Dual valuation and hedging of Bermudan options.

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Abstract. Some years ago now, a different characterisation of the value of a Bermudan option was discovered, which can be thought of as the viewpoint of the seller of the option, in contrast to the conventional characterisation which took the viewpoint of the buyer. Since then, there has been a lot of interest in finding numerical methods which exploit this dual characterisation. This paper presents a pure dual algorithm for pricing and hedging Bermudan options.

1 Introduction.

This paper derives an algorithm for valuing and hedging an Bermudan[1] option, from a 'pure dual' standpoint. Apart from various changes of names, pricing a Bermudan option is the same as solving an optimal stopping problem, arguably the simplest possible stochastic optimal control problem. For as long as derivatives have been priced within the Black-Scholes paradigm, the traditional value-function approach, and the associated Bellman equations, have been widely used in the attempt to price Bermudan options; the whole area has been a mathematical playground, because of the scarcity of a closed-form solutions, and the consequent need for approximations, estimates and asymptotics to come up with prices.

During the last century, the value-function approach was in effect the *only* method available, but in recent years another quite different 'dual' approach has been discovered: see Rogers [3], Haugh & Kogan [2]. The main result is that if the reward process[2] is denoted Z,

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[1]While we shall discuss only discrete-time problems in this paper, the methodology could be applied to the pricing of American options after discretizing the time.

[2]Some minor regularity condition needs to be imposed on Z; it is sufficient that $\sup_{0\le t\le T}|Z_t| \in L^p$ for some $p > 1$.

then the value Y_0^* of the optimal stopping problem can be alternatively expressed as

$$Y_0^* = \sup_{\tau\in\mathcal{T}} E[Z_\tau] = \min_{M\in\mathcal{M}_0} E\Big[\sup_{0\le t\le T}(Z_t - M_t)\Big], \tag{1.1}$$

where $\mathcal{T}$ is the set of stopping times bounded by T, the time horizon for the problem, and $\mathcal{M}_0$ is the set of uniformly-integrable martingales vanishing at zero. The minimum *is* attained, by the martingale M^* of the Doob-Meyer decomposition of the Snell envelope process of Z, and in that case

$$\sup_{0\le t\le T}(Z_t - M_t^*) = Y_0^* \quad \text{almost surely.} \tag{1.2}$$

The traditional approach via the value function and the Bellman equations takes the viewpoint of the *buyer* of the option, who seeks to choose the best stopping time at which to exercise; the first expression for Y_0^* in (1.1) embodies this. The dual approach is the solution of the problem from the viewpoint of the *seller* of the option, who seeks a hedging martingale $Y_0^* + M_t$, whose value at all times will be at least the value of the reward process; and the result (1.2) shows that for the perfect choice of $M = M^*$ this does indeed happen.

Since the dual approach was discovered, there have been various attempts to apply it in practice, with mixed success; choosing a good martingale is at least as difficult as choosing a good stopping time! The early paper of Andersen & Broadie [1] uses a numerical approximation to the value function to suggest a good martingale to use, and in this way obtains quite tight bounds on both sides for a number of test examples. However, at a conceptual level, it has been an outstanding issue to derive a 'pure' dual method, which solves the optimal stopping problem *without need to calculate a value function,* using only the dual characterization of (1.1). Let us amplify the distinction: pure primal methods are well understood - indeed, virtually all solutions of the optimal stopping problem (and *all* solutions prior to the discovery of the dual characterization in the early 21^{st} century) are of this type; there are *hybrid* methods, such as that of Andersen & Broadie; but where is the pure dual method? It is the purpose of this short note to demonstrate how the solution may be derived by purely dual methods akin to the backward recursion of dynamic programming. The key observation, obvious from (1.1), is that the value of the optimal stopping problem is left unaltered if Z is replaced by $Z - M$, where $M \in \mathcal{M}_0$.

2 The algorithm.

In this Section, we specify the algorithm by which the given Bermudan option is to be hedged. We are given a reward process $(Z_t)_{t=0,\dots,T}$ adapted to the filtration $(\mathcal{F}_t)_{t=0,\dots,T}$, and the aim is to find some martingale $M \in \mathcal{M}_0$ such that (1.1) holds. If (1.1) holds, then by the earlier result of [3], [2] we also have (1.2). The construction is based on two very simple observations:

1. The value of the stopping problem for Z is the same as the value of the stopping problem for $Z + N$, where N is any martingale in $\mathcal{M}_0$;

2. Adding a constant to Z adds a constant to the value.

The proof of the following result shows how to solve the problem recursively, by constructing a sequence of martingales which do an ever better job of hedging. The idea is that the pathwise maximum must become a constant random variable - see (1.2); while it not obvious how we shall achieve this in one go, we can easily see how to ensure that the final value of Z is a constant, by subtracting a martingale which is equal to Z_T at time T. The inductive proof constructs martingales which are constant on some interval $[T-k, T]$ for ever bigger k.

Proposition 1. *There exists a sequence of constants a_j and a sequence of martingales $N^{(j)} \in \mathcal{M}_0$, $j = 1, \ldots, T+1$ such that*

$$\max_{T-j<i\leq T} Z_i^{(j)} = a_j, \tag{2.1}$$

where

$$Z_t^{(j)} \equiv Z_t - N_t^{(j)} \tag{2.2}$$

Proof. The proof proceeds by induction on j. To start the induction off, we consider the martingale

$$M_t^{(1)} = E_t[Z_T]$$

and set

$$N_t^{(1)} = M_t^{(1)} - E[M_T^{(1)}],$$

which is clearly in $\mathcal{M}_0$, and equally clearly achieves (2.1) for $j = 1$, with $a_1 = E[M_T^{(1)}] = E[Z_T]$.

Now suppose that (2.1) is true for $j \leq k$, and consider the non-negative martingale

$$M_t^{(k+1)} = E_t[\ \{Z_{T-k}^{(k)} - a_k\}^+\ ]. \tag{2.3}$$

This martingale is constant in $[T-k, T]$; *we shall subtract it from $Z^{(k)}$,* to form the process $\tilde{Z}_t^{(k)} \equiv Z_t^{(k)} - M_t^{(k+1)}$. Two cases then need to be considered:

1. if $Z_{T-k}^{(k)} > a_k$, then we have $\tilde{Z}_{T-k}^{(k)} = a_k$, and $\tilde{Z}_t^{(k)} \leq Z_t^{(k)}$ for all $t > T-k$, since $M^{(k+1)}$ is non-negative. Thus
$$\max_{T-k\leq t\leq T} \tilde{Z}_t^{(k)} = a_k;$$

2. if $Z_{T-k}^{(k)} \leq a_k$, then $M^{(k+1)}$ is zero in $[T-k, T]$, and by the inductive hypothesis
$$\max_{T-k\leq t\leq T} \tilde{Z}_t^{(k)} = \max_{T-k<t\leq T} Z_t^{(k)} = a_k;$$

either way, the conclusion is the same, namely that

$$\max_{T-k \le t \le T} \tilde{Z}_t^{(k)} = a_k. \tag{2.4}$$

We now define

$$\begin{aligned} N^{(k+1)} &= N^{(k)} + M^{(k+1)} - E[M_T^{(k+1)}], && (2.5) \\ a_{k+1} &= a_k + E[M_T^{(k+1)}], && (2.6) \end{aligned}$$

so that

$$\begin{aligned} Z_t^{(k+1)} &\equiv Z_t - N_t^{(k+1)} \\ &= \tilde{Z}_t^{(k)} + E[M_T^{(k+1)}] \end{aligned}$$

satisfies (2.1) taking $j = k+1$.

$\square$

Remarks. (i) Applying the Proposition in the case $j = T+1$, we learn that

$$\max_{0 \le i \le T} Z_i^{(T+1)} \equiv \max_{0 \le i \le T} \left\{ Z_i - N_i^{(T+1)} \right\} = a_{T+1}. \tag{2.7}$$

Hence the value of the optimal stopping problem with reward process $Z^{(T+1)}$ is equal to a_{T+1}, and by the first observation, this is actually the value of the optimal stopping problem for the original reward process Z.
(ii) The recursive construction generates an increasing sequence $a_1 \le a_2 \le \ldots \le a_{T+1}$ increasing to the value of the problem, as well as the hedging martingale.
(iii) Observe that by adding equations (2.5) and (2.6) we learn that

$$N^{(k)} + a_k = \sum_{i=1}^{k} M^{(i)}. \tag{2.8}$$

From (2.3) and (2.2) we deduce that

$$\begin{aligned} M_t^{(k+1)} &= E_t[\ \{Z_{T-k}^{(k)} - a_k\}^+\ ] \\ &= E_t[\ \{Z_{T-k} - N_{T-k}^{(k)} - a_k\}^+\ ] \\ &= E_t[\ \{Z_{T-k} - \sum_{i=1}^{k} M_{T-k}^{(i)}\}^+\ ], \end{aligned}$$

and taking $t = T-k$ leads to the conclusion

$$M_{T-k}^{(k+1)} = \{Z_{T-k} - \sum_{i=1}^{k} M_{T-k}^{(i)}\}^+. \tag{2.9}$$

The statement and proof of Proposition 1 is 'pure dual'; there is no mention of the value of the stopping problem, we only talked about the hedging martingales $M^{(k)}$. To tie things together, we shall now show how the constructs from the proof of Proposition 1 relate to the more familiar value process of the optimal stopping problem,

$$Y_t^* \equiv \sup_{\tau \in \mathcal{T}, \tau \geq t} E_t[\, Z_\tau \,]. \tag{2.10}$$

As is well known, Y^* is the Snell envelope process of the reward process Z, and Y_t^* is interpreted as the best that can be done if by time t the process has not been stopped.

Proposition 2. *For all $k = 0, 1, \ldots, T$ we have*

$$\sum_{i=1}^{k+1} M_{T-k}^{(i)} = Y_{T-k}^* \tag{2.11}$$

PROOF. The proof is by induction on k. Clearly the statement is true if $k = 0$, for both sides of (2.11) are equal to Z_T. Suppose now that the statement is true for all $k < n$, and consider

$$\begin{aligned}
\sum_{i=1}^{n+1} M_{T-n}^{(i)} &= M_{T-n}^{(n+1)} + \sum_{i=1}^{n} M_{T-n}^{(i)} \\
&= M_{T-n}^{(n+1)} + E_{T-n}[Y_{T-n+1}^*] \qquad \text{by inductive hypothesis} \\
&= \{Z_{T-n} - \sum_{i=1}^{n} M_{T-n}^{(i)}\}^+ + E_{T-n}[Y_{T-n+1}^*] \qquad \text{using (2.9)} \\
&= \{Z_{T-n} - E_{T-n}[Y_{T-n+1}^*]\}^+ + E_{T-n}[Y_{T-n+1}^*] \qquad \text{by inductive hypothesis} \\
&= \max\{\, Z_{T-n}, E_{T-n}[Y_{T-n+1}^*] \,\} \\
&= Y_{T-n}^*,
\end{aligned}$$

as required. □

REMARKS. (i) From (2.6) and Proposition (2) we see that $a_k = E[Y_{T-k+1}^*]$, which explains why the sequence a_k is increasing, and why a_{T+1} is the value of the problem.
(ii) If we restrict attention to problems where there is some underlying Markovian structure, it is not hard to see that the martingales $M^{(k)}$ constructed are characterized[3] as $M_{T-k}^{(k+1)} = \varphi_k(X_{T-k})$ for all $k = 0, 1, \ldots, T$. In trying to use (2.9) to determine the functions φ_k recursively, we are faced with the two steps, conditional expectation and pointwise maximization, which are central to the standard dynamic programming approach. Thus it seems unlikely that this pure dual approach will lead to any numerical methodology based on Markovian structure which would differ significantly from existing approaches.

[3]We know that $M^{(1)}$ has this form, and from (2.9) and the Markovian structure it is obvious by induction that this form persists.

References

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