Decomposing the branching Brownian path

by

Kalvis M. Jansons & L.C.G. Rogers

Summary. We consider a real-valued branching Brownian motion where particles are killed at rate μ and split at rate $\lambda \leq \mu$ into two independent offspring particles. The process dies out almost surely, so reaches some lowest level. We prove a decomposition of the branching Brownian path at its minimum. The post-minimum path is just branching Brownian motion conditioned never to go beneath its initial point. The pre-minimum piece is independent of the post-minimum piece, and has the same law as the post-minimum piece reweighted by a functional of the endpoints of the tree. Applications to branching polymers are discussed.

Abbreviated title: Decomposing the branching path.

Decomposing the branching Brownian path

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1. Introduction. We shall consider a real-valued branching Brownian motion with drift $c \in \mathbb{R}$, which is killed at rate μ and gives birth to single offspring at rate $\lambda \leq \mu$. This ensures that the process dies out in finite time almost surely, so the path of the branching Brownian motion is bounded below. In this paper, we state and prove a decomposition of the branching Brownian motion at its minimum, analogous to the decomposition at its minimum of an upward-drifting Brownian motion (Williams [10]), or more closely analogous to the decomposition at its minimum of an exponentially-killed piece of Lévy process (Greenwood and Pitman [4]). Indeed, in the limiting situation where the birth rate λ drops to zero, we recover the well-known decomposition at its minimum of an exponentially-killed piece of Brownian path.

If one started the branching Brownian motion (BBM) at 0, one could condition it never to enter $(-\infty, -\epsilon]$, where $\epsilon > 0$ is small (so one is conditioning every particle of the process to remain in $(-\epsilon, \infty)$ until death); in this way one would obtain a branching diffusion process in $(-\epsilon, \infty)$, which converges to a non-degenerate limit process as $\epsilon \downarrow 0$. If P_* denotes the law of this limit branching diffusion, *then P_* is the law of the post-minimum piece of the BBM.* It is a homogeneous Markov branching diffusion, with position-dependent splitting and birth rates; see Jansons & Rogers [5] for the exact form of the process, and a number of calculations.

The description of the pre-minimum piece of the BBM is not so obvious. For one thing, the parametrisation of the pre-minimum piece in terms of time is inappropriate and clumsy; one should be parametrising in terms of the 'time through the tree'. To understand this, start at the minimum, and trace back towards the origin of time; after a while, you come to a place where the particle splits, and you could follow either of the branches, one now going *forward* in real time, the other going *backward* in real time. Thus in terms of 'time through the tree', going back from the minimum we see a branching diffusion process not unlike the post-minimum part of the BBM. An obvious

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and appealing conjecture is that this process has the same law as the post-minimum piece, and is independent (given the height of the minimum). This conjecture turns out to be false, but something very similar is true. If V is the number of ends of the branching diffusion, and if $x_1, \ldots, x_V$ are the heights above the minimum of the endpoints of the branching diffusion shape, then the law of the pre-minimum piece of the BBM P^* is related to P_* by

(1.i) $\qquad dP^* \equiv \frac{Z}{E_* Z} dP_*,$

(1.ii) $\qquad Z \equiv \sum_{j=1}^{V} \exp(-2cx_j).$

(Informally, the reason for the appearance of the sum of V terms is that each of the V ends could have been the root of the tree.) In contrast to the post-minimum process, under P^* the branching shape is *not* a Markov branching diffusion.

Thus the path decomposition result says that *the pre- and post-minimum pieces of the BBM are independent (given the height of the minimum), with laws P^* and P_* respectively.* An accurate statement and proof of this result occupy §2.

Thus the pre- and post-minimum parts of the tree are similar, but differ by a reweighting of the measure; the pre-minimum piece is 'bigger'. Indeed, if N_1 is the number of internal vertices in the pre-minimum piece, and N_2 is the number in the post-minimum piece, we compute EN_1 and EN_2 for the case $c = 0$ and show that as $\mu \downarrow \lambda$, $EN_1 \to \infty$, but $EN_2 \to \frac{1}{2}$. One can understand qualitatively why the pre-minimum piece should be bigger, because in the pre-minimum piece a particle J may split off from the parent particle and then go diffusing and splitting while the parent particle heads down toward the minimum; if the minimum is a reasonable distance below the place where the particle J was born, the requirement that J and its descendants do not go below the minimum is not unduly restrictive, so the pre-minimum piece can have quite big branches which are unlikely for the post-minimum piece.

There are at least two areas where the branching Brownian motion and its decomposition are of interest. The first is in the modelling of branching polymers in a fluid; see De'Bell, Lookman & Zhao [2] for analysis of aspects of a discrete model of branching polymers, Silver [8], Parsons & Subjeck [6], and Viitala & Järnefelt [9] for examples of branching polymers in biology, and Jansons & Rogers [5] for an exposition of probabilistic methods in polymer physics, including applications to conditioned

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branching Brownian motion. A branching polymer does not have any distinguished root, or direction of time running through it, so to model these things, we have to quotient out this part of the BBM description. In §3 we do this, showing that it is possible to put a measure on the space of branching polymer shapes with the property (stated here for $c = 0$) that *if one takes a random branching polymer shape, picks an end of it at random and uses this as the root of a rooted branching-diffusing process, then the process one obtains is just BBM.* We then give a description of a model of a branching polymer attached to a wall by one of its ends. It is worth pointing out that the law of this process is neither P_* or P^*, but is in fact *a slightly different reweighting of P_*.* The framework we have set up allows us to compute many of the things we might want to know about because of the connection with BBM; as an example, we compute the expected number of ends of a branching polymer attached by one of its ends to a wall.

The second area where branching Brownian motion is extensively studied is measure-valued diffusions and superprocesses; as a first entrance to a vast and growing literature, we refer the reader to Dawson, Iscoe & Perkins [1], Dynkin [3], Perkins [7]. The decomposition at the minimum of a BBM splits the process into two; how is this reflected in the measure-valued limiting process?

Finally, we remark that the methods and results of this paper could be generalised to any subcritical branching Lévy process.

§**2. Decomposing branching Brownian motion**. Ignoring the motion of the particles, we see a binary branching tree with a distinguished root, internal vertices (where a particle gives birth to an offspring) and external vertices (where a particle dies). We consider the root to be an external vertex, so that there are always two more external vertices than internal.

Now consider what happens on one of the edges of the tree; there is a drifting Brownian motion X running along until an independent $\exp(\lambda + \mu)$ time T, say. Define

$$\rho \equiv (c^2 + 2\lambda + 2\mu)^{\frac{1}{2}},$$

$$\alpha \equiv \rho - c, \quad \beta \equiv \rho + c.$$

Then it is well known (see, for example, Greenwood & Pitman [4]) that the path $(X_t)_{0 \le t \le T}$ can be decomposed at its minimum in the following way.

LEMMA 1. *On some probability space, take independent random elements*

(i) ξ *with* $\exp(\beta)$ *distribution;*

(ii) ξ' *with* $\exp(\alpha)$ *distribution;*

(iii) $(Y_t)_{t \ge 0}$ *and* $(Y_t')_{t \ge 0}$ *both Brownian motions with drift* $-\rho$.

If now $\tau = \inf\{u : Y_u = -\xi\}, \tau' \equiv \inf\{u : Y_u' = -\xi'\}$, *and we define*

$$Z_t = \begin{cases} Y_t, & 0 \le t \le \tau \\ Y_{\tau+\tau'-t}' & +\xi' - \xi, \end{cases} \qquad \tau \le t \le \tau + \tau'$$

then $(Z_t)_{0 \le t \le \tau + \tau'}$ *has the same distribution as* $(X_t)_{0 \le t \le T}$.

The feature of Lemma 1 which we shall exploit is this; given the value $X_0, J \equiv \inf\{X_s : 0 \le s \le T\}, X_T$, we could reconstruct (the distribution of) $(X_t)_{0 \le t \le T}$ by running a Brownian motion with drift $-\rho$ until it hit J, and then by running an independent Brownian motion with drift $-\rho$ backward down from X_T until it hit J. *This recipe gives a regular conditional distribution for* $(X_t)_{0 \le t \le T}$ *given* (X_0, J, X_T).

The probabilistic structure of the branching Brownian motion can thus be described informally by the following three steps:

(2.i) choose the 'family tree', assigning an independent geometric (q) number of offspring to each individual;

(2.ii) decide the heights of each of the vertices of the tree, and the minimum height of each edge, recursively by choosing the minimum Y on an edge which starts at height x_0 according to the density

$$\beta \exp(-\beta(x_0 - y)) \, I(x_0 > y) dy$$

and then, given the value of the minimum on the edge, choosing the final value X_1 according to the density

$$\alpha \exp(-\alpha(x_1 - y)) \, I_{(x_1 > y)} dx_1;$$

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(2.iii) knowing the heights of each vertex, and the smallest value on each edge, use the regular conditional probability for the path given the initial, final and minimal values to fill in the behaviour of the particle on each edge of the tree.

We can make this precise by defining the set $\mathcal{E}$ of edges with heights to be

$$\mathcal{E} = \{(x_0, y, x_1); \quad x_0 > y, x_1 > y\},$$

then taking the set $\mathcal{J}_1(x)$ to be

$$\mathcal{J}_1(x) = \{(x_0, y, x_1) \in \mathcal{E}; \quad x_0 = x\},$$

and then defining for $n > 1$

$$\mathcal{J}_n(x) = \bigcup_{j=1}^{n-1} \{(\tau, \xi, \tau') : \xi \equiv (x, y, x') \in \mathcal{E}, \tau \in \mathcal{J}_j(x'), \tau' \in \mathcal{J}_{n-j}(x')\}.$$

Finally, let the set $\mathcal{J}(x)$ of rooted family trees with heights (RFTH) started at x be defined by

$$\mathcal{J}(x) \equiv \bigcup_{n \geq 1} \mathcal{J}_n(x).$$

We shall shortly define our probability measure on $\mathcal{J}(x)$, but first we stress that these are *family* trees; when a particle gives birth, the new one is considered to be distinct from the old. This is to be contrasted with the situation where particles split into two indistinguishable particles; while the family tree can be converted into one of the latter by ignoring the distinction between new and old particles, it is essential to some of the counting arguments that we *retain the distinction*, at least until the analysis has proceeded far enough to allow us to drop the distinction without harm. Just to be quite explicit, if $(\tau, \xi, \tau') \in \mathcal{J}_n(x)$, with $\xi \equiv (x, y, x') \in \mathcal{J}_1(x)$, and $\tau \in \mathcal{J}_j(x')$, $\tau' \in \mathcal{J}_{n-j}(x')$, then we consider τ to be the future of the parent particle, τ' to be the future of the offspring particle.

Now we specify the distribution to be put on $\mathcal{J}(x_0)$. If we let

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$$\phi_-(x, y) \equiv \beta e^{-\beta(x-y)} I_{(x>y)}$$

$$\phi_+(y, x) \equiv \alpha e^{-\alpha(x-y)} I_{(x>y)},$$

and if T denotes the random RFTH, then

$$P_{x_0}(T = (x_0, Y, X_1) \in \mathcal{J}_1(x_0), \ Y \in A, \ X_1 \in B)$$

$$= q \int_{A \times B} \phi_-(x_0, y) \phi_+(y, x_1) dy \ dx_1$$

and recursively, for $\tau \in \mathcal{J}_j$, $\tau' \in \mathcal{J}_{n-j}$,

$$P_{x_0}(T = (\tau, \xi, \tau'), \ \xi = (x_0, Y, X_1), \ \tau \in \Gamma, \tau' \in \Gamma', \ Y \in A, X_1 \in B)$$

$$= p \int_{A \times B} \phi_-(x_0, y) \phi_+(y, x_1) dy \ dx_1 \ P_{x_1}(\tau \in \Gamma) P_{x_1}(\tau' \in \Gamma').$$

Informally, but more intuitively, if the edges of the RFTH are indexed by S, edge $e \in S$ starting at vertex $i(e)$ and ending at vertex $j(e)$, and if the height at vertex i is ξ_i, minimum height on edge e is η_e, then the probability density of the tree is

$$\left(\prod_{e \in S} \phi_-(\xi_{i(e)}, \eta_e) \phi_+(\eta_e, \xi_{j(e)}) d\xi_{j(e)} d\eta_e \right) p^m q^{\ell-1}$$

where m is the number of internal vertices in the tree (i.e. with valence 3), ℓ is the number of external vertices (i.e. with valence 1), to include the root. Now the root plays a distinguished rôle which is a hindrance in our study of branching polymers, and even in the decomposition of branching Brownian motion. It is needed only to provide some reference altitude on the path, and for what follows it is better to use the *minimum* of the path to provide this reference height, which we shall shift to 0. Thus if the minimum of a RFTH occurs on edge e_*, we define

$$x_i \equiv \xi_i - \eta_{e_*}, \qquad y_e \equiv \eta_e - \eta_{e_*}$$

and the probability density of the tree can be expressed as

$$\tilde{\Pi}_- \tilde{\Pi}_+ \quad \phi_-(x_{i(e_*)}, 0) dx_{i(e_*)} \phi_+(0, x_{j(e_*)}) dx_{j(e_*)} \ p^m q^{\ell-1},$$

where

$$\tilde{\Pi}_- = \prod_{e \in S_-} \phi_-(x_{i(e)}, y_e)\phi_+(y_e, x_{j(e)})dx_{i(e)}dy_e$$

$$\tilde{\Pi}_+ = \prod_{e \in S_+} \phi_-(x_{i(e)}, y_e)\phi_+(y_e, x_{j(e)})dy_e\,dx_{j(e)},$$

and where S_- is the set of edges before e_*, S_+ the set of edges after e_*.

Let us notice that

$$\phi_-(x_i, y)\phi_+(y, x_j) = \alpha\beta e^{-\rho(x_i - y + x_j - y)}e^{c(x_j - x_i)}I_{\{x_i, x_j > y\}}$$

so that the density can be written as

$$(3) \qquad \prod_e r(x_{i(e)}, y_e, x_{j(e)}). \quad \exp\left\{\sum_{j \text{ external}} cx_j - \sum_{i \text{ internal}} cx_i\right\}e^{-2cx_0} \, p^m \, q^{\ell-1},$$

where 0 is the initial node, and

$$r(x, y, x') \equiv 2(\lambda + \mu)\exp(-\rho(x + x' - 2y))I_{(x, x' > y)}.$$

The point of this is that *the distinguished rôle of the root has largely disappeared from the expression for the density*; indeed, the root is only distinguished by the factor e^{-2cx_0} (which is 1 in the special case $c = 0$, for example).

We need next to consider branching Brownian motions with drift c, started at some point $x > 0$ and conditioned never to go into $(-\infty, 0]$. If we set

$$\psi(x) = P_x \quad \text{(no particle ever enters } (-\infty, 0])$$

then by considering what happens at the first birth or death we see that

$$(4) \qquad\qquad\qquad \psi = R^\partial_{\lambda+\mu}(\lambda\psi^2 + \mu),$$

where $(R^\partial_\alpha)_{\alpha > 0}$ is the resolvent of the drifting Brownian motion killed at 0, from which easily ψ satisfies

$$(5) \qquad\qquad \tfrac{1}{2}\psi'' + c\psi' + (1 - \psi)(\mu - \lambda\psi) = 0, \quad \psi(0) = 0, \quad 0 \le \psi \le 1.$$

The solution can be found explicitly in the case $c = 0$; see the Appendix.

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Now suppose we take $\epsilon > 0$ as the starting point of a RFTH and insist that no part of the RFTH goes below 0. The probability density of a tree with m internal vertices and ℓ external is then

$$\left(\prod_e \tilde{\phi}_-(\xi_{i(e)}, \eta_e) \tilde{\phi}_+(\eta_e, \xi_{j(e)}) d\eta_e \right) \ p^m q^{\ell-1},$$

where

$$\tilde{\phi}_\pm(x, y) \equiv \phi_\pm(x, y) I_{(x>0, y>0)}$$

If edge 0 from vertex 0 to vertex 1 is the initial edge, recalling that $\xi_0 = \epsilon$, if we integrate out η_0 in this density we get

$$\left(\int_0^\epsilon \tilde{\phi}_-(\epsilon, \eta) \tilde{\phi}_+(\eta, \xi_1) d\eta \right) d\xi_1 \left(\prod_{e \neq 0} \tilde{\phi}_-(\xi_{i(e)}, \eta_e) \tilde{\phi}_+(\eta_e, \xi_{j(e)}) d\xi_{j(e)}, d\eta_e \right) p^m \ q^{\ell-1}$$

$$\sim \beta \epsilon \tilde{\phi}_+(0, \xi_1) d\xi_1 \left(\prod_{e \neq 0} \tilde{\phi}_-(\xi_{i(e)}, \eta_e) \tilde{\phi}_+(\eta_e, \xi_{j(e)}) d\xi_{j(e)} d\eta_e \right) p^m \ q^{\ell-1}.$$

Thus if we were to take the RFTH started at ϵ, condition it not to go below 0, and then let $\epsilon \downarrow 0$, we shall get a conditioned rooted family tree with heights (CRFTH), which has density

$$\text{(6)} \qquad \frac{\beta}{\psi'(0)} \tilde{\phi}_+(0, \xi_1) d\xi_1 \left(\prod_{e \neq 0} \tilde{\phi}_-(\xi_{i(e)}, \eta_e) \tilde{\phi}_+(\eta_e, \xi_{j(e)}) d\xi_{j(e)} d\eta_e \right) p^m \ q^{\ell-1}.$$

The CRFTH is a slightly different object from the ones we have been studying before; the first edge has just an initial height, 0, and a final height specified, whereas all others have initial, final and minimum heights specified. Let us call such a tree a positive rooted family tree with heights (PRFTH); every CRFTH is a PRFTH, and a CRFTH is a PRFTH chosen at random according to the density (6).

The CRFTH is, not surprisingly, the piece of the branching Brownian motion after the minimum. To explain the piece of the branching Brownian motion *before* the minimum, it is helpful to introduce the notion of a *branching* tree, as distinct from a *family* tree. A (rooted) branching tree with heights (RBTH) which has m internal vertices is formally

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the set, containing 2^m elements, of those RFTHs which are the same if the distinction between new and old particles is ignored; it is an equivalence class under the equivalence relation of ignoring new/old distinctions.

Thus, for example, the probability density for a CRBTH is

$$(7) \qquad \frac{\beta}{\psi'(0)}\tilde{\phi}_+(0,\xi_1)d\xi_1\left(\prod_{e\neq 0}\tilde{\phi}_-(\xi_{i(e)},\eta_e)\tilde{\phi}_+(\eta_e,\xi_{j(e)})d\xi_{j(e)}d\eta_e\right)(2p)^m q^{\ell-1}.$$

Before we state the main result, we set up some notation. When we take a RBTH, shift the minimum to level zero, and split at the minimum, we get a post-minimum PRBTH T'', and before the minimum we see the RBTH T' started at the original root. If we took the pre-minimum RBTH T' and took the minimum as the new root, moving through the tree from the minimum, we should see a PRBTH $\tilde{T}$, starting at vertex $\tilde{0}$, the minimum.

THEOREM. *The decomposition of the branching Brownian motion is as follows:*

(8.i) *the trees $\tilde{T}$ and T'' are independent;*

(8.ii) *the tree T'' is a CRBTH;*

(8.iii) *the law of $\tilde{T}$ is the same as the of T'' but weighted by the density*

$$(9) \qquad \frac{\psi'(0)^2}{\beta^2 q}\sum_{j \text{ external}} e^{-2cx_j}.$$

Proof. It is clear that we only need to explain the decomposition at the minimum of the RBTH associated with the drifting branching Brownian motion.

To prove the theorem, let us take PRBTH's τ'' and $\tilde{\tau}$ and compute the probability (density) of T'' being τ'', and $\tilde{T}$ being $\tilde{\tau}$. Suppose that T'' (respectively, $\tilde{T}$) has m'' internal vertices, and ℓ'' external vertices, 0 included (respectively, $\tilde{m}, \tilde{\ell}$). The PRBTH τ'' can arise as a PRFTH in exactly $2^{m''}$ ways, corresponding to the $2^{m''}$ choices of new/old distinctions possible at the internal vertices. On the other hand, the PRBTH $\tilde{\tau}$ can arise in *exactly* $(\tilde{\ell}-1)2^{\tilde{m}}$ ways, corresponding to the different choices of new/old at the internal vertices, *and to the $\tilde{\ell}-1$ possible external vertices which could have*

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been the root of the original branching Brownian motion. Not all of these $(\tilde{\ell} - 1)2^{\tilde{m}}$ possibilities will be equally likely (except in the case $c = 0$).

We thus have

(10)
$$P(T = \tau, \tilde{T} = \tilde{\tau}) = 2^{\tilde{m}+m''}\left(\sum_{\substack{j \text{ external} \\ \text{vertex of } \tilde{\tau}}} e^{-2cx_j}\right)\tilde{\Pi}.\Pi''.q^{-1}$$

where

$$\Pi'' = \phi_+(0, x_1'')dx_1''$$
$$\left(\prod_{e''\neq 0}\phi_-(x_{i(e'')}, y_{e''})\phi_+(y_{e''}, x_{j(e'')})dy_{e''}dx_{j(e'')}\right)p^{m''}q^{\ell''-1}$$

(11)
$$= e^{-\rho x_1''}dx_1''\left(\prod_{e''\neq 0} r(x_{i(e'')}, y_{e''}, x_{j(e'')})dy_{e''}dx_{j(e'')}\right)$$
$$\exp\left\{\sum_{j'' \text{ external}} cx_{j''} - \sum_{i'' \text{ internal}} cx_{i''}\right\}p^{m''}q^{\ell''-1}$$

and

$$\tilde{\Pi} = \phi_+(0, \tilde{x}_1)d\tilde{x}_1\left(\prod_{\tilde{e}\neq 0}\phi_-(x_{i(\tilde{e})}, y_{\tilde{e}})\phi_+(y_{\tilde{e}}, x_{j(\tilde{e})})dy_{\tilde{e}}dx_{j(\tilde{e})}\right)p^{\tilde{m}}q^{\tilde{\ell}-1}$$

is the analogous expression for the PRBTH $\tilde{\tau}$. Do notice the extra factor of q^{-1} in (10); it is here because in the original RBTH there are $\tilde{\ell} + \ell'' - 3$ external nodes apart from the root, so there are $\tilde{\ell} + \ell'' - 3$ places where a particle dies. We re-express (10) as

(12)
$$\left(\frac{\psi'(0)}{\beta q}\left(\sum_{\substack{j \text{ external} \\ \text{vertex of } \tilde{\tau}}} e^{-2cx_j}\right)\tilde{\Pi}.2^{\tilde{m}}\right)\left(\frac{\beta}{\psi'(0)}\Pi''2^{m''}\right),$$

the point being that the second factor *is the probability (density) of the CRBTH.* This completes the proof of the theorem. $\square$

One obvious corollary of the Theorem is that the expectation of the density (9) with respect to the CRBTH law must be 1. We provide an independent verification of this for the case $c = 0$ in the Appendix.

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It turns out to be quite easy to compute now the distributions of the number of internal vertices both before and after the minimum in the case $c = 0$. Indeed, if N_1 is the number of internal vertices in the pre-minimum piece, and N_2 is the number in the post-minimum piece, and $\phi_i(s) = Es^{N_i}$, $i = 1, 2$, then if $N = N_1 + N_2$ is the total number of internal vertices, we have

$$\phi_1(s)\phi_2(s) = \phi(s) \equiv Es^N = (2sp)^{-1}[1 - (1 - 4spq)^{\frac{1}{2}}],$$

since ϕ satisfies

$$\phi = q + sp\phi^2.$$

But we know from the theorem that

$$E(s^{N_1}) = E[(1 + N_2)s^{N_2}]/E(1 + N_2),$$

because the number of external vertices (excluding the root) of the post-minimum tree is just $1 + N_2$, and to get the law of the pre-minimum piece, we simply weight the post-minimum part by the number of external vertices.

Thus

$$\phi_1(s) = (E(1 + N_2))^{-1}\frac{d}{ds}[s\phi_2(s)].$$

Writing $u(s) \equiv s\phi_2(s)$, $b \equiv E(1 + N_2)$, we deduce that

$$2bs\phi(s) = \frac{d}{ds}[u(s)^2]$$

implying that

$$u(s)^2 = A + \frac{b}{p}\left\{ s + \frac{(1 - 4spq)^{3/2}}{6pq} \right\}$$

where A is a constant of integration, fixed by the obvious requirements $u(1) = 1$, $u(0) = 0$. These together determine that

(13.i) $$b \equiv E(1 + N_2) = \frac{3q}{3q - p} = \frac{3\mu}{3\mu - \lambda},$$

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(13.ii) $$A = -1/(2p(3q - p)).$$

[The first of these confirms the argument leading to (A3).] To summarise, then,

(14)
$$\phi_2(s) = \frac{1}{s}\left[\frac{3q}{(3q-p)p}\left\{s + \frac{(1-4spq)^{3/2} - 1}{6pq}\right\}\right]^{\frac{1}{2}}.$$

It is easy to compute

$$EN = \frac{p}{q - p} = \frac{\lambda}{\mu - \lambda},$$

either from ϕ, or directly. Using (13.i), we conclude that

(15)
$$EN_2 = \frac{p}{3q - p}, \quad EN_1 = \frac{2pq}{(q - p)(3q - p)}$$

which highlights the asymmetry between the pre- and post-minimum parts of the tree; as $q \downarrow \frac{1}{2}$, one sees that

$$EN_2 \to \tfrac{1}{2}, \quad EN_1 \to \infty.$$

§**3. Tree shapes with heights.** The RFTH was a useful notion for setting up, but gave way soon enough to RBTH, where the new/old distinction of particles was forgotten. We shall now coarsen the specification further, to give what really matters for polymer modelling, namely the tree shape with heights (TSH). We identify two RBTH if

(i) one is obtained from the other by shifting all the heights by the same amount;

or

(ii) one is obtained from the other by nominating a different external vertex as root.

Without loss of generality, the minimum of a TSH will be at zero, and the TSH is then specified by its 'shape' and the heights of each vertex, together with the minimum height on each edge. Thus the probability (density) of a TSH is

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(16)
$$\left(\sum_{\substack{j \text{ external} \\ \text{vertices}}} e^{-2cx_j}\right)\left\{\prod_e r(x_{i(e)}, y_e, x_{j(e)})\right\} \exp\left\{\sum_{j \text{ external}} cx_j - \sum_{i \text{ internal}} cx_i\right\}(2p)^m q^{\ell-1}$$

with respect to $\prod_i dx_i \prod_{e \neq e_*} dy_e$, where the meanings of the symbols are as before. This specifies the law in use when we take a 'random branching polymer'. It is the natural choice to take when one considers the following fact, now obvious by comparing (3) and (16); if one chooses a random branching polymer, then picks at random one of its external vertices according to the distribution

$$p_j \equiv e^{-2cx_j}\left\{\sum_{i \text{ external}} e^{-2cx_i}\right\}^{-1}$$

and uses this external vertex as the root of the tree, then *the random RBTH obtained is just branching Brownian motion with drift.*

To illustrate how various 'conditional' polymer shapes can be analysed, we shall consider the law of a TSH conditioned to have the minimum at one of the external vertices. We can think of this as the law of a *branching polymer attached to a wall by one end.* It is the limit as $\varepsilon \downarrow 0$ of the law of a TSH conditioned to have one end in $[0, \varepsilon)$. Comparing (11) and (15), it is not too hard to see that if

$$Z \equiv \sum_{j \text{ external}} e^{-2cx_j},$$

(*all* external vertices are counted, including the one where the minimum is achieved) then the law $\tilde{P}_*$ of the TSH conditioned to have its minimum at one of the external vertices is given by

$$\frac{d\tilde{P}_*}{dP_*} = \frac{Z}{E_*(Z)}.$$

In the special case $c = 0$, Z is just the total number of external vertices, which is 2 greater than the number of internal vertices. But we computed (18) that

$$EN_2 = \frac{p}{3q-p}, \quad EN_1 = \frac{EN_2(1+N_2)}{E(1+N_2)} = \frac{2pq}{(q-p)(3q-p)},$$

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where N_2 is the number of internal vertices in the CRBTH, $N_2 = Z - 2$. From this, then, we can easily compute the expected number of external vertices on the branching polymer with one external vertex attached to a wall; it is

$$\text{(19)} \qquad \tilde{E}_* Z = \frac{36q^3 - 45pq^2 + 16p^2q - p^3}{(6q - p)(3q - p)(q - p)}.$$

Appendix. We shall assume here that $c = 0$, and that $0 < \lambda < \mu$. Thus if

$$\psi(x) \equiv P_x \quad (\text{no particle ever enters } (-\infty, 0))$$

we have

$$\psi = R^{\partial}_{\lambda+\mu}(\lambda\psi^2 + \mu)$$

(see (3) and (4)) from which

$$\psi(x) = 1 - \sinh^2 \gamma \, \operatorname{cosech}^2(\gamma + akx)$$

where

$$a = \left(3(\mu - \lambda)/2\lambda\right)^{\frac{1}{2}}, \quad \gamma = \sinh^{-1}(a), \quad k = \sqrt{\lambda/3}.$$

As $\mu \downarrow \lambda$, we obtain

$$\psi(x) = 1 - (1 + kx)^{-2}.$$

For general $\lambda \leq \mu$, and $c = 0$, we have

$$\psi'(0) = 2\left[\frac{3\mu - \lambda}{6}\right]^{\frac{1}{2}}.$$

Our aim now is to compute

$$g(x) \equiv E_x[\text{number of ends in the tree; no particle enters } (-\infty, 0)].$$

By considering the first birth or death,

$$g = R^{\partial}_{\lambda+\mu}(2\lambda g\psi + \mu),$$

where g must solve

$(A1) \qquad \frac{1}{2}g'' - (\lambda + \mu)g + 2\lambda g\psi + \mu = 0, \quad g(0) = 0, \quad g \geq 0, \quad \text{bounded}.$

Boundedness is assured because $\mu > \lambda > 0$. One solution of the differential equation can be verified to be

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$$g_0(x) = \frac{2\mu \sinh^2(\gamma + akx)}{5(\mu - \lambda)},$$

so the task is now to find the general solution to the homogeneous d.e.

$(A2)$
$$\tfrac{1}{2}g'' - (\lambda + \mu)g + 2\lambda g\psi = 0$$

and match the boundary conditions on g. Setting $g(x) = h(\gamma + akx)$, the equation (A2) becomes

$$h''(y) = 4(1 + \mathrm{cosech}^2 y) \ h(y),$$

which has

$$h_1(y) \equiv e^{2y} \coth y(\coth y - 1)^2 = \cosh y \ \mathrm{cosech}^3 y$$

as one solution. The other solution is of the form

$$h_2(y) = h_1(y)v(y),$$

where

$$v' = h_1^{-2},$$

so

$$v(y) = \int_0^y \frac{\sinh^6 x}{\cosh^2 x} dx$$
$$= \frac{15y}{8} - \tfrac{1}{2}\sinh 2y + \frac{1}{32}\sinh 4y - \tanh y,$$

whence

$$h_2(y) = \frac{\cosh y}{\sinh^3 y}\left[\frac{15y}{8} - \tfrac{1}{2}\sinh 2y + \frac{1}{32}\sinh 4y - \tanh y\right].$$

Writing y for $\gamma + akx$, we obtain the general solution to (A2) as

$$\frac{2\mu}{5(\mu - \lambda)}\sinh^2 y + Ah_1(y) + Bh_2(y),$$

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and for boundedness at infinity, we must have

$$B = -\frac{8\mu}{5(\mu - \lambda)}.$$

The boundary condition at $x = 0$ now implies that

$$A = \frac{2\mu}{5h_1(\gamma)(\mu - \lambda)}\{4h_2(\gamma) - \sinh^2\gamma\}.$$

Straightforward calculations lead to

$$g'(0) = ak\frac{2\mu \tanh\gamma}{(\mu - \lambda)}.$$

Now we have

$$
\begin{aligned}
\frac{g(x)}{\psi(x)} &= E_x \text{ (number of ends in tree} \mid \text{no particle enters } (-\infty, 0)) \\
&\to E_0 \text{ (number of ends in tree} \mid \text{no particle enters } (-\infty, 0)) \\
(A3) \qquad &= g'(0)/\psi'(0) \\
&= \frac{\mu}{\mu - \lambda}\tanh^2\gamma \\
&= \frac{3\mu}{3\mu - \lambda}.
\end{aligned}
$$

In order that the weighting (11) should convert the law of CRBTH into a probability measure, we must have that

$$\frac{\psi'(0)^2}{\beta^2 q}\cdot\frac{g'(0)}{\psi'(0)} = 1.$$

A few lines of calculation now confirm that this is the case.

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