The squared Ornstein-Uhlenbeck market *

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Abstract

We study a complete market containing J assets, each asset contributing to the production of a single commodity at a rate which is a solution to the squared Ornstein-Uhlenbeck (Cox-Ingersoll-Ross) SDE. The assets are owned by K agents with CRRA utility functions, who follow feasible consumption/investment regimes so as to maximize their expected time-additive utility from consumption. We compute the equilibrium for this economy and determine the state-price density process from market clearing. Reducing to a single (representative) agent, and exploiting the relation between the squared-OU and squared-Bessel SDE's, we obtain closed-form expressions for the values of bonds, assets, and options on the total asset value. Typical model parameters are estimated by fitting bond price data, and we use these parameters to price the assets and options numerically. Implications for the total asset price itself as a diffusion are discussed. We also estimate implied volatility surfaces for options and bond yields.

Key Words: Dynamic Equilibrium, Consumption/Investment, Squared Bessel Process, Hypergeometric Function.

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1 Introduction

Characterizations of equilibrium prices in market models with intertemporal consumption abound. Unfortunately, however, the computation of these equilibria, or the application of the pricing mechanisms themselves (e.g. solutions to partial differential equations, or fixed points of certain operators) is not only not straightforward but in most cases outright impossible unless several simplifications are made. In this paper, we study an equilibrium model that is simple enough that equilibrium pricing expressions become explicit (and involve at most numerical integration), but also rich enough that these prices possess interesting behaviour that one can study.

We compute in detail the equilibrium for an economy having as primitives (i) shares in firms that produce a single good paid out in the form of dividends modelled by diffusion processes, and (ii) market agents who consume the good and who trade shares at market prices in order to maximize expected time-additive utility of lifetime consumption. In equilibrium markets clear, and share and bond prices adjust so that all output of the good is consumed and shares of the firms are in unit net supply.

By making judicious choices for a representative agent utility and for the dividend processes in (i) above, we derive explicit expressions for the equilibrium prices of bonds and for investing in the firms. In particular, we assume a representative utility of constant relative risk aversion (CRRA) form, and that the flows of the good from the different firms occur at rates that are independent Cox-Ingersoll-Ross (CIR) diffusion processes (see Cox, Ingersoll and Ross (1985b), hereafter referred to as CIR (1985b)). Importantly, under the latter assumption, the aggregate flow is itself a CIR process. Our model is simple: we do not model the supply side of the economy[1], endowment consists of only one good, we solve only the representative agent's problem, and our market is automatically complete because firms' production is driven by independent Brownian Motions. Because of this simplicity (or in spite of it !) our model allows us to characterize the equilibrium total market value of output of the good as a diffusion with interesting properties (one whose asymptotic behaviour we can exhibit analytically, for instance). We also obtain endogenously a one-factor model for the real interest rate that is new, so far as is known to us.

In general terms, our methodology is close in spirit to that of Karatzas, Lehoczky and Shreve (1990) (hereafter referred to as KLS (1990)) and Duffie and Zame (1989). In these papers, the idea is to derive the market state-price density from a representative agent's marginal utility evaluated at the aggregate consumption level, which is equal to the total output of the economy by market clearing. Prices of all market assets then follow from martingale representation with respect to gains processes of the assets. Such pricing formulas were obtained by Lucas (1978) in a discrete-time setting, but his approach was to study fixed points of the Bellman operator of a dynamical program, rather than martingale representation. Considering only one market agent, he derives explicit prices only for cases where one asset pays out a sequence of independent identically distributed dividends, or where agents are identical with linear utility. Aase (2002) derives equilibrium interest rates in models with CRRA (respectively CARA) utility agents and lognormal (respectively Gaussian) endowments with constant coefficients. As expected, the interest rates are constant. In similar settings, Aase (2002) also obtains Black-Scholes-like equilibrium pricing formulas for call options on dividend-paying assets. 'Explicit' equilibrium solutions are given in KLS (1990) for a market with identical agents having power-law utility, and for a heterogeneous market where the aggregate production is a constant. These authors also claim that a two-agent equilibrium for agents with logarithmic and square-root utilities can be computed. This is true only because for such a pair of utilities, the state price density solves a quadratic equation, and even then, computing the equilibrium weights involves taking expectations of a complicated function of the aggregate endowment process.

As the examples mentioned above show, the difficulty of computing explicit equilibria in a multi-agent economy is well-appreciated. The representative agent utility for such an economy, (and hence the state-price density derived therefrom) is a weighted sum of the individual agents' utilities. Proving equilibrium is tantamount to exhibiting weights that correspond to the individual agents attaining their optimal levels of consumption. This issue is what KLS (1990) deals with; its resolution depends on fixed-point arguments which are non-constructive. In practice, if any of this is to be done explicitly, the only way to solve for the representative agent weights explicitly is to render them irrelevant by studying a market with identical agents; this is what we do here, and this is what is done in all examples assuming a one-agent market. In contrast to the examples already mentioned, however, our model yields prices which are far from trivial and which exhibit interesting characteristics. Our expression for the interest spot rate, Equation (4.5), and our pricing recipe (Equation (3.2)) are consistent with the general formulas in Duffie and Zame (1989), KLS (1990), and also Aase (2002).

Duffie and Huang (1985) and Duffie (1986) were the first to apply martingale representation technology to show how the pricing function in the classic Arrow-Debreu equilibrium (see, e.g. Arrow and Debreu (1954)) can be characterized as an expectation. By this important result, equilibrium is attainable by trading in a finite number of market securities. Indeed, the driving force behind the general pricing relations obtained in KLS (1990, Th. 8.2) is martingale representation with respect to gains processes of productive assets.

Building on Duffie and Huang (1985) and Duffie (1986), Huang (1987) showed how equilibrium is consistent with a representative agent who, endowed with the aggregate dividend output from a set of market securities maximizes expected time-additive utility from consumption. He also proved that if the consumption process attains its essential infimum only on a set of measure zero, then each individual agent's optimal consumption is a smooth function of the aggregate consumption. This is key to applying the fixed-point arguments employed by KLS (1990), as becomes very clear in the examples considered by these authors.

While we can prove explicitly what the equilibrium is for our market, our model primitives do not concord with several conditions that are sufficient for the general results in KLS (1990). For example, in KLS (1990), the aggregate endowment process in the economy is assumed to have bounded diffusion coefficients, an assumption which we do not make. Also, the martingale change of measure is forced to be well behaved by the restrictive condition that its diffusion coefficient be bounded. This condition does not hold for our specific model; in fact our martingale change of measure is unbounded, but a simple criterion involving the parameters of the model ensures that the martingale property still goes through (see Appendix A). Finally, in the KLS (1990) model, agents' utilities satisfy a condition that is equivalent to the relative risk aversion coefficients of the agents (which vary with their optimal consumption processes) being bounded above by 1. This condition is required to ensure uniqueness of equilibrium. The CRRA utility functions that we consider have (constant) risk aversion coefficient R, and we require only that R be positive. Uniqueness of equilibrium in our model is proved explicitly. In fact, in a simplified version of the KLS (1990) paper, Karatzas et al. (1991), CRRA utilities are singled out as a class of utilities for which the quite restrictive condition on the risk aversion is not necessary for uniqueness.

In a celebrated paper, Cox, Ingersoll and Ross (1985a) (hereafter cited as CIR (1985a)) develop an equilibrium model for a production economy. In their model, technological change in production is modelled by a state variable, and consumption depends on the model uncertainty only through this state variable. There is a single good, and this can be consumed or invested in one of several production processes whose output depends on the technology state variable as well as amount of good invested. Equilibrium in this model involves choosing levels of investment that maximize a given expected time-additive utility for consumption. Within this framework, CIR (1985a) obtain expressions for the equilibrium rate of interest and for the optimal rate of return from production, and also derive a differential equation that prices of contingent claims must satisfy. Because their model is based on returns in raw production, rather than on the market price for shares in the production process, their budget equation differs from ours. However, in their model, the value of a share in a firm that invests in production can also be viewed as a claim to a dividend stream flowing at rate equal to the rate of return for the firm's investment (with the difference that shares are in net positive supply). The price of a firm that invests in production therefore satisfies the pricing differential equation; at equilibrium, the firm's value must equal the value of the supply of the good that the firm owns. In this sense, asset prices as derived in our model are implicit in the differential equations given by CIR (1985a). The viewpoint of Sundaresan (1984) is more aligned with ours; his budget equation involves market prices of assets rather than wealth invested in production. For the special and simple case of production returns with Cobb-Douglas drifts and constant volatility, and zero technological change, he derives simple expressions for the interest rate at equilibrium.

CIR (1985b) obtained their well-known model for term structure of interest rates by specializing the model presented in CIR (1985a). To do this, they assume that the single state variable (technological change) is a CIR (squared Ornstein-Uhlenbeck) diffusion[2] and that the means and variances of the rates of return on production are proportional to the level of the state process. There is one agent with logarithmic utility that depends on the state variable only through consumption; from this, the equilibrium spot rate process is determined to be also a CIR diffusion. By this, the density function for the law of the spot rate can be written in closed form, allowing CIR (1985b) to derive closed-form prices for bonds and for options thereon. All the analysis in our model also hinges on knowledge of the CIR diffusion, in particular its relation to the squared Bessel process.

The rest of the paper is organized as follows. Section 2 describes the primitives of our economy, for which we derive an explicit equilibrium in Section 3. In Section 4, we derive the equilibrium spot rate and martingale change of measure for the particular case of a single representative agent with CRRA utility. As described above, the development is quite conventional, and has much in common with KLS (1990), Aase (2002) and Breeden (1979); the state-price density process is determined from market clearing, and assets are priced from that. There is nothing particularly new here at a general theoretical level, but our model assumptions are sufficiently specific that we have the rare pleasure of being able to compute various prices *in closed form.*

Section 5 contains some computations involving squared Bessel processes, which we then use in Section 6 to obtain expressions for the prices of bonds, assets and options in the market. In Section 7 we calibrate the model using observed bond price data, and use typical parameters to evaluate and study numerically the market prices of assets and of options on the total asset.

2 The Model

We consider a market containing one unit of each of J productive assets, the j'th of which produces the single commodity of the economy at rate $\delta^j \equiv (\delta^j_t)_{t\geq 0}$ satisfying the (Cox-Ingersoll-Ross) SDE

$$d\delta^j_t = \sigma\sqrt{\delta^j_t}dW^j_t + (a_j - \beta\delta^j_t)dt, \quad 1 \leq j \leq J. \tag{2.1}$$

Here, $\sigma > 0$, $\beta > 0$, and $a_j > 0$ are constants and $W \equiv (W^j)_{1\leq j\leq J}$ is a standard J-dimensional Brownian motion.[3] *Note that the processes δ^j are independent, with common volatility parameter σ and mean-reversion parameter β.* These assumptions are restrictive but essential; the smallest variation destroys the analysis.

We shall write $\delta = (\delta^j)_{1\leq j\leq J}$ for the $\mathbb{R}^J$-valued rate-of-production process. Now because of our assumptions, the total production rate $\Delta \equiv (\Delta_t := \sum_{j=1}^J \delta^j_t)_{t\geq 0}$ satisfies an SDE of a type similar to (2.1):

$$d\Delta_t = \sigma\sqrt{\Delta_t}dB_t + (A - \beta\Delta_t)dt, \tag{2.2}$$

where $A = \sum_{j=1}^J a_j$ and the one-dimensional Brownian Motion B is related to W via $\sqrt{\Delta_t}dB_t = \sum_{j=1}^J \sqrt{\delta^j_t}dW^j_t$. We shall maintain the standing assumption[4] :

$$\frac{2A}{\sigma^2} \geq 1. \tag{2.3}$$

The assets in the market are owned by K agents; agent k has C^1 utility function $U_k : (0,\infty) \times (0,\infty) \to \mathbb{R}$, which is increasing and strictly concave in its second argument, and satisfies the Inada conditions at 0 and ∞. Agent k begins with $\alpha^j_k(0)$ units of asset j, and aims to consume according to a feasible non-negative process $c_k \equiv (c_k(t))_{t\geq 0}$ so as to maximize the objective function

$$\mathbb{E}\Big[\int_0^\infty U_k(t, c_k(t))dt\Big], \quad 1 \leq k \leq K. \tag{2.4}$$

In the next Section, we will compute the equilibrium for this economy. Our SDE (2.2) is closely related to the squared-Bessel SDE (see Revuz and Yor (2001) for the most important facts on these); in particular, it is possible to obtain a closed-form expression for the transition density of the diffusion δ, and this is the key to the various pricing expressions we will derive in Section 6.

3 Equilibrium

Suppose that at time t agent k consumes at rate $c_k(t)$. His wealth may be invested in the assets available on the market, or in a riskless bank account bearing interest at instantaneous rate r_t. If $\alpha_k = (\alpha_k^j(t))_{t\geq 0}$, $j = 1, \ldots, J$, denotes the (J-vector) process of his holdings of the asset, then his wealth $X_k = (X_k(t))_{t\geq 0}$ will evolve according to the dynamics

$$(3.1) \qquad dX_k(t) = r_t X_k(t)dt + \alpha_k(t)\cdot[dS_t - r_t S_t dt] + [\alpha_k(t) \cdot \delta_t - c_k(t)]dt;$$
$$X_k(0) = \alpha_k(0) \cdot S_0.$$

Here, S is the (J-vector) price-process of the assets; this and the instantaneous rate of interest r are *a priori* unknown, but will be obtained from equilibrium considerations. The only constraint on the agent's investment and consumption decisions is that his wealth X_k should remain non-negative at all times (to prevent him consuming unboundedly by running up ever larger debts.) If c_k^* denotes the optimal[5] consumption process for agent k with objective (2.4), wealth dynamics (3.1) and the non-negativity constraint on wealth, then agent k's marginal price for a cashflow $(\varepsilon_t)_{t\geq 0}$ is simply given by

$$(3.2) \qquad E\left[\int_0^\infty \zeta_k(t)\varepsilon_t \, dt \right] / \zeta_k(0),$$

where $\zeta_k(t) \equiv U_k'(t, c_k^*(t))$ is agent k's state-price density process: see, for example, KLS (1990). Since we have a complete market, all agents will value a given cashflow the same, which implies that the state-price density processes are all multiples of one another: for some constants λ_k,

$$(3.3) \qquad U_k'(t, c_k^*(t)) = \lambda_k \zeta_t \quad \forall k, \ \forall t.$$

Turning this around, we have for all k that

$$(3.4) \qquad c_k^*(t) = I_k(t, \lambda_k \zeta_t), \quad 1 \leq k \leq K.$$

On the other hand, we have the *market clearing condition*, that all of the commodity must be exactly consumed as it is produced. Thus

$$(3.5) \qquad \sum_{k=1}^K c_k(t) = \sum_{j=1}^J \delta^j(t) = \Delta_t,$$

which together lead to the relation

$$(3.6) \qquad \Delta_t = \sum_{k=1}^K I_k(t, \lambda_k \zeta_t).$$

If the constants λ_k were known, then, this equation (3.6) would determine the state-price density ζ (and hence market prices of all cashflows) from the data Δ and from the agents' preferences. For example, and in particular, the asset prices (which appeared in (3.1) as unknowns) would be given by

$$(3.7) \qquad \zeta_t S(t) = \mathbb{E}\left[\int_t^\infty \zeta_u \delta(u) du \middle| \mathcal{F}_t \right]$$

In general, the constants λ_k are determined from the initial wealths of the agents, but it seems in practice that virtually the only case where we can solve for the constants is in the case of a single representative agent.

4 Representative agent equilibrium

In this section, we restrict ourselves to the case $(K = 1)$ where there is only one agent in the market. We can think of this as a market with all agents having identical utilities, or as a market where one representative agent acts as proxy for all the agents. We further assume[6] that there are positive constants $R \neq 1$ and ρ in terms of which $U(t,x) = e^{-\rho t}x^{1-R}/(1-R)$, so that

$$U'(t,x) = e^{-\rho t}x^{-R}. \tag{4.1}$$

Dropping the now irrelevant subscripts k that previously identified the agents, in equation (3.6) we have $c^*(t) = \Delta_t$, and the equation (3.3) then reduces to

$$e^{-\rho t}\Delta_t^{-R} = \zeta_t, \tag{4.2}$$

where we have without loss of generality taken $\lambda_1 = 1$. Thus we have ζ_t explicitly as a smooth function of t and Δ_t; applying Itô's lemma to ζ from equation (4.2) therefore gives

$$d\zeta_t = \zeta_t\Big[\frac{-R\sigma}{\sqrt{\Delta}}dB_t - \big\{\rho - \beta R + \frac{R(A - \sigma^2(R+1)/2)}{\Delta}\big\}dt\Big], \quad \zeta_0 = \Delta_0^{-R}. \tag{4.3}$$

From this we can read off the change-of-measure process which converts the reference measure $\mathbb{P}$ to the pricing measure $\tilde{\mathbb{P}}$, as well as the interest rate process. Indeed, these satisfy

$$dZ_t = -\frac{Z_t R\sigma}{\sqrt{\Delta}}dB_t, \quad \text{where } Z_t = \mathbb{E}[d\tilde{\mathbb{P}}/d\mathbb{P}|\mathcal{F}_t], \quad Z_0 = 1; \tag{4.4}$$

and

$$r_t = \rho - \beta R + \frac{R(A - \sigma^2(R+1)/2)}{\Delta}, \tag{4.5}$$

for $t \geq 0$. Under what conditions will the change-of-measure process Z actually be a martingale and not just a local martingale? The criterion is simple and complete:

Lemma 4.1 *The process Z defined by (4.4) will be a martingale if and only if*

$$\frac{2A}{\sigma^2} \geq 2R + 1. \tag{4.6}$$

PROOF. See Appendix A. ■

Notice that the spot rate process will be bounded below provided

$$A > \frac{\sigma^2(R+1)}{2}, \tag{4.7}$$

a condition implied by (4.6).

We will henceforth assume that inequality (4.6) holds. Now if we denote $\alpha = \rho - \beta R$ and $\gamma = R(A - (\sigma^2/2)(R+1))$ and write $r_t = \alpha + \frac{\gamma}{\Delta}$ from equation (4.5), then it is easy to see that in the measure $\tilde{\mathbb{P}}$, the spot rate process $(r_t)_{t\geq 0}$ satisfies the SDE

$$dr_t = (r_t - \alpha)\Big\{\Big[(\sigma^2(R+1) - A)\Big(\frac{r_t - \alpha}{\gamma}\Big) + \beta\Big]dt - \sigma\sqrt{\frac{r_t - \alpha}{\gamma}}d\tilde{B}\Big\} \tag{4.8}$$

where $\tilde{B}$ satisfies $d\tilde{B} = dB + (\sigma R/\sqrt{\Delta})dt$ and is a $\tilde{\mathbb{P}}$-Brownian Motion.

The SDE (4.8) becomes considerably neater for the case $\alpha = 0$ ($\rho = \beta R$); we then have

$$\frac{dr_t}{r_t} = \Big[\frac{(\sigma^2(R+1) - A)}{\gamma}r_t + \beta\Big]dt - \frac{\sigma}{\sqrt{\gamma}}\sqrt{r_t}d\tilde{B}. \tag{4.9}$$

We have determined (4.2) the candidate state-price density ζ, so from this and from the pricing relation (3.2) we expect[7] that

$$S^j_t \equiv S^j(\delta^j_t, \Delta_t) = \frac{1}{\zeta_t}\,\mathbb{E}\Big[\int_t^\infty \delta^j_u \zeta_u\, du \,\Big|\, \mathcal{F}_t\Big] \tag{4.10}$$

It remains to prove that what we suspect is an equilibrium for the economy actually is. To spell out what is required, we have to show that if we suppose that the price processes S^j are given by (4.10) and the spot rate process by (4.5), then the optimal consumption / investment policy for the representative agent whose wealth evolves as (3.1) is to take $\alpha^j(t) \equiv 1$ for all $t \geq 0$, for all $j = 1, \ldots, J$. The proof of this is a straightforward Lagrangian sufficiency argument. Firstly, note from (4.10) that

$$\zeta_t S^j_t + \int_0^t \zeta_u \delta^j_u\, du \quad \text{is a martingale.} \tag{4.11}$$

In particular, the initial wealth X_0 of the representative agent who at time 0 holds all of the asset will be

$$X_0 = \zeta_0^{-1}\mathbb{E}\Big[\int_0^\infty \zeta_u \Delta_u\, du\Big] \tag{4.12}$$

Moreover, we see from(3.1) that holding 1 unit of each of the assets for all time and consuming at rate Δ_t is a feasible strategy, with corresponding non-negative wealth process $\Sigma \equiv (\Sigma_t := \sum_{j=1}^J S^j_t)_{t\geq 0}$.

From (4.11) and from (3.1) we deduce that

$$\zeta_t X_t + \int_0^t \zeta_u c_u\, du \quad \text{is a local martingale} \tag{4.13}$$

for any feasible consumption process c. Non-negativity of X then implies that

$$\zeta_0 X_0 \geq \mathbb{E}\Big[\int_0^\infty \zeta_u c_u\, du\Big]. \tag{4.14}$$

Thus

$$\begin{aligned}
\mathbb{E}\Big[\int_0^\infty U(t,c_t)\,dt\Big] &\le \mathbb{E}\Big[\int_0^\infty \{U(t,c_t)-\zeta_t c_t\}\,dt\Big]+\zeta_0 X_0\\
&\le \mathbb{E}\Big[\int_0^\infty \{U(t,c_t^*)-\zeta_t c_t^*\}\,dt\Big]+\zeta_0 X_0\\
&= \mathbb{E}\Big[\int_0^\infty U(t,c_t^*)\,dt\Big].
\end{aligned}$$

The key point is the second line here, which follows precisely because $\zeta_t = U'(t,c_t^*) = U'(t,\Delta_t)$.

This establishes the claim that the conjectured equilibrium holds for this economy. In the following Sections, we shall proceed to apply the general pricing recipe (3.2) to compute prices of bonds, of the assets, and of options on the total assets for the particular model studied here.

5 Bessel Processes

We show in this section that the solutions δ^j and Δ to the SDE's (2.1) and (2.2) are simple transformations of squared Bessel processes. This fact will then be used in the next Section to derive expressions for market prices of bonds and assets.

Let $W \equiv (W_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion, let $\eta \geq 0$, and consider the SDE

$$dX_t = 2\sqrt{|X_t|}dW_t + \eta dt, \quad X_0 = x \geq 0. \tag{5.1}$$

The (pathwise unique) exact solution to (5.1) is called a squared Bessel process of dimension η started at x, and denoted by $BESQ^\eta(x)$. See Revuz and Yor (2001) for the basic properties of squared Bessel processes.

The parameter η is called the dimension of the process X. The transition density of X involves the Bessel function of index $\nu = \eta/2 - 1$; we shall write $BESQ^{(\nu)}(x)$ when it is more convenient to characterize the squared Bessel process by the index rather than dimension.

For later reference, we recall that the $BESQ^\eta(x)$ process has a density given by (see Revuz and Yor (2001))

$$q_t^\eta(x,y) = \frac{1}{2t}(\frac{y}{x})^{\nu/2} \exp\left(-(x+y)/2t\right) I_\nu(\sqrt{xy}/t) \tag{5.2}$$

$$= \frac{1}{2t} e^{-(x+y)/2t} \sum_{k\geq 0} \frac{(\frac{x}{2t})^k (\frac{y}{2t})^{k+\nu}}{k!\,\Gamma(k+\nu+1)} \tag{5.3}$$

valid for non-negative x and y, and for $t > 0$. Here, $I_\nu(\cdot)$ is the modified Bessel function of the first kind of index ν, where ν is related to η by

$$\nu = \eta/2 - 1.$$

The Laplace transform of X_t is

$$\mathbb{E}^x[\exp(-\lambda X_t)] \equiv \phi^\eta(\lambda, t, x) = (1+2\lambda t)^{-\eta/2} \exp(-\lambda x/(1+2\lambda t)), \qquad \lambda > 0,\ t > 0,\ x > 0. \tag{5.4}$$

We now give two results, involving expectations of functions of squared Bessel processes, that are the basis for our calculations in Section 6.

Lemma 5.1 *Let $X \equiv (X_t)_{t\geq 0}$ be a $BESQ^\eta(x)$ process, denote its law by $\mathbb{P}^x$, and let $R > 0$. If the condition*

$$\nu + 1 - R > 0 \tag{5.5}$$

holds, then

$$\mathbb{E}^x(X_t^{-R}) = \frac{\Gamma(\nu+1-R)}{\Gamma(\nu+1)} \exp(-x/2t)(2t)^{-R}\ {}_1F_1(\nu+1-R, \nu+1, x/2t), \tag{5.6}$$

where $\nu = \eta/2 - 1$ and where the function ${}_1F_1(\cdot,\cdot,\cdot)$ is the confluent hypergeometric function defined by

$${}_1F_1(a,b,z) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+j)} \frac{z^j}{j!}. \tag{5.7}$$

PROOF. Consider $x > 0$ first. Let $q_t^\eta(x,\cdot)$ be the transition density of $BESQ^\eta(x)$ given in equation (5.2). Then

$$\begin{aligned}\mathbb{E}^x(X_t^{-R}) &= \int_0^\infty y^{-R} q_t^\eta(x,y)dy \\ &= \int_0^\infty y^{-R}(y/x)^{\nu/2}(2t)^{-1}\exp(-(x+y)/2t) I_\nu(\sqrt{xy}/t)dy \\ &= \int_0^\infty y^{-R}(y/x)^{\nu/2}(2t)^{-1}\exp(-(x+y)/2t)\sum_{j=0}^\infty \frac{\left(\sqrt{xy}/2t\right)^{2j+\nu}}{j!\,\Gamma(j+\nu+1)}dy.\end{aligned}$$

The integrand here is non-negative; after interchanging the order of summation and integration and making the substitution $z = y/2t$, we recognize part of the integrand as a gamma density. Because of the condition (5.5), we can integrate this out, which leaves us with

$$\begin{aligned}\mathbb{E}^x(X_t^{-R}) &= \exp(-x/2t)(2t)^{-R}\sum_{j=0}^\infty (x/2t)^j \frac{\Gamma(\nu+j+1-R)}{j!\,\Gamma(\nu+j+1)} \\ &= \frac{\Gamma(\nu+1-R)}{\Gamma(\nu+1)}\exp(-x/2t)(2t)^{-R}\ {}_1F_1(\nu+1-R,\nu+1,x/2t),\end{aligned}$$

as we want.

The case $x = 0$ is similar but easier, so we omit details. ■

In the next Lemma we use the explicit form (5.4) for the Laplace transform of the squared Bessel transition density to compute the expectation of a function of two independent squared Bessel Processes. This computation can be used to derive expressions for prices of the single assets.

Lemma 5.2 *Let $X \equiv (X_t)_{t\geq 0}$ (resp. $Y \equiv (Y_t)_{t\geq 0}$) be $BESQ^\eta(x)$ (resp. $BESQ^\mu(y)$), two independent squared Bessel processes, let $\mathbb{P}^{(x,y)}$ denote their joint law, and let $R > 0$. Then*

$$\begin{aligned}\mathbb{E}^{(x,y)}\left[Y_t(X_t+Y_t)^{-R}\right] &= -\frac{\partial}{\partial\theta}\left[\int_0^\infty \phi^\eta(\lambda,t,x)\phi^\mu(\lambda+\theta,t,y)\frac{\lambda^{R-1}}{\Gamma(R)}d\lambda\right]\Bigg|_{\theta=0} \\ &= \int_0^\infty (1+2\lambda t)^{-\frac{\eta+\mu}{2}-1} e^{-\frac{\lambda(x+y)}{1+2\lambda t}}\left(\mu t + \frac{y}{1+2\lambda t}\right)\frac{\lambda^{R-1}}{\Gamma(R)}d\lambda. \qquad (5.8)\end{aligned}$$

This integral is finite if

$$\frac{\eta+\mu}{2} + 1 - R > 0. \qquad (5.9)$$

PROOF. For $\lambda > 0$, $\theta > 0$, consider the joint Laplace transform

$$\mathbb{E}^{(x,y)}\left[\exp(-\lambda(X_t+Y_t) - \theta Y_t)\right] = \phi^\eta(\lambda,t,x)\phi^\mu(\lambda+\theta,t,y). \qquad (5.10)$$

Multiplying both sides of this equation by $\lambda^{R-1}/\Gamma(R)$ and integrating with respect to λ, we get

$$\begin{aligned}&\int_0^\infty \mathbb{E}^{(x,y)}\exp(-\theta Y_t)\exp(-\lambda(X_t+Y_t))\lambda^{R-1}\frac{d\lambda}{\Gamma(R)} \\ &\qquad = \int_0^\infty \phi^\eta(\lambda,t,x)\phi^\mu(\lambda+\theta,t,y)\lambda^{R-1}\frac{d\lambda}{\Gamma(R)}. \qquad (5.11)\end{aligned}$$

On the left hand side, changing the order of integration transforms the expression to

$$\mathbb{E}^{(x,y)}\big[\exp(-\theta Y_t)(X_t+Y_t)^{-R}\big]. \tag{5.12}$$

Differentiating this expression, and the right side of (5.11), with respect to θ, using the Laplace transform given in (5.4), gives (5.8).

Let us now verify the integrability condition (5.9). In equation (5.8), make the substitution $\alpha = (1+2\lambda t)^{-1}$, to get

$$\mathbb{E}^{(x,y)}\big[Y_t(X_t+Y_t)^{-R}\big] = \exp\big(-(x+y)/2t\big)\frac{1}{(2t)^R\Gamma(R)}$$

$$\times\int_0^1 \alpha^{(\eta+\mu)/2-1}\exp\big((x+y)\alpha/2t\big)(\mu t+y\alpha)(1/\alpha-1)^{R-1}d\alpha. \tag{5.13}$$

For small α, the integrand is proportional to

$$\alpha^{(\eta+\mu)/2-R}\exp\big(\alpha(x+y)/2t\big)(\mu t+y\alpha),$$

which shows why we need the condition (5.9). ■

We now exhibit the solution Δ to the SDE (2.2) as a transformation of a squared Bessel process. Thus, let $(\Delta_t)_{t\geq 0}$ be a solution to (2.2). A simple Itô calculation verifies that the process defined by $\tilde{Y}_t = \exp(\beta t)\Delta_t$ satisfies the SDE

$$d\tilde{Y}_t = \exp(\beta t/2)\sigma\sqrt{\tilde{Y}_t}dB_t + A\exp(\beta t)dt,$$

with $\tilde{Y}_0 = \Delta_0$. This says that for $f\in C^2$, the process defined by

$$\tilde{M}_t = f(\tilde{Y}_t) - f(\tilde{Y}_0) - \int_0^t \tilde{\mathcal{G}}f_s ds$$

is a martingale, where $\tilde{\mathcal{G}}f_t \equiv \exp(\beta t)\big[(\sigma^2/2)\tilde{Y}_t f''(\tilde{Y}_t) + Af'(\tilde{Y}_t)\big]$. If we now change the time scale via the deterministic clock $A_t \equiv \int_0^t \lambda\exp(\beta s)ds = (\lambda/\beta)(e^{\beta t}-1)$ with continuous inverse $\tau_t = \inf\{u : A_u > t\}$, so that $Y_t \equiv \tilde{Y}_{\tau_t}$, then $\tilde{M}$ time-changes to the martingale

$$M_t \equiv \tilde{M}_{\tau_t} = f(Y_t) - f(Y_0) - \int_0^t \mathcal{G}f(Y_s)ds,$$

where $\mathcal{G}f(y) = (\sigma^2/2\lambda)yf''(y) + (A/\lambda)f'(y)$. Thus, the process Y satisfies the SDE

$$dY_t = \frac{\sigma}{\sqrt{\lambda}}\sqrt{Y_t}dB_t + (A/\lambda)dt. \tag{5.14}$$

Choosing

$$\lambda = \sigma^2/4,$$

we recognise (5.14) as the $BESQ^\eta$ SDE, with

$$\eta = 4A/\sigma^2.$$

Thus, we have the following result.

Lemma 5.3 *The process* Δ *satisfying SDE (2.2) can be written as*

$$\Delta_t = \exp(-\beta t)\tilde{Y}_t = \exp(-\beta t) Y_{A_t}, \tag{5.15}$$

where $A_t = (\lambda/\beta)(e^{\beta t} - 1)$, *with* $\lambda = \sigma^2/4$ *and where* Y *is* $BESQ^\eta(\Delta_0)$, *with* $\eta = 4A/\sigma^2$.

Obviously, an analogous result holds for the processes δ^j, $1 \leq j \leq J$. Now we are able to read off the distribution of the process Δ. Indeed, using the characterisation in Lemma 5.3 and recalling the form of the Laplace transform of the squared Bessel process given by expression 5.4, we obtain the Laplace transform of Δ as

$$\mathbb{E}^x\left[\exp\left(-\alpha\Delta_t\right)\right] = \left(1 + 2A_t\alpha e^{-\beta t}\right)^{-\eta/2} \exp\left(\frac{-x\alpha e^{-\beta t}}{1 + 2A_t\alpha e^{-\beta t}}\right). \tag{5.16}$$

This is in fact the Laplace transform of a non-central chi-squared random variable, though we shall not make any explicit use of this identification of the law.

6 Bond and asset prices

In this section, we apply the various results just established to derive expressions for bond prices, for the price of the total assets in the market, and for options on the assets. These are not of course as explicit as the prices in the Black-Scholes model, but they are perfectly tractable numerically.

6.1 Bond prices

The price $P(0,T)$ of the zero-coupon bond maturing at T is

$$\begin{aligned} P(0,T) &= \frac{1}{\zeta_0}\mathbb{E}[\zeta_T|\mathcal{F}_0] \\ &= e^{-\rho T} z^R \mathbb{E}^z[\Delta_T^{-R}], \end{aligned} \tag{6.1}$$

where $z \equiv \Delta_0$. But from Lemma 5.3, we can compute the expectation above as

$$\begin{aligned} P(0,T) &= e^{-\rho T} z^R \mathbb{E}^z[e^{\beta RT} Y_{A_T}^{-R}] \\ &= z^R e^{-\rho T} e^{\beta RT} e^{-z/2A_T} (2A_T)^{-R} \frac{\Gamma(\nu+1-R)}{\Gamma(\nu+1)}\ {}_1F_1(\nu+1-R, \nu+1, z/2A_T), \end{aligned} \tag{6.2}$$

where we have used (5.15) to write Δ in terms of the $BESQ^\eta(z)$ process Y, with the clock A_t being as in Lemma 5.1, $\eta = 4A/\sigma^2$, $\nu = \eta/2 - 1$.

6.2 Bond Yield Volatilities

Notice from Equation (6.2) that the price now of a bond maturing in T time units from now is a function of the level of Δ now only (apart from the model parameters, of course). If we take $z = \Delta_t$ in Equation (6.2), then $P(0,T) \equiv g(z)$ is the price of a bond written at t with lifetime T. Therefore, the SDE for the yield of such a bond is

$$d(-\log g(\Delta_t)/T) = -\frac{1}{Tg(\Delta_t)}\left[\left(g'(\Delta_t)(A-\beta\Delta) + \frac{1}{2}g''(\Delta_t)\sigma^2\Delta_t\right)dt + \left(g'(\Delta_t)\sigma\sqrt{\Delta_t}dB_t\right)\right]. \tag{6.3}$$

By computing bond prices for different levels z, we can estimate numerically the values of the volatility coefficient

$$\frac{g'(z)}{Tg(z)}\sigma\sqrt{z} \tag{6.4}$$

in the above SDE. This is done in Section 7.4.

6.3 The Asset price processes

Since the processes δ^j are independent, we can write $\Delta \equiv (\bar{\Delta}^j + \delta^j)$ as the sum of two independent squared Bessel processes. Specifically, we represent

$$\delta^j(s) = \exp(-\beta s)Y_{A_s}, \quad \bar{\Delta}^j(s) = \exp(-\beta s)X_{A_s}$$

where

$$X \text{ is } BESQ^{\eta}(x), \quad \eta = 4(A - a_j)/\sigma^2,$$

and

$$Y \text{ is } BESQ^{\mu}(y), \quad \mu = 4a_j/\sigma^2$$

and $y = \delta_0^j$, $x = \Delta_0 - \delta_0^j$. Using the pricing relation (3.7) and the representation of Lemma 5.3,

$$S_0^j \zeta_0 = S_0^j (x+y)^{-R} = \int_0^\infty \mathbb{E}^{(x,y)} \left[e^{-\rho s} e^{(R-1)\beta s} Y_{A_s} (Y_{A_s} + X_{A_s})^{-R} \right] ds, \tag{6.5}$$

where $\mathbb{P}^{(x,y)}$ denotes the joint law of (x, y). Using Lemma 5.2, we obtain an expression for the price of the j'th asset:

(6.6)

$$S_0^j = (x+y)^R \int_0^\infty e^{((R-1)\beta - \rho)s} \left\{ \int_0^\infty (1 + 2\lambda A_s)^{-\frac{\mu+\eta}{2} - 1} e^{-\frac{\lambda(x+y)}{1+2\lambda A_s}} \left(\mu A_s + \frac{y}{1 + 2\lambda A_s} \right) \frac{\lambda^{R-1}}{\Gamma(R)} d\lambda \right\} ds.$$

This expression can be and has to be computed numerically, but in the sequel we focus attention on the sum of the asset prices $\sum_{j=1}^J S^j$.

Writing $\Sigma = (\Sigma_t)_{t \geq 0}$ for the price process of the total assets, the pricing relation (3.7) gives us

$$\begin{aligned} \zeta_0 \Sigma_0 &= \mathbb{E}\Big[\int_0^\infty \Delta_u \zeta_u du \Big] \\ &= \int_0^\infty e^{-\rho u} \mathbb{E}^x (\Delta_u^{1-R}) du, \end{aligned} \tag{6.7}$$

with $x = \Delta_0$. From this, using again the representation of Lemma 5.3 and the computation in Lemma 5.1, we get

(6.8)

$$\Sigma_t = z^R \int_0^\infty e^{-\rho u} e^{\beta(R-1)u} \exp(-z/2A_u)(2A_u)^{1-R} \frac{\Gamma(\nu + 2 - R)}{\Gamma(\nu + 1)} \, {}_1F_1(\nu + 2 - R, \nu + 1, z/2A_u) du,$$

where now $z = \Delta_t$. The change of variable $s = 2A_u$ followed by $v = z/s$ gives an equivalent expression for Σ_t:

$$\Sigma_t = f(z) \equiv z^2 \frac{1}{2\lambda} \int_0^\infty \left(1 + \frac{\beta}{2\lambda} \frac{z}{v} \right)^{-\theta} e^{-v} v^{R-3} \frac{\Gamma(c)}{\Gamma(d)} \, {}_1F_1(c, d, v) dv, \tag{6.9}$$

where $c \equiv \nu + 2 - R$, $d \equiv \nu + 1$, and $\theta \equiv 2 + (\rho/\beta) - R$.

The finiteness of these integrals is guaranteed (see Lemma A.2) if we have

$$\nu + 2 - R > 0. \tag{6.10}$$

From (6.8), we see that the time-t price Σ_t is a function of only Δ_t. Thus, $\Sigma_t = f(\Delta_t)$ where the function f is defined by the right side of (6.8) or of (6.9) and can be computed numerically. Even more, Σ is a diffusion satisfying the SDE

$$\begin{aligned} d\Sigma_t &= \{f'(\Delta_t)(A-\beta\Delta) + \frac{1}{2}f''(\Delta_t)\sigma^2\Delta\}dt + f'(\Delta_t)\sigma\sqrt{\Delta_t}dB_t \\ &=: a(\Sigma_t)dt + b(\Sigma_t)dB_t, \end{aligned} \tag{6.11}$$

where we can in principle write Δ in terms of Σ as $\Delta_t = f^{-1}(\Sigma_t)$. Numerical estimation of derivatives of f by finite differencing allows us to characterize the diffusion Σ, and details of this are given in Section 7.3.

6.4 Asymptotic behaviour of f near zero

Consider again the expression (6.9) (which we recall is finite for all $z > 0$ as long as the condition (6.10) holds):

$$\begin{aligned} f(z) &= \frac{z^2}{2\lambda}\int_0^\infty \left(1+\frac{\beta}{2\lambda}\frac{z}{v}\right)^{-\theta} e^{-v}v^{R-3}\frac{\Gamma(c)}{\Gamma(d)}\,{}_1F_1(c,d,v)dv \\ &= \frac{z^2}{2\lambda}\sum_{k\geq 0}\int_0^\infty \left(1+\frac{\beta}{2\lambda}\frac{z}{v}\right)^{-\theta} e^{-v}v^{R+k-3}\frac{\Gamma(c+k)}{k!\,\Gamma(d+k)}dv, \\ &\equiv \frac{z^2}{2\lambda}\sum_{k\geq 0}\frac{\Gamma(c+k)}{k!\,\Gamma(d+k)}f_k(z) \end{aligned} \tag{6.12}$$

say, where of course

$$f_k(z) \equiv \int_0^\infty \left(1+\frac{\beta z}{2\lambda v}\right)^{-\theta} e^{-v}v^{R+k-3}dv.$$

We shall determine the asymptotics of $f(z)$ as $z \downarrow 0$ for different parameter regimes[8]. By monotone convergence, the limit as $z \downarrow 0$ of $f_k(z)$ is finite if and only if $R+k-2>0$, and the limit value is then $\Gamma(R+k-2)$. We therefore have a complete resolution of:
Case 1: $2 < R < \nu + 2$[9]. We obtain

$$f(z) \sim \frac{z^2}{2\lambda}\sum_{k\geq 0}\frac{\Gamma(c+k)\Gamma(R+k-2)}{\Gamma(k+1)\Gamma(d+k)},$$

the sum being convergent because the terms decay as k^{-2}.
The next case is
Case 2: $R = 2 < \nu + 2$. In this case, the terms $f_k(z)$ are convergent to finite limits except for $k = 0$, where we have

$$f_0(z) = \int_0^\infty \left(1+\frac{\beta z}{2\lambda v}\right)^{-\theta} e^{-v}\frac{dv}{v} \sim C.\log(1/z) \quad (z\downarrow 0)$$

for some constant C which varies from place to place. Accordingly,

$$f(z) \sim C\; z^2\log(1/z) \quad (z\downarrow 0)$$

in this case.
The final case is
Case 3: $R < \min\{2, \nu+2\}$. Once again, it is the term $f_0(z)$ which dominates, and if we write $\varepsilon = 2 - R > 0$, by change of variables in the integral we have

$$\begin{aligned} f_0(z) &= \int_0^\infty \left(1 + \frac{\beta}{2\lambda u}\right)^{-\theta} (zu)^{-1-\varepsilon} e^{-zu} z du \\ &\sim z^{-\varepsilon} \int_0^\infty \left(1 + \frac{\beta}{2\lambda u}\right)^{-\theta} u^{-1-\varepsilon} du \quad (z \downarrow 0), \end{aligned}$$

the final integral being convergent because $\theta = \varepsilon + \rho/\beta$. We deduce the asymptotics:

$$f(z) \sim C z^R \quad (z \downarrow 0).$$

In practice, the risk aversion coefficient $R > 2$, so that we expect the asset price to be quadratic in Δ when Δ is close to zero. See Section 7.3 for numerical evidence supporting this.

6.5 A closed-form expression for the function f

If with $R > 2$ we set $\theta = 0$, equivalent to the special choice of parameter

$$\rho = \beta(R-2),$$

the asymptotic form derived above for Case 1 ($2 < R < \nu + 2$) is in fact valid for all values $z \geq 0$. Thus,

$$\begin{aligned} f(z) &= \frac{z^2}{2\lambda} \sum_{k \geq 0} \frac{\Gamma(c+k)\Gamma(R+k-2)}{\Gamma(k+1)\Gamma(d+k)} \\ &= \frac{z^2}{2\lambda} \frac{\Gamma(c)\Gamma(R-2)}{\Gamma(d)} \, {}_2F_1(R-2, c; d; 1) \quad \forall z \geq 0, \end{aligned}$$

where

$${}_2F_1(a, b; k; x) = \frac{\Gamma(k)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(k+n)} \frac{x^n}{n!}$$

is the Gauss hypergeometric series discussed in, for example, Ch. 15 of Abramowitz and Stegun (1964).

If we denote

$$\gamma := \frac{1}{2\lambda} \frac{\Gamma(c)\Gamma(R-2)}{\Gamma(d)} \, {}_2F_1(R-2, c; d; 1),$$

then $f(z) = \gamma z^2$ and the SDE (6.11) specializes to

$$d\Sigma_t = 2\gamma^{1/4} \sigma \Sigma_t^{3/4} dB_t + \gamma[(\sigma^2 + 2A)\sqrt{\Sigma_t/\gamma} - 2(\beta/\gamma)\Sigma_t] dt. \tag{6.13}$$

In Section 7.3, we compare this analytical result with numerical estimates.

6.6 Prices of options

Given that we can compute the total asset price, the price of an option on the total asset price is only an integration away, because the transition density of the underlying process Δ is known to us by virtue of the Lemma 5.3. So consider a European call option on the total asset price S written at time $t = 0$ (when the value of Δ is Δ_0) with expiry date T and strike price K. The time-0 value of the option is then

$$\frac{1}{\zeta_0}\mathbb{E}\Big[\zeta_T(\Sigma_T - K)^+\Big|\mathcal{F}_0\Big].$$

Writing $\Sigma_T = f(\Delta_T)$ explicitly in terms of the underlying process Δ, and recalling the expression (4.2) for the state price density, the above expectation becomes

$$\begin{aligned} & x^R e^{-\rho T}\mathbb{E}^x\Big[\Delta_T^{-R}(f(\Delta_T) - K)^+\Big] \\ &= e^{(R\beta-\rho)T} x^R \mathbb{E}^x\Big[Y(A_T)^{-R}\big(f(e^{-\beta T}Y(A_T)) - K\big)^+\Big] \\ &= e^{(R\beta-\rho)T} x^R \int_0^\infty y^{-R}\Big(f(ye^{-\beta T}) - K\Big)^+ q_{A(T)}(x,y)dy \end{aligned} \tag{6.14}$$

where, with $x \equiv \Delta_0$ and $\nu = 2A/\sigma^2 - 1$, Lemma (5.3) has been used to write Δ in terms of the $BESQ^{(\nu)}(x)$ process X whose transition density $q_t(x,\cdot)$ is given from (5.2). We discuss evaluation of option prices in Section 7.4.

7 Calibration and Numerical Results

We now discuss the numerical evaluation of the various pricing expressions derived in the previous Sections, and also explain how typical parameter values for the model were obtained by using bond price data.

The model is parametrized by A, σ, β, R, and ρ. We recall that A and β control the mean reversion, and σ the volatility, of the ergodic diffusion Δ. R is the coefficient of relative risk aversion, assumed constant, and ρ is the discount factor for the utility of the agent. Apart from these parameters, the time-t prices of bonds and of the total asset as given in expressions (6.2) and (6.8) depend only on the value $z = \Delta_t$.

7.1 Numerical evaluation of model prices for bonds

Given model parameters and a time-t level $z = \Delta_t$ for Δ, computing time-t bond prices from equation (6.2) involves evaluating the hypergeometric function ${}_1F_1(a, b, x)$ for arguments $a = \nu + 1 - R = (2A/\sigma^2) - R$, $b = \nu + 1 = 2A/\sigma^2$ and $x = z/2A_T$. Because $R > 0$ and because of the condition (5.5), we shall have $b > a > 0$; the function ${}_1F_1(a, b, x)$ is then well-defined for all values of x that are contingent on the maturity date T.

Unfortunately, no general method exists to evaluate the hypergeometric function for a wide range of argument values. As the time of maturity varies over the entries of the vector M, the argument x will vary greatly, and numerical problems occur for short maturities when the argument x can become very large.

Muller (2001) advocates choosing a method depending on the values of $R_1 \equiv ax/b$ and $R_2 \equiv a(b-a)/x$, and we found this to work well with some modifications.

Muller's Method 1 involves adding a finite number of terms of the series defining ${}_1F_1(a, b, x)$. We use this method when $R_1 < 30$.

If $R_2 < 1$ and $x > 400$, we use an asymptotic series in x^{-1}, given in formula 13.5.1 of Abramowitz and Stegun (1964). This is also Method 2 of Muller, who suggests using it if $R_2 < 1$ and $x > 50$. In simulation tests, we found that this cutoff value of 50 for x is not large enough, which is why we increased it to 400.

The most reliable method seems to be the rational approximation of Luke, referred to as Method 5 in Muller (2001). We implemented this method using the freely available[10] SAS code of Muller adapted to the Scilab environment, and used it whenever the criteria for x, R_1 and R_2 described above were not satisfied.

7.2 Calibration Procedure

In order to calibrate the model, we searched for optimal parameter vectors $\theta = (A, \sigma, \beta, R, z, \rho)$ such that the prices for bonds, as computed from the expression (6.2) are close, in some appropriate sense, to actually observed bond prices. The level z of Δ has therefore been treated as one of the parameters in the fitting procedure.

The data consisted of a time series of $N + 1 = 278$ daily consecutive prices for zero coupon bonds on the US dollar, for maturities which we shall denote by the vector $M = (1/12, 1/4, 1/2, 1, 2, 5, 7, 10)$ whose entries are in years. We shall denote the data by (Y_i^n), where for $0 \leq n \leq N$ and $1 \leq i \leq 8$, Y_i^n represents the price observed on the n'th day for the bond maturing in $M(i)$ years from that day. The corresponding prices obtained from the model equation (6.2) shall be denoted by $(P_i^n(\theta))$.

We adopted as error-of-fit criterion the function

$$MAD_n(\theta) := \frac{1}{8}\sum_{i=1}^{8}\left|\frac{Y_i^n - P_i^n(\theta)}{Y_i^n}\right|, \tag{7.1}$$

defined for each day n of data, $0 \leq n \leq N$.

As a first attempt at calibration, we took the function (7.1) as our objective and estimated a time series of parameters $(\hat{\theta}_n)$ such that for each n, $\hat{\theta}_n$ minimizes $MAD_n(\cdot)$ subject to the parameter conditions (5.5) and (A.1). On each day, the starting iterate for the minimization was chosen to be some fixed vector θ which gives a reasonable fit to typical values in our data set.

Although reasonably good fits were obtained the parameter time series $(\hat{\theta}_n)$ was not as stable as one would wish.[11] To remedy this, we redid the minimization procedure using the cost function

$$MADV_n(\theta) := MAD_n(\theta) + |\theta - \hat{\theta}_{n-1}|^2, \tag{7.2}$$

which penalizes day-to-day variation in the parameter vector θ.

The form of the penalty term here is not the most natural one to choose. In particular, because typical parameter values are of different orders of magnitude, the weightings for parameter variation implied in the cost function (7.2) are unequal. The results of using an equally weighted cost function, (by penalizing changes in parameter values *relative* to values that are the result of the previous day's optimization, say) were qualitatively similar, and in general did not improve quality of fit. We also tried a likelihood approach, as follows. We suppose that rather than observing true bond prices we observe bond prices plus an independent noise term, and also suppose that parameters are conditionally Gaussian. The process Δ is allowed to vary from day to day according to its (known) transition density, and parameter fits are obtained by maximizing an appropriate likelihood. This approach is more flexible and intuitive, because weights can now be interpreted as variances of error terms. However, we found that quality of fit and parameter stability were inferior to those obtained with cost function (7.2).

In Table 7.1 we report the descriptive statistics for quality of fit (expressed through the criterion $MAD(\cdot)$) resulting from using $MAD(\cdot)$ (left column) and $MADV(\cdot)$ (right column) as cost functions. As expected, a slight loss of quality of fit results from using $MADV(\cdot)$ but this is a small price to pay for the appreciable gain in parameter stability. We see in the first two columns of Table 7.2 that parameter variance is reduced by several orders of magnitude in some cases[12] when using $MADV(\cdot)$ as opposed to $MAD(\cdot)$ as objective function. In the right column of the same table, we present values of parameters on the day of best fit for one of several calibration runs that we performed. This parameter set was used for all numerical studies of asset prices presented below.

7.3 Total Asset Price

Fixing model parameters A, σ, β, R, and ρ, the time-t total asset price $\Sigma_t = f(\Delta_t)$, as discussed in Section 6.3. Here, the function f defining Σ in terms of Δ is the right hand side

Table 7.1: Description of $MAD(\cdot)$ errors-of-fit for the two cost funtions.

	MAD	MADV
	(basis points)	
Min.	4.092	4.670
1st Qu.	7.301	7.899
Median	9.297	10.076
Mean	10.589	10.726
3rd Qu.	12.982	12.507
Max.	22.707	23.422

Table 7.2: Variances of parameter estimates resulting from minimizing the two cost functions, and best values obtained from minimizing $MADV(\cdot)$ cost function.

	MAD	MADV	'Best' values
	(units of 10^{-6})		
A	455.1479	9.4801	0.5189612
σ	126.0241	46.3808	0.1483002
β	86.0588	4.5066	0.207032
R	139.6877	0.1777	3.04367
ρ	10.7768	8.0131	0.078836

of the expression (6.8) and can be evaluated numerically.

We set model parameters as in the rightmost column of Table 7.2. The parameter z, which we recall is the time-t level of Δ, is now considered to be a variable, rather than a fitted value, in order that we can study the dependence of Σ on Δ. We evaluated $f(z)$ on a sparse grid spanning z-values from 0 up to three standard deviations beyond the mean of the stationary law of Δ. Interpolation was then used to obtain values of f on a much finer grid.

The resulting prices are plotted in the first frame in Figure 7.1. and on a log-log scale in the second frame. The remaining two plots exhibit the form of the drift and volatility coefficients $a(\cdot)$ and $b(\cdot)$, respectively, in the SDE (6.11). The slope of the log-log plot for the volatility coefficient $b(\cdot)$ is estimated at 0.8065, and this is close to the value 0.75 which is valid asymptotically as $\Delta \to 0$ (cf. Section 6.4).

For the special case $\rho = \beta(2-R)$, the function f is known exactly in closed form, as explained in section 6.5. Figure 7.2 shows the same characteristics for the diffusion Σ, evaluated numerically, for parameter choices as those described above, but with $\rho = \beta(2-R)$. The estimate of the form of the volatility is as expected, the exponent 0.75 now being valid for all values of Δ. The constant γ defined in Section 6.5 is estimated (from the log-linear relationship in the third frame) as 1.93, which is to be compared to its analytical value 1.92987.

Figure 7.3 shows how total asset price varies as a function of σ, the volatility coefficient of the aggregate rate of output, and the risk aversion R. We would generally expect that as σ rises, the asset price would fall. This indeed happens when R is large enough, but for smaller values of R the opposite happens. Why should this be? Looking at the expression (4.5) for the spot rate, we see that it decreases with σ; this therefore depresses the value of a bond, and explains qualitatively why the share price rises. Similarly, rising R makes agents less inclined to invest in risky assets, so the fact that markets nevertheless clear requires the share value to rise, as we see for smaller values of σ. But once again, the direction is reversed when σ gets too large. Again, the interest rates have fallen as σ rises, and so the bond is a less attractive investment; market clearing requires that the share price also falls to keep zero net supply of the less attractive bond.

7.4 Option and Bond Yield Volatilities

In Section 6.6 we exhibited the value of a (European) option as an integral with respect to the density function of a certain squared Bessel process. Computationally, evaluating the integral involves the integration of the modified Bessel function of order ν.

For our choice of parameters, the index ν lies in the region of 40, and for small values of the maturity time T, we are dealing here with evaluating the modified Bessel function at arguments of the order of $10^3 - 10^4$, where the function available in Scilab fails. To get around this problem, we used the relation, given in formula 13.6.3 of Abramowitz and Stegun (1964), between the modified Bessel function and the confluent hypergeometric function:

$$e^{-z} I_\nu(z) = (z/2)^\nu \frac{1}{\Gamma(\nu+1)} e^{-2z} \, {}_1F_1(\nu + 1/2, 2\nu + 1, 2z). \tag{7.3}$$

Computing the right side of this formula using the hypergeometric function calculation method described in section 7.1 works well for values $z > 1300$.

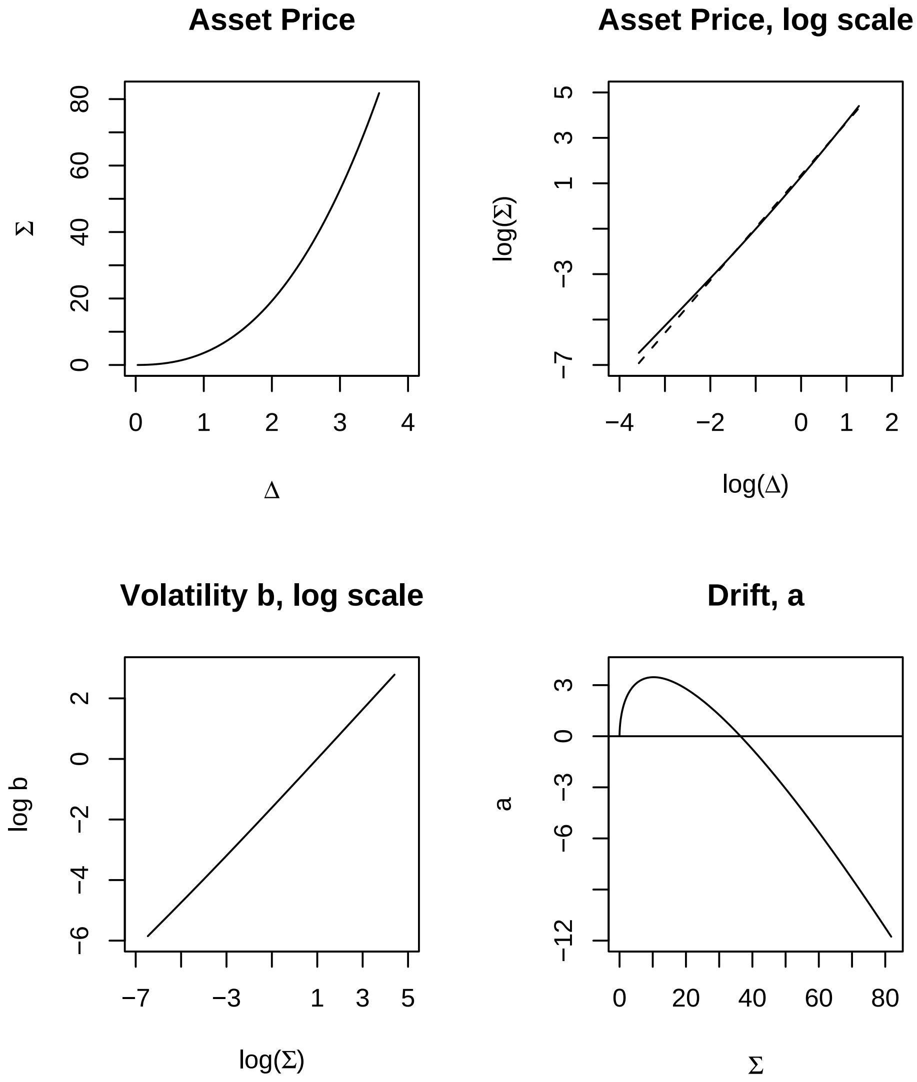

Figure 7.1: Characteristics of the diffusion Σ. Model parameters are $A = 0.5189612$, $\sigma = 0.1483002$, $\beta = 0.207032$, $R = 3.04367$, $\rho = 0.07883598$. Estimated Regression model for asset price, shown as dotted line, is $f(z) = 3.9041\ z^{2.3169}$. Estimated volatility is $b(s) = 0.4566\ s^{0.8065}$.

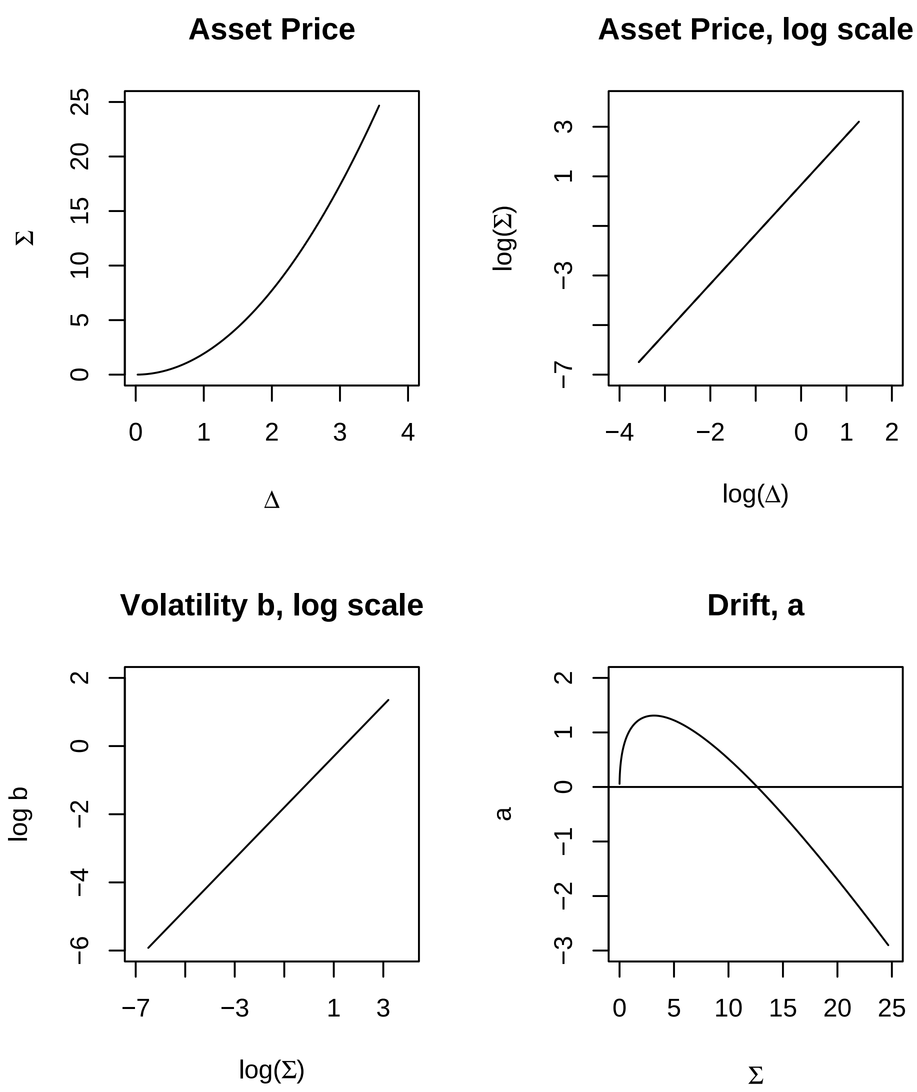

Figure 7.2: Characteristics of the diffusion Σ. Model parameters are $A = 0.5189612$, $\sigma = 0.1483002$, $\beta = 0.207032$, $R = 3.04367$, $\rho = \beta(R-2)$. Estimated Regression model agrees perfectly with analytical form; $f(z) = 1.93\ z^2$, $b(s) = 0.5724\ s^{0.75}$.

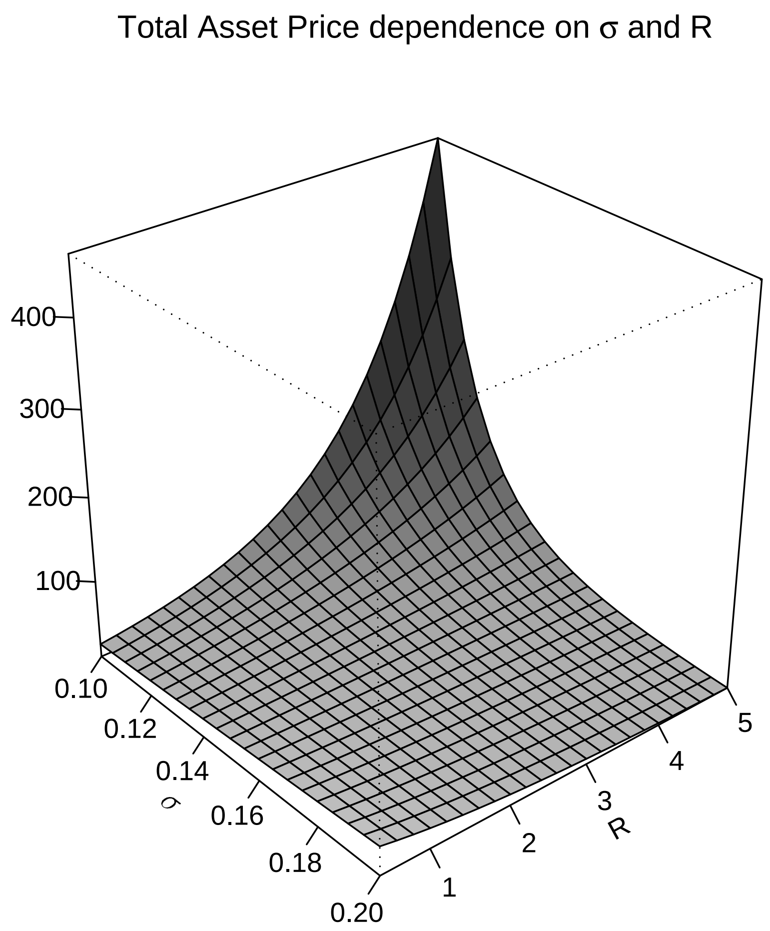

Figure 7.3: The surface represents the total asset value as a function of risk aversion, R, and of the volatility, σ, of the aggregate output Δ. Risk aversion ranges from 0.4 to 5, while σ varies between 0.1 and 0.2. Remaining model parameters are as in Table 7.2, and the level of Δ is fixed at 2.5.

For purposes of illustration, we fixed the strike price $K = f(A/\beta)$ to correspond to the market level when Δ is at the mean of its stationary law, and computed call prices, put prices, and the implied volatility surface. The latter is shown in Figure 7.4. For comparison, we also plot volatility of yields of bonds of same lifetimes as the options (Figure 7.5), as given by the expression (6.4).

The put-call parity relation for our model is, in obvious notation,

$$\begin{aligned} \mathrm{Call}(\Sigma_0, T, K) - \mathrm{Put}(\Sigma_0, T, K) &= \mathbb{E}^{\Delta_0}[\zeta_T \Sigma_T/\zeta_0] - K\mathbb{E}^{\Delta_0}[\zeta_T/\zeta_0] \\ &\equiv \tilde{\Sigma}_0 - K \exp(-\tilde{r}T). \end{aligned}$$

In the Black-Scholes model, the second line here would be the difference in price between a call and a put option written when the underlying is at $\tilde{\Sigma}_0$, with expiry time T and constant rate of interest $\tilde{r}$. In computing implied volatilities for our model, we therefore solved for the volatility parameter in the Black-Scholes pricing formula with $\tilde{\Sigma}_0$ as starting value for the asset, with $\tilde{r}$ being the yield of a bond expiring with the option, and with the dividend rate being zero.[13]

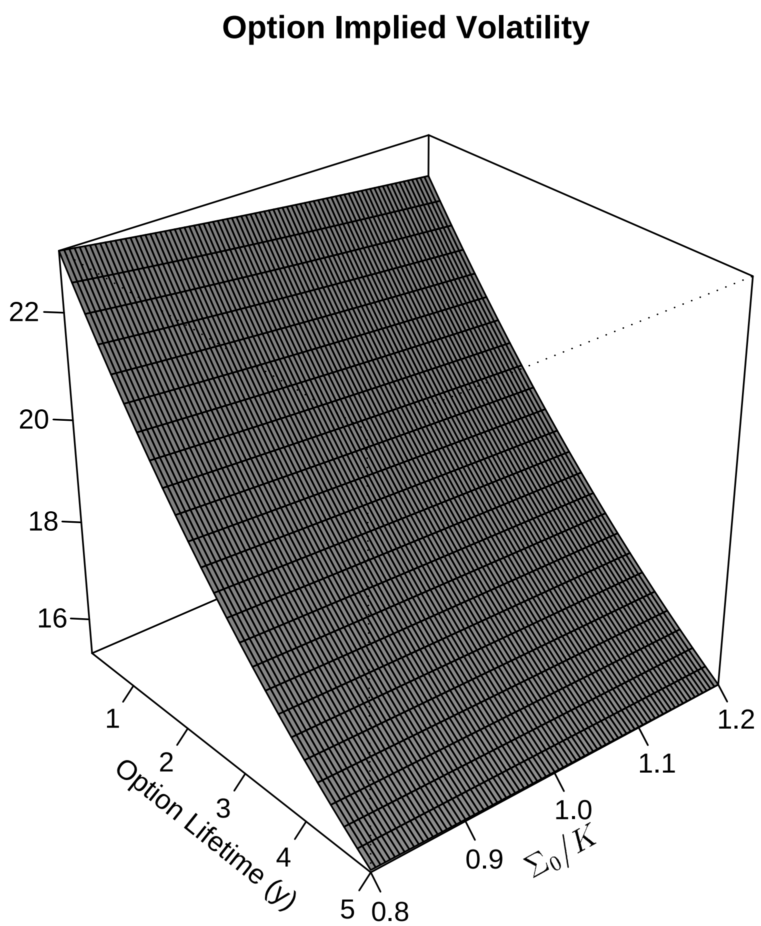

Figure 7.4: Implied Volatility (%) surface for European options on the total asset value Σ. Model Parameters are as in Table 7.2. Strike $K = 35$; this value is close to the total asset price when Δ is at the mean of its stationary law. Option lifetimes range from 0.2 to 5 years, and the moneyness of the option varies from 80% to 120%.

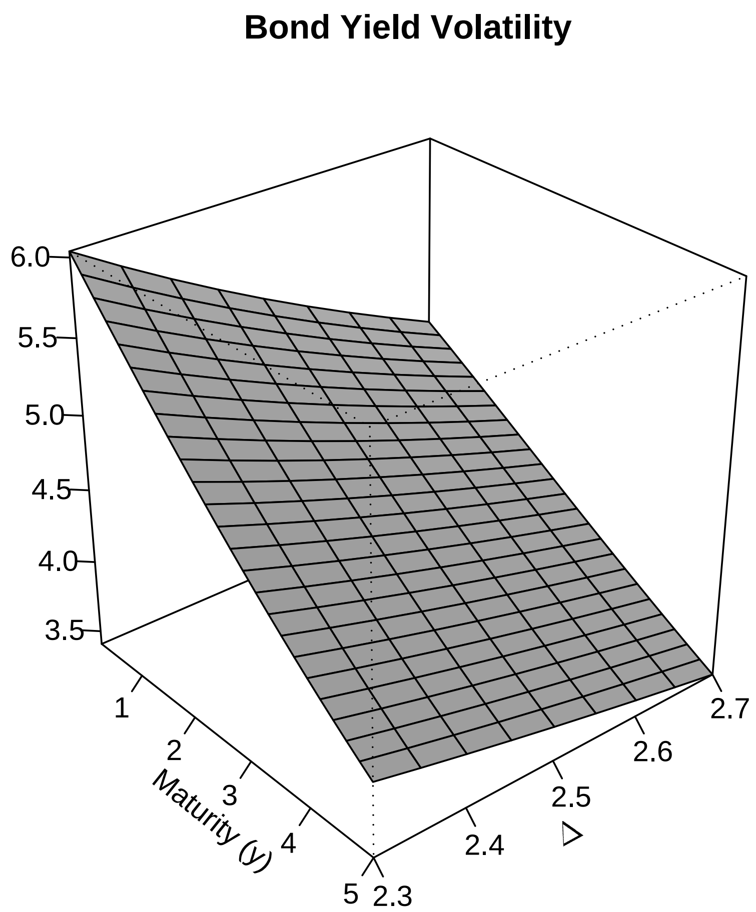

Figure 7.5: Volatility (%) of bond yields as a function of Δ and maturity. Δ varies within one standard deviation away from the mean of its stationary law, and maturity times range from 0.2 to 5 years. Model parameters are as in Table 7.2.

8 Conclusions

We have taken a simple and quite explicit model for a multi-asset single-agent economy in which the prices of bonds and shares can be computed in closed form, and simple recipes can be provided for pricing effectively any European option. The one-factor interest-rate model implied by the model is of an apparently novel form, and we have fitted the model to yield curve data.

The quality of the fit is comparable with other one-factor models, and the stability of parameter estimates is good. Our principal aim in calibrating the model was to obtain typical parameter values to work with when investigating the implications of the model assumptions on asset price dynamics. However, in principle, the time series of parameters obtained in fitting the model might also give information about the dynamics of risk aversion and the market price of risk over the period fitted. Indeed, the values for R and ρ that we obtain are sensible.

We have computed implied volatilities for European call options, and find that these typically exhibit a skew, not unlike actual data. The basic model has features in common with the CEV stock model with exponent 3/4, and such skewed implied volatility curves typically arise for CEV models.

The model assumptions are very restrictive; independence of the productive assets, and common volatility and mean-reversion parameters are quite severe. Nonetheless, under these assumptions we get a long way: we have built a consistent complete market model for multiple shares and the riskless rate. The assumed CRRA utility of the representative agent can be relaxed a little. Indeed, we could as easily compute prices using as a state-price density

$$\zeta_t = a_1 e^{-\rho_1 t} \Delta_t^{-R_1} + a_2 e^{-\rho_2 t} \Delta_t^{-R_2}$$

for positive constants a_1, a_2, R_1, R_2, ρ_1, ρ_2. This would implicitly define the agent's utility via

$$U'(t, \Delta_t) = \zeta_t;$$

although it is not possible to write the utility now in closed form, all the pricing calculations of the paper can be carried through with minor modifications. Such an extension would allow for different coefficients of relative risk aversion for large and small consumption levels.

A Appendix

We prove the result, referred to in Section 4, that under some mild conditions on the model parameters, the change-of-measure process induced by the state price density ζ is a true martingale.

Lemma A.1 *The condition*

$$\frac{2A}{\sigma^2} \geq 2R+1 \tag{A.1}$$

is necessary and sufficient for the local martingale Z defined at (4.4) to be a martingale.

PROOF. Firstly, suppose that the local martingale Z is actually a martingale. The effect of the change of measure is to add a drift to dB:

$$dB = d\tilde{B} - \frac{R\sigma}{\sqrt{\Delta}}\, dt$$

where $\tilde{B}$ is a $\tilde{\mathbb{P}}$-Brownian motion, so that Δ solves the SDE

$$d\Delta = \sigma\sqrt{\Delta} d\tilde{B} + (A - R\sigma^2 - \beta\Delta)dt \tag{A.2}$$

in the probability $\tilde{\mathbb{P}}$. This SDE for Δ is exact, and is of the same general (CIR) form as the original SDE. Because of the standing assumption (2.3), we have for any $T > 0$ that

$$\mathbb{P}[\Delta_t > 0 \text{ for all } 0 \leq t \leq T] = 1,$$

and since $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ on any $\mathcal{F}_T$, we have to have

$$\tilde{\mathbb{P}}[\Delta_t > 0 \text{ for all } 0 \leq t \leq T] = 1.$$

Routine scale function calculations give (A.1) as the necessary and sufficient condition for this.

Conversely, suppose that condition (A.1) is satisfied. The only problem with defining a new measure using the exponential local martingale Z is that the drift $-R\sigma/\sqrt{\Delta}$ becomes unbounded near zero; if this local martingale genuinely were a martingale, then under the transformed measure the process Δ would satisfy (A.2). However, it is useful result (see Hobson & Rogers (1998) that as long as the solution of the SDE which would arise by change of measure never reaches the singularity set of the drift, then the local martingale *is* a martingale, and everything goes through as desired. The scale function s of the solution to (A.2) is characterised by

$$s'(x) = \exp(-\int^x \frac{2(A - R\sigma^2 - \beta y)}{\sigma^2 y}\, dy) = x^{-2(A-R\sigma^2)/\sigma^2} \exp(2\beta x/\sigma^2)$$

and so 0 is an inaccessible boundary point if (A.1) holds. ■

Lemma A.2 *The solution to (2.2)*

$$d\Delta_t = \sigma\sqrt{\Delta_t}dB_t + (A - \beta\Delta_t)dt$$

is an ergodic diffusion on $(0, \infty)$ *with invariant law* $\Gamma(2A/\sigma^2, 2\beta/\sigma^2)$. *The expectation*

$$\mathbb{E}^x \int_0^\infty e^{-\rho t}\Delta_t^\theta \, dt \tag{A.3}$$

is finite for all $x > 0$ *if and only if*

$$\theta + \frac{2A}{\sigma^2} > 0 \tag{A.4}$$

PROOF. The invariant density π of Δ solves the adjoint equation

$$\mathcal{G}^*\pi \equiv D^2\left[\frac{1}{2}\sigma^2 x\pi(x)\right] - D\left[(A - \beta x)\pi(x)\right] = 0$$

and it is a simple exercise to solve this for π to obtain a density

$$\pi(x) = x^{-1+2A/\sigma^2}e^{-2\beta x/\sigma^2}/\Gamma(2A/\sigma^2).$$

For the final statement, it is clear that the expectation (A.3) is either finite for all $x > 0$ or for no $x > 0$, since the diffusion is regular. Now assuming (A.4) holds,

$$\int_0^\infty \left\{\mathbb{E}^x \int_0^\infty e^{-\rho t}\Delta_t^\theta \, dt\right\} \pi(x) \, dx = \frac{1}{\rho} \, \mathbb{E}^\pi \Delta_0^\theta < \infty, \tag{A.5}$$

so the expectation (A.3) must be finite for all $x > 0$. Conversely, if the condition (A.4) fails, then the expectation (A.5) is infinite and so every expectation (A.3) must be infinite by a coupling argument. ■

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Notes

[1] The processes in (i) can be thought of as the output at equilibrium from firms in which optimally behaving market agents invest labour as well as units of the good. See section 2 of Breeden (1979) for more on this.

[2] This is the same process that the *consumption flows* in our model follow.

[3] The probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is the usual augmentation of the filtration generated by W - see, for example, Rogers and Williams (2001).

[4] ... equivalent to the statement that $\mathbb{P}(\Delta_t > 0 \quad \forall t > 0) = 1$ ...

[5] ... assumed for the moment to exist: this point will be dealt with later for our explicit example.

[6] for the moment: more involved examples are discussed later.

[7] The finiteness of the expectation in (4.10) is not immediately obvious; it turns out that the expectation is finite if $-R + 2A/\sigma^2 > 0$, which is implied by the assumed condition (4.6). See Lemma A.2 for more on this.

[8] We thank Alexander Cherny (personal communication) for the first proof of these asymptotics, using methods of sample-path estimation.

[9] The inequality $R < \nu + 2$ is simply (6.10).

[10] http://www.bios.unc.edu/~muller.

[11] If the model is correct, then we logically expect the time series $(\hat{\theta}_n)$ to stay constant from one day to the next.

[12] especially for R and A which are large compared to remaining model parameters and are thus more heavily penalized by the choice of penalty term in Equation (7.2).

[13] ... dividends having already been taken into account implicitly in computing $\tilde{\Sigma}_0$.