Does the behaviour of the asset tell us anything about the option price formula? A cautionary tale.

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Abstract

If $Y \equiv (Y_1, \ldots, Y_N)$ are the log-returns of an asset on succeeding days, then under the assumptions of the Black-Scholes option pricing formula, these are independent normal random variables with common mean and variance in the risk-neutral measure. If we can show empirically that Y does not have these properties in the real-world measure, does this mean that the Black-Scholes option pricing formula fails? It does not; as we show in this note, so long as the joint distribution of Y in the real-world measure has *a strictly positive density*, then the Black-Scholes option price formula may still be correct. We conclude that attempts to argue that the Black-Scholes formula must fail because observed log returns appear to be fat-tailed, or appear to have non-constant volatility, or appear to have serial correlation are fallacious.

1. Introduction Ever since the original Black-Scholes option-pricing formula was published (Black & Scholes, [1973]), there have been many variants offered, based on different assumptions concerning the distribution (in a/the risk-neutral measure) of the asset returns; models with continuous paths and non-constant volatility (for example, Merton [1973]), models with continuous paths and stochastic volatility (for example, Hull & White [1987], Hobson & Rogers [1996]), models with discontinuous paths and various specifications of the jump process (usually in terms of an added Lévy process - see, for example, Merton [1976], - or in terms of a continuous process subordinated by some increasing process - see, for example, Madan and Seneta [1990],), discrete-time models such as the ARCHipelago of stationary conditionally-heteroskedastic processes (for example, Duan [1995]) and even various 'fractal' processes which are not semimartingales (for example, Peters [1991], Bouchaud, Iori & Sornette [1996] - see also Maheswaran & Sims [1993] and Rogers [1996] for a critique of such models). Indeed much of the above work has followed similar developments in the modelling of asset returns; for example, different specifications of the stochastic volatility were used involving either discrete or continuous processes by Merton [1976], Agkiray & Booth [1987,1988], amongst many others. These led to a vast literature on stochastic volatility options,including Wiggins [1987], Johnson & Shanno [1987], Hull & White [1987], Melino & Turnbull [1990]. An important subgroup of discrete time stochastic volatility models is the GARCH class of models; see Engle [1982], Bollerslev et al. [1992] and many hundreds of other authors. These in turn led to GARCH option pricing models such as Engle & Mustafa [1992], Duan [1995] and Satchell & Timmermann [1995a,1995b] among others. An exhaustive list of all the developments and the empirical papers which have motivated them would involve a bibliography with hundreds of entries.

It is sometimes argued that because the empirically-observed returns do not have the independent lognormal distributions assumed for the Black-Scholes formula, it is therefore necessary to consider such extensions in order to price options accurately. The purpose of this note is to show that this is a fallacious conclusion; so long as the distribution of the log returns $Y \equiv (Y_1, \ldots, Y_N)$ has a *strictly positive density*, the Black-Scholes formula may be correct. The fallacy lies in linking distributional properties of the returns in the risk-neutral and objective worlds, which can be very different.

This is not to say that all studies of different models for asset returns are redundant. Firstly, the frequently-reported presence of volatility smiles and skews in option prices shows that in practice the Black-Scholes option-pricing formula is violated; *this* is conclusive evidence of a failure of the Black-Scholes assumptions, even if distributional properties of returns cannot provide such evidence. Thus we *have* to study different models of the (risk-neutral) distribution of returns. Secondly, even if the Black-Scholes pricing formula is correct, the estimate of volatility which we obtain from data may need to be reinterpreted in the light of the specific model assumed (in the real-world measure) for the asset returns; Lo and Wang [1995] study one particular modelling framework where this happens and where, although the Black-Scholes formula is identical, the σ involved is no longer the sample volatility of daily log returns but involves an autocorrelation term as well.

Section 2 gives the main result; the proof is very simple but somewhat abstract. In Section 3, we discuss the special case of $N = 1$ in a more concrete way, which may shed light on the result of Section 2.

2. The main result. To state the main result of this paper, we shall need some notation. We have already introduced the vector $Y \equiv (Y_1, \ldots, Y_N)$ of log returns, which we shall suppose has a strictly positive density f with respect to Lebesgue measure on $\mathbb{R}^N$. Now let $\Omega \equiv C([0, N])$ be the space of continuous functions from $[0, N]$ to $\mathbb{R}$, starting at 0; we shall let the canonical process [1] X be the log price process of the asset we are considering. Now fix some positive volatility parameter σ. The aim is to construct measures $\mathbb{P}^*$ (the risk-neutral measure) and $\tilde{\mathbb{P}}$ (the real-world measure) on Ω such that under $\tilde{\mathbb{P}}$, $(X_1, \ldots, X_N)$ has density f, while under $\mathbb{P}^*$, X has the same distribution as $(\sigma W_t + \mu t)_{0 \le t \le N}$, where W is a standard Brownian motion, and $\mu \equiv r - (\sigma^2/2)$. We assume that the spot-rate r is constant, but this is only for convenience of exposition. It is clear that if we can construct the two measures for $\sigma = 1$, we can achieve it in general (just multiply the returns Y by σ^{-1}!), so henceforth we assume that $\sigma = 1$.

Let $\mathbb{P}$ be Wiener measure on Ω, and let φ be the density of $(X_1, \ldots, X_N)$ under $\mathbb{P}$ (so that φ is a zero-mean Gaussian density). We are going to define $\tilde{\mathbb{P}}$ on Ω by the recipe

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \tilde{Z} \equiv \frac{f}{\varphi}\,(X_1, \ldots, X_N).$$

It is easy to see that under $\tilde{\mathbb{P}}$, $(X_1, \ldots, X_N)$ has density f.

Now consider the (Girsanov change-of-measure) martingale

$$\tilde{Z}_t \equiv \mathbb{E}[\tilde{Z} | \mathcal{F}_t].$$

Since $\tilde{Z}$ is positive and integrates to 1, it is clear that this does define a martingale. As is well known (see, for example, Rogers & Williams [1987], Theorem IV.38.5) under $\tilde{\mathbb{P}}$ the process X is a Brownian motion with a drift:

$$dX_t = d\tilde{W}_t + \alpha_t dt,$$

for some predictable process α, where $\tilde{W}$ is a $\tilde{\mathbb{P}}$-Brownian motion. We define $\mathbb{P}^*$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = Z^* \equiv \exp(\mu X_N - \frac{1}{2}\mu^2 N),$$

[1] defined by $X_t(\omega) \equiv \omega(t)$ for all $0 \le t \le N$ and $\omega \in \Omega$

so that under $\mathbb{P}^*$, X is a Brownian motion with drift μ. We have immediately the following result.

THEOREM. *Under the real-world measure $\tilde{\mathbb{P}}$, $(X_1, \ldots, X_N)$ has density f, and under the risk-neutral measure $\mathbb{P}^*$, X is a Brownian motion with drift $r - (1/2)$.*

Remarks. (i) The density of $\mathbb{P}^*$ with respect to $\tilde{\mathbb{P}}$ is $Z^*/\tilde{Z}$, which requires f to be strictly positive in order to be well defined.

(ii) In the risk-neutral world, Black-Scholes prices apply, since the discounted asset price process is a log Brownian motion with constant volatility.

(iii) The diversity of possible (real-world) distributions of returns is limitless; we could have fat-tailed returns, non-independent returns, returns with different second moments each day, returns generated by a GARCH process, or by a stationary Gaussian time series, we could even have independent returns with a common Gaussian distribution with variance π! One class of return distributions which we could not capture would be where the returns were a compound Poisson process, but for more general Lévy processes, the law is often absolutely continuous.

3. Construction in the case $N = 1$. In the case $N = 1$, the change-of-measure martingale $(\tilde{Z}_t)$ has the simple representation

$$\begin{aligned} \tilde{Z}_t &\equiv \mathbb{E}[\tilde{Z}|\mathcal{F}_t] \\ &= \mathbb{E}[\,\frac{f}{\varphi}\,(X_1)|\mathcal{F}_t] \\ &= \int p_{1-t}(y)\,\frac{f}{\varphi}\,(X_t + y)dy \\ &\equiv h(t, X_t) \end{aligned}$$

say, where $p_t(x) \equiv (2\pi t)^{-1/2} \exp(-x^2/2t)$ is the Brownian transition density.

Applying Itô's formula to this martingale, we obtain

$$\begin{aligned} d\tilde{Z}_t &= h_x(t, X_t) dX_t \\ &= \frac{h_x(t, X_t)}{h(t, X_t)} \tilde{Z}_t dX_t. \end{aligned}$$

A familiar consequence of Girsanov's Theorem (Rogers & Williams [1987], Theorem IV.38.5 again) is that under $\tilde{\mathbb{P}}$ the process X can be expressed as

$$dX_t = d\tilde{W}_t + \nabla \log h(t, X_t) dt. \tag{1}$$

This tells us that in the real-world probability $\tilde{\mathbb{P}}$, the log-price process X must solve the stochastic differential equation (1). Thus by imposing a suitable drift on the constant-volatility log-price process, we can make the return at time 1 any law with a strictly positive density! The recipe detailed above shows how to determine the drift required to achieve any given law. Now we could in fact extend the above argument by induction to the case of general N, using the step just given to show how to construct the required conditional distribution of the return Y_{N+1} given $\mathcal{F}_N$; this would then tell us how we should make the log-price process drift during all the time intervals. However, such a proof seems clumsy in comparison with the one we gave in Section 2.

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