminated from both countries

where Q and B are constant $d \times d$ matrices with Q symmetric and positive definite, and c is a constant d-dimensional vector.
If we denote by G the infinitesimal generator of the diffusion, the function $g = (\alpha - G)f$ is then given by

$$g(x) = \alpha\gamma - \frac{1}{2}\mathrm{tr}\, Q + \frac{1}{2}\alpha c^T Qc + \frac{1}{2}(x-v)^T S(x-v) - \frac{1}{2}v^T Sv \tag{3.10}$$

where

$$S = \alpha Q + B^T Q + QB \tag{3.11}$$

and

$$v = S^{-1}(\alpha Qc + B^T Qc) \tag{3.12}$$

If we choose γ so that

$$\gamma = \frac{\mathrm{tr}\, Q + v^T Sv}{2\alpha} - \frac{1}{2}c^T Qc \tag{3.13}$$

then the spot rate process is given by

$$r(X_t) = \frac{g(X_t)}{f(X_t)} = \frac{(X_t - v)^T S(X_t - v)}{2\gamma + (X_t - c)^T Q(X_t - c)} \quad , \tag{3.14}$$

the zero-coupon bond prices are given by

$$P(0,t) = \frac{\exp\{-\alpha\, t\}}{f(x_0)}(\gamma + \frac{1}{2}(\mathrm{tr}(Q\, V_t) + \mu_t^T Q\mu_t)) \tag{3.15}$$

where

$$\mu_t = \exp\{-tB\}\, x_0 - c, \tag{3.16}$$

$$V_t = \int_0^t \exp\{-sB\}(\exp\{-sB\})^T\, ds, \tag{3.17}$$

and the state price density is given by

$$\zeta_{0,t} = \exp\{-\alpha\, t\}\frac{\gamma + \frac{1}{2}(X_t - c)^T Q(X_t - c)}{\gamma + \frac{1}{2}(X_0 - c)^T Q(X_0 - c)}. \tag{3.18}$$

3.2 Exponential quadratic

Let us consider a different model (see example 2 in [Rog95]); the function $f : R^d \to R$ is defined by

$$f(x) = \exp(\frac{1}{2}(x-c)^T Q(x-c)) \tag{3.19}$$

and the diffusion $(X_t, t \geq 0)$ is the solution of the linear stochastic differential equation 3.9.
In this case the function $g = (\alpha - G)\, f$ is given by

$$g(x) = f(x)\,(\frac{1}{2}(x - S^{-1}v)^T S(x - S^{-1}v)) \tag{3.20}$$

where

$$S = B^T Q + QB - Q^2, \quad v = (B^T - Q)Qc \tag{3.21}$$

and the parameter α has been chosen to be

$$\alpha = \frac{1}{2}(\mathrm{tr}(Q) + |Qc|^2 + v^T S^{-1} v) \tag{3.22}$$

These choices give a squared-Gaussian spot rate process

$$r_t = \frac{1}{2}(X_t - S^{-1}v)^T S(X_t - S^{-1}v) \tag{3.23}$$

and explicit formulas for the computation of the T-forward measure (see [Rog95]) and of the zero-coupon bond prices that for a maturity t are given by

$$P(0,t) = \exp\{-\alpha\, t\}\ \det(I - QV_t)^{-\frac{1}{2}}\ \exp(\frac{1}{2}\mu_t^T (I - QV_t)^{-1} Q\mu_t - \frac{1}{2}\mu_0^T Q\mu_0) \tag{3.24}$$

where

$$\mu_t = \exp\{-tB\}\, x_0 - c, \tag{3.25}$$

$$V_t = \int_0^t \exp\{-sB\}(\exp\{-sB\})^T\, ds, \tag{3.26}$$

and I denotes the identity matrix. Moreover if we consider several countries at once and we assume the same diffusion and the same values for the entries of the matrix Q for all of the countries, then we get that the exchange rates between two countries i and j are log-Brownian processes

$$Y_t^{ij} = Y_0^{ij} \frac{\zeta_t^j}{\zeta_t^i} = Y_0^{ij} \exp((\alpha^i - \alpha^j)\, t + (c^i - c^j)Q(X_t - X_0)) \tag{3.27}$$

4 Day-by-day Fits

4.1 Quadratic

We are interested in modelling the interest rate of the two countries (US and UK) and the exchange rate of the currencies. We assume that there is only one

two-dimensional diffusion driving the evolution of the three processes (same B and X, $d = 2$) and that Q is diagonal and the same for both countries.
The exchange rate process is obtained as the quotient of the state price densities in the single countries (see 1.2).
We have 12 parameters $\theta = [\lambda_1, \lambda_2, \beta_1, \beta_2, q_1, q_2, c_1^{US}, c_2^{US}, c_1^{UK}, c_2^{UK}, \alpha^{US}, \alpha^{UK}]'$ and a 2-dimensional diffusion X. The first 6 parameters characterize the two matrices B and Q: q_1 and q_2 are the diagonal elements of Q; λ_1 and λ_2 are the eigenvalues of B, and β_1 and β_2 are angles such that if we define

$$R = \begin{bmatrix} \cos\beta_1 & \cos\beta_2 \\ \sin\beta_1 & \sin\beta_2 \end{bmatrix} \tag{4.28}$$

then

$$B = R \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R^{-1} \tag{4.29}$$

Let us denote by $y_{ni}^{US}(\theta, X)$ and $y_{ni}^{UK}(\theta, X)$ the model value of the yield at the i-th maturity in the n-th day for the US and the UK respectively, and by $y_n^{FX}(\theta, X)$ the logarithm of the ratio of the state price densities in the two countries

$$y_n^{FX}(\theta, X) = \log\left\{\frac{\zeta_{t_{n-1},t_n}^{UK}}{\zeta_{t_{n-1},t_n}^{US}}\right\} \tag{4.30}$$

The fitting is obtained by a minimization of the function

$$\begin{aligned} F_n(\theta, X) &= \textstyle\sum_{i=1}^{9} w_i^{US}(y_{ni}^{US} - y_{ni}^{US}(\theta, X))^2 \\ &+ \textstyle\sum_{i=1}^{9} w_i^{UK}(y_{ni}^{UK} - y_{ni}^{UK}(\theta, X))^2 + w^{FX}(y_n^{FX} - y_n^{FX}(\theta, X))^2 \end{aligned} \tag{4.31}$$

with respect to (θ, X) using the NAG routine E04JAF, repeated for the 200 days under consideration (i.e. $n = 1, .., 200$), where the weights are chosen to be $w_i^{US} = w_j^{UK} = 15000$, $(i, j = 1, .., 9)$ and $w^{FX} = 450000$.
Figures **??** and **??** show the modulus of the residuals of the fitting for every maturity in the 200 days for the two countries; the mod-residuals are sorted by order of magnitude to give a clearer idea of the errors. Figure **??** shows the observed data and the fitting curve for the foreign exchange rate (there are indeed two curves in the top picture). The model fitted is completely non-rigid; different parameters are fitted each day. If we follow the procedure described in the Introduction, we must expect the quality of the fit to deteriorate.

4.2 Exponential Quadratic

We have 12 parameters $\theta = [\lambda_1, \lambda_2, \beta_1, \beta_2, q_1^{US}, q_2^{US}, q_1^{UK}, q_2^{UK}, c_1^{US}, c_2^{US}, c_1^{UK}, c_2^{UK}]'$ and a 2-dimensional diffusion X. The first 8 parameters characterize the matrices B, Q^{US} and Q^{UK}: we assume Q^{US} and Q^{UK} to be diagonal matrices

and let $q_1^{(\cdot)}$ and $q_2^{(\cdot)}$ be the values of the elements on the diagonal of $Q^{(\cdot)}$; λ_1, λ_2, β_1 and β_2 have been defined in the previous section.
The functions y are defined (using the current model) as in the previous section and the day-by-day fitting is obtained by a minimization of the function

$$\begin{aligned} F_n(\theta, X) &= \textstyle\sum_{i=1}^9 w_i^{US}(y_{ni}^{US} - y_{ni}^{US}(\theta, X))^2 \\ &+ \textstyle\sum_{i=1}^9 w_i^{UK}(y_{ni}^{UK} - y_{ni}^{UK}(\theta, X))^2 + w^{FX}(y_n^{FX} - y_n^{FX}(\theta, X))^2 \end{aligned} \tag{4.32}$$

with respect to (θ, X) using the NAG routine E04JAF, repeated for the 200 days under consideration, where the weights are chosen to be $w_i^{US} = w_j^{UK} = 30000$, $(i, j = 1, .., 9)$ and $w^{FX} = 94000$
Figures **??** and **??** show the ordered mod-residuals of the fitting for every maturity in the 200 days for the two countries; Figure **??** shows the observed data and the fitting curve for the foreign exchange rate.

5 Approximate Kalman Filtering

Although it may look appealing to perform the fitting procedure as illustrated at the end of the previous section, it is nevertheless inconsistent with the theoretical model. We are in fact allowing the parameters to move from day to day, while the model was requiring constant parameters. We need therefore to constrain the parameters in the minimization by adding a penalty for fluctuations of their values.
We do this by using the approximate Kalman filtering approach that we outline in what follows.
(i) If

$$\begin{pmatrix} Z \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \nu \end{pmatrix}, \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}\right) \tag{5.33}$$

is a multivariate Gaussian vector, then it is well known that

$$E(Z|Y) = \mu + BD^{-1}(Y - \nu) \tag{5.34}$$

and

$$\mathrm{var}(Z|Y) = A - BD^{-1}B^T \tag{5.35}$$

(ii) It is an easy exercise to prove that the minimization problem

$$\min_x \phi(x, y), \tag{5.36}$$

where

$$\phi(x, y) \equiv \frac{1}{2}\begin{pmatrix} x - \mu \\ y - \nu \end{pmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}^{-1} \begin{pmatrix} x - \mu \\ y - \nu \end{pmatrix}, \tag{5.37}$$

is solved by

$$x^* = \mu + BD^{-1}(y - \nu), \tag{5.38}$$

and that the Hessian of ϕ with respect to x at the minimizing value x^* (and, indeed, everywhere) is

$$(A - BD^{-1}B^T)^{-1} \tag{5.39}$$

(iii) *Thus the conditional distribution of Z given Y can also be obtained by solving the minimization of the quadratic loss.*
(iv) Suppose that we know $Z_{(n-1)\delta} \sim N(\mu_{n-1}, V_{n-1})$ and that Z solves the SDE

$$dZ_t = b(Z_t)\,dt + \sigma\,dW_t, \tag{5.40}$$

where b is a smooth function, and σ is constant. (The constancy of σ is not critical to the argument, but we make this assumption because it is all we shall need, and it simplifies the development.) If $\delta > 0$ is small, we have (neglecting second order terms in $\xi = Z_{n\delta-\delta} - \mu_{n-1}$)

$$Z_{n\delta} = Z_{n\delta-\delta} + \delta\, b(Z_{n\delta-\delta}) + \sigma\,(W_{n\delta} - W_{n\delta-\delta}) + o(\delta^{\frac{3}{2}}) \tag{5.41}$$

$$= \mu_{n-1} + \delta\, b(\mu_{n-1}) + (I + \delta\, D\, b(\mu_{n-1}))\,\xi + \sigma\,(W_{n\delta} - W_{n\delta-\delta}) + o(\delta^{\frac{3}{2}})$$

so that now we have approximately

$$Z_{n\delta} \sim N(\tilde{\mu}_n, \tilde{V}_n) \tag{5.42}$$

where

$$\tilde{\mu}_n \equiv \mu_{n-1} + \delta\, b(\mu_{n-1}) \tag{5.43}$$

$$\tilde{V}_n \equiv (I + \delta\, D\, b(\mu_{n-1}))\, V_{n-1}\, (I + \delta\, D\, b(\mu_{n-1}))^T + \delta\,\sigma\,\sigma^T \tag{5.44}$$

(v) If we now observe

$$Y_n = a + K\, Z_{n\delta} + \epsilon \tag{5.45}$$

where ϵ is some independent zero-mean Gaussian noise, with covariance V_ϵ, we would find the maximum likelihood estimate of Z by finding

$$\min_x \frac{1}{2}(Y_n - a - K\,x)^T V_\epsilon^{-1}(Y_n - a - K\,x) + \frac{1}{2}(x - \tilde{\mu}_n)^T \tilde{V}_n^{-1}(x - \tilde{\mu}_n) \tag{5.46}$$

The minimizing value μ_n will be the conditional mean of Z_n, and the Hessian evaluated at μ_n will be V_n^{-1}.
(vi) If we observe

$$Y_n = f(Z_{n\delta}) + \epsilon, \tag{5.47}$$

where f is no longer necessarily linear, we assume that the analogue of step (v) may be used; we find

$$\min_x \frac{1}{2}(Y_n - f(x))^T V_\epsilon^{-1}(Y_n - f(x)) + \frac{1}{2}(x - \tilde{\mu}_n)^T \tilde{V}_n^{-1}(x - \tilde{\mu}_n) \qquad (5.48)$$

and set μ_n to be the minimizing value, and V_n^{-1} to be the Hessian at μ_n.

6 Fits of Constrained Models

6.1 Quadratic case

In this Section we apply the approach, described in the previous Section, to the models that are considered in this paper. We suppose that the observations on the prices are not exact and that the values of the components of θ are slowly moving. We therefore have a 14-dimensional random process $(Z_t = (\theta_t, X_t), t \geq 0)$ whose dynamics is given by (compare with equation 5.40)

$$\begin{cases} d\theta_t = \bar{\sigma}\, d\bar{W}_t \\ dX_t = -B\, X_t\, dt + dW_t \end{cases} \qquad (6.49)$$

and a 19-dimensional random process $(Y_t, t \geq 0)$ that models the price observations as actual price plus Gaussian noise given by $\epsilon \sim N(0, V_\epsilon)$ (compare with equation 5.47).
Note that, in order to apply this method, we must assume some values for $\bar{\sigma}$ and V_ϵ. The choice that we make influences the outcome of the procedure: there is indeed a trade off between accuracy of the fitting and stability of the parameters (the latter is after all one of the reasons for introducing this approach).
We apply the procedure to the quadratic model assuming V_ϵ and $\bar{\sigma}$ diagonal with $(\text{diag}(V_\epsilon))_i = (15000)^{-1}$, $(j = 1, .., 18)$; $(\text{diag}(V_\epsilon))_{19} = (500000)^{-1}$, $\text{diag}(\bar{\sigma})_i = 200^{-1}$, $(i = 1, .., 10)$, and $\text{diag}(\bar{\sigma})_i = 10^{-1}$, $(i = 11, 12)$.
Figures **??**, **??**, and **??** show the results of the fitting for the constrained quadratic case and Figure **??** shows the values of the parameters in the period that has been considered.
As a by-product of this approach we obtain also confidence intervals for each yield. Using a first order Taylor expansion we see that an observable $f(z)$ is approximately a normal random variable with mean $f(\mu_n)$ and variance $(\nabla f(\mu_n))^T\, V_n\, (\nabla f(\mu_n))$ where the conditional mean μ_n and the conditional covariance V_n are obtained through the minimization as illustrated in section

5. We can therefore compute

$$\left((\nabla f(\mu_n))^T \, V_n \, (\nabla f(\mu_n)) \right)^{\frac{1}{2}} \tag{6.50}$$

and obtain the half length of the one standard deviation confidence interval. Figures **??**, **??** show the data, the fitting curves and the confidence intervals for the bonds with 3 months, 1, 3, and 10 years maturities in the two countries. The dashed line is the point estimate of the yield, the middle solid line is actual yield, and the outer solid lines are the ends of the confidence intervals.

6.2 Exponential quadratic case

We now apply the approximate Kalman filtering method described above to the exponental quadratic model assuming V_ϵ and $\bar{\sigma}$ diagonal with $(\mathrm{diag}(V_\epsilon))_i = (30000)^{-1}$, $i = 1,..,18$; $(\mathrm{diag}(V_\epsilon))_{19} = (120000)^{-1}$; $(\mathrm{diag}(\bar{\sigma}))_i = 150^{-1}$, $i = 1,..,12$.
Figures **??**, **??**, and **??** show the results of the fitting for the constrained exponential quadratic case and Figure **??** show the values of the parameters.
We can again find confidence intervals using (6.50). Figures **??**, **??** show the data, the fitting curves and the confidence intervals for the 3 months, and for the 1, 3, 10 years maturities in the two countries.

Comparing the constrained and unconstrained fits, we find that the fit of the constrained model to the yield curves is 3-4 times as bad. Four times out of five the day-by-day fits get within 6 bp in the US and 4 bp in the UK. There appears to be no marked preference for quadratic as against exponential quadratic in the fit of the yields, but the quadratic seems to do a bit better on exchange rates.
As for parameter stability, most of the parameters of the quadratic model display remarkable stability; only $B(1,2)$ and q_2 seem a bit unsteady, but this can be understood when we notice that both are quite small ($q_2 \simeq 10^{-2}\, q_1$, $|B(1,2)| \simeq 2 * 10^{-2}\, B(2,2)$) . By contrast, the exponential quadratic model displays less impressive parameter stability.
The confidence intervals generally cover the actual data very effectively, though they are quite wide (30-70 bp for the quadratic model, 60-120 bp for the exponential model). This is a confirmation of the integrity of the procedure, though a tighter fit would be desirable.

7 Conclusions

We have fitted simple two-factor potential models to yield curve data in the US and the UK, and to the exchange rates between them. The fit is not, of course, perfect, but it is similar to what one obtains when fitting a time-homogeneous one factor model to one country's yield curves. A perfect fit can only be guaranteed by using a time-inhomogeneous model and fitting it afresh each day; the consistency of this is highly questionable, but is conventionally ignored by those who practice it. Time-inhomogeneous versions of potential models can be devised; for example, Flesaker and Hughston [FH96], study (amongst others) a class of models which can be described by writing the state-price density as $\zeta_t = a(t) + b(t)\, M_t$, where a and b are positive decreasing functions, and M is a positive log-Brownian martingale. It is probably preferable to think of a pricing routine as a package which outputs a *range* of prices (i.e. a confidence interval) rather than a single price. After all, we only have estimates, never true values, and a procedure which fits exactly to input data and pretends there is no error is liable to be severely misleading.
There are three clear directions for further research. The first is to use the fitted models to price derivatives, and this we intend to do once we can get hold of clean derivative price data. The second is to extend to more than two factors; the fit we have obtained with two factors is reasonable but not marvellous, and we can hope for better with a more flexible model. The third direction is to involve more countries in the model, and this must be the eventual test of the usefulness of the approach. Even with more factors we do not necessarily expect a miraculous fit, because of the nature of the data, but if such models can adequately explain the systematic part of the data, we may then address the actual data as a *perturbation* of an underlying time-homogeneous model, and this is structurally more satisfactory than day-by-day fitting.

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