This article appeared in *Seminar on Stochastic Processes 1990*, 275–283, eds. P. J. Fitzsimmons and R. J. Williams. Birkhäuser, Boston 1991.

$A(t, B_t)$ is not a semimartingale

by

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1. Introduction. Let $(B_t)_{t\geq 0}$ be Brownian motion on $\mathbb{R}$, $B_0 = 0$, and for each real x define

$$A(t,x) \equiv \int_0^t I_{(-\infty,x]}(B_s)ds = \int_{-\infty}^x L(t,y)dy,$$

where $\{L(t,y) : t \geq 0, y \in \mathbb{R}\}$ is the local time process of B. The process $A(t,x)$ enters naturally into the study of the Brownian excursion filtration (see Rogers & Walsh [1],[2], and Walsh [3]). In [2], it was necessary to consider the occupation density of the process $Y_t \equiv A(t, B_t)$, which would have been easy if Y were a semimartingale; it is not, and the aim of this paper is to prove this.

To state the result, we need to set up some notation. Let $(X_t)_{0\leq t\leq 1}$ be the process $A(t, B_t) - \int_0^t L(u, B_u)dB_u$, and define for $j, n \in \mathbb{N}$

$$\Delta_j^n \equiv X(j2^{-n}) - X((j-1)2^{-n}), \qquad j \leq 2^n,$$

and

$$V_p^n \equiv \sum_{j=1}^{2^n} |\Delta_j^n X|^p.$$

THEOREM 1. *For any* $p > 4/3$,

$$(1) \qquad V_p^n \overset{L^1}{\underset{\text{a.s.}}{\longrightarrow}} 0 \qquad (n \to \infty).$$

For any $p < 4/3$,

$$(2) \qquad \limsup_{n\to\infty} V_p^n = +\infty \quad \text{a.s.}.$$

This proves conclusively that X (and hence Y) cannot be a semimartingale, because if it were, it could be written as $X = M + A$, where M is a local martingale, A is a finite-variation process (both continuous since X is; see Rogers & Williams [4], VI.40). Now since $V_2^n \xrightarrow{\text{a.s.}} 0$, M must be zero, and $X = A$; but $\overline{\lim} V_1^n = +\infty$ rules out the possibility that X is finite-variation, as we shall see.

In outline, the proof runs as follows. Firstly, we estimate $E|\Delta_j^n X|^p$ above and deduce from this that $EV_p^n \to 0$ for any $p > 4/3$; in fact, the L^1 convergence is sufficiently rapid that $V_p^n \xrightarrow{\text{a.s.}} 0$. Next we estimate $E|\Delta_j^n X|^p$ below, and combine the estimates to prove that $EV_{4/3}^n$ is bounded away from 0 and from infinity. The upper bound allows us to prove that $\{V^n_{4/3} : n \geq 1\}$ is uniformly integrable, and hence that $P(\limsup V^n_{4/3} > 0) > 0$. From this, by Hölder's inequality, we prove that for any $p < 4/3$, $P[\limsup V^n_p = +\infty] > 0$. Finally, an application of the Blumenthal $0-1$ law allows us to conclude.

In the forthcoming paper, we analyse the exact 4/3-variation of X completely, and prove that it is $\gamma \int_0^t L(s, B_s)^{2/3} ds$, from which the present conclusions (and more) follow. (Here, γ is $4\pi^{-\frac{1}{2}}\Gamma(7/6)E(\int L(1,x)^2 dx)^{2/3}$.) The proof of this is a great deal more intricate, however, and this paper shows how to achieve the lesser result with less effort.

2. Upper bounds. To lighten the notation, we are going to perform a scaling so that there is only one parameter involved. It is elementary to prove that for any $c > 0$, the following identities in law hold:

$$(L(t,x))_{t\geq 0, x\in\mathbb{R}} \overset{\mathcal{D}}{=} \left(cL\left(\frac{t}{c^2}, \frac{x}{c}\right)\right)_{t\geq 0, x\in\mathbb{R}}; \tag{3}$$

$$(A(t,x))_{t\geq 0, x\in\mathbb{R}} \overset{\mathcal{D}}{=} \left(c^2A\left(\frac{t}{c^2}, \frac{x}{c}\right)\right)_{t\geq 0, x\in\mathbb{R}}; \tag{4}$$

$$(X_t)_{t\geq 0} \overset{\mathcal{D}}{=} \left(c^2 X_{t/c^2}\right)_{t\geq 0}. \tag{5}$$

Hence $V^n_p \overset{\mathcal{D}}{=} N^{-p}\sum_{j=1}^N |X_j - X_{j-1}|^p$, where $N \equiv 2^n$. We can write the increment $X_{j+1} - X_j$ in the form

$$\begin{aligned} X_{j+1} - X_j &= \int_j^{j+1} I_{\{B_u \leq B_{j+1}\}} du + \int_{B_j}^{B_{j+1}} L(j,x)dx - \int_j^{j+1} L(s,B_s)dB_s \\ &= \int_j^{j+1} I_{\{B_u \leq B_{j+1}\}} du + \int_{B_j}^{B_{j+1}} \{L(j,x) \\ &\qquad - L(j,B_j)\}dx - \int_j^{j+1} \{L(s,B_s) - L(j,B_j)\}dB_s\,. \end{aligned} \tag{6}$$

Let us write

$$\int_j^{j+1} I_{\{B_u \leq B_{j+1}\}} du \equiv Z_{j,1},$$

$$\int_{B_j}^{B_{j+1}} \{L(j,x) - L(j,B_j)\}dx \equiv Z_{j,2},$$

$$\int_j^{j+1} \{L(s,B_s) - L(j,B_s)\}dB_s \equiv Z_{j,3},$$

$$\int_j^{j+1} \{L(j,B_s) - L(j,B_j)\}dB_s \equiv Z_{j,4},$$

so that

$$X_{j+1} - X_j = Z_{j,1} + Z_{j,2} - Z_{j,3} - Z_{j,4}. \tag{7}$$

We now estimate various terms. For $p \geq 2$, with c denoting a variable constant

(i)
$$|Z_{j,1}| \equiv |\int_j^{j+1} I_{\{B_u \leq B_{j+1}\}} du| \leq 1;$$

(ii)
$$\begin{aligned} E|Z_{j,3}|^p &\equiv E|\int_j^{j+1} (L(j, B_s) - L(s, B_s)) dB_s|^p \\ &\leq cE(\int_j^{j+1} (L(j, B_s) - L(s, B_s))^2 ds)^{p/2} \\ &\leq cE\int_j^{j+1} |L(j, B_s) - L(s, B_s)|^p ds \\ &= c\int_0^1 EL(u, 0)^p du, \end{aligned}$$

by reversing the Brownian motion from (s, B_s);

$$\leq c.$$

(iii) By Tanaka's formula,

$$L(t, x) - L(t, 0) = |B_t - x| - |x| - |B_t| - \int_0^t (\text{sgn}(B_s - x) - \text{sgn}(B_s)) dB_s,$$

and

$$||B_t - x| - |x| - |B_t|| \leq 2(|B_t| \wedge |x|),$$

so we have the estimation

$$E|L(t, x) - L(t, 0)|^p \leq c\{|x|^p \wedge t^{p/2} + E|\int_0^t I_{\{0 < B_s < |x|\}} ds|^{p/2}\};$$

but

$$\begin{aligned} E|\int_0^t I_{(0 < B_s < |x|)} ds|^{p/2} &= E|\int_0^{|x|} L(t, y) dy|^{p/2} \\ &= t^{p/2} E(\int_0^{|x|/\sqrt{t}} L(1, y) dy)^{p/2}, \end{aligned}$$

using the scaling relationship (3);

$$\begin{aligned} &\leq t^{p/2} \left(\frac{|x|}{\sqrt{t}}\right)^{p/2-1} E\int_0^{|x|/\sqrt{t}} L(1, y)^{p/2} dy \\ &\leq ct^{p/2} \left(\frac{|x|}{\sqrt{t}}\right)^{p/2-1} \frac{|x|}{\sqrt{t}} \\ &= c|x|^{p/2} t^{p/4}. \end{aligned}$$

Hence for $p \geq 2$

$$E|L(t,x) - L(t,0)|^p \leq c\{|x|^p \wedge t^{p/2} + |x|^{p/2}t^{p/4}\}. \tag{8}$$

(iv)
$$\begin{aligned} E|Z_{j,2}|^p &\equiv |\int_{B_j}^{B_{j+1}} \{L(j,x) - L(j,B_j)\}dx|^p \\ &= E|\int_0^{W_1} \{L(j,x) - L(j,0)\}dx|^p, \end{aligned}$$

where W is a Brownian motion independent of $(B_s)_{0 \leq s \leq j}$;

$$\begin{aligned} &= E|\int_0^{|W_1|} \{L(j,x) - L(j,0)\}dx|^p \\ &\leq E(\int_0^\infty I_{(x \leq |W_1|)}|L(j,x) - L(j,0)|^p dx |W_1|^{p-1}) \\ &= \int_0^\infty dx E|L(j,x) - L(j,0)|^p E(|W_1|^{p-1}; |W_1| > x), \end{aligned}$$

and the function $\Phi_p(x) \equiv E(|W_1|^{p-1}; |W_1| > x)$ decreases rapidly, so

$$\begin{aligned} &\leq c\int_0^\infty ((|x| \wedge \sqrt{j})^p + |x|^{p/2}j^{p/4})\Phi_p(x)dx, \qquad \text{by (iii)} \\ &\leq c(1 + j^{p/4}). \end{aligned}$$

(v)
$$\begin{aligned} E|Z_{j,4}|^p &\equiv E|\int_j^{j+1} (L(j,B_s) - L(j,B_j))dB_s|^p \\ &\leq cE(\int_0^1 (L(j,W_s) - L(j,0))^2 ds)^{p/2}, \end{aligned}$$

where W is a Brownian motion independent of $(B_s)_{0 \leq s \leq j}$;

$$\begin{aligned} &\leq cE\int_0^1 |L(j,W_s) - L(j,0)|^p ds \\ &= c\int g_1(y)E|L(j,y) - L(j,0)|^p dy, \end{aligned}$$

where g_1 is the Green function of Brownian motion on $[0,1]$;

$$\begin{aligned} &\leq c\int g_1(y)\{(|y| \wedge \sqrt{j})^p + |y|^{p/2}j^{p/4}\}dy, \qquad \text{by (iii);} \\ &\leq c(1 + j^{p/4}). \end{aligned}$$

Thus of the four terms in (7) making up $X_{j+1} - X_j$, the p^{th} moments of $Z_{j,1}$ and $Z_{j,3}$ are bounded, and the p^{th} moments of $Z_{j,2}$ and $Z_{j,4}$ grow at most like $1 + j^{p/4}$. (Notice that the bounds for the p^{th} moments, proved only for $p \geq 2$, extend to all $p > 0$ by Hölder's inequality.) We shall soon show that this is the true growth rate. Firstly, though, we complete the upper bound estimation by replacing $X_{j+1}-X_j$ by something more tractable, namely

$$\xi_j \equiv \int_{B_j}^{B_{j+1}} L(j,x)dx - \int_j^{j+1} L(j,B_s)dB_s \tag{9}$$
$$\equiv \int_{B_j}^{B_{j+1}} \{L(j,x)-L(j,B_j)\}dx - \int_j^{j+1} \{L(j,B_s)-L(j,B_j)\}dB_s.$$

To see that this is negligibly different from $X_{j+1}-X_j$, observe the elementary inequality valid for all $p \geq 1$, and $a,b \in \mathbb{R}$:

$$||b|^p - |a|^p| \leq |b-a|p(|a|^{p-1} \vee |b|^{p-1}). \tag{10}$$

Now since $\xi_j = Z_{j,2} - Z_{j,4} = X_{j+1} - X_j - Z_{j,1} + Z_{j,3}$, we conclude from (10) that

$$\begin{aligned} E||\xi_j|^p - |X_{j+1}-X_j|^p| &\leq pE\{|Z_{j,1}-Z_{j,3}|(|\xi_j|^{p-1} \vee |X_{j+1}-X_j|^{p-1})\} \\ &\leq p(E|Z_{j,1}-Z_{j,3}|^a)^{1/a}(E\{|\xi_j|^{b(p-1)} + |X_{j+1}-X_j|^{b(p-1)}|)^{1/b} \\ &\quad \text{for any } a,b>1 \text{ such that } a^{-1}+b^{-1}=1; \\ &\leq c(1+j^{(p-1)/4}), \end{aligned}$$

using the estimates (i), (ii), (iv) and (v). Thus since $V_p^n \stackrel{\mathcal{D}}{=} N^{-p}\sum_{j=1}^N |X_j - X_{j-1}|^p$, we have for $p>1$

$$\begin{aligned} &E|N^{-p}\sum_{j=0}^{N-1}(|\xi_j|^p - |X_{j+1}-X_j|^p)| \\ &\leq cN^{-p}\sum_{j=0}^{N-1}(1+j^{(p-1)/4}) \\ &\leq c(1+N^{-3(p-1)/4}) \\ &\to 0 \qquad \text{as } N\to\infty, \end{aligned}$$

so for each $p>1$, $V_p^n - \tilde{V}_p^n \to 0$ in L^1, where

$$\begin{aligned} \tilde{V}_p^n &\equiv \sum_{j=1}^N |\int_{B((j-1)2^{-n})}^{B(j2^{-n})} L((j-1)2^{-n},x)dx - \int_{(j-1)2^{-n}}^{j2^{-n}} L((j-1)2^{-n},B_s)dB_s|^p \\ &\stackrel{\mathcal{D}}{=} N^{-p}\sum_{j=1}^N |\xi_{j-1}|^p. \end{aligned}$$

Henceforth, we shall concentrate on $\tilde{V}_p^n$, that is, on the ξ_j. Notice that we can say immediately that for $p>4/3$

$$\begin{aligned} EV_p^n &= N^{-p}E\sum_{j=1}^N |X_j - X_{j-1}|^p \\ &\leq cN^{-p}\sum_{j=1}^N (1+j^{p/4}) \\ &\leq CN^{-p}(1+N^{1+p/4}) \\ &\leq cN^{-3p/4+1} \end{aligned}$$

so that not only does $V_p^n \to 0$ in L^1, but also the convergence is geometrically fast in n, so there is even almost sure convergence. This proves the statement (1) of Theorem 1.

3. Lower bounds. We can compute

$$
\begin{aligned}
E(\xi_j|\mathcal{F}_j) &= E\Big[\int_{B_j}^{B_j+1} L(j,x)dx|\mathcal{F}_j\Big] \\
&= \int_0^\infty \{L(j,B_j+x) - L(j,B_j-x)\}\overline{\Phi}(x)dx,
\end{aligned}
$$

where $\overline{\Phi}(x) \equiv P(B_1 > x)$ is the tail of the standard normal distribution;

$$
\begin{aligned}
&\overset{\mathcal{D}}{=} \int_0^\infty \{L(j,x) - L(j,-x)\}\overline{\Phi}(x)dx \\
&= \int_0^\infty (|B_j - x| - |B_j + x|)\overline{\Phi}(x)dx \\
&\qquad + 2\int_0^\infty \Big(\int_0^j I_{[-x,x]}(B_s)dB_s\Big)\overline{\Phi}(x)dx
\end{aligned}
$$

by Tanaka's formula.

We estimate the p^{th} moment of each piece in turn, the first being negligible in comparison with the second. Indeed, since $||B_j - x| - |B_j + x|| \le 2|x|$, the first term is actually bounded, and for the second we compute

$$
\int_0^\infty \Big(\int_0^j I_{[-x,x]}(B_s)dB_s\Big)\overline{\Phi}(x)dx = \int_0^j f(B_s)dB_s,
$$

where $f(x) \equiv \int_{|x|}^\infty \overline{\Phi}(y)dy$, so that by the Burkholder-Davis-Gundy inequalities, the p^{th} moment of the second term is equivalent to

$$
\begin{aligned}
E\Big(\int_0^j f(B_s)^2 ds\Big)^{p/2} &= E\Big(\int f(x)^2 L(j,x)dx\Big)^{p/2} \\
&= j^{p/4} E\Big(\int f(x)^2 L(1,x/\sqrt{j})dx\Big)^{p/2} \\
&\sim j^{p/4} E\Big(\int f(x)^2 L(1,0)dx\Big)^{p/2}
\end{aligned}
$$

as $j \to \infty$. Thus we have for each $p \ge 1$ that

$$
E|\xi_j|^p \ge E|E(\xi_j|\mathcal{F}_j)|^p \ge c_p j^{p/4}, \tag{11}
$$

which, combined with the bounds of §2, implies that for each $p \ge 1$ there are constants $0 < c_p < C_p < \infty$ such that for all $j \ge 0$

$$
c_p \le \frac{E|\xi_j|^p}{1 + j^{p/4}} \le C_p. \tag{12}
$$

Hence in particular

$$0 < \liminf_{n\to\infty} EV^n_{4/3} \le \limsup_{n\to\infty} EV^n_{4/3} < \infty, \tag{13}$$

and for each $p < 4/3$

$$\lim_{n\to\infty} EV^n_p = +\infty, \tag{14}$$

making the conclusion of the Theorem look very likely.

4. The final steps. We shall begin by proving that $\{V^n_{4/3} : n \ge 0\}$ is uniformly integrable. Indeed, for each $p \ge 1$

$$\begin{aligned}\|V^n_p\|_2 &= \|N^{-p}\sum_{j=1}^N |\xi_{j-1}|^p\|_2 \\ &\le N^{-p}\sum_{j=1}^N \||\xi_{j-1}|^p\|_2 \\ &\le cN^{-p}\sum_{j=1}^N (1+j^{p/4})\end{aligned}$$

by (12). Hence for $p = 4/3$, the sequence (V^n_p) is bounded in L^2, therefore uniformly integrable. Hence

$$P(\limsup_n V^n_{4/3} > 0) > 0, \tag{15}$$

because otherwise $V^n_{4/3} \to 0$ a.s., and hence in L^1 (by uniform integrability), contradicting (13). Now define

$$V^n_p(t) \equiv \sum_{j=1}^{[2^n t]} |\Delta^n_j X|^p,$$

and let

$$F_k \equiv \{\limsup_{n\to\infty} \sum_{j=1}^{2^{n-k}} |\Delta^n_j X|^{4/3} > 0\},$$

an event which is $\mathcal{F}(2^{-k})$-measurable. Notice that $F_{k+1} \subseteq F_k$; and by Brownian scaling, all the F_k have the same probability, which is positive by (15). By the Blumenthal $0-1$ law, $P(F_k) = 1$ for every k, and hence for each $t > 0$

$$P\left[\limsup_{n\to\infty} V^n_{4/3}(t) > 0\right] = 1. \tag{16}$$

Now suppose that X were of finite variation, so that there exist stopping times $T_k \uparrow 1$ such that $V_1(T_k) \equiv\uparrow \lim_{n\to\infty} V^n_1(T_k) \le k$. Choose $a > 1 > \alpha > 0$ such that $4a\alpha/3 = 1$, and let b be the conjugate index to a ($b^{-1} + a^{-1} = 1$). By Hölder's inequality,

$$V^n_{4/3}(T_k) \le (V^n_1(T_k))^{1/a}(V^n_{4b(1-\alpha)/3}(T_k))^{1/b}$$

and since $4b(1-\alpha)/3 > 4/3$, the second factor on the right-hand side goes to zero a.s. as $n \to \infty$. The first factor remains bounded as $n \to \infty$, by definition of T_k. Hence $V^n_{4/3}(T_k) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, which is only consistent with (16) if each T_k is zero a.s., which is impossible since $T_k \uparrow 1$.

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