Trading to stops

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Abstract

The use of trading stops is a common practice in financial markets for a variety of reasons: it reduces the frequency of trading and thereby transaction costs; it provides a simple way to control losses on a given trade, while also ensuring that profit-taking is not deferred indefinitely; and it allows opportunities to consider reallocating resources to other assets. In this study, we try to explain why the use of stops may be desirable, by proposing a simple objective to be optimized. We investigate a number of possible rules for the placing and use of stops, either fixed or moving, with fixed costs, showing how to identify optimal levels at which to set stops, and compare the performance of different rules.

1 Introduction.

When a trader enters a position in a risky asset, it is common to set stops at which he will come out of the position; for example, he may decide to come out of the position when the value has either risen by 0.1 or fallen by 0.03. Such a fixed-stop trading rule is the simplest to describe, but there are other possibilities, where perhaps the lower stop rises as the value of the position rises, thereby locking in any gain, while allowing the position to continue to rise in value. In this paper, we shall study some simple explicit instances of trading to stops, and try to answer two questions: Is it a good idea to trade to stops in some way? Given that we intend to trade to stops in some way, how would we go about placing them?

To answer the second of these questions, we shall propose a simple objective which must be maximized over the parameters defining the stopping rule. The answer to the first question is more subtle. If we (just for now) restrict the discussion to rules which trade to fixed stops, what we find is that in most instances the best thing to do is to put the lower stop at $-\infty$, which is counter-intuitive.

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It is counter-intuitive, because one of the reasons to use stops is to prevent the trade running up huge losses, and yet it seems from the theory that this is exactly what we should be doing. However, the theoretical predictions are based on very precise assumptions about the dynamics of the risky asset; if we relax these strong assumptions, we find a different picture emerging. Specifically, we shall assume that the value of the position evolves as a Brownian motion with constant drift and constant volatility; the volatility will always be assumed to be known, but we will relax the assumption that the drift is known with certainty to the more realistic assumption that we have some (finite atomic) prior over the possible values of the drift. Given this, we find that there is good reason to place stops, either fixed or moving, as a means to protect against model uncertainty, and we compare various different ways of placing the stops.

2 Model set-up.

We shall suppose that a trader enters a position at time 0, requiring the commitment of unit capital. The trading gains of this position at time t is $X_t = \sigma W_t + \mu t$, where W is a standard Brownian motion[1]. Now this gain is not realized until the position is closed out, at some stopping time $T = T_1$, when the trader will be able to book a gain equal to X_T, which may be negative. We shall suppose that when the position is closed, a constant cost equal to c will be paid. Having closed out the position, we will suppose that the trader repeats the process, once again investing unit capital in the position, and using the same stopping rule applied to the rebased process $(X(T_1 + t) - X(T_1))_{t\geq 0}$. Thus the stopping times T_n (which are the times at which the position gets closed and immediately re-opened) form a renewal process. The time-0 value of this repeated trading activity will be

$$\varphi \equiv E \sum_{n\geq 0} e^{-\rho T_{n+1}}\, U(X(T_{n+1}) - X(T_n) - c\,) \tag{2.1}$$

where ρ is the (constant) rate of discounting , and the utility U is some smooth concave strictly increasing function. If we cared only about the net present value of all the gains from trade over time, we would take $\rho = r$, the riskless rate of interest, and $U(x) = x$, and take expectations with respect to the pricing measure. However, this is not the only possible case of interest. Indeed, we shall see that we must allow strict concavity of U to explain why an agent would wish to place stops; when it comes to studying this, we shall always take

$$U(x) = 1 - \exp(-\gamma x) \tag{2.2}$$

for some $\gamma > 0$, the coefficient of absolute risk aversion. The (risk-neutral) case of linear U is regarded as a limiting case, using the limit as $\gamma \downarrow 0$ of $\gamma^{-1}(1 - e^{-\gamma x})$

The following simple result reduces the calculation of φ to two simpler calculations.

[1]It might be considered more natural to use geometric Brownian motion as the asset model, taking a CRRA utility to express the agent's preferences. However, it turns out that in the case of fixed stops the optimization problem results in an uninteresting solution; the optimal placing of the upper stop is either at infinity, or at the starting value.

Proposition 1. *The value φ of the trading strategy is*

$$\varphi = \frac{E[e^{-\rho T}\, U(X_T - c)]}{1 - Ee^{-\rho T}} \tag{2.3}$$

where $T \equiv T_1$.

PROOF. By the strong Markov property, by decomposing the objective (2.1) at the first time $T = T_1$ that the position gets closed out we see that

$$\begin{aligned}
\varphi &= E[e^{-\rho T} U(X_T - c)] + E \sum_{n \geq 1} e^{-\rho T_{n+1}} U(X(T_{n+1}) - X(T_n) - c) \\
&= E[e^{-\rho T} U(X_T - c)] + E\big\{e^{-\rho T} E[\sum_{n \geq 1} e^{-\rho(T_{n+1} - T)} U(X(T_{n+1}) - X(T_n) - c) | \mathcal{F}_T]\big\} \\
&= E[e^{-\rho T} U(X_T - c)] + Ee^{-\rho T} \varphi.
\end{aligned}$$

Rearrangement gives the result (2.3).

□

To set the stage, we now offer a few natural examples which we will study in more detail later.

Example 1: fixed stops. This is the easiest example of all. We take $a > 0$, $b > 0$ and set

$$T \equiv \inf\{t : X_t = -a \text{ or } X_t = b\}. \tag{2.4}$$

Example 2: one rising stop. Fix $a > 0$ and let $\bar{X}_t \equiv \sup_{0 \leq s \leq t} X_s$. Then we use the stopping time

$$T \equiv \inf\{t : \bar{X}_t - X_t = a\}. \tag{2.5}$$

This example is dealt with by Glynn & Iglehart [1].

Example 3: one rising stop, one fixed stop. This time we fix $a > 0$ and $b > 0$, and set

$$T \equiv \inf\{t : \bar{X}_t - X_t = a \text{ or } X_t = b\}, \tag{2.6}$$

which gives the rising stop of Example 2 but with a take-profit stop at $b > 0$.

Example 4: converging stops. Fix $a > 0$ and $\varepsilon > 0$. Then we use the stopping time

$$T \equiv \inf\{t : (1 + \varepsilon)\bar{X}_t - X_t = a\}. \tag{2.7}$$

In this situation, it is easy to see that the trade stops out before X first hits a/ε; it has similarities to Example 3, and in the special case $\varepsilon = 0$ we recover Example 2.

Since our main interest is in the case of CARA utility U (2.2), we see that the value of the problem can be expressed as

$$\begin{aligned} \varphi &= \frac{E[e^{-\rho T}] - e^{\gamma c} E[e^{-\rho T - \gamma X_T}]}{1 - E[e^{-\rho T}]} \\ &= \frac{L(\rho, 0) - e^{\gamma c} L(\rho, \gamma)}{1 - L(\rho, 0)} \end{aligned} \tag{2.8}$$

where

$$L(s, z) \equiv E[e^{-sT - zX_T}] \tag{2.9}$$

is the joint Laplace transform of the time and place of stopping. Thus the first objective is to identify the joint Laplace transform L as explicitly as possible in each of the examples under investigation. As we shall see, this is not the end of the story, merely the start.

3 Analysis of the examples.

In this Section, we shall analyse the examples presented in Section 2 and derive explicit solutions for the joint Laplace transform L in each case. The first example is solved using differential equations techniques, which we can think of as an application of Itô calculus. Similar techniques may also be used to solve the other examples, but as the state variable is no longer one-dimensional, the construction of the correct functions is not as simple or transparent. For this reason, we prefer to derive the answers using Itô excursion theory, introduced by Itô in [2]; see [3] or [4] for accessible accounts.

3.1 Example 1: fixed stops.

We write

$$\mathcal{L} \equiv \tfrac{1}{2}\sigma^2 \frac{d^2}{dx^2} + \mu \frac{d}{dx} - \rho \tag{3.1}$$

for the generator of the diffusion X with killing rate ρ. If $f : \mathbb{R} \mapsto \mathbb{R}$ is C^2 and satisfies $\mathcal{L}f = 0$, then by an application of Itô's formula we have that

$$M_t \equiv e^{-\rho t} f(X_t) \text{ is a local martingale}$$

which is bounded on the interval $[0, T]$, and therefore[2] $(M(t \wedge T))_{t \geq 0}$ is a martingale. By the Optional Sampling Theorem, it follows[3] that

$$f(0) = E^0[e^{-\rho T} f(X_T)] \tag{3.2}$$

[2]Here, of course, T is given by (2.4).

[3]The notation E^x denotes expectation under the initial condition $X_0 = x$.

so in order to compute the numerator and denominator in (2.3) it is enough to solve the ODE $\mathcal{L}f = 0$ in $[-a, b]$ with the appropriate boundary conditions.

If we let $-\alpha < 0 < \beta$ be the roots of the quadratic

$$\tfrac{1}{2}\sigma^2 z^2 + \mu z - \rho = 0, \tag{3.3}$$

then the solution to the ODE

$$\mathcal{L}f = 0, \quad f(-a) = A, \quad f(b) = B$$

is

$$f(x) = \frac{(Ae^{\beta b} - Be^{-\beta a})e^{-\alpha x} + (Be^{\alpha a} - Ae^{-\alpha b})e^{\beta x}}{e^{\alpha a+\beta b} - e^{-\alpha b-\beta a}}.$$

Evaluating at $x = 0$ simplifies to

$$f(0) = \frac{A(e^{\beta b} - e^{-\alpha b}) + B(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a+\beta b} - e^{-\alpha b-\beta a}}. \tag{3.4}$$

If we now take $A = \exp(\gamma a)$ and $B = \exp(-\gamma b)$ we read off the joint Laplace transform L_1 for this first example:

$$L_1(\rho, \gamma) = \frac{e^{\gamma a}(e^{\beta b} - e^{-\alpha b}) + e^{-\gamma b}(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a+\beta b} - e^{-\alpha b-\beta a}}. \tag{3.5}$$

Substituting the form of L_1 into the expression (2.8) gives the value φ for this stopping rule. The dependence of the right-hand side on ρ is of course through the dependence of α, β on ρ as solutions to (3.3). The mean of the hitting time can be derived from the Laplace transform as

$$\begin{aligned} E[T] &= -\frac{\partial L_1}{\partial \rho}(0, 0) \\ &= \frac{b(e^{ka} - 1) - a(1 - e^{-kb})}{\mu(e^{ka} - e^{-kb})} \end{aligned} \tag{3.6}$$

after some calculations, where $k \equiv 2\mu/\sigma^2$.

3.2 Example 3: one rising stop, one fixed stop.

We deal with this example first, and read off the solution to Example 2 as the special case $b = \infty$. Recall that we take the stopping time

$$T \equiv \inf\{t : \bar{X}_t - X_t = a \text{ or } X_t = b\}, \tag{3.7}$$

where $\bar{X}_t \equiv \sup_{0 \le s \le t} X_s$. The process $Y \equiv X - \bar{X}$ is a continuous strong Markov process with values in $\mathcal{X} \equiv (-\infty, 0]$, and 0 is a recurrent point for this process. The Itô theory of excursions [2] applies to this process, and we will make use of it. Let U denote the space of all excursions of Y away from 0, that is, continuous functions $f : \mathbb{R}^+ \to \mathcal{X}$ with the property that for some $\zeta = \zeta(f) \in (0,\infty]$, the *lifetime* of the excursion, the set $f^{-1}((-\infty,0))$ is of the form $(0,\zeta)$. Regarding U as a subset of $C(\mathbb{R}^+,\mathbb{R})$ induces the subset topology on U, and in fact U is a Polish space; see, for example, [4] for definitions and basic properties. The process $\bar{X}$ is a continuous homogeneous additive functional of Y, growing only when $Y = 0$, and acts as the local time at zero for Y. The open set $Y^{-1}((-\infty,0))$ is the disjoint union of countably many excursion intervals I_j, and the point process $\Pi \equiv \{(L_j,\xi^j) : j \in \mathbb{Z}\}$ is a Poisson point process in $(0,\infty) \times U$, where

$$\begin{aligned} L_j &= \bar{X}(I_j), \\ \xi^j &= Y|_{I_j} . \end{aligned}$$

The mean measure of Π is Leb $\times\, n$, where n is the σ-finite *excursion measure*: see Itô [2]. The key to effective use of Itô excursion theory is an explicit characterization of the excursion measure n. Once the excursion has escaped from 0, it evolves like the diffusion $X - \bar{X}$ until it first hits zero, and it leaves 0 according to an entrance law.

We shall use excursion theory to calculate for any $\theta \geq 0$ the expectation

$$L(\rho,\theta) \equiv E[\, \exp(-\rho T - \theta X_T)\,]; \tag{3.8}$$

evidently, once we have this, we can obtain the numerator and denominator in (2.3) by suitable substitutions and combinations. As explained in [3], we deal with expectations such as (3.8) by introducing an independent $\exp(\rho)$ time τ, and writing

$$E[\, \exp(-\rho T - \theta X_T)\,] = E[\, e^{-\theta X_T} : T < \tau]. \tag{3.9}$$

The way this is handled by excursion theory is to think of τ as being the first event time τ_1 in a Poisson process on $\mathbb{R}^+$ of intensity ρ, with event times $\tau_1 < \tau_2 < \ldots$. This Poisson process of times can be dealt with by *marking* the excursions of Y, each independently of all others, according to a Poisson process of intensity ρ. The excursion point process Π gets modified to the marked excursion point process $\tilde{\Pi}$, where each excursion ξ^j gets augmented to $\tilde{\xi}^j \equiv (\xi^j, N^j)$, where N^j is an increasing $\mathbb{Z}^+$-valued path, representing the path of the marking process restricted to the excursion ξ^j. We observe the marked excursion process $\tilde{\Pi}$ until *either* local time $\bar{X}$ reaches b; *or* we see an excursion which gets to $-a$ before any mark; *or* we see an excursion which gets marked before it reaches $\{0,-a\}$. To set some notation, let

$$\begin{aligned} A &\equiv \{\text{excursions which are marked before reaching 0 or } -a\}; && (3.10) \\ B &\equiv \{\text{excursions which get to } -a \text{ with no mark before reaching } -a\,\}. && (3.11) \end{aligned}$$

We shall calculate $n(A)$ and $n(B)$ quite simply, but for this we need to characterize the excursion measure effectively. Let $-\alpha < 0 < \beta$ be the roots of the quadratic $\frac{1}{2}\sigma^2 t^2 + \mu t - \rho$; then routine calculations lead to the conclusion that for any $-a < x < 0$

$$\begin{aligned} E^x[1 - e^{-\rho\, H_0 \wedge H_{-a}}] &= \frac{1 - e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}(1 - e^{-\alpha x}) + \frac{e^{\alpha a} - 1}{e^{\alpha a} - e^{-\beta a}}(1 - e^{\beta x}) && (3.12) \\ E^x[e^{-\rho H_{-a}} : H_{-a} < H_0] &= \frac{e^{-\alpha x} - e^{\beta x}}{e^{\alpha a} - e^{-\beta a}} && (3.13) \end{aligned}$$

where $H_z \equiv \inf\{t : X_t = z\}$ is the hitting time of z. Since the measure of excursions which reach $-\varepsilon$ is asymptotic to ε^{-1} as $\varepsilon \downarrow 0$ (see Williams' decomposition of the Brownian excursion law [5], II.67), we conclude that

$$n(A) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}[1 - e^{-\rho\, H_0 \wedge H_{-a}}] = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a} - (\alpha + \beta)}{e^{\alpha a} - e^{-\beta a}}, \tag{3.14}$$

$$n(B) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}[e^{-\rho H_{-a}} : H_{-a} < H_0] = \frac{\alpha + \beta}{e^{\alpha a} - e^{-\beta a}}. \tag{3.15}$$

The first excursion in $A \cup B$ comes at local time rate

$$\nu \equiv n(A \cup B) = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}. \tag{3.16}$$

We shall stop the point process either at the first time we see an excursion in $A \cup B$, or when local time reaches b, whichever comes sooner.

Now we come back to the expectation (3.9), and consider how the event $T < \tau$ could happen: this could *either* be because $\bar{X}$ reaches b before the first excursion in $A \cup B$; *or* because the first excursion in $A \cup B$ happens before $\bar{X}$ reaches b, and is in fact an excursion in B. By simple properties of Poisson processes, we discover after a little thought that

$$\begin{aligned} L_3(\rho, \gamma) &\equiv E[\, \exp(-\rho T - \theta X_T)\,] \\ &= E[e^{-\theta X_T} : T < \tau] \\ &= e^{-\nu b - \theta b} + \int_0^b \nu e^{-\nu y}\, \frac{n(B)}{\nu}\, e^{-\theta(y-a)}\, dy \\ &= e^{-(\nu+\theta)b} + \frac{n(B) e^{\theta a}}{\nu + \theta}(1 - e^{-(\nu+\theta)b}). \end{aligned}$$

We evaluate the objective (2.8) by setting $\theta = 0, \gamma$ to calculate the numerator and denominator. After some calculations we obtain

$$E[e^{-\rho T}] = \frac{(\alpha + \beta)(1 - e^{-\nu b}) + \nu e^{-\nu b}(e^{\alpha a} - e^{-\beta a})}{\nu(e^{\alpha a} - e^{-\beta a})}, \tag{3.17}$$

As before, the mean of T can be computed by differentiating the Laplace transform (3.17) with respect to ρ at zero. We find[4] that

$$E[T] = \frac{\sigma^2}{2\mu^2}(e^{ka} - 1 - ka)(1 - e^{-mb}) \tag{3.18}$$

where $k = 2\mu/\sigma^2$ as before, and $m = k/(e^{ka} - 1)$.

[4]The calculations were carried out by a symbolic mathematics package, and by traditional methods.

The simpler expressions which obtain when $b = \infty$ reduce to

$$L_2(\rho,\theta) = \frac{n(B)e^{\theta a}}{\nu+\theta}, \qquad E[e^{-\rho T}] = \frac{(\alpha+\beta)}{\nu(e^{\alpha a} - e^{-\beta a})}, \qquad E[T] = \frac{\sigma^2}{2\mu^2}(e^{ka} - 1 - ka). \quad (3.19)$$

The second of these agrees with the result of Glynn & Iglehart [1], equation (3.2), after translation of notation.

3.3 Example 4: converging stops.

In this example, the stopping time is given by (2.7):

$$T \equiv \inf\{t : (1+\varepsilon)\bar{X}_t - X_t = a\}.$$

The analysis of this example is quite similar to Example 3, except that the excursion measure of the excursions which stop the process now depends on how much local time has elapsed. When local time $\bar{X}$ has reached ℓ, then any excursion which either contains a mark, or reaches $-a+\varepsilon\ell$ will stop the Poisson point process. Exactly as at (3.14), (3.15), the intensity of excursions which are marked before reaching $-a+\varepsilon\ell$ or zero is

$$n_A(\ell) \equiv \frac{\beta e^{\alpha(a-\varepsilon\ell)} + \alpha e^{-\beta(a-\varepsilon\ell)} - (\alpha+\beta)}{e^{\alpha(a-\varepsilon\ell)} - e^{-\beta(a-\varepsilon\ell)}}\ , \quad (3.20)$$

and the intensity of excursions which get to $-a+\varepsilon\ell$ before getting marked is

$$n_B(\ell) \equiv \frac{\alpha+\beta}{e^{\alpha(a-\varepsilon\ell)} - e^{-\beta(a-\varepsilon\ell)}}\ . \quad (3.21)$$

So in total, the intensity of excursions which stop the Poisson point process is

$$n_{A\cup B}(\ell) = \frac{\beta e^{\alpha(a-\varepsilon\ell)} + \alpha e^{-\beta(a-\varepsilon\ell)}}{e^{\alpha(a-\varepsilon\ell)} - e^{-\beta(a-\varepsilon\ell)}}\ . \quad (3.22)$$

We can now calculate

$$\begin{aligned}
\bar{F}(t) &\equiv P(\bar{X} \text{ reaches } t \text{ before the stopping excursion}) \\
&= \exp\Big[-\int_0^t n_{A\cup B}(s)\, ds \Big] \\
&= \exp\Big\{ -\beta t - \varepsilon^{-1}\log\Big(\frac{1-e^{-(\alpha+\beta)a}}{1-e^{-(\alpha+\beta)(a-\varepsilon t)}} \Big) \Big\} \\
&= e^{-\beta t}\Big(\frac{1-e^{-(\alpha+\beta)(a-\varepsilon t)}}{1-e^{-(\alpha+\beta)a}} \Big)^{1/\varepsilon},
\end{aligned}$$

which we notice is decreasing with t, and vanishes when $t = a/\varepsilon$ as it must. Using this, we deduce after some calculations that

$$\begin{aligned} E[e^{-\theta X_T - \rho T}] &= \int_0^{a/\varepsilon} e^{-\theta((1+\varepsilon)x-a)}\, n_B(x)\bar{F}(x)\, dx \\ &= \int_0^{a/\varepsilon} e^{-\theta((1+\varepsilon)x-a)}\, \frac{\alpha+\beta}{e^{\alpha(a-\varepsilon x)} - e^{-\beta(a-\varepsilon x)}}\, \bar{F}(x)\, dx \\ &= \frac{1}{\varepsilon}\left(\frac{e^{-(\theta+\beta)a}}{1-e^{-(\alpha+\beta)a}}\right)^{1/\varepsilon} \int_{\exp\{-(\alpha+\beta)a\}}^{1} (1-t)^{(1-\varepsilon)/\varepsilon}\, t^{-c}\, dt \end{aligned} \tag{3.23}$$

where $c = (\theta+\beta)(1+\varepsilon)/\varepsilon(\alpha+\beta)$. The answer is therefore available in terms of incomplete beta functions.

4 Placing of the stops.

The identification of the joint Laplace transform of T and X_T in each of the previous examples now allows us to evaluate the objective φ (2.8), and by varying the parameters a and b we are able to optimize φ. However, numerical investigation shows that in many cases it is optimal to let $a \to \infty$. If this happens, then there would be no reason to place a lower stop, which is somewhat unexpected. We can analyse this phenomenon quite completely for the case of fixed stops, which we shall now do. The other examples are more complicated, and we have not pursued the analysis of this phenomenon in those instances; numerical investigations show similar behaviour. In any case, since we observe that often for fixed stops the best thing is to use *no* lower stop, we are forced to re-assess the modelling assumptions.

Accordingly, we will until further notice restrict attention to the fixed-stops example, Example 1. The joint Laplace transform L_1 of T and X_T has been found (3.5), and so we are able to obtain an explicit expression for the value φ using (2.8). Since we are concerned with the behaviour of this as $a \to \infty$ with all other parameters fixed, we shall use the (local) notation $\varphi(a)$, where we have explicitly

$$\begin{aligned} \varphi(a) &= \frac{L(\rho,0) - e^{\gamma c}L(\rho,\gamma)}{1 - L(\rho,0)} \\ &= -1 + \frac{1 - e^{\gamma c}L(\rho,\gamma)}{1 - L(\rho,0)} \\ &= -1 + \frac{e^{\alpha a} - B_1 e^{-\beta a} - (1-B_1)e^{\gamma(a+c)} - B_2 e^{\gamma c}(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a} - B_1 e^{-\beta a} - (1-B_1) - B_3(e^{\alpha a} - e^{-\beta a})} \end{aligned} \tag{4.1}$$

where $B_1 = e^{-(\alpha+\beta)b}$, $B_2 = e^{-(\gamma+\beta)b}$, $B_3 = e^{-\beta b}$, all positive constants less than 1. The large-a behaviour of this expression is determined in the following little result.

Proposition 2. *Consider the behaviour of the objective* (2.8) *in the case of fixed stops* (2.4) *as* $a \to \infty$*, with* b *fixed.*

(i) If $\gamma > \alpha$ *then*

$$\lim_{a \to \infty} \varphi(a) = -\infty \tag{4.2}$$

(ii) If $\alpha > \gamma$ *and* $b > c$ *then*

$$\varphi(a) < \varphi(\infty) \tag{4.3}$$

for all $a > 0$*.*

PROOF. The proof is given in the Appendix A.

REMARKS. It is easy to understand the content of Proposition 2. In the case where $\gamma > \alpha$, it is *not* advantageous to let $a \to \infty$ because although the expectation

$$E[e^{-\rho T} : X_T = -a] \sim e^{-\alpha a} \tag{4.4}$$

is getting exponentially small, the utility when this event happens is getting large negative exponentially, and *at a greater rate*. In contrast, if $\gamma < \alpha$, the exponential decay of the expectation (4.4) beats the growth of the penalty, and the agent can ignore the penalty for stopping at a low negative level. The condition $b > c$ is needed for the proof, but has a natural interpretation; if $b < c$, we are certain to be losing money every time we review our portfolio, so we would never consider entering this trade.

For a reasonable solution, then, it seems that we require $\gamma > \alpha$. However, in typical examples, this can lead to coefficients γ of absolute risk aversion so high that the value φ is always negative, so we would never engage in this trade! The point is that $-\alpha$ solves the quadratic (3.3), and if $\mu > 0$, we will always have $\alpha > 2\mu/\sigma^2$, a lower bound which need not be small. So for a solution with realistic values of γ, and with a rationale for a lower stop at a finite position, it seems that we are forced to consider situations where μ is negative. But if the growth rate of the trade was negative, and we are paying transaction costs, we would certainly never want to enter into it!

The resolution of these seemingly inconsistent requirements is to suppose that *we are not certain of the true value of* μ*.* If we have some prior distribution over possible μ values which allows positive probability that μ is negative, we will find that even for small values of γ the punishment for stopping at very low levels really hurts, and we will want to use a finite lower stop. On the other hand, if the probability of decently positive values of μ is quite high, we will be emboldened to take part in the trade.

So now we briefly explain how the value is calculated in the situation where there is a prior distribution over μ, given by the probability measure m. We will denote the value[5] φ if the true drift

[5]We now remove the restriction that we are discussing Example 1 only; what follows applies generally.

is μ by φ_μ. This is calculated as explained in Section 2. Then by conditioning on the value of μ, it is clear that the overall value $\bar{\varphi}$ of the trade when are uncertain of μ is simply

$$\bar{\varphi} = \int \varphi_\mu \, m(d\mu). \tag{4.5}$$

Ideally, we would tell some story where what we learn about μ from the outcomes of the successive trades would be incorporated into our beliefs about μ, but as in practice such learning takes place very slowly, we are safe to ignore it.

5 Numerical study.

We shall compare the stopping rules of Section 3 in just two examples. In all cases, we shall assume that $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, and $\rho = 0.1$. We have explored various other examples, and the behaviour which we report in these two appears to be quite typical. A further comparison we make is with a fixed-revision rule, where the investor chooses $T > 0$ fixed, and then revises his position at multiples of T, regardless of the performance of the asset. The objective is once again given by (2.3), though now of course T is constant. We find the agent's best choice of fixed T and compare the performance of this rule with the various rules determined by stops.

In the first example, we assume that the agent knows $\mu = 0.15$ with certainty. There are four stopping rules to be considered now, and the results are given in the following table.

	Best a	Best b	Objective	τ
Fixed stops	∞	0.0184	4.1553	0.1224
Rising lower stop	0.0894		0.5314	0.0983
Rising lower, fixed upper stop	∞	0.0184	4.1553	0.1224
Fixed exit time			0.8724	0.3780

Notice how with fixed stops or with a fixed upper stop and a rising lower stop, the best choice of a is $a = \infty$; it always pays to push the lower stop all the way down. If this is done, then of course the two stopping rules amount to stopping at b, and so it is no surprise that the values, the optimal choices of b, and the mean time per trade all agree. The value φ as a function of a and b is displayed in Figure 1; for finite a, the pictures for Examples 1 and 3 are in principle different, but in this example they are not visibly different. Notice that the value for a fixed upper and rising lower stop is substantially higher than for a rising lower stop only; this is of course to be expected, as we have optimized over a larger set, but the magnitude of the improvement is noteworthy. The rising lower stop example, Example 2, is quite different in character, with a much shorter mean time in trade. The fixed revision rule performs very poorly relative to the two-sided stops rules, Examples 1 and 3.

To illustrate the point made in Section 4 for the reason why we may want to use finite stops, we take as our second example the situation where we do not suppose that μ is known, but rather that there is a prior $N(\mu_0, \sigma_\mu^2)$ distribution for μ. We suppose that $\mu_0 = 0.15$, with $\sigma_\mu = 0.3$. The results obtained are reported in the following table.

	Best a	Best b	Objective	τ
Fixed stops	0.2159	0.0470	0.8416	0.0998
Rising lower stop	0.0603		0.2397	0.0439
Rising lower, fixed upper stop	0.2375	0.0464	0.8398	0.0984
Fixed exit time			0.5627	0.0670

The calculated values are displayed in Figures 3, 4 and 5. The values of all the rules have dropped, particularly the stops trading rules. As with the certain growth rate, the two-stops rules do substantially better than either the lower stop or the fixed time to revision. Mean times in trades have fallen in all cases. As before, there is no appreciable difference between Examples 1 and 3; the rising lower stop has very little effect.

6 Conclusions.

There are at least three reasons why we might in practice wish to trade to stops in some way. The first is to reduce transaction costs: trading strategies which rebalance infrequently are always to be preferred, and in some asset classes, such as EM currencies where costs might be typically 40bp, daily rebalancing will quickly eliminate any profitable tendencies. The second reason for wishing to trade to stops in some way is that until a position has been closed out, no profits can be booked to it. There is therefore an incentive not to let a position run indefinitely, but to take profits at some point. Following from this is a third reason; if our choice of stops rests on current estimates of asset dynamics, then it is important that we do not sit in the trade long after the parameter estimates have wandered away, otherwise the expected performance may not materialize.

In this study, we have investigated several possible rules for placing fixed or moving stops, and compared their performance. We have found that uncertainty over the growth rate of the asset is an essential feature of choosing stops; if we know that the asset is drifting up, we would never want to place a lower stop. The possibility that the drift might be negative is what makes us want to put in lower stops. If we use a fixed upper stop, we have found that there is little difference between using a fixed or rising lower stop, but the mean time in the trade is smaller using a rising stop. Thus we would recommend of the three rules studied here, the best to use is a fixed upper stop, with a rising lower stop.

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A Proof of Proposition 2.

PROOF OF PROPOSITION 2. The case $\gamma > \alpha$ is easy; the dominant term in (4.1) is the term $k_1 e^{\gamma(a+c)}$ in the numerator, and this makes it obvious[6] that $\varphi(a) \to -\infty$ as $a \to \infty$.

The second case is more delicate. The limit of $\varphi(a)$ is easily seen to be

$$\varphi(\infty) = -1 + \frac{1 - B_2 e^{\gamma c}}{1 - B_3}.$$

If we now consider $\varphi(\infty) - \varphi(a)$, we find a rational expression whose denominator is positive, and whose numerator is (a positive multiple of)

$$H \equiv (1 - B_3)z - (B_2 e^{\gamma c} - B_3)y - (1 - B_2 e^{\gamma c}), \tag{A.1}$$

where we set $z \equiv e^{\gamma(a+c)}$, $y \equiv e^{-\beta a}$ for brevity. Thus it will be sufficient to prove that the expression H is non-negative.

Since $b > c$, we may write $\varepsilon = b - c > 0$, and then H becomes

$$\begin{aligned} H &= (1 - B_3)z + B_3(1 - e^{-\gamma - \varepsilon})y - (1 - B_3 e^{-\gamma\varepsilon}) \\ &= (1 - B_3)(z - 1) + B_3(1 - e^{-\gamma\varepsilon})y - B_3(1 - e^{-\gamma\varepsilon}) \\ &= (1 - B_3)(z - 1) - B_3(1 - e^{-\gamma\varepsilon})(1 - y). \end{aligned} \tag{A.2}$$

It is clear from the final equation that if we now hold $a > 0$ fixed, and consider H as a function of γ, then H is convex, and vanishes as $\gamma \downarrow 0$. To prove non-negativity of H, we now investigate the gradient of H with respect to γ, which is

$$\begin{aligned} \frac{\partial H}{\partial \gamma} &= (1 - B_3)(a + c)e^{\gamma(a+c)} - \varepsilon B_3(1 - y)e^{-\gamma\varepsilon} \\ &= e^{-\gamma\varepsilon}\big[(1 - B_3)(a + c)e^{\gamma(a+b)} - (1 - y)B_3(b - c)\big]. \end{aligned}$$

As $\gamma \downarrow 0$, we obtain the limit

$$\begin{aligned} \frac{\partial H}{\partial \gamma}(0) &= (1 - B_3)(a + c) - (1 - y)B_3(b - c) \\ &= (1 - e^{-\beta b})(a + c) - e^{-\beta b}(b - c)(1 - e^{-\beta a}) \\ &= e^{-\beta b}\big[\,(a + c)e^{\beta b} + (b - c)e^{-\beta a} - (a + b)\,\big] \\ &= (a + b)e^{-\beta b}\left[\,\frac{a + c}{a + b}\,e^{\beta b} + \frac{b - c}{a + b}\,e^{-\beta a} - 1\,\right] \\ &\geq (a + b)e^{-\beta b}\big[\,e^{\beta c} - 1\,\big] \\ &> 0, \end{aligned}$$

[6] The denominator is asymptotic to $e^{\alpha a}(1 - B_3)$ which is certainly positive.

where we have used convexity of the exponential function for the first inequality. Since H is convex, and its derivative at zero is positive, it follows that H is increasing, and therefore is everywhere non-negative, since it is zero at $\gamma = 0$.

$\square$

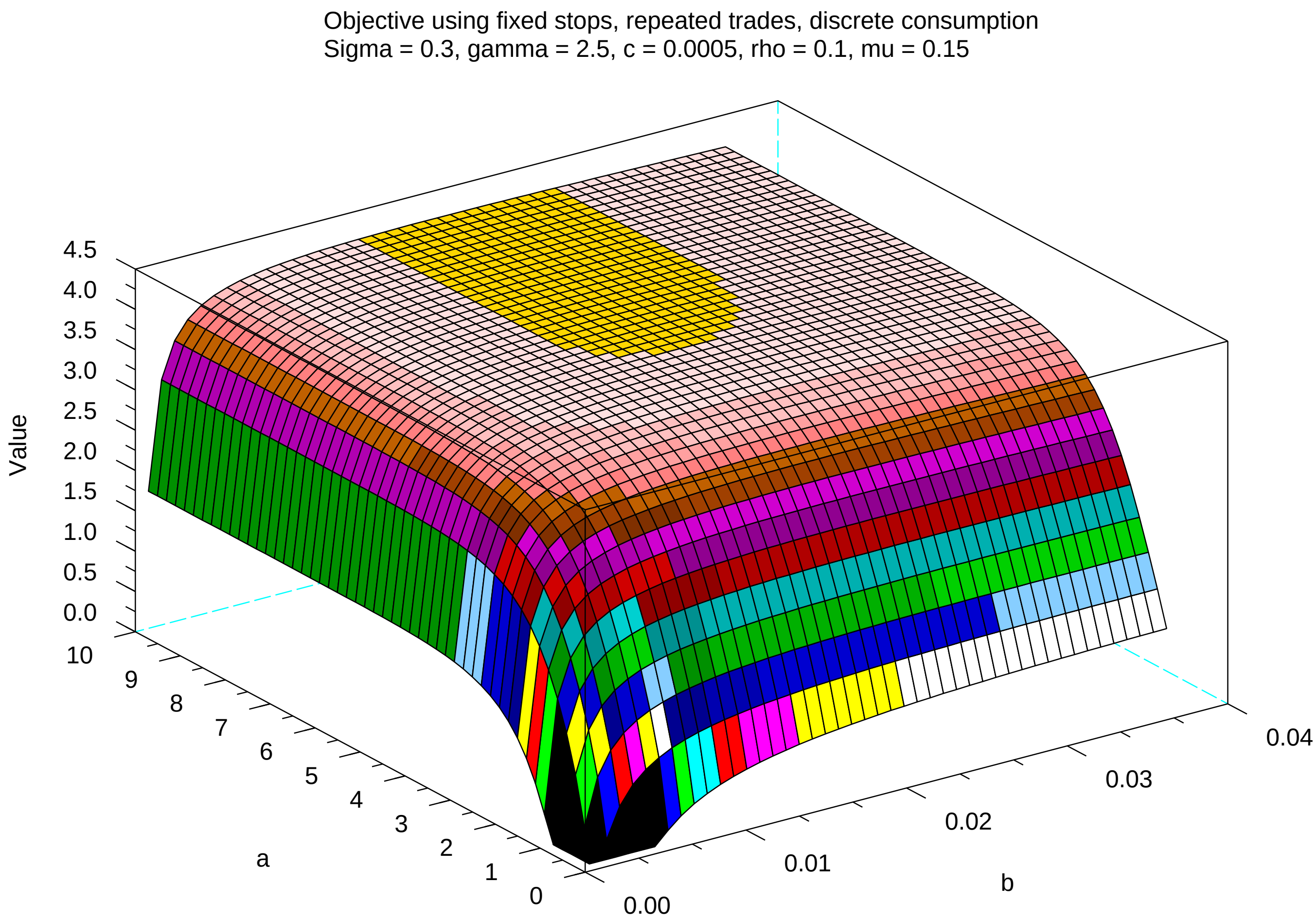

Figure 1: Example with known $\mu = 0.15$.

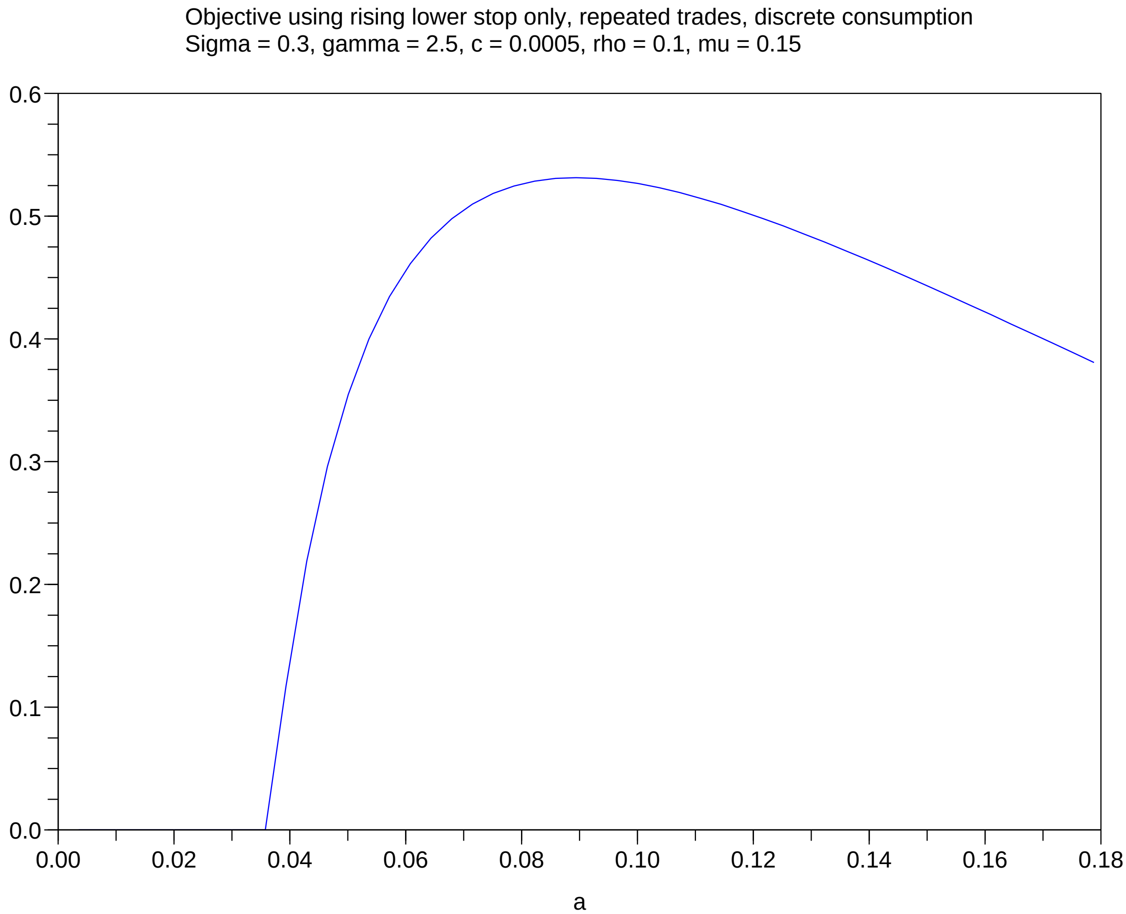

Figure 2: Example with known $\mu = 0.15$, rising lower stop only.

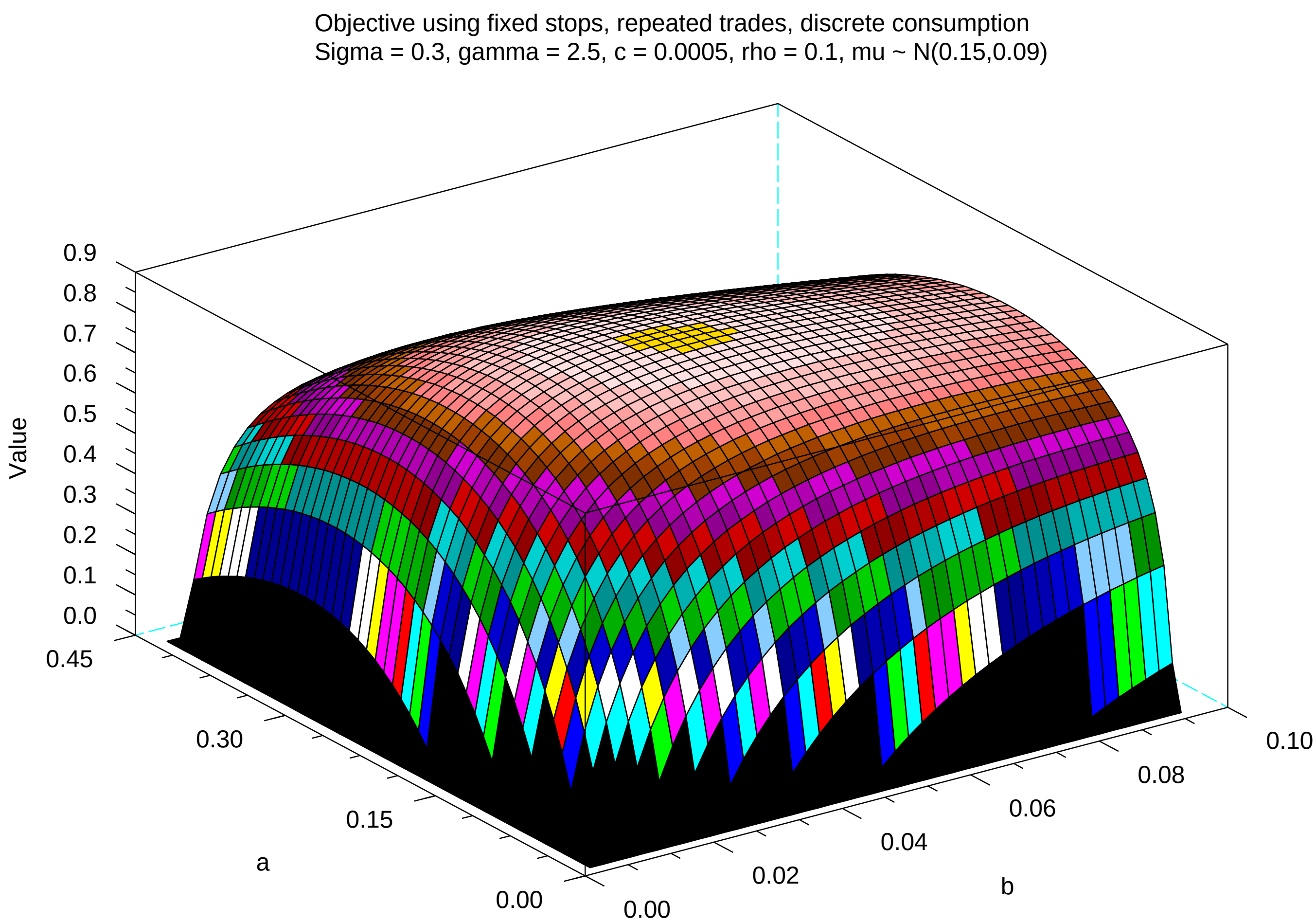

Figure 3: Example with $\mu \sim N(0.15, \sigma^2)$.

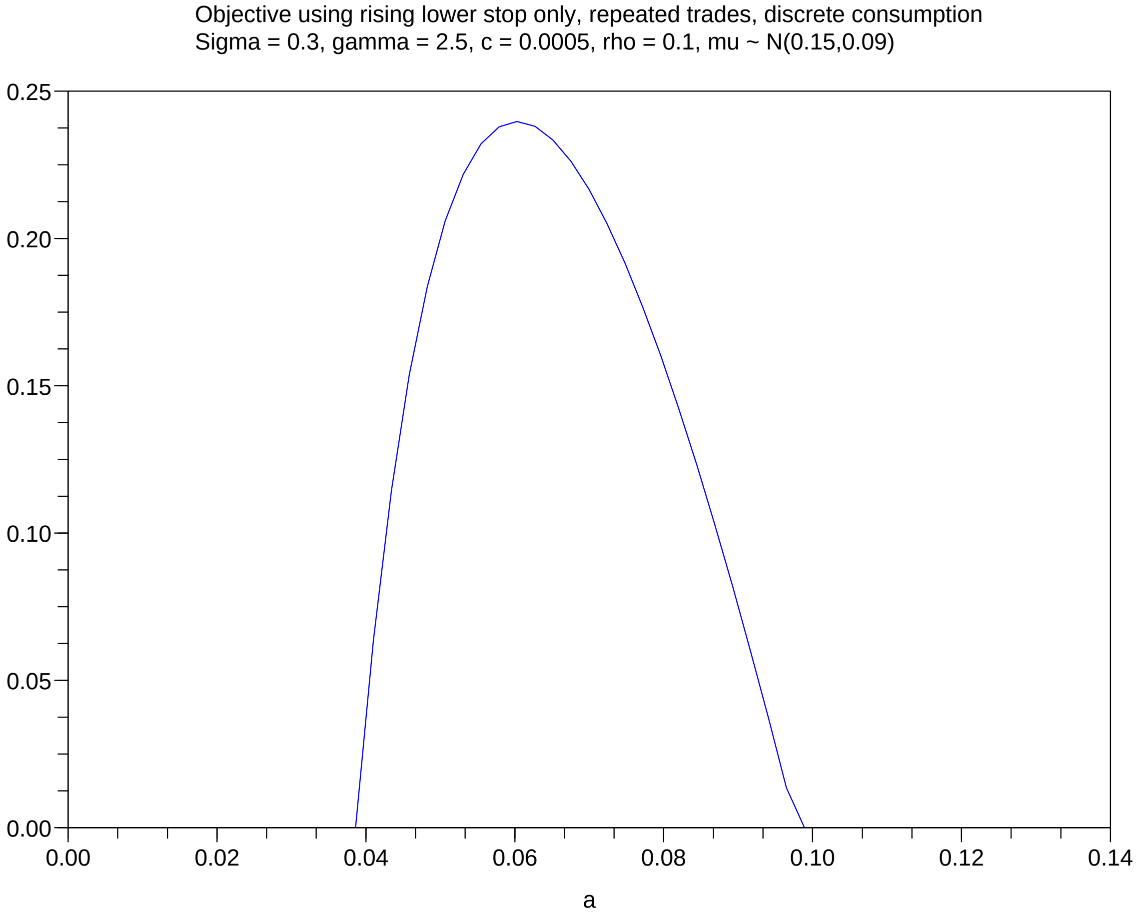

Figure 4: Example with $\mu \sim N(0.15, \sigma^2)$, rising lower stop only.

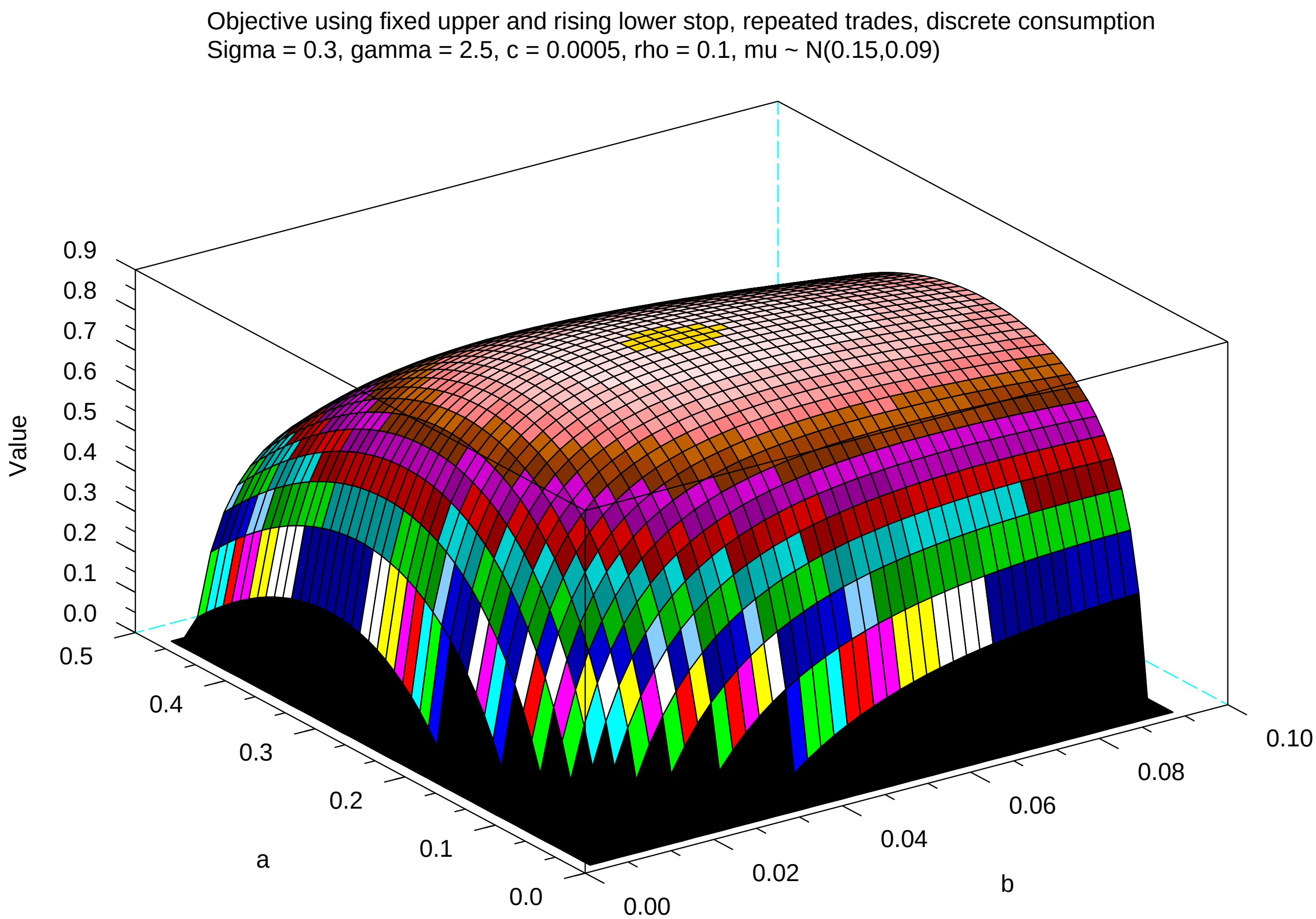

Figure 5: Example with $\mu \sim N(0.15, \sigma^2)$, rising lower stop and fixed upper stop.