Changing the law of the scaled Brownian excursion (12/4/87).

1/ Let P be the law on $C([0,1], \mathbb{R}^+)$ of the scaled Brownian excursion, let ρ denote the canonical process, equal in law to $\{(1-t)\,R(t/1-t) : 0 \le t \le 1\}$, where R is a $BES_0(3)$. If one changes measure to $\tilde{P}$, where

$$\frac{d\tilde{P}}{dP} = c.\left(\sup_{0\le t\le 1} \rho_t\right)^{\alpha},$$

where c is a normalizing constant, and α is $-\frac{1}{2}$, then according to David, certain expectations with respect to $\tilde{P}$ can be expressed in terms of the Riemann zeta f^n, or related f^{ns}. If one could obtain explicitly the induced drift, this could be very useful in evaluating such expectations.

2/ Define $\psi : (0,\infty) \times C([0,1], \mathbb{R}^+) \to \mathcal{U} \equiv C(\mathbb{R}^+, \mathbb{R}^+)$ by

$$\psi(\tau, f) = \sqrt{\tau}\, f(\cdot/\tau).$$

Thus the product measure $(2\pi\tau)^{-3/2} d\tau \times dP$ on $(0,\infty)\times C([0,1], \mathbb{R}^+)$ goes under ψ to the Brownian excursion law n. If $\tilde{n}$ is the image of the law got by replacing P by $\tilde{P}$, we have that

$$\frac{d\tilde{n}}{dn} = c\left(\bar{\xi}/\sqrt{\zeta}\right)^{\alpha},$$

where $\bar{\xi} \equiv \sup_t \xi_t$, and ζ is the lifetime of the excursion ξ. So we can ask equivalently, "What is the induced drift got by taking $\tilde{n}$ in place of n?" To answer this, we want the C-M martingale reasonably explicitly, so we need to find

$$\varphi(t, x, s) \equiv E^x\left[\left(s \vee \bar{B}(\tau)/(t+\tau)^{\frac{1}{2}}\right)^{\alpha}\right] \qquad 0 \le x \le s,\ 0 \le t,$$

where $\bar{B}_t \equiv \sup(B_u : u \le t)$ and $\tau \equiv \inf\{u : B_u = 0\}$. This is the problem of solving the PDE

$$\begin{cases} \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial\varphi}{\partial t} = 0 \\ \frac{\partial\varphi}{\partial s} = 0 \quad \text{when } s = x \end{cases}$$

with bcs

$$\lim_{x\downarrow 0} \varphi(t,x,s) = \left(s/\sqrt{t}\right)^{\alpha}$$

and

$$\lim_{s\to\infty} s^{-\alpha}\varphi(t,x,s) = E^x\left[(t+\tau)^{-\alpha/2}\right] = \int_0^\infty \frac{x\,e^{-x^2/2u}}{(2\pi u^3)^{\frac{1}{2}}}\,(t+u)^{-\alpha/2}\,du.$$

The final property precludes any particularly nice closed-form solution.
Whatever the drift, it is going to depend on S_t in some way.
One natural approach is to try to represent

$$\varphi(t,x,s) = \int \mu(s, d\theta) \exp(\theta x - \tfrac{1}{2}\theta^2 t).$$

Biane + Yor, in a big paper on Cauchy's principal values of local times etc... also observe the connection with the zeta f^n.

The process $Y_t \equiv A(t, B_t)$ (15/4/87)

In the study of Brownian local time sheet, the processes $A(t,x) \equiv \int_0^t I_{\{B_s \leq x\}} ds$ enter in various ways; in particular, when trying to make stochastic integrals with respect to the intrinsic local time sheet, we find ourselves looking at $Y_t \equiv A(t, B_t)$.

(i) Y has finite quadratic variation; more precisely, $Y_t - \int_0^t L(u, B_u) dB_u$ has zero quadratic variation.

To see this,

$$Y_{t+h} - Y_t - \int_t^{t+h} L(u, B_u)\, dB_u$$

$$= \int_{-\infty}^{B_{t+h}} L(t,x)\, dx - \int_{-\infty}^{B_t} L(t,x)\, dx - \int_t^{t+h} L(u, B_u)\, dB_u - \int_t^{t+h} I_{\{B_s \leq B_{t+h}\}} ds$$

$$(*) \qquad = \int_{B_t}^{B_{t+h}} \{L(t,x) - L(t,B_t)\}\, dx - \int_t^{t+h} \{L(u, B_u) - L(t, B_t)\}\, dB_u - \int_t^{t+h} I_{\{B_s \leq B_{t+h}\}} ds.$$

The final term is $\leq h$, so is negligible. For the second term,

$$E\left(\int_t^{t+h} \{L(u,B_u) - L(t,B_t)\}\, dB_u\right)^2 = E \int_t^{t+h} \{L(u,B_u) - L(t,B_t)\}^2 du$$

$$\leq 2 \int_t^{t+h} E\{L(u,B_u) - L(t,B_u)\}^2 du + 2\int_t^{t+h} E\{L(t,B_u) - L(t,B_t)\}^2 du.$$

Now $L(u, B_u) - L(t, B_u) \stackrel{\mathcal{D}}{=} L(u-t, 0)$, just by considering time reversed from u, so the first term is

$$2 \int_t^{t+h} (u-t)\, du = h^2,$$

so this is OK. The other term is simplified by noting that

$$L(t, B_u) - L(t, B_t) \stackrel{\mathcal{D}}{=} L(t, Y) - L(t, 0), \text{ where } Y \sim N(0, u-t)$$

indept. of B. For $y \in \mathbb{R}^+$,

$$L(t,0) - L(t,y) = |B_t| - |B_t - y| + |y| + \int_0^t (\mathrm{sgn}(B_s - y) - \mathrm{sgn}(B_s))\, dB_s$$

$$= |B_t| - |B_t - y| + |y| - 2\int_0^t I_{\{0 < B_s \leq y\}} dB_s,$$

so

$$E\{L(t,0) - L(t,y)\}^2 \leq 2 \cdot 4y^2 + 2 \cdot 4 \int_0^t E(I_{\{0 < B_s \leq y\}}) ds$$

$$\leq 8y^2 + 8 \int_0^t \frac{y}{\sqrt{2\pi s}}\, ds \leq C\,(y^2 + y\sqrt{t}\,).$$

Thus

$$E \int_t^{t+h} \{L(t,B_u) - L(t,B_t)\}^2\, du$$

$$\leq C \int_t^{t+h} du \int_{-\infty}^{\infty} \frac{e^{-y^2/2(u-t)}}{\sqrt{2\pi(u-t)}}\, dy\, (y^2 + |y|\sqrt{t}\,)$$

$$\leq C \int_0^h du\, (u + \sqrt{t}\sqrt{u}\,)$$

$$\leq C\,(h^2 + \sqrt{t}\, h^{3/2}).$$

Thus the expectation of the square of the second term in (*) is bounded by $C\, h^{3/2}$.

The first term of (*) is now estimated as follows. For $y > 0$,

$$E\left[\left(\int_0^y (L(t,x) - L(t,0))\, dx \right)^2 \right]$$

$$\leq y\, E \int_0^y (L(t,x) - L(t,0))^2\, dx$$

$$\leq y \cdot \int_0^y C\,(z^2 + z\sqrt{t}\,)\, dz \leq C\,(y^4 + \sqrt{t}\, y^3)$$

Now taking average over y with a $N(0,h)$ law gives

$$E\left[\left(\int_{B_t}^{B_{t+h}} \{L(t,x) - L(t,B_t)\}\, dx \right)^2 \right] \leq C\,(h^2 + \sqrt{t}\, h^{3/2}).$$

Thus

$$E\left[\left(Y_{t+h} - Y_t - \int_t^{t+h} L(u,B_u)\, dB_u \right)^2 \right]$$

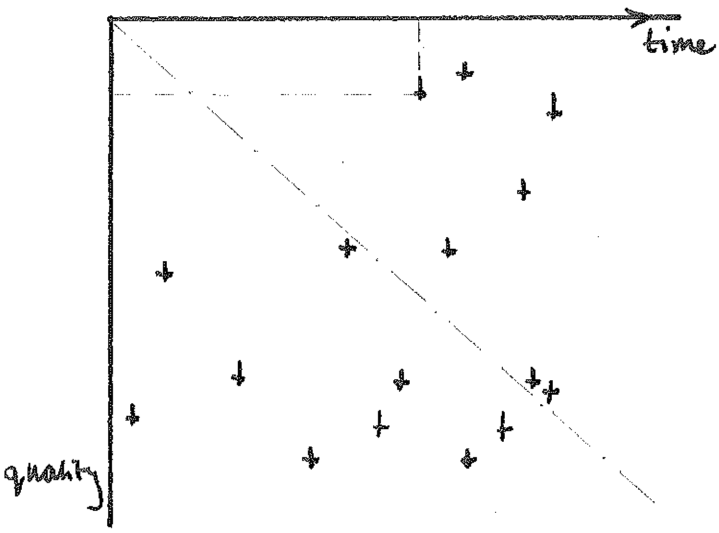
$$\leq C_T\, h^{3/2} \qquad \text{for all } h \in [0,1],\ t \leq T,$$

from which zero quadratic variation follows.

<u>Best-Choice problems etc.</u> (21/4/87).

1/ The business of choosing a 'non-informative' prior on the number of options to be viewed looks like it can be represented by the following device, which allows a quick proof of the fact that the number of records in $(t,1]$ is Poisson, independent of the number of records in $(0,t)$. It also can be used to give a quick proof of Ignatov's theorem in the continuous case.

2/ Take a Poisson point process Π in $(0,\infty)\times(-\infty,0)$ with intensity measure equal to Lebesgue measure. Think of the first coordinate as measuring time, the second as measuring the quality of options; a realisation of the point process tells us when options arrived, and of what quality they were. Of course, infinitely many arrive in any open interval, but most of them are of dreadfully low quality!!



Fix some $-K<0$, and define for $-K\le x<0$

$R_x \equiv$ no. of records whose quality is in $(-K,x]$.

Then I claim that

$$(1)\qquad R_x - \int_{-K}^{x}\frac{dy}{|y|} = R_x - \log\left(\frac{K}{|x|}\right)$$

is a martingale, and that for $-b<-a<0$,

$$(2)\qquad R(-a)-R(-b) \sim \mathcal{P}\left(\log\frac{|b|}{|a|}\right).$$

The easiest way to prove this is to restrict attention to those options with quality $\ge -K$, and apply the function $x\mapsto\log(K/|x|)$ to the qualities of such options; we end up with a Poisson point process in $(0,\infty)\times(0,\infty)$ with intensity measure $K\,dt\times e^{-x}dx$, so we are observing i.i.d. $\exp(1)$ r.v.'s arriving at the points of a Poisson process in time of rate K. The record process for this is evidently a Poisson process of rate 1, and statements (1) and (2) follow from this.

Return to the original Poisson process Π in $(0,\infty)\times(-\infty,0)$. A record is a point (t,x) in the point process such that

$$\Pi\left((0,t]\times[x,0)\right) = 1$$

– there are no points of the point process to the left and above (t,x). Now the law of Π is invariant under the reflection $(t,x)\mapsto(-x,-t)$ in the line $t=-x$, and it's easy to see that the image of the record process under this reflection is the record process of the reflected point process!

Hence the number of records occurring at times between t and 1 has the same dist^{n} as the number of records with values between -1 and $-t$, and is independent of the number of records occurring between times 0 and t!

Thus the no. of records occurring in the time interval $(t,1]$ is Poisson with mean $-\log t$, independent of the number of records in $(0,t]$.

If one takes as one's rule "Take the first record which arrives after time τ", where $\tau\in(0,1)$ is a parameter to be chosen, then you win if there is exactly one record arriving in $(\tau,1)$, otherwise you lose.

$$P\left(\text{exactly 1 record arriving in } (t,1)\right) = \left(\log\tfrac{1}{t}\right)e^{-(\log\frac{1}{t})} = -t\log t,$$

which is maximised when $t=e^{-1}$.

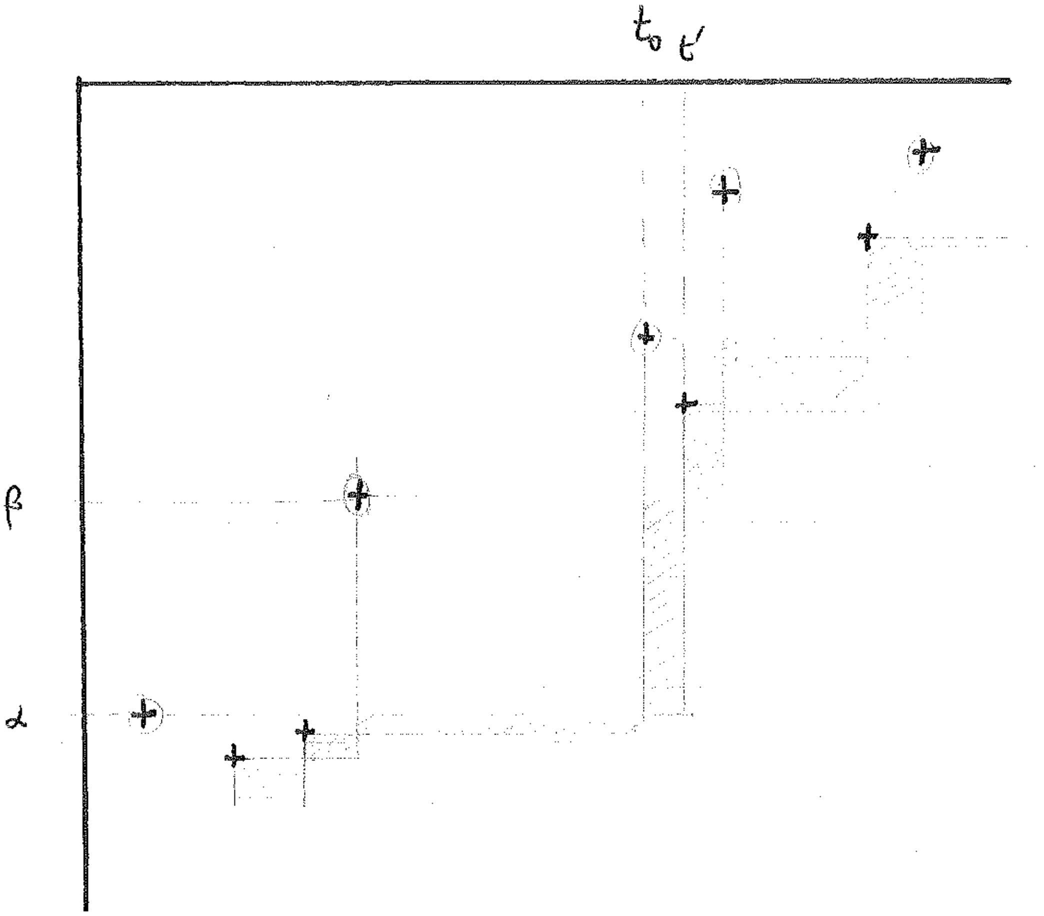
3 Ignatov's Theorem

Call a point (t,x) in the point process a k-record if

$$\Pi\left((0,t]\times[x,0)\right) = k.$$

The k-record process is the point process on $(-\infty,0)$ obtained by marking all the values of the k-records. Ignatov's thm states that the k-record processes, $k=1,2,\dots$, are i.i.d. point processes.

The picture below illustrates the situation when the 1-records and 2-records have been marked in. Only in the shaded rectangles is it possible for 3-records to appear. If we consider one of these rectangles, with lower edge at level α, upper edge at level β, the 3-record process restricted to (α,β) can be thought of as arising in the following manner. Wipe out the whole



of Π to the left of t_0; then the 3-records in (α,β) are just the 1-records of the remaining process which have values in (α,β). The necessary independence is at least intuitively obvious, though writing out a proof would kill the result. Vervaat's proof is in any case short and pretty, but this is quite nice

Note: by the same reflection tricks, we can say that for $0 < s < t$, the numbers Y_k of k-records which occur in the time interval $(s,t]$ are i.i.d. $P(\log t/s)$ r.v.'s, which allows us to analyse ~~[illegible]~~ 'one of best k' choice models!

Uniform integrability via transforms? (12/5/87)

The statement that a family of r.v.s is UI is a statement about the distributions of the r.v.s, and so should be expressible in terms of Laplace or Fourier transforms.

1/ Let F be the dist^{n} f^{n} of some integrable non-negative r.v.,

$$\tilde{F}(\lambda) \equiv \int_0^\infty e^{-\lambda t} F(dt).$$

Then

$$1-\tilde{F}(\lambda) = \int_0^\infty \frac{1-e^{-\lambda t}}{t} \, t F(dt)$$

$$= \int_0^\infty \Big[\lambda - \int_0^t \frac{ds}{s^2} \, e^{-\lambda s}(e^{\lambda s}-1-\lambda s) \Big] \, t F(dt)$$

$$= \lambda \int_0^\infty t F(dt) - \int_0^\infty \frac{ds}{s^2} \{1-(1+\lambda s)e^{-\lambda s}\} \int_s^\infty t F(dt)$$

$$\therefore \quad \boxed{\frac{\tilde{F}(\lambda)-1-\lambda \tilde{F}'(0)}{\lambda} = \int_0^\infty \frac{du}{u^2} \{1-e^{-u}(1+u)\} \int_{u/\lambda}^\infty t F(dt) \geq 0}$$

Let's suppose that $\{F_n : n \geq 0\}$ is a sequence of dist^{ns} on $\mathbb{R}^+$, let

$$\tau_n(a) \equiv \int_{(a,\infty)} t F_n(dt),$$

$$\tau(a) \equiv \sup_n \tau_n(a).$$

Then τ is right-continuous, decreasing, and $\{F_n\}$ is UI iff $\tau(a) \to 0$ as $a \to \infty$.
From the boxed equation,

$$\boxed{\{F_n\} \text{ is UI} \iff \sup_n \left\{ \frac{\tilde{F}_n(\lambda)-1-\lambda \tilde{F}_n'(0)}{\lambda} \right\} \to 0 \text{ as } \lambda \downarrow 0}$$

2/ If the distribution F on $\mathbb{R}$ has characteristic function φ, we have that

$$\int_{-\infty}^\infty e^{-\lambda |x|} F(dx) = \int \frac{\lambda \, d\theta}{\pi(\lambda^2+\theta^2)} \varphi(\theta) \equiv \tilde{F}(\lambda),$$

the Laplace transform of $|X|$. This in principle allows a characterisation of uniform integrability in terms of characteristic functions.

3/ $\{F_n\}$ uniformly integrable $\Rightarrow$ (assuming all are zero mean)

$$|\varphi_n'(\theta)| = \left| \int ix(e^{i\theta x}-1) F_n(dx) \right|$$

$$\leq 2\int_{|x|\geq\lambda} |x| F_n(dx) + \int_{-\lambda}^{\lambda} |x| \, |e^{i\theta x}-1| \, F_n(dx)$$

$$\leq 2\int_{|x|\geq\lambda} |x| F_n(dx) + \sup_{|x|\leq\lambda} |e^{i\theta x}-1| \, \|X_n\|_1$$

$$\Rightarrow \quad \sup_n |\varphi_n'(\theta)| \to 0 \quad \text{as } \theta \to 0.$$

In particular, since $\varphi_n(\theta) = 1 + \theta\varphi_n'(\sigma\theta)$ for some $\sigma \in (0,1)$,

$$\boxed{\{F_n\} \text{ UI} \Rightarrow \sup_n \left| \frac{\varphi_n(\theta)-1}{\theta} \right| \to 0 \quad \text{as } \theta \to 0}$$

4/ Let G_n denote the symmetrised dist^{n} put onto $\mathbb{R}^+$: $G_n(dx) = F_n(dx) + F_n(-dx)$.
Notice that

$$\int_0^\infty e^{-\lambda\theta} \frac{1-\cos\theta x}{\theta^2} \, d\theta = \int_0^\infty e^{-\lambda\theta} \left(\int_0^x \frac{\sin\theta y}{\theta} dy \right) d\theta$$

$$= \int_0^\infty d\theta \, e^{-\lambda\theta} \int_0^x dy \int_0^y \cos\theta v \, dv$$

$$= \int_0^x dy \int_0^y dv \, \frac{\lambda}{\lambda^2+v^2},$$

so that if we integrate $G_n(dx)$, we obtain

$$\int_0^\infty e^{-\lambda\theta} \frac{1-\operatorname{Re}\varphi_n(\theta)}{\theta^2} \, d\theta = \int_0^\infty \frac{\lambda \, dv}{\lambda^2+v^2} \int_v^\infty \bar{G}_n(y) \, dy$$

$$= \int_0^\infty \frac{du}{1+u^2} \int_{\lambda u}^\infty \bar{G}_n(y) \, dy.$$

It's not hard to see that

$$\{G_n\} \text{ is } UI \iff h(t) \equiv \sup_n \int_t^\infty \bar{G}_n(y)\,dy \to 0,$$

one way being obvious, and for "$\Leftarrow$", we define $\psi : R^+ \to R^+$ by $\psi(0) = 0$, $\psi'(t) = \log[h(0)/h(t)] \uparrow \infty$, so that ψ is convex, $\psi(x)/x \uparrow \infty$, and $\sup_n E\,\psi(|X_n|) < \infty$, which does it.

Thus
$$\boxed{\{G_n\} \text{ is } UI \iff \sup_n \int_0^\infty e^{-\lambda\theta}\, \frac{1 - \operatorname{Re}\varphi_n(\theta)}{\theta^2}\, d\theta \to 0 \quad (\lambda \to \infty).}$$

From this, we can easily deduce that

$$\boxed{\begin{aligned}\{F_n\} \text{ is } UI \iff &(i)\quad \sup_n \left| \frac{1 - \operatorname{Re}\varphi_n(\theta)}{\theta} \right| \to 0 \quad \text{as } \theta \downarrow 0\\ &(ii)\ \sup_n \left| \int_0^\varepsilon \frac{1 - \operatorname{Re}\varphi_n(\theta)}{\theta^2}\, d\theta \right| \to 0 \quad \text{as } \varepsilon \downarrow 0.\end{aligned}}$$

An Application.

If we take (X_n) independent, $P(X_n = n) = P(X_n = -n) = (n \log n)^{-1}$ for $n \geq 2$, with $X_n = 0$ otherwise. Then $\{X_n\}$ is a UI family with zero means, yet BC-II implies that i.o. $|X_n| > n$, so that $n^{-1} S_n$ is convergent with probability 0. This shows that we can't expect extension of SLLN to sequences of independent non-identically distributed r.v's.

Nonetheless, we have the following:

THEOREM. <u>Suppose that $(X_n)_{n\geq 1}$ is a sequence of independent zero mean random variables, which is UI. Then</u>

$$\frac{1}{n}(X_1 + \cdots + X_n) \longrightarrow 0 \quad \underline{\text{in}}\ L^1.$$

<u>Proof</u> Evidently $Y_n \equiv \frac{1}{n} S_n$ is bounded in L^1, and given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $P(A) < \delta \Rightarrow \forall n, \int_A |Y_n|\, dP < \varepsilon$, since the same is true of (X_n).

Hence $(Y_n)_{n \geq 0}$ is UI, and so for the result it will be sufficient to prove that $Y_n \xrightarrow{\mathcal{D}} 0$.

The characteristic function of Y_n is

$$\psi_n(t) = \prod_{k=1}^{n} \varphi_k\left(\frac{t}{n}\right).$$

Because of UI assumption,

$$R(\theta) \equiv \sup_n |1-\varphi_n(\theta)|/|\theta| \to 0 \quad \text{as } \theta \to 0,$$

and so for $t \in \mathbb{R}$ fixed, for all large enough n, $|1-\varphi_k(\frac{t}{n})| \leq \frac{1}{2}$, so

$$\log \psi_n(t) = \sum_1^n \log\left[1-(1-\varphi_k(\tfrac{t}{n}))\right]$$

$$= \sum_1^n \left\{1-\varphi_k(\tfrac{t}{n}) + O((1-\varphi_k(\tfrac{t}{n}))^2)\right\}$$

But $\left|\sum_1^n\{1-\varphi_k(\frac{t}{n})\}\right| \leq \sum_1^n |\frac{t}{n}|.\, R(\frac{t}{n}) \leq |t|\, R(\frac{t}{n}) \to 0$ as $n \to \infty$

and hence $\psi_n(t) \longrightarrow 1$ as $n \to \infty$, and $Y_n \xrightarrow{\mathcal{D}} 0$.

<u>Remark</u> This example of George O'Brien shows that we cannot expect $Y_n \xrightarrow{L^1} 0$ if we assume merely that the X_n are bounded in L^1.

Take

$$P(X_n = n) = P(X_n = -n) = 1/2n$$
$$P(X_n = 0) = 1 - 1/n.$$

Now consider $\quad V_k = \sum_{n=2^{k-1}+1}^{2^k} X_n \equiv S_{2^k} - S_{2^{k-1}}$.

This is zero mean and the expected number of non-zero terms in the sum is

$$\sum_{2^{k-1}+1}^{2^k} \frac{1}{n} \sim \log 2.$$

Thus the number of non-zero terms is $\sim P(\log 2)$ approximately, and if there is exactly one non zero term, then

$$\left|\frac{S_{2^k}}{2^k} - \frac{S_{2^{k-1}}}{2^{k-1}}\right| = 2^{-k}\left|(S_{2^k} - S_{2^{k-1}}) - S_{2^{k-1}}\right|$$

$$\geq \frac{1}{2} - 2^{-k}|S_{2^{k-1}}|$$

which proves that there can be no L^1 convergence to zero.

Remark on UI via transforms (30/7/87)

Returning to p 10, the distⁿ G_n is the distⁿ of $|X_n|$, and $\{|X_n|\}$ is UI iff $\{X_n\}$ is UI. It's easy to see that the condition of the first box on p 10 holds iff

$$\sup_n \int_0^\varepsilon \frac{1-\operatorname{Re}\varphi_n(\theta)}{\theta^2}\, d\theta \longrightarrow 0 \qquad (\varepsilon\downarrow 0)$$

so we have in fact

$$\boxed{\{F_n\} \text{ is UI} \iff \sup_n \int_0^\varepsilon \{1-\operatorname{Re}\varphi_n(\theta)\}\frac{d\theta}{\theta^2} \to 0 \quad (\varepsilon\downarrow 0)}$$

Comments on a paper of N.U. Prabhu "Wiener-Hopf factorisation of Markov semigroups I: the countable state space case" MR 867442 (3/8/87).

Prabhu considers a stable conservative non-exploding Markov chain on $\mathbb{Z}$, with $S\sim \exp(\theta)$ independent of X. He wants to find Q-matrices Q^+, Q^- of non-decr. (resp. non-increasing) chains such that

$$(I-\theta^{-1}Q) = (I-Q^-)(I-Q^+).$$

This says in other terms

$$\theta R_\theta = (I-Q^+)^{-1}(I-Q^-)^{-1};$$

Prabhu gives formulae for candidate $Q^\pm$, but no explanation of their probabilistic meaning.

If

$$T_j \equiv \inf\{u : X_u \geq j\},$$

and defining

$$\mathfrak{I}^\ell_{jk} \equiv P_j\,(X(T_\ell)=k,\quad T_\ell < S) \qquad (j\le \ell\le k),$$

$$\delta_j \equiv P_j(S<T_{j+1}) = 1-\sum_{k>j}\mathfrak{I}^{j+1}_{jk},$$

then Prabhu says that

$$\begin{aligned} q^+_{jk} &= \mathfrak{I}^{j+1}_{jk}/\delta_j && (k>j)\\ &= \sum_{k>j} q^+_{jk} && (k=j)\\ &= 0 && k<j. \end{aligned}$$

If one lets $Z_1, Z_2, \dots$ be the sequence of ascending ladder values, then the (Z_i) constitute a Markov chain with (defective) ~~transition~~ matrix

$$p^+_{jk} = P_j(X(T_{j+1}) = k,\ T_{j+1} < S) \equiv \zeta^{j+1}_{jk} \qquad (k > j)$$

$$= 0 \qquad (k \le j),$$

and, if $D \equiv \operatorname{diag} \delta_j$, we have that $P^+1 + D1 = 1$, and

$$Q^+ = D^{-1}(P^+ - I) + I,$$

whence

$$(I - Q^+)^{-1} = (I - P^+)^{-1} D,$$

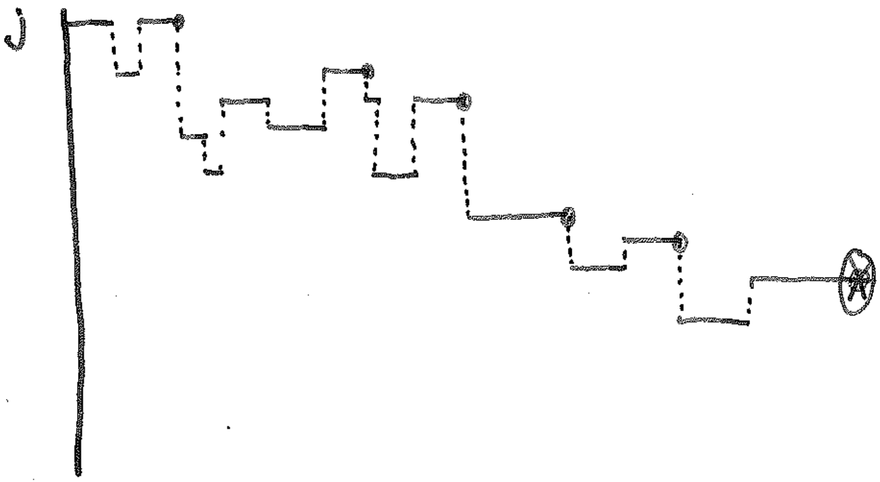
so

$$\boxed{(I - Q^+)^{-1}_{jk} = P_j(Z \text{ dies in state } k) = P_j\left[\sup_{t \le S} X_t = k\right].}$$

This explains the Q^+ part of the factorisation, and hence the Q^- part of the factorisation must be

$$(I - Q^-)^{-1}_{jk} = P_j\left(X_S = k \mid S < T_{j+1}\right) \qquad (k \le j).$$

To see how the decreasing Markov chain enters, consider the Markov chain $\{X_t : 0 \le t \le S\}$ with $X_0 = j$, and conditioned by $S < T_{j+1}$.



The process $\tilde{Y}_t \equiv \sup_{t \le u \le S} X_u$ is decreasing, with $\tilde{Y}_0 = j$, $\tilde{Y}_S = X_S$; let $Y_0, Y_1, Y_2, \dots$ be the sequence of values through which $\tilde{Y}$ passes before extinction. THEN (Y_i) IS A MARKOV CHAIN; let's define it for all time by setting $Y_n = -\infty$ after extinction.

The rate of excursions out of j whose sup is $k<j$ is

$$\sum_{l \le k} q_{jl} \, J^k_{lk} \, \delta_k$$

which in Prabhu's notation is abbreviated to θq^-_{jk}. If α_j is the rate of those excursions out of j which return to j before S, and if π_j is the probability that $X_S = j$ (the chain dies in its starting state), then

$$\pi_j = \frac{\theta + \alpha_j \pi_j}{\theta + \alpha_j + \sum_{k<j} \theta q^-_{jk}}$$

whence
$$\pi_j = \frac{1}{1 + \sum_{k<j} q^-_{jk}} \equiv \frac{1}{1+q^-_j}$$

which is the probability that the chain with Q-matrix Q^- started in j and killed at rate 1 should die in state j. That is, $\pi_j = P_j(Y_1 = -\infty)$.

Similarly,
$$P_j(Y_1 = k) = q^-_{jk} / (1+q^-_j), \qquad k<j,$$

and the appearance of the decreasing chain is explained in some measure.

It's really just a pastiche of familiar W-H ideas: we could reverse the killed chain and obtain a time-homogeneous chain whose ladder process is the Y process.

Example where $\Gamma X \sim X$, yet law of X isn't invariant (19/8/87)

If Γ is a random elt of a metizable topological gp acting on the left on a Polish space S, if X is a random elt. of S independent of Γ, then Tim Brown + I have been looking at the question "Suppose $\Gamma X \sim X$; is $gX \sim X$ for all $g \in G$?"

A nice example from Revuz "Markov Chains" Ch 5 ex 1.10 gives the following example.

Take $G = Sl(2,\mathbb{R})$. Any $A \in G$ can be written

$$A = R(\theta)\, T,$$

where T is upper triangular, in a unique way. Let $G_0 \equiv \{A \in G : A \text{ is upper triangular}\}$.

Then $S^1 \cong G/G_0$, and G acts on S^1. It's not hard to show that the action is given as follows. Identifying $\theta \in S^1$ with $(\cos\theta \ \sin\theta)^T$, we define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \Big/ \| \text{same} \|$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. So just let A act on the vector $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, and then renormalise.

Since G contains everything of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, it's clear that there can be no invariant law on S^1.

But if μ is any diffuse law on G, any weak limit point π of the measures $n^{-1} \sum_1^n (\mu^{*n}) * \varepsilon_1$, i.e. the laws of $\frac{1}{n} \sum_1^n \Gamma_1 \cdots \Gamma_k \cdot 1$, will satisfy $\mu * \pi = \pi$.

<u>Angular part of the Krylov example</u> (26/8/87)

The Krylov example is dealt with in § V.29 of the book; it's an example of a mg in $\mathbb{R}^d$ with bdd uniformly elliptic covariance, but hits points. It solves

$$dX = \sigma(X)\, dB$$

where $\sigma(x) = uu^T + \delta(I - uu^T)$, $u \equiv x/|x|$, $\delta > 0$ a constant ≤ 1.

Then

$$d\,|X_t| = U_t \cdot dB_t + \frac{dt}{2|X_t|}(k-1)$$

where $k \equiv 1 + \delta^2(d-1)$ is the effective dimension of the radial part. Now a bit of Itô calculus gives us

$$dU = \frac{\delta}{|X|}(I - UU^T)\, dB - \frac{\alpha U}{2|X|}\, dt,$$

where

$$\alpha \equiv \delta^2(d-1),$$

so that

$$\boxed{U_t = \tilde{U}\left(\int_0^t \delta^2 |X_s|^{-2} ds \right)} \qquad \text{where } \tilde{U} \text{ is a BM on } S^{d-1}$$

Usual sort of arguments show that the clock explodes as one approaches zero, so <u>there is no limiting direction of approach to zero for the Krylov martingale.</u>

An approach to local time? (12/9/87).

If B is $BM(\mathbb{R})$, then if we could prove that $E\,|\int_0^1 e^{i\theta B_t}dt|$ was integrable, it would immediately follow that B had a continuous local time.

This approach cannot possibly work. We have

$$\theta^{-2}(e^{i\theta B_1}-1) = i\theta^{-1}\int_0^1 e^{i\theta B_s}dB_s - \tfrac{1}{2}\int_0^1 e^{i\theta B_s}ds.$$

Hence $E|\int_0^1 e^{i\theta B_t}dt|$ is integrable iff $\frac{1}{\theta}E|\int_0^1 e^{i\theta B_s}dB_s|$ is integrable iff $\frac{1}{\theta}E|\int_0^1 \sin\theta B_s dB_s|$ and $\frac{1}{\theta}E|\int_0^1 \cos\theta B_s dB_s|$ are integrable.

Let $Y_\theta \equiv \int_0^1 \sin\theta B_s\,dB_s$. The family $\{Y_\theta : \theta\in\mathbb{R}\}$ is UI. Now

$Y_\theta = \beta(\int_0^1 \sin^2\theta B_s\,ds)$ for some BM β, and, I claim, $Y_\theta \xrightarrow{\mathcal{D}} N(0,\tfrac{1}{2})$ as $\theta\to\infty$.

This is because

$$
\begin{aligned}
&E\left(\int_0^1 \sin^2\theta B_s\,ds - \tfrac{1}{2}\right)^2\\
&= \tfrac{1}{4}\,E\left(\int_0^1 \cos 2\theta B_s\,ds\right)^2\\
&= \tfrac{1}{2}\,E\left[\int_0^1 ds \int_s^1 dt\, \cos 2\theta B_s \cos 2\theta B_t\right]\\
&= \tfrac{1}{2}\,E\left[\int_0^1 ds\, \cos^2 2\theta B_s \int_s^1 dt\, e^{-2\theta^2(t-s)}\right]\\
&\to 0 \quad \text{as} \quad \theta\to\infty,
\end{aligned}
$$

so that the 'clock' is converging L^2 (and so in prob$^{\underline{y}}$) to $\tfrac{1}{2}$, and now it's routine to prove $Y_\theta \xrightarrow{\mathcal{D}} N(0,\tfrac{1}{2})$. Hence, because of UI,

$$E\,|Y_\theta| \longrightarrow E\,|N(0,\tfrac{1}{2})| > 0$$

and $\displaystyle\int \frac{d\theta}{\theta} E|Y_\theta| = +\infty.$

<u>Question</u>: Is it true that $\displaystyle\int_{|\theta|\geq 1} |\theta|^{-1}\left|\int_0^1 e^{i\theta B_t} dB_t\right| d\theta = +\infty$ a.s.?

Self-intersection loc. times for BM($\mathbb{R}^2$) from case of independent BM($\mathbb{R}^2$)s. (17/9/87)

Geman, Horowitz + Rosen (Ann Prob 12, 86–107) prove that if $W_1, \dots, W_N$ are indept BM($\mathbb{R}^2$), then

$$\int_A d\underline{t}\ f(W_1(t_1) - W_2(t_2), \dots, W_{N-1}(t_{N-1}) - W_N(t_N)) = \int_{\mathbb{R}^{2N-2}} f(x)\, \alpha(x, A)\, dx,$$

where the process α has the property that $(x,t) \longmapsto \alpha(x, Q_t)$ is jointly continuous. $[Q_t \equiv \prod_{i=1}^N [0, t_i]]$.

Fixing $\delta > 0$, and letting $C = \{(t_1, \dots, t_N) \in (\mathbb{R}^+)^N : |t_i - t_j| \geq \delta \ \forall\, i \neq j\}$, then I claim that $\exists\ \check{\alpha}$ s.t.

$$\boxed{\int_{A \cap C} d\underline{t}\ f(W_{t_1} - W_{t_2}, \dots, W_{t_{N-1}} - W_{t_N}) = \int_{\mathbb{R}^{2N-2}} f(x)\, \check{\alpha}(x, A)\, dx}$$

and such that $(x,t) \longmapsto \check{\alpha}(x, Q_t)$ is jointly cts. (This is all for the same BM($\mathbb{R}^2$), W).

Rosen (Ann Prob 12, 108–119) proves this from scratch as Prop. 2, but it's easy to deduce it from the GHR result, using the sort of ideas of Le Gall's SP XIX paper.

Restrict attention to $A \subseteq S = [0, K\delta]^N$, K some large integer, let $I_j \equiv ((j-1)\delta, j\delta]$ let $I_{\underline{j}} = \prod_{r=1}^N I_{j_r}$ and notice that

$$S \cap C \subseteq \bigcup{}' I_{\underline{j}}$$

where the dash denotes union over all sequences $(j_1, \dots, j_N)$ of distinct elts of $\{1, \dots, K\}$, and is a disjoint union. Thus

$$\int_{A \cap C} f(W_{t_1} - W_{t_2}, \dots, W_{t_{N-1}} - W_{t_N})\, dt = \sum{}' \int_{A \cap C \cap I_{\underline{j}}} f(\text{——})\, dt. \qquad (1)$$

Abbreviate $W((j-1)\delta)$ to y_j, $W_j(t) \equiv W(t + (j-1)\delta) - y_j$. Then a typical summand on the RHS of (1) can be expressed as

$$V_{\underline{j}} \equiv \int_{A \cap C \cap I_{\underline{j}}} f(W_1(t_1 - (j_1-1)\delta) - W_2(t_2 - (j_2-1)\delta) + y_{j_1} - y_{j_2}, \dots)\, dt$$

$$\equiv \int_{\psi_{\underline{j}}(A \cap C \cap I_{\underline{j}})} f(W_1(u_1) - W_2(u_2) + y_{j_1} - y_{j_2}, \dots)\, du,$$

where $\psi_{\underline{j}}(t_1, \dots, t_N) \equiv (t_1 - (j_1-1)\delta, \dots, t_N - (j_N-1)\delta)$.

Thus $V_{\underset{\sim}{j}} = \int f(x_1+y_{j_1}-y_{j_2},\dots,x_{N-1}+y_{j_{N-1}}-y_{j_N})\, \underbrace{\alpha_{\underset{\sim}{j}}(x, \psi_{\underset{\sim}{j}}(A\cap C_n\cap I_{\underset{\sim}{j}}))}\, dx$

$\alpha_{\underset{\sim}{j}}$ exists by GHR result, since the W_i are indept BM($\mathbb{R}^d$).

$= \int f(x)\, \tilde{\alpha}_{\underset{\sim}{j}}(x, A)\, dx$

where $\tilde{\alpha}_{\underset{\sim}{j}}(x,A) \equiv \alpha_{\underset{\sim}{j}}\left((x_1-y_{j_1}+y_{j_2},\dots,x_{N-1}-y_{j_{N-1}}+y_{j_N}),\ \psi_{\underset{\sim}{j}}(A\cap C_n\cap I_{\underset{\sim}{j}})\right)$.

It is easy to see from properties of $\alpha_{\underset{\sim}{j}}$ that $(x,t)\longmapsto \tilde{\alpha}_{\underset{\sim}{j}}(x, Q_t)$ is jointly continuous.

Weak law for UI martingale difference sequences. (1/11/87)

Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be a UI martingale difference sequence, $S_n = X_1+\dots+X_n$. Then $(S_n/n)_{n\geq 1}$ is a UI family, and

$$\frac{S_n}{n} \xrightarrow{P} 0 \quad \Leftrightarrow \quad \frac{S_n}{n} \xrightarrow{L^1} 0.$$

In fact, both of these hold, as are easily seen as follows. Given $\varepsilon>0$, pick λ so large that $E[|X_n| : |X_n|>\lambda] < \varepsilon$ $\forall n$, and let $Y_n' \equiv X_n I_{(|X_n|\leq\lambda)}$, $Z_n' \equiv X_n - Y_n'$, with $Z_n \equiv Z_n' - E(Z_n'/\mathcal{F}_{n-1})$, $Y_n \equiv Y_n' - E(Y_n'/\mathcal{F}_{n-1})$.

Then the Y_n are a sequence of bdd martingale differences, and

$$E\left|\frac{S_n}{n}\right| \leq \frac{1}{n} E\left|\sum_1^n Y_k\right| + \frac{1}{n} E\left|\sum_1^n Z_k\right|$$

$$\leq \frac{1}{n}\left(E\left(\sum_1^n Y_k\right)^2\right)^{1/2} + \frac{1}{n} E \sum_1^n \{|Z_k'| + |E(Z_k'/\mathcal{F}_{k-1})|\}$$

$$\leq \frac{1}{n}\left(\sum_1^n E Y_k^2\right)^{1/2} + \frac{2}{n} E\left[\sum_1^n |X_k| I_{(|X_k|\geq\lambda)}\right].$$

Now $\frac{1}{n^2}\sum_1^n EY_k^2 \to 0$, since $\sum_1^\infty k^{-2} EY_k^2 < \infty$, so we can apply Kronecker's Lemma. The last term on the RHS is small because of choice of λ.

This argument is similar to that of J. Elton, Ann. Prob. 9 405-412, 1981, who requires that the mg difference sequence is identically distributed, a far more stringent condition.

Lindeberg condition and CLT (1/11/87)

(a) If $X_1, X_2, \dots$ are indept zero-mean, $EX_k^2 = \sigma_k^2$, $s_n^2 = \sigma_1^2+\dots+\sigma_n^2$, $S_n = X_1+\dots+X_n$ then the Lindeberg condition is

(L) for each $\varepsilon>0$, $\displaystyle \frac{1}{s_n^2}\sum_{k=1}^n \int_{\{|x|\geq\varepsilon s_n\}} x^2\, F_k(dx) \to 0 \quad (n\to\infty)$.

Condition (L) implies (AN) $\sup_{k\le n} \sigma_k/s_k \to 0$ $(k\to\infty)$, and if (AN) holds, CLT $\Leftrightarrow$ (L).
Interpret this as follows

<u>Lemma</u> If (AN) holds, then any weak limit of (S_n/s_n) is infinitely divisible.

<u>Proof</u>. If $\varphi_k(t) \equiv E e^{itX_k}$, we have

$$\left|\varphi_k\left(\frac{t}{s_n}\right) - 1\right| = \left|\int \left(e^{itx/s_n} - 1 - itx/s_n\right) F_k(dx)\right| \le \frac{1}{2}\int \frac{t^2}{s_n^2} x^2 F_k(dx) = \frac{\sigma_k^2 t^2}{2 s_n^2}$$

Thus $\prod_{k=1}^n \varphi_k\left(\frac{t}{s_n}\right) = \exp \sum_{k=1}^n \log\left(1 + \varphi_k\left(\frac{t}{s_n}\right) - 1\right)$ provided $\left|\varphi_k\left(\frac{t}{s_n}\right) - 1\right| < 1$, which, by (AN), will be true if n is big enough

$$= \exp\left[\sum_{k=1}^n \left(\varphi_k\left(\frac{t}{s_n}\right) - 1\right) + \sum_{k=1}^n R_{k,n}\right]$$

where $|R_{k,n}| \le \varepsilon \left|\varphi_k\left(\frac{t}{s_n}\right) - 1\right|$ if $\left|\varphi_k(t/s_n) - 1\right|$ is small enough, i.e., if n is large enough.
Thus the remainder terms are negligible, and what really matters is

$$(*) \qquad \exp\left[\sum_{k=1}^n \int \left(e^{itx/s_n} - 1 - itx/s_n\right) F_k(dx)\right].$$

The assertion follows easily from this.

<u>Remark</u>. It's not necessary to have second moments to have asymptotic negligibility.

(b) Now if we define μ_n by $\int f(y)\,\mu_n(dy) \equiv \sum_{k=1}^n \int f\left(\frac{x}{s_n}\right) F_k(dx)$, then (L) implies

$$\int_{(|y|\ge\varepsilon)} \mu_n(dy) \equiv \sum_{k=1}^n \int_{|x|\ge \varepsilon s_n} \frac{x^2}{s_n^2} F_k(dx) \to 0 \quad \text{as } n\to\infty.$$

Thus there is the CLT.

If CLT holds, and (AN) then $\sum_{k=1}^n \int \left(e^{itx/s_n} - 1 - it\frac{x}{s_n} + \frac{1}{2}t^2\frac{x^2}{s_n^2}\right) F_k(dx) \to 0$, so the real part $\to 0$, and, since $\cos x - 1 + \frac{1}{2}x^2 \ge 0$, for each t

$$\sum_{k=1}^n \int \left(\cos\frac{tx}{s_n} - 1 + \frac{1}{2}\frac{t^2x^2}{s_n^2}\right) F_k(dx) \to 0$$

But since $\inf_{x\ge 1} (\cos x - 1 + \frac{1}{2}x^2) \equiv \eta > 0$, we get (L) easily.

(c) I thought that (L) was more-or-less the same as "$\{(S_n/s_n)^2 : n\ge 1\}$ is UI". This latter statement can be shown equivalent to

"$\frac{1}{t^2}\left(E\, e^{itS_n/s_n} - 1\right) \to -\frac{1}{2}$ uniformly in n as $t\to 0$."

While (L) implies this condition, the converse is not true, as the following example shows. X_k is symmetric, values in $\{-\sqrt{k}, 0, \sqrt{k}\}$, $P(X_k = \sqrt{k}) = 1/2k$, so $\sigma_k^2 = 1$, and we have

(AN). Also, $\varphi_k\left(\frac{t}{s_n}\right) = 1 - \frac{1}{k}(1-\cos t\sqrt{k/n})$, so, fixing n, we want to consider $t^{-2}(Ee^{itS_n/s_n}-1)$ as $t \to 0$, or again, we want to consider

$$t^{-2} \sum_{k=1}^{n} \log\left\{1 - \frac{1}{k}\left(1-\cos t\sqrt{\tfrac{k}{n}}\right)\right\}$$

and show that it $\to -\frac{1}{2}$ uniformly in n. But given $\varepsilon > 0$, can pick t_0 so small that $|t| \le t_0 \Rightarrow \frac{1}{k}(1-\cos t\sqrt{k/n}) < \varepsilon \ \forall k, n$, and so we get the bounds

$$-\frac{1}{t^2}\sum_1^n \frac{1}{k}\left(1-\cos t\sqrt{\tfrac{k}{n}}\right) \ge \frac{1}{t^2}\sum_1^n \log\left\{1-\frac{1}{k}\left(1-\cos t\sqrt{\tfrac{k}{n}}\right)\right\} \ge -\frac{1}{t^2}\sum_{k=1}^n \left\{\frac{1}{k}\left(1-\cos t\sqrt{\tfrac{k}{n}}\right)(1+\varepsilon)\right\}$$

The statement that $\{(S_n/s_n)^2 : n \ge 1\}$ is UI follows.

However, $\displaystyle\sum_{k=1}^n \left(\varphi_k\left(\tfrac{t}{s_n}\right)-1\right) = -\sum_{k=1}^n \frac{1}{k}\left(1-\cos t\sqrt{k/n}\right) \to \int_0^1 dx\ \frac{1-\cos t\sqrt{x}}{x}$,

so there's no CLT, and therefore no (L).

<u>Self-intersection local times of cts semimgs in $\mathbb{R}^d$</u> (15/11/87)

Let X be a cts semimg in $\mathbb{R}^d$, and let's begin by restricting attention just to double-point local time. So we investigate

(1) $\quad \alpha_\varepsilon(z) = \displaystyle\int_B ds\,dt\ p_\varepsilon(X_t - X_s - z)$,

where B is some rectangle in the time domain $(\mathbb{R}^+)^2$, and p_ε is the Gaussian transition density. Now since

$$p_\varepsilon(x) \equiv \frac{e^{-|x|^2/2\varepsilon}}{(2\pi\varepsilon)^{d/2}} = \int e^{-i\theta.x - \frac{1}{2}\varepsilon|\theta|^2}\,d\theta\,(2\pi)^{-d},$$

we can express

$$\alpha_\varepsilon(z) = \int \frac{d\theta}{(2\pi)^d}\, e^{-\frac{1}{2}\varepsilon|\theta|^2} \int_B e^{-i\theta.(X_t - X_s - z)}\,ds\,dt$$

$$= \int_0^\infty c_d\, r^{d-1} e^{-\frac{1}{2}\varepsilon r^2}\,dr \int_{S^{d-1}} dv \int_B e^{-ir\,v.(X_t - X_s - z)}\,ds\,dt,$$

which leads us to consider

$$\int_{s_0}^{s_1} ds \int_{t_0}^{t_1} dt\, e^{-ir\,v.(X_t - X_s - z)} \qquad \left(B \equiv [s_0,s_1]\times[t_0,t_1]\right)$$

(2) $$= \int_{s_0}^{s_1} e^{+irv.X_s}\,ds \int_{t_0}^{t_1} e^{-ir\,v.(X_t - z)}\,dt.$$

Let us now <u>assume that for</u> $v \in S^{d-1}$, $d\langle v.X\rangle_t/dt \equiv \gamma_t^v$ <u>exists and is</u>

continuous in t, and bounded away from zero. Thus if we set

$$\ell(t,x;v) \equiv \int_0^t L^{v.X}(ds,x) \,/\, \gamma_s^v ,$$

we have that $\displaystyle\int_{t_0}^{t_1} f(v.X_t)\,dt = \int f(x)\,\{\ell(t_1,x;v) - \ell(t_0,x;v)\}\,dx$. Thus if we abbreviate $\ell(t_1,x;v) - \ell(t_0,x;v)$ to $\ell([t_0,t_1],x;v)$, and even abbreviate $[t_0,t_1]$ to I, $[s_0,s_1]$ to J, we have an expression for (2) as

$$\int dx\; e^{irx}\,\ell(J,x;v) \int dx\; e^{-irx}\,\ell(I,x+v.z;v)$$

$$= \int e^{irx}\,dx \left(\int \ell(J,y;v)\,\ell(I,\,y-x+v.z\,;v)\,dy \right).$$

So what sort of regularity properties in x does this convolution integrand have? Let's drop the parameter v from the notation, write $\theta_s \equiv \gamma_s^{-1}$, and assume that θ is a semimartingale. Then for $f \in C_K^\infty$, (suppose the time interval I is $[0,1]$),

[think of f standing in place of $\ell(J,y)$]

$$\int f(y) \int_0^1 \theta_s\,L(ds,y-x)\,dy \equiv \int f(y)\,\ell(I,y-x)\,dy$$

$$= \int f(y+x) \left(\int_0^1 \theta_s\,L(ds,y)\right) dy$$

which has derivative

$$\int f'(y+x)\left(\int_0^1 \theta_s\,L(ds,y)\right) dy$$

$$= \int_0^1 f'(v.X_s + x)\,ds$$

$$= \int_0^1 f'(v.X_s + x)\,\theta_s\,d\langle X.v\rangle_s$$

$$= 2\left[\int_0^1 \theta_s\,d\big(F(v.X_s+x)\big) - \int_0^1 f(v.X_s+x)\,\theta_s\,dX_s.v \right] \qquad (F \equiv \textstyle\int f)$$

$$= 2\left[\theta_1 F(v.X_1+x) - \theta_0 F(v.X_0+x) - \int_0^1 F(v.X_s+x)\,d\theta_s - \int_0^1 f(v.X_s+x)\,\theta_s\,dX_s.v \right]$$

which is perfectly meaningful even for $f \in C_K$. So this makes us believe that the convolution is in fact differentiable!

Note the links with R. Bass's paper Stoch Proc Appn $\underset{\sim}{17}$.

More on the law of the maximum of a scaled Brownian excursion. (18/2/88)

Let $\{e_s : 0 \le s \le 1\}$ be a scaled (non-neg) Brownian excursion, and let $M \equiv \sup e_s^2$. For each $c>0$, $\{\sqrt{c}\, e(t/c) : 0 \le t \le c\}$ is a Brownian excursion of lifetime c, so for each $\lambda > 0$,

$$E M^\lambda = c^{-\lambda} E\Big[\max_{0\le t\le c} \big(\sqrt{c}\, e(\tfrac{t}{c})\big)^{2\lambda}\Big]$$

so if we mix over c with the law of the lifetime of the Brownian excursion, we get

$$E(M^\lambda)\int_U n(dp)(1-e^{-\alpha\zeta}) = \int_U n(dp)(1-e^{-\alpha\zeta})\, \zeta^{-\lambda} \max_{0\le s\le\zeta} p_s^{2\lambda}$$

$$= \int_0^\infty \frac{dx}{x^2}\, x^{2\lambda}\, E\big[\zeta^{-\lambda}(1-e^{-\alpha\zeta}) \mid \max p = x\big]$$

$$= \int_0^\infty x^{2\lambda-2}\,dx \int_0^\infty t^{\lambda-1}\frac{dt}{\Gamma(\lambda)}\, E\big[e^{-t\zeta}(1-e^{-\alpha\zeta}) \mid \max p = x\big]$$

$$= \int_0^\infty \frac{t^{\lambda-1}\,dt}{\Gamma(\lambda)}\int_0^\infty x^{2\lambda-2}\Big[\Big(\frac{x\sqrt{2t}}{\sinh x\sqrt{2t}}\Big)^2 - \Big(\frac{x\sqrt{2t+2\alpha}}{\sinh x\sqrt{2t+2\alpha}}\Big)^2\Big]dx \qquad (x = y/\sqrt{2t})$$

$$= \int_0^\infty \frac{t^{-\frac12}\,dt}{\Gamma(\lambda)}\, 2^{-(\lambda-\frac12)}\int_0^\infty y^{2\lambda-2}\Big[\frac{y^2}{\sinh^2 y} - \frac{y^2(1+\alpha/t)}{\sinh^2 y\sqrt{1+\alpha/t}}\Big]dy \qquad (t = \alpha/v)$$

$$= \int_0^\infty \frac{\sqrt{\alpha}\, v^{-\frac32}\,dv}{\Gamma(\lambda)}\, 2^{-(\lambda-\frac12)}\int_0^\infty y^{2\lambda}\big[\operatorname{cosech}^2 y - (1+v)\operatorname{cosech}^2 y(1+v)^{\frac12}\big]dy$$

Hence

$$\boxed{E(M^\lambda) \equiv E\big(\sup_{s\le 1} e_s\big)^{2\lambda} = \int_0^\infty \frac{v^{-\frac32}\,dv}{\Gamma(\lambda)}\, 2^{-\lambda}\int_0^\infty y^{2\lambda}\big[\operatorname{cosech}^2 y - (1+v)\operatorname{cosech}^2 y(1+v)^{\frac12}\big]dy}$$

According to Biane + Yor (Bull. Sci. Math. 111, 23–101, 1987) p 72,

$$E[M^s] = \Big(\frac{\pi}{2}\Big)^s 2\xi(2s) \equiv \Big(\frac{\pi}{2}\Big)^s 2s(2s-1)\,\pi^{-s}\,\Gamma(s)\,\zeta(2s)$$

$$= 2^{-s}\, 2s(2s-1)\,\zeta(2s)\,\Gamma(s),$$

where ζ is the Riemann zeta fn.

More on $A(t, B_t)$ (18/2/88).

The aim is to investigate the order-p variation of $X_t \equiv A(t, B_t) - \int_0^t L(s, B_s)\, dB_s$.
As before, for fixed $t > 0$, $h > 0$

$$\Delta X \equiv X_{t+h} - X_t = -\int_t^{t+h} L(u, B_u)\, dB_u + \int_{B_t}^{B_{t+h}} L(t,x)\, dx + O(h)$$

Thus if $\lambda(x) \equiv L(t,x)$ for short, $\Lambda(x) = \int^x \lambda(y)\, dy$, we have

$$\Delta X = \left[\Lambda(B_{t+h}) - \Lambda(B_t) - \int_t^{t+h} \lambda(B_u)\, dB_u \right] + \int_t^{t+h} \{ L(t, B_u) - L(u, B_u) \}\, dB_u$$

$$+ O(h).$$

The interesting bit is the bit in square brackets, because

$$E \left| \int_t^{t+h} \{ L(t, B_u) - L(u, B_u) \}\, dB_u \right|^p$$

$$= E \left| \int_0^h L(u, B_u)\, dB_u \right|^p$$

$$\sim E \left(\int_0^h L(u, B_u)^2\, du \right)^{p/2}$$

$$= E \left(h \int_0^1 L(hv, B_{hv})^2\, dv \right)^{p/2}$$

$$= E \left(h^2 \int_0^1 L(u, B_u)^2\, du \right)^{p/2}$$

$$= \text{const}.\ h^p,$$

which, for $p \geq 1$, is going to be negligible as $h \downarrow 0$. Thus if Δ' is the bit in square brackets, we have (assuming λ is differentiable) that

$$E(\Delta' / \mathcal{F}_t) = E\left(\int_0^h \tfrac{1}{2} \lambda'(B_{t+u})\, du \,/\, \mathcal{F}_t \right)$$

$$= \int_0^h du \int_{-\infty}^{\infty} \frac{e^{-y^2/2u}}{\sqrt{2\pi u}}\, \tfrac{1}{2} \lambda'(y + B_t)\, dy$$

$$= \tfrac{1}{2} \int_0^h du \int_{-\infty}^{\infty} \lambda(y + B_t)\, \frac{y e^{-y^2/2u}}{\sqrt{2\pi u^3}}\, dy$$

$$= \tfrac{1}{2} \int_0^\infty \{ \lambda(B_t + y) - \lambda(B_t - y) \}\, P(HN \geq y/\sqrt{h})\, dy$$

and we see that the differentiability of λ isn't really necessary – we can always convolute

λ to smooth it, and then fix it that way. Thus

$$E\,|E(\Delta'/\mathcal{F}_t)|^p = E\,\Big|\int_0^\infty \{L(t,y)-L(t,-y)\}\,\bar\Phi\Big(\frac{y}{\sqrt h}\Big)dy\,\Big|^p.$$

By Tanaka's formula,

$$L(t,y) - L(t,-y) = |B_t-y| - |B_t+y| + 2\int_0^t I_{[-y,y]}(B_s)\,dB_s$$

and

$$E\,\Big|\int_0^\infty \{|B_t-y| - |B_t+y|\}\,\bar\Phi(y/\sqrt h)\,dy\Big|^p$$

$$= h^{p/2}\,E\,\Big|\int_0^\infty (|B_t - y\sqrt h| - |B_t + y\sqrt h|)\,dy\,\bar\Phi(y)\Big|^p$$

$$\leq \text{const}.\,h^p,$$

which will not contribute, because the stochastic integral is of larger order as $h\downarrow 0$; indeed,

$$E\,\Big|\int_0^\infty\int_0^t I_{[-y,y]}(B_s)\,dB_s\;\bar\Phi(y/\sqrt h)\,dy\Big|^p$$

$$= E\,\Big|\int_0^t f\Big(\frac{B_s}{\sqrt h}\Big)dB_s\Big|^p h^{p/2} \qquad \text{where } f(x) \equiv \int_0^\infty I_{(y>|x|)}\,\bar\Phi(y)\,dy;$$

$$\sim h^{p/2}\,E\Big(\int_0^t f(B_s/\sqrt h)^2\,ds\Big)^{p/2}$$

$$= h^{p/2}\,E\Big(\int_{-\infty}^\infty f\Big(\frac{x}{\sqrt h}\Big)^2 L(t,x)\,dx\Big)^{p/2}$$

$$= h^{p/2}\,E\Big(\int_{-\infty}^\infty f(y)^2\,L(t,y\sqrt h)\,dy\,\sqrt h\Big)^{p/2}$$

$$\sim h^{3p/4}\,E\Big(\int_{-\infty}^\infty f(y)^2\,dy\,.\,L(t,0)\Big)^{p/2}$$

$$= \text{const}.\;h^{3p/4} \qquad \text{as } h\to 0,$$

which shows that

$$\boxed{E\,|E(\Delta X\,|\,\mathcal{F}_t)|^p \sim \text{const}\;h^{3p/4} \quad \text{as } h\downarrow 0}$$

Now

$$\Delta' - E(\Delta' | \mathcal{F}_t) = \int_{-\infty}^{\infty} \left(I_{(x \le Z)} - P(x \le Z) \right) \{ L(t, x+B_t) - L(t, B_t) \} \, dx$$

$$- \int_t^{t+h} \{ L(t, B_u) - L(t, B_t) \} \, dB_u ,$$

where $Z \equiv B_{t+h} - B_t$. The expectation of the s.i. to the p is $\sim h^{3p/4}$, and all the calculations lead me to believe that the same is true of the first piece, so the difference appears very hard to handle.

Coping with the variation of a process (22/2/88)

Let X be a process with R-paths, say, and let $\Delta_j^n X \equiv X(j2^{-n}) - X(\overline{j-1}2^{-n})$. Let's suppose that

$$\lim_{n \to \infty} \sum_{j=1}^{2^n} E \, |\Delta_j^n X|^p \quad \begin{array}{lll} = 0 & & (p > \gamma) \\ = c & \in (0,\infty) & (p = \gamma) \\ = +\infty & & (p < \gamma). \end{array}$$

Then if $\{ \sum_{j=1}^{2^n} |\Delta_j^n X|^\gamma \, ; \, n \in \mathbb{Z}^+ \}$ is uniformly integrable,

$$P\Big(\limsup_n \sum_1^{2^n} |\Delta_j^n X|^p = +\infty \Big) > 0 \quad \underline{\text{for each}} \quad p < \gamma.$$

Proof

If the family is UI, then it cannot tend to zero a.s, because it doesn't tend to zero in L^1. For $1 > a > 1/p = 1 - 1/q$

$$\sum_1^{2^n} |\Delta_j^n X|^\gamma \le \Big(\sum_1^{2^n} |\Delta_j^n X|^{\gamma a p} \Big)^{1/p} \Big(\sum_1^{2^n} |\Delta_j^n X|^{\gamma(1-a)q} \Big)^{1/q} .$$

Down some subsequence with pos prob^y the LHS has a positive limit. Taking a sub-subsequence if need be, the first factor on RHS $\xrightarrow{a.s.} 0$, which gives result.

Finally, we can prove that $A(t,B_t)$ is not a semimartingale (26/2/88)

1/ If $X_t \equiv A(t,B_t) - \int_0^t L(u,B_u)\,dB_u$, the claim is that X has zero order-p variation for $p > 4/3$, and infinite order-p variation for $p < 4/3$.
We have

$$X_{t+h} - X_t = \int_{B_t}^{B_{t+h}} \{L(t,x) - L(t,B_t)\}\,dx - \int_t^{t+h} \{L(u,B_u) - L(t,B_t)\}\,dB_u + O(h)$$

(see p.3). As before, we see that this differs negligibly from the same expression with $L(u,B_u)$ replaced by $L(t,B_u)$ in the stochastic integral. So let's define

$$\Delta_j^n X \equiv \int_{B_t}^{B_{t+h}} \{L(t,x) - L(t,B_t)\}\,dx - \int_t^{t+h} \{L(t,B_u) - L(t,B_t)\}\,dB_u$$

where $h = 2^{-n}$, $t = (j-1)2^{-n}$, and define

$$V_p^n \equiv \sum_{j=1}^{2^n} |\Delta_j^n X|^p.$$

2/ To begin with, we estimate very much as on p.3-4 an upper bound for $E\,|\Delta_j^n X|^p$.
We have for $p \geq 2$

$$E\,\Big|\int_t^{t+h} (L(t,B_u) - L(t,B_t))\,dB_u\Big|^p$$

$$\leq c\, E\left(\int_t^{t+h} \{L(t,B_u) - L(t,B_t)\}^2\,du\right)^{p/2}$$

$$= c\, E\left(\int_0^h (L(t,W_u) - L(t,0))^2\,du\right)^{p/2} \qquad \text{where } W \text{ is BM indept of } B;$$

$$\leq c\, h^{(p-2)/2}\; E\int_0^h |L(t,W_u) - L(t,0)|^p\,du.$$

Estimating from Tanaka's formula again gives

$$E\,|L(t,y) - L(t,0)|^p \leq c|y|^p + c\,E\left(\int_0^t I_{\{0<B_s\leq y\}}\,ds\right)^{p/2}$$

$$\leq c\left(|y|^p + |y|^{p/2}\, E(L_1^*)^{p/2}\right)$$

$$\leq c\left(|y|^p + |y|^{p/2}\right),$$

from which (for $p \geq 2$ still) we obtain

$$E\left|\int_t^{t+h}(L(t,B_u)-L(t,B_t))\,dB_u\right|^p \le c\,h^{(p-2)/2}\int_0^h du\int_{-\infty}^{\infty}\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy\left(|y\sqrt{u}|^{p/2}+|y\sqrt{u}|^{p}\right)$$

$$\le c\;h^{(p-2)/2}\,h\,(c_1 h^{p/4}+c_2\,h^{p/2})$$

$$\le c\;h^{3p/4}.$$

Next, for $p\le 2$, we estimate

$$E\left|\int_t^{t+h}(L(t,B_u)-L(t,B_t))\,dB_u\right|^p \le c\;E\left(\int_t^{t+h}(L(t,B_u)-L(t,B_t))^2\,du\right)^{p/2}$$

$$\le c\left[E\int_t^{t+h}(L(t,B_u)-L(t,B_t))^2\,du\right]^{p/2}$$

$$\le c\;h^{3p/4},$$

using the $p=2$ case of the first estimation. This copes with the stochastic integral part of $\Delta_j^n X$.

As for the other part, we have for $p\ge 1$

$$E\left|\int_0^y(L(t,B_t+x)-L(t,B_t))\,dx\right|^p$$

$$\le |y|^{p-1}\;E\int_0^y|L(t,x)-L(t,0)|^p\,dx$$

$$\le |y|^{p-1}\,c\int_0^y\left(|x|^p+|x|^{p/2}\right)dx$$

$$\le c\;|y|^{2p}+\;c\;|y|^{3p/2}$$

and now mixing over y with a normal law mean zero variance h gives

$$E\left|\int_{B_t}^{B_{t+h}}\{L(t,x)-L(t,B_t)\}\,dx\right|^p\le c\;h^{3p/4}.$$

Hence
$$\boxed{E\;|\Delta_j^n|^p\;\le\;c\;h^{3p/4}\;=\;c\left(2^{-n}\right)^{3p/4}.}$$

We have also that for $p\ge 1$

$$\boxed{E\;|\Delta_j^n|^p\;\ge\;E\left(\left|E(\Delta_j^n\,|\,\mathcal{F}_{j-1}^n)\right|^p\right)\;\ge\;c\;h^{3p/4}}$$ (see p 24)

3/ This gives us $\quad E\, V_p^n \longrightarrow 0 \quad$ as $n \to \infty \quad$ for $p > 4/3$

$$E\, V_p^n \longrightarrow \infty \quad \text{as } n \to \infty \quad \text{for } p < 4/3$$

and for $p = 4/3$, $\; E\, V_p^n$ remains bounded away from 0 and from $+\infty$.

Now
$$E\,(V_p^n)^2 = E \sum_{i,j=1}^{2^n} |\Delta_i^n|^p \, |\Delta_j^n|^p$$
$$\le \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} E\left(|\Delta_i^n|^2 \; |\Delta_j^n|^2\right)^{p/2} \qquad \text{if } p \le 2$$
$$\le \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \left(E\,|\Delta_i^n|^4 \;\; E\,|\Delta_j^n|^4\right)^{p/4}$$
$$\le c \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \left(h^6\right)^{p/4} \qquad \text{by the previous estimates.}$$

Thus if $p = 4/3$, we have the bound
$$c \sum \sum h^2 = C$$
which implies that <u>the family</u> $\{ V_{4/3}^n : n \in \mathbb{N} \}$ <u>is bdd in L^2</u>, <u>therefore UI</u>.

Thus the argument on p. 25 implies that for each $p < 4/3$,
$$P\left(\limsup_{n\to\infty} V_p^n = +\infty\right) > 0,$$
which certainly establishes that $A(t, B_t)$ is not a semimartingale.

4/ By scaling, we can show that
$$\{\Delta_i^n \; ; \, i = 1, \dots, 2^n\} \overset{\mathcal{D}}{=} \{ 2^{kp}\, \Delta_i^{n+k} \; ; \, i = 1, \dots, 2^n \},$$

and hence
$$P\left[\limsup_{m\to\infty} \sum_{j=1}^{2^{m-k}} |\Delta_j^m X|^p = +\infty \text{ for infinitely many } k \right] > 0$$
so by 0-1 law,
$$P\left[\limsup_{m\to\infty} \sum_{j=1}^{2^{m-k}} |\Delta_j^m X|^p = +\infty \text{ for infinitely many } k \right] = 1,$$

so $\quad P\left(\limsup_n V_p^n = +\infty\right) = 1 \quad$ for each $p < \frac{4}{3}$.

Distribution of x^1 coordinate of point uniform on S^d (10/3/88)

If $(X_1,\dots,X_d)$ is uniform on S^{d-1}, and $(Y_1,\dots,Y_d)$ are indept $N(0,1)$, then for $t>0$

$$P(X_1>t) = P(X_1^2>t^2) = P(Y_1^2 > t^2(Y_1^2+\dots+Y_d^2))$$

$$= P\left[(t^{-2}-1)Y_1^2 > Y_2^2+\dots+Y_d^2\right]$$

$$= \int_0^{(t^{-2}-1)} v^{(d-3)/2}(1+v)^{-d/2}\,dv\ \Gamma(\tfrac{d}{2})/\Gamma(\tfrac12)\Gamma(\tfrac{d-1}{2})$$

so that X_1 has density

$$P(X_1\in dt)/dt = \frac{2\Gamma(\frac d2)}{\Gamma(\frac12)\Gamma(\frac{d-1}{2})}(1-t^2)^{(d-3)/2}.$$

Heuristics on self-repellent r.w. in 1 dimension (21/3/88)

Consider the SDE

$$dX_t = dB_t + \int_0^t f(X_t-X_s)\,ds.$$

If for large t we have $X_t\sim\xi_t\uparrow\infty$, then it looks reasonable that

$$\xi_t' \sim \int_0^t f(\xi_t-\xi_s)\,ds$$

so if ρ is the inverse to ξ, we expect

$$\rho_t' \equiv (\xi'(\rho_t))^{-1} \sim \left[\int_0^{\rho_t} f(\xi_{\rho_t}-\xi_s)\,ds\right]^{-1} \qquad s=\rho_u$$

$$= \left[\int_0^t f(t-u)\,\rho_u'\,du\right]^{-1},$$

which says that

$$(f*\rho')(t)\sim 1/\rho_t'.$$

Now suppose that $\rho_t' = t^\alpha L(t)$, where L is slowly varying, $\alpha>-1$. (In fact, we also impose $\alpha\le 0$ else there is no solution). Then

$$\tilde\rho'(\lambda)\sim \lambda^{-1-\alpha}\,\Gamma(1+\alpha)\,L(\tfrac1\lambda) \quad\text{as } \lambda\downarrow 0,$$

and so we obtain

$$\tilde f(\lambda)\sim\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\lambda^{2\alpha}L(\tfrac1\lambda)^{-2} \quad\text{as } \lambda\downarrow 0,$$

which implies that

$$\boxed{f(t) \sim \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)\,\Gamma(2\alpha)}\; t^{-1-2\alpha}\; L(t)^{-2}.} \quad (*)$$

Alternatively, if $\xi_t \sim t\mu$, we get $\rho_t \sim \mu^{-1} t$, and

$$\boxed{\mu^2 = \int_0^\infty f(x)\,dx}$$

This feels suspect; in this case, neglecting the push down on X_t due to time spent above X_t must be wrong, though if $xf(x) \geq 0$ the μ given above should be an upper bound on $\limsup (X_t/t)$. It seems to fit reasonably well with simulation though.

<u>Self-repellent r.w. in the special case $f(x) = cx$.</u> (12/5/88)

Consider
$$X_t = B_t + \int_0^t c\left(\int_0^s (X_s - X_u)\,du\right) ds,$$

Firstly for the case $c>0$.
If we set $Y_t \equiv X_t - B_t$, we get

$$\begin{aligned} dY_t &= c\int_0^t \{Y_t - Y_u + B_t - B_u\}\,du\,dt \\ &= \left[c\int_0^t (Y_t - Y_u)\,du + c\int_0^t s\,dB_s\right] dt \end{aligned}$$

So if $\dot{Y}_t \equiv V_t$, we have
$$V_t = c\left\{t Y_t - \int_0^t Y_u\,du + c\int_0^t s\,dB_s\right\}$$

and so
$$dV_t = c\,t\,V_t\,dt + c\,t\,dB_t$$

$$\therefore \quad d\left(e^{-ct^2/2} V_t\right) = ct\,e^{-ct^2/2}\,dB_t$$

$$\therefore \quad \boxed{V_t = e^{ct^2/2}\,\tilde{B}\left(\int_0^t c^2 s^2 e^{-cs^2}\,ds\right)}$$

$$\sim e^{ct^2/2}\,U, \quad \boxed{\text{where } U \sim N\left(0, \tfrac{1}{4}\sqrt{c\pi}\right)}.$$

(*) This suggests that if $f(t) \sim \operatorname{sgn}(t)\, t^{-\gamma} L(t)$, then the growth of X_t is like t^β, where $\beta = \frac{2}{1+\gamma}$ (assuming $0 \leq \gamma < 1$)

Next we take the case $c = -a < 0$, and consider

$$Z_t \equiv tX_t - \int_0^t X_u\,du = \int_0^t s\,dX_s = \int_0^t (X_t - X_u)\,du.$$

We have that

$$dZ_t = t\,dX_t = t\,dB_t - a\,t\,Z_t\,dt$$

$$\therefore\quad d\left(e^{at^2/2} Z_t\right) = t e^{at^2/2}\,dB_t$$

$$\therefore\quad \boxed{Z_t = e^{-at^2/2}\,\tilde{B}\left(\int_0^t s^2 e^{as^2}\,ds\right).}$$

By the law of the iterated logarithm,

$$\limsup_{t\to\infty} |Z_t| / \psi(t) = 1,$$

where

$$\psi(t) \equiv \left(e^{-at^2}\, 2\int_0^t s^2 e^{as^2}\,ds\ \log\log\left(\int_0^t s^2 e^{as^2}\,ds\right)\right)^{\frac{1}{2}}.$$

Now

$$e^{-at^2}\int_0^t s^2 e^{as^2}\,ds = e^{-at^2}\int_0^{t^2} e^{au}\,u^{\frac{1}{2}}\,\frac{du}{2}$$

$$= \int_0^{t^2} e^{-av}\sqrt{t^2 - v}\,\frac{dv}{2}$$

$$\sim t/2a,$$

so, in particular, $|Z_t| / t^{\frac{1}{2}+\varepsilon} \to 0$ for any $\varepsilon > 0$ $(t \to \infty)$, and, when we take $\varepsilon = \frac{1}{2}$, we see that

$$\boxed{X_t - \frac{1}{t}\int_0^t X_u\,du \to 0}$$

Now

$$d\left(\frac{1}{t}\int_0^t X_u\,du\right) = \left(\frac{X_t}{t} - \frac{1}{t^2}\int_0^t X_u\,du\right)dt = \frac{Z_t}{t^2}\,dt,$$

so

$$\frac{1}{t}\int_0^t X_u\,du = \int^t \frac{Z_u}{u^2}\,du = \int^t \frac{Z_u}{u^{3/4}}\,\frac{du}{u^{5/4}}$$

which is obviously convergent. Hence

$$\boxed{\lim_{t\to\infty} X_t \text{ exists a.s.}}$$

A regular one-dimensional diffusion is strong Feller (23/5/88)

Take a regular diffusion in natural scale on the interval I. The aim is to prove that if f is a bounded measurable function on I, then for each $t>0$ the map $x \mapsto P_t f(x)$ is continuous.

Case 1: $I = \mathbb{R}$. Fix $t>0$, $x \in \mathbb{R}$ and $\varepsilon > 0$, and choose $\eta > 0$ so small that

$$P^x (\langle X \rangle_t > \eta) \geq 1 - \varepsilon .$$

Next choose $h > 0$ so small that

$$P^0 \Big(\sup_{s \leq \eta} B_s > h, \ \inf_{s \leq \eta} B_s < -h \Big) \geq 1 - \varepsilon .$$

Then for $x' \in (x-h, x+h)$, if we start the independent copy X' of X from x', and couple X and X' when they meet,

$$\begin{aligned}
&P(X, X' \text{ have not coupled by time } t) \\
&= P(X_s - X'_s \neq 0 \quad \forall s \leq t) \\
&\leq P(X_s - X'_s \neq 0 \quad \forall s \leq t, \text{ and } \langle X \rangle_t > \eta) + \varepsilon \\
&= P(B(\langle X - X' \rangle_s) + x - x' \neq 0 \ \forall s \leq t \text{ and } \langle X \rangle_t > \eta) + \varepsilon \\
&\leq P(B_u + x - x' \neq 0 \quad \forall u \leq \eta) + \varepsilon \\
&\leq 2 \varepsilon .
\end{aligned}$$

Hence for $x' \in (x-h, x+h)$, $\| f \|_\infty \leq 1$,

$$| P_t f(x) - P_t f(x') | \leq 2 \varepsilon .$$

Case 2: $I = [0, \infty)$. We represent X as $X_t = W(\langle X \rangle_t) + \lambda_t$, where λ grows only when X is at zero.

Fix $t > 0$, and suppose firstly that x is not an absorbing endpoint. Choose η, h as before. For $x' \in (x, x+h)$,

$$\begin{aligned}
&P(X, X' \text{ have not coupled by time } t) \\
&\leq P(\lambda'_t = 0, \ W'(\langle X' \rangle_s) - W(\langle X \rangle_s) - \lambda_s > 0 \quad \forall s \leq t)
\end{aligned}$$

$$\leq P\left(W'(\langle X'\rangle_s) - W(\langle X\rangle_s) > 0 \ \forall s \leq t\right)$$

which we handle as before. The case $x' \in (x-h, x)$ is similar. If x is an absorbing endpoint, then $\langle X\rangle_t = 0 \ \forall t$. But this case is easy to deal with directly (for example, the argument of Rogers-Williams Prop V.50.1 will do)

Case 3 : $I = (0,\infty)$. This is just like case 1

Case 4: I is bounded. Combine earlier methods suitably

Right-continuity of the Brownian excursion filtration. (1/8/88)

Let $(\mathcal{E}^\circ_x)_{x\in\mathbb{R}}$ be the natural (unaugmented) filtration of the Brownian excursion process, so that $\mathcal{E}^\circ_x \equiv \sigma(\tilde{B}(t,x);\ t \geq 0)$. Let $\mathcal{N}$ be the σ-field of null sets and their complements, and let $\mathcal{N} \vee \mathcal{E}^\circ_x \equiv \tilde{\mathcal{E}}_x$.

Notice that $(\mathcal{E}^\circ_x)$ is not right-continuous. This is because each $\mathcal{E}^\circ_{x+1/n}$ contains a lot of extra information; for example, for each excursion into (x,∞), the amount of local time elapsed at x when the excursion begins can be found from $\mathcal{E}^\circ_{x+1/n}$, but this information is not in $\mathcal{E}^\circ_x$.

Nonetheless, $\tilde{\mathcal{E}}_x$ is right-continuous. Suppose not. Then $\exists\, a \in \mathbb{R}$, $Y \in L^\infty(\tilde{\mathcal{E}}_{a+})$ such that $E(Y|\tilde{\mathcal{E}}_a) \neq Y$, and

$$E\left(\left(Y - E(Y|\tilde{\mathcal{E}}_a)\right)^2\right) > 0.$$

Now consider $\bar{Y}_x$, the right-continuous regularisation of $E(Y|\mathcal{E}^\circ_x)$. This is a martingale in the usual augmentation $(\mathcal{E}_x)$ of $(\mathcal{E}^\circ_x)$, and from my paper "Continuity of martingales in the Brownian excursion filtration", the martingale $\bar{Y}$ must be continuous.

However, $\bar{Y}_x = \text{a.s.} \lim_{q\downarrow x} E(Y|\mathcal{E}^\circ_q)$ and so $\bar{Y}_a = Y$ a.s. and for $x<a$, $E\,\bar{Y}_x^2 \leq E\left(E(Y|\mathcal{E}_a)^2\right) < E\,Y^2 = E\,\bar{Y}_a^2$. Hence $E\,\bar{Y}_{a-}^2 < E\,\bar{Y}_a^2$ and $\bar{Y}$ fails to be cts.

John Walsh sketches a proof, but it seems to me to be far from complete; if one accepted it, then it also proves continuity of all mgs!

An astonishing consequence of this is the following. Suppose that for each i,

N_t^i is a Poisson pr. of rate 1, indept for different i, $N_i(A)$ = no. of points of N^i in A, $N(A) = \sum N^i(A)$. Then the random measure N is trivial; any event about N has probability 0 or 1!!

Energy criterion for transience of a Markov chain (8/8/88).

Consider an irreducible infinite state-space discrete-time Markov chain. The following is a necessary and sufficient condition for it to be transient:

> There exists f which is not identically zero such that
>
> $$f_b = 0, \quad \sum p_{ij} f_j = f_i \quad (i \neq b) \quad \text{and} \quad \sum\sum \mu_i p_{ij} (f_j - f_i)^2 < \infty.$$

Here, b is some state in I, and μ is an invariant measure.

Proof (i) Suppose that the chain is recurrent, and that there exists such an f. Then the invariant measure μ is unique, and

$$E^b\Big[\sum_{r=0}^{T-1} I_{\{j\}}(X_r)\Big] = \mu_j/\mu_b$$

where $T = \inf\{n>0 : X_n = b\}$. Now $(f(X_{n\wedge T}))_{n\geq 1}$ is a P^b-martingale, so

$$E^b\Big[\sum_{r=1}^{T-1}(f(X_{r+1}) - f(X_r))^2\Big] \leq E^b\Big[\sum_{r=0}^{T-1}(f(X_{r+1}) - f(X_r))^2\Big]$$

$$= \frac{1}{\mu_b}\sum_j \mu_j \sum_k p_{jk}(f_j - f_k)^2 < \infty.$$

Thus $f(X_{n\wedge T})$ is an L^2-bounded martingale with terminal value 0 – and hence identically zero ✗.

(ii) Suppose that the chain is transient, $h_j \equiv P^j(T<\infty), j\neq b, \theta \equiv P^b(T=\infty)$. Consider the L^2-bounded martingale

$$M_n = h(X_n) + \theta\sum_{r=0}^{n-1} I_{(X_r = b)} \qquad (h_b \equiv 1)$$

We have for any $\xi \neq b$

$$\sum_{j,k} G_{\xi j}\, p_{jk}(h_k - h_j)^2 = E^\xi\Big[\sum_{n\geq 0}(h(X_{n+1}) - h(X_n))^2\Big]$$

$$= E^\xi\Big[\sum_{n\geq 0}(M_{n+1} - M_n - \theta I_{(X_n = b)})^2\Big]$$

$$= E^\xi \left[\sum_{n\ge 0} (M_{n+1}-M_n)^2 + \theta^2 \sum_{n\ge 0} I_{(X_n=b)} \right]$$

$$= E^\xi (M_\infty^2) - h_\xi^2 + \theta^2 G_{\xi b} .$$

But $M_\infty^2 = \theta^2 \left(\sum_{n\ge 0} I_b(X_n)\right)^2$ so $E^\xi(M_\infty^2) = \theta^2 \dfrac{G_{\xi b}}{G_{bb}} E^b \left(\sum I_b(X_n)\right)^2$

Hence

$$\sum_{j,k} G_{\xi j}\, p_{jk} (h_k - h_j)^2 = G_{\xi b} \left\{ \frac{\theta^2}{G_{bb}} \, \frac{2-\theta}{\theta^2} - h_\xi \frac{1}{G_{bb}} + \theta^2 \right\}$$

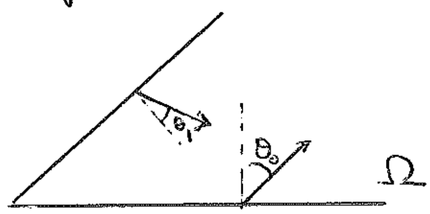
Now divide by $G_{\xi b}$ and let $\xi \to \infty$ in such a way that $G_{\xi j}/G_{\xi b} \to \mu_j$. From Fatou we have, using the fact that $G_{bb} = \frac{1}{\theta}$,

$$\boxed{\sum \mu_j\, p_{jk} (h_k - h_j)^2 \le 2\theta \le 2 .}$$

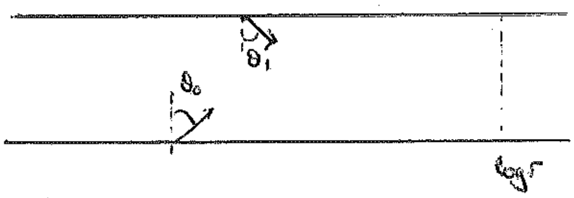
There appears to be no analogue of the "flow of finite energy" criterion for symmetrisable chains.

Exit law for Brownian motion in the wedge with skew reflection (9/8/88)

Let's consider BM in a wedge with skew reflection on the boundaries. In cases where the corner is hit, in order to show that there is at most one extension we need to consider the place of first exit from



$\Omega \cap \{ |z| < r \}$. Conformal transformation (taking logs) maps Ω to the strip:



and to find the exit law, we take another conformal transformation and seek the exit distribution from $\mathbb{H}$ with skew reflection at the boundary as shown:

To this end, we define

$$\theta(x) = \begin{cases} \theta_0 & (x < -1) \\ 0 & (-1 < x < 1) \\ -\theta_1 & (1 < x) \end{cases}$$

We shall assume we are dealing with a case where the corner can be hit: $\theta_0 + \theta_1 < 0$. Let $\alpha_0 \equiv \frac{1}{2} + \theta_0/\pi$, $\alpha_1 \equiv \frac{1}{2} + \theta_1/\pi$, and consider the function

$$\psi(z) \equiv (-1-z)^{\alpha_0} \frac{1}{\xi - z} (1-z)^{\alpha_1} \cdot \frac{1}{\pi} (1+\xi)^{-\alpha_0} (1-\xi)^{-\alpha_1} e^{i\pi\alpha_0}$$

where $\xi \in (-1,1)$ is fixed, and whenever we write z^α, we are taking the branch of the f^n which is cut along $(-\infty, 0]$ and fixes $\mathbb{R}^+$. Then ψ is analytic in $\mathbb{H}$, and along $(-\infty,-1)$ the argument is $\pi\alpha_0 = \pi/2 + \theta_0$, along $(1,\infty)$ the argument is $\pi - \pi\alpha_1 = \frac{\pi}{2} - \theta_1$. Writing $\psi = u + iv$ we therefore have

$$\begin{cases} u \cos\theta_0 + v \sin\theta_0 = 0 & \text{on } (-\infty,-1) \\ u \cos\theta_1 - v \sin\theta_1 = 0 & \text{on } (1,\infty), \end{cases}$$

from which

$$\begin{cases} \frac{\partial u}{\partial x} \cos\theta_0 + \frac{\partial v}{\partial x} \sin\theta_0 = \frac{\partial v}{\partial y} \cos\theta_0 + \frac{\partial v}{\partial x} \sin\theta_0 = 0 & \text{on } (-\infty,-1) \\ \frac{\partial v}{\partial y} \cos\theta_1 - \frac{\partial v}{\partial x} \sin\theta_1 = 0 & \text{on } (1,\infty) \end{cases}$$

and so the imaginary part of ψ satisfies the boundary conditions in $\mathbb{R} \setminus [-1,1]$ and

$\psi(X_{t\wedge\tau})$ <u>is a local martingale</u>, $\tau \equiv \inf\{u : X_u \in [-1,1]\}$.

To make everything rigourous, we would now multiply by $g \in C_K^\infty(-1,1)$ and integrate $d\xi$, but the end conclusion is

$$\boxed{P^z\left[X_\tau \in d\xi\right]/d\xi = \operatorname{Im}\left\{\frac{(-1-z)^{\alpha_0}(1-z)^{\alpha_1} e^{i\pi\alpha_0}}{\xi - z}\right\}\left\{\pi (1+\xi)^{\alpha_0}(1-\xi)^{\alpha_1}\right\}^{-1}}$$

The most interesting thing is to see what this does as $z_n = R_n e^{i\theta_n} \to \infty$. It goes like

$$R_n^{(\theta_0+\theta_1)/\pi} \operatorname{Im}\left[e^{i\alpha_0\theta_n + i\alpha_1(\theta_n - \pi) + i(\pi - \theta_n)}\right] \left\{\pi(1+\xi)^{\alpha_0}(1-\xi)^{\alpha_1}\right\}^{-1}$$

$$= R_n^{(\theta_0+\theta_1)/\pi} \sin\left(\frac{\theta_0+\theta_1}{\pi}\theta_n + \frac{\pi}{2} - \theta_1\right) \left\{\pi(1+\xi)^{\alpha_0}(1-\xi)^{\alpha_1}\right\}^{-1}.$$

There are two interesting features: (i) given that we do hit $[-1,1]$, the law of the place where we hit it is asymptotically independent of the starting point (ii) the probability that we hit $[-1,1]$ retains some dependence on θ_n.

Itô excursion theory via resolvents: some after-thoughts. (16/9/88)

When we express the new resolvent

$$R_\lambda f(x) = R^\partial_\lambda f(x) + \psi_\lambda(x)\, R_\lambda f(a)$$

with

$$R_\lambda f(a) = \frac{\gamma f(a) + n_\lambda f}{\lambda\gamma + \delta + \lambda n_\lambda 1},$$

one wants to know "Is this the decomposition of $(R_\lambda)_{\lambda>0}$ at the first hit on a?"

We begin by investigating local time at a

(i) The function ψ_α is uniformly α-excessive iff $\lim_{\lambda\to\infty}(\lambda\gamma + \lambda n_\lambda 1) = +\infty$.

We have

$$\lambda R_{\lambda+\alpha}\psi_\alpha = \lambda R^\partial_{\lambda+\alpha}\psi_\alpha + \psi_{\lambda+\alpha}\, \lambda R_{\lambda+\alpha}\psi_\alpha(a)$$

$$= \psi_\alpha - \psi_{\lambda+\alpha} + \psi_{\lambda+\alpha}\,\frac{\lambda(\gamma + n_{\lambda+\alpha}\psi_\alpha)}{(\lambda+\alpha)\gamma + \delta + (\lambda+\alpha)n_{\lambda+\alpha}1}$$

$$= \psi_\alpha - \psi_{\lambda+\alpha}\{\alpha\gamma + \delta + \alpha n_\alpha 1\}/\{(\lambda+\alpha)\gamma + \delta + (\lambda+\alpha)n_{\lambda+\alpha}1\}.$$

Hence if $\lambda\gamma + \lambda n_\lambda 1 \to \infty$, the uniform α-excessive property is immediate, and if it fails, we have that ψ_α is not even α-excessive; there is not convergence at a.

(ii) It is easy to think of examples where $\lambda\gamma + \lambda n_\lambda 1$ is bounded and the decomposition of R_λ is not the decomposition at the first hit on a. Let's now assume that $\lambda\gamma + \lambda n_\lambda 1$ is unbounded, and therefore ψ_α is uniformly α-excessive. We have that ψ_α is the α-potential of some PCHAF, and we can approximate the PCHAF by

$$\int_0^t g_\lambda(X_s)\,ds,$$

where

$$g_\lambda = \lambda(I - \lambda R_{\lambda+\alpha})\psi_\alpha, \qquad R_\alpha g_\lambda = \lambda R_{\lambda+\alpha}\psi_\alpha \equiv f_\lambda, \text{ say}.$$

[This is slightly different from III.33 in DW, but it works just as well, for any α-excessive function, in fact] The essential part of the problem is concerned with $\lambda n_\lambda 1$, so let's suppose $\gamma = \delta = 0$ to ease notation; they can be included if need be later.

Thus we have

$$g_\lambda = \lambda\psi_{\lambda+\alpha}\frac{\alpha n_\alpha 1}{(\lambda+\alpha)\, n_{\lambda+\alpha} 1}$$

(iii) Unfortunately, the analysis of this defeats me. However, we can in this case guess an appropriate family of approximating α-potentials: if

$$h_\lambda \equiv \frac{\alpha n_\alpha 1}{n_\lambda 1} \psi_\lambda$$

then we have

$$R_\alpha h_\lambda = \left\{ R_\alpha^\partial \psi_\lambda + \psi_\alpha \; R_\alpha \psi_\lambda(a) \right\} \frac{\alpha n_\alpha 1}{n_\lambda 1}$$

$$= \left\{ \frac{\psi_\alpha - \psi_\lambda}{\lambda - \alpha} + \psi_\alpha \frac{\lambda n_\lambda 1 - \alpha n_\alpha 1}{(\lambda-\alpha)\; \alpha n_\alpha 1} \right\} \frac{\alpha n_\alpha 1}{n_\lambda 1}$$

which converges uniformly to ψ_α. The argument in §III.33 of DW now clinches the convergence of the corresponding PCHAFs. However this doesn't answer the underlying question.

<u>Good behaviour of a Pick function from its boundary behaviour</u> (25/1/89)

Suppose that $\theta : \mathbb{R} \to \mathbb{R}$ is bounded and Hölder (1),

$$|\theta(x)| \leq K, \qquad |\theta(x) - \theta(y)| \leq C\, |x-y| \qquad \forall\, x, y \in \mathbb{R}.$$

Consider now the function $f : \mathbb{H} \to \mathbb{C}$ defined by

$$f(z) \equiv \int_{-\infty}^{\infty} \frac{\theta(x)}{\pi}\, dx \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\}.$$

This function is analytic, and has imaginary part bounded by K. The real part is <u>also</u> very well behaved: in fact

$$\boxed{\; |\operatorname{Re} f(a+ib)| \leq \text{Const} + \frac{2K}{\pi} \log(1+a^2+b^2) \;}$$

<u>Proof</u> Let's split the real part of $f(z)$ into two pieces

$$I_1 \equiv \int_{-\infty}^{\infty} \frac{\theta(x)}{\pi}\, dx \left[\frac{x-a}{(x-a)^2+b^2} - \frac{x}{x^2+b^2} \right], \quad I_2 \equiv \int_{-\infty}^{\infty} \frac{\theta(x)}{\pi}\, dx \left[\frac{x}{x^2+b^2} - \frac{x}{x^2+1} \right],$$

and work on these separately.

A bit of variable changing on the first to begin with: if $a = 2\alpha b$,

$$I_1 = \int_{-\infty}^{\infty} \theta(b\alpha + bv)\,\frac{dv}{\pi}\left\{\frac{v-\alpha}{(v-\alpha)^2+1} - \frac{v+\alpha}{(v+\alpha)^2+1}\right\}.$$

Now we continue. Wlog $\alpha > 0$, and

$$\left|\int_{2\alpha}^{\infty} \theta(b\alpha + bv)\,\frac{dv}{\pi}\left\{\frac{v-\alpha}{(v-\alpha)^2+1} - \frac{v+\alpha}{(v+\alpha)^2+1}\right\}\right|$$

$$= \left|\int_{2\alpha}^{\infty} \theta(b\alpha+bv)\,\frac{dv}{\pi}\,\frac{2\alpha\,(v^2-1-\alpha^2)}{(1+(v-\alpha)^2)(1+(v+\alpha)^2)}\right|$$

$$\leq K\int_{2\alpha}^{\infty}\frac{dv}{\pi}\,\frac{2\alpha\,(v^2+1+\alpha^2)}{(1+(v-\alpha)^2)(1+(v+\alpha)^2)}$$

$$= K\int_{2}^{\infty}\frac{du}{\pi}\,\frac{2\alpha^4u^2 + 2\alpha^2(1+\alpha^2)}{(1+\alpha^2(u-1)^2)(1+\alpha^2(u+1)^2)}$$

which is increasing in α, with limit $(2K/\pi)\int_2^\infty \frac{(1+u^2)\,du}{(u-1)^2(u+1)^2} < \infty$.
Also well-behaved as $\alpha \downarrow 0$.

So all the interest is in the behaviour in $(-2\alpha, 2\alpha)$. We deal with the case $(0, 2\alpha)$, the other bit being handled similarly. (*) see over!

$$\left|\int_{0}^{2\alpha} \theta(b\alpha+bv)\,\frac{dv}{\pi}\left\{\frac{v-\alpha}{(v-\alpha)^2+1} - \frac{v+\alpha}{(v+\alpha)^2+1}\right\}\right|$$

$$= \left|\int_{-\alpha}^{\alpha}\theta(2b\alpha+bt)\,\frac{dt}{\pi}\left\{\frac{t}{1+t^2} - \frac{t+2\alpha}{1+(t+2\alpha)^2}\right\}\right|$$

$$\leq \left|\int_0^\alpha \frac{t\,dt}{\pi(1+t^2)}\left\{\theta(2b\alpha+bt) - \theta(2b\alpha-bt)\right\}\right| + \int_{-\alpha}^{\alpha}\frac{K\,dt}{\pi}\,\frac{t+2\alpha}{1+(t+2\alpha)^2}$$

$$\leq \int_0^\alpha \frac{t\,dt}{\pi(1+t^2)}\,(2K)\wedge(2cbt) + \frac{K}{\pi}\int_\alpha^{3\alpha} du\,\frac{u}{1+u^2}$$

$$= \int_0^{K/cb}\frac{t^2\,dt}{\pi(1+t^2)}\,2cb + \int_{K/cb}^{\alpha}\frac{2K}{\pi}\,\frac{t\,dt}{(1+t^2)} + \frac{K}{2\pi}\log\left(\frac{1+9\alpha^2}{1+\alpha^2}\right)$$

b: $\alpha > K/cb$, that is, α [illegible]

$$= \frac{2cb}{\pi}\Big[t - \tan^{-1}(t)\Big]_0^{K/cb} + \frac{K}{\pi}\left\{\log(1+\alpha^2) - \log\left(1+\left(\frac{K}{cb}\right)^2\right)\right\} + \frac{K}{2\pi}\log\left(\frac{1+9\alpha^2}{1+\alpha^2}\right)$$

$$\leq \frac{2K}{\pi} - \frac{2cb}{\pi}\tan^{-1}\left(\frac{K}{cb}\right) + \frac{K}{\pi}\log\left(\frac{1+\alpha^2}{1+K^2/c^2b^2}\right) + \frac{K}{2\pi}\log 9$$

$$\leq \text{const} + \frac{K}{\pi}\log(1+\alpha^2)/(1+K^2/c^2b^2)$$

As for the other integral, we have

$$I_2 = \int_0^\infty \left(\frac{x}{x^2+b^2} - \frac{x}{x^2+1}\right)\left(\theta(x) - \theta(-x)\right)\frac{dx}{\pi}$$

so that

$$|I_2| \leq \int_0^\infty \left|\frac{x}{x^2+b^2} - \frac{x}{x^2+1}\right| (2K)\wedge(2cx)\,\frac{dx}{\pi}$$

$$\leq \int_0^{K/c} 2c\left|\frac{x^2}{x^2+b^2} - \frac{x^2}{x^2+1}\right|\frac{dx}{\pi} + \int_{K/c}^\infty 2K\frac{dx}{\pi}\left|\frac{x}{x^2+b^2} - \frac{x}{x^2+1}\right|$$

$$\leq \text{Const} + \frac{K}{\pi}\left|\Big[\log(x^2+b^2) - \log(x^2+1)\Big]_{K/c}^\infty\right|$$

$$= \text{const} + \frac{K}{\pi}\left|\log\frac{K^2+c^2b^2}{K^2+c^2}\right|$$

$$\leq \text{const} + \frac{K}{\pi}\log(1+b^2). \qquad (*)$$

The integral $\int_{-\infty}^0$ contributes another term the same.

Remarks (i) If we pick θ to be odd, $\theta(x) = (\log\frac{1}{x})^{-1}$ for $0<x<e^{-1}$, θ smooth in $(0,\infty)$, zero outside $(0,1)$, then looking at $f(ib)$ is just like taking the estimation of I_2, and letting $b \downarrow 0$ gives explosion!!

(ii) Consider next the case where $\theta(x) = \operatorname{sgn}(x)\,|x|^\alpha$, where $0<\alpha<1$, at least for $|x|\geq 1$, with a smooth interpolation in $[-1,1]$.

Then $f(ib) = 2\int_0^\infty \frac{dx}{\pi}\, x^\alpha \left\{\frac{x}{x^2+b^2} - \frac{x}{x^2+1}\right\}$ + harmless bit from the integral $(-1,1)$

How does this behave as $b\to\infty$?

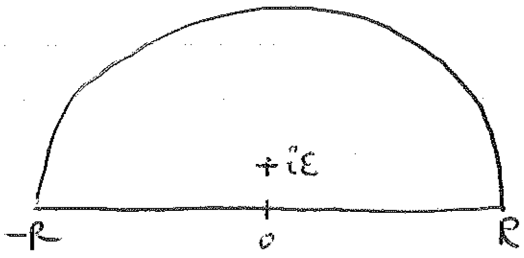
$$\int_0^\infty x^\alpha\,dx\left(\frac{x}{x^2+1} - \frac{x}{x^2+b^2}\right) = (b^2-1)\int_0^\infty \frac{x^{\alpha+1}\,dx}{(x^2+1)(x^2+b^2)}$$

$$= (b^2-1)\int_0^\infty \frac{u^{\alpha+1}\,du}{(1+b^2u^2)(1+u^2)}\, b^\alpha$$

$$\sim b^\alpha \int_0^\infty \frac{u^{\alpha+1}\,du}{1+u^2} \quad \text{as } b\to\infty$$

so this grows worse than logarithmically. Moral: both the boundedness and the Lipschitz conditions are needed!

Exit from a semicircle (30/1/89)



The map $z \mapsto -4R^2 \left(\frac{1}{2R} - \frac{1}{z+R} \right)^2 \equiv f(z)$

takes this semicircle to $\mathbb{H}$, and takes the curved part of the boundary to $\mathbb{R}^+$. Thus

$$P^{i\varepsilon}(\text{hit radius } R \text{ before } \mathbb{R}) = P^{f(i\varepsilon)}(\text{BM leaves } \mathbb{H} \text{ in } \mathbb{R}^+)$$

$$= P(\alpha + \beta U > 0) \quad \text{where } f(i\varepsilon) \equiv \alpha + i\beta = \frac{-\{(R^2-\varepsilon^2)^2 - 4\varepsilon^2R^2 - 4i\varepsilon R(R^2-\varepsilon^2)\}}{(R^2+\varepsilon^2)}$$

$$= P(U > -\alpha/\beta) = \frac{1}{\pi} \tan^{-1}\left\{ \frac{4\varepsilon R(R^2-\varepsilon^2)}{(R^2-\varepsilon^2)^2 - 4\varepsilon R^2} \right\}$$

$$\sim 4\varepsilon / R\pi \qquad \text{as } R \to \infty.$$

Back to Pick functions (1/2/89)

Returning to the situation on p. 38, we can say certain things about

$$f'(z) = \int_{-\infty}^{\infty} \frac{\theta(x)\,dx}{\pi} (x-z)^{-2}.$$

$$= \int \theta(x+a) \frac{dx}{\pi} \frac{x^2 - b^2 + 2ibx}{(x^2+b^2)^2}.$$

Thus

$$\text{Re } f'(z) = \int \{\theta(x+a) - \theta(a)\} \frac{dx}{\pi} \frac{x^2-b^2}{(x^2+b^2)^2}$$

$$= \int \frac{\theta(bx+a) - \theta(a)}{b} \frac{dx}{\pi} \frac{x^2-1}{(x^2+1)^2}$$

$$\therefore |\text{Re } f'(z)| \le \int \frac{1}{b} (2K) \wedge |bx| \frac{|x^2-1|}{(x^2+1)^2} \frac{dx}{\pi}$$

$$\le \int \left(\frac{2K}{b} \wedge |x| \right) \frac{|x^2-1|}{(1+x^2)^2} \frac{dx}{\pi}$$

$$\boxed{\therefore |\text{Re } f'(z)| \le \text{const} \left\{ \frac{1}{1+b} + \log\left(1 + \frac{1}{b}\right) \right\}}$$

$$\operatorname{Im} f'(z) = \int \theta(x+a)\frac{dx}{\pi}\,\frac{2bx}{(x^2+b^2)^2} = \int \theta(bx+a)\,\frac{dx}{\pi b}\,\frac{2x}{(1+x^2)^2}$$

$$= \int_0^\infty \frac{\theta(a+bx)-\theta(a-bx)}{bx}\;\frac{2x^2}{(1+x^2)^2}\;\frac{dx}{\pi}$$

which is thus globally bounded by C, and converges for a.e. a to $\theta'(a)$ as $b \to 0$.

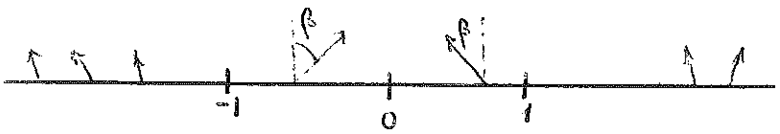
Thus $\operatorname{Im} f'(\cdot)$ is the Poisson integral of θ'. This implies in particular that

$$f'(z) = \int \frac{\theta'(x)\,dx}{\pi}\left\{\frac{1}{x-z} - \frac{x}{1+x^2}\right\} + \text{real const}$$

<u>Brownian motion in wedge with skew-reflection</u> (8/2/89)

The problem is to decide whether one can hit the corner, when the direction of reflection is allowed to <u>vary</u> along the sides. Since it only matters what happens in a neighbourhood, let's suppose that outside some neighbourhood of 0 the direction of reflection varies smoothly, and for $|z| \geq 1$ the direction of reflection is a constant $-\beta < 0$. This makes the process go out to ∞, so the process will <u>either</u> go into 0 <u>or</u> go out to ∞. So we may as well reduce to this picture:

(the direction of reflection varies outside $[-1, 1]$)



(i) <u>How does the probability of escaping to ∞ behave for a starting point near 0?</u>

Let $h(z) \equiv P^z(\text{motion escapes to } \infty)$.

Consider $x \in (-1, 1)$: $h(x) \leq P^x(\text{reach unit circle before } 0)$, which we can handle as before. Specifically, $z \mapsto \log z$ maps $\mathbb{H} \cap \{|z| \leq 1\}$ to $\{x+iy : x \leq 0,\ 0 \leq y \leq \pi\}$, and then $z \mapsto \cosh z$ opens this out to $\mathbb{H}$ again, and we can use results on p 35–36 for this:

$$P^x(\text{reach unit circle before } 0) = \int_{-1}^{1} P^\xi(X_\tau \in d\xi) \qquad \left(\xi \equiv \cosh(\log x) = \tfrac{1}{2}(x + x^{-1})\right)$$

$$= \int_{-1}^{1} \mathrm{Im}\left(\frac{(-1-\zeta)^{\alpha_0}(1-\zeta)^{\alpha_0}}{\xi - \zeta} e^{i\pi\alpha_0} \right) \frac{d\xi}{\pi(1-\xi^2)^{\alpha_0}} \qquad \alpha_0 \equiv \tfrac{1}{2} - \beta/\pi$$

$$= (\zeta^2-1)^{\alpha_0} \frac{\cos\beta}{\pi} \int_{-1}^{1} \frac{d\xi}{(\zeta-\xi)(1-\xi^2)^{\alpha_0}}$$

$$\sim \text{const}.\ \zeta^{2\alpha_0 - 1} = \text{const}\ \zeta^{-2\beta/\pi} \qquad \text{as } \zeta \to \infty$$

Hence $\quad h(x) \leq P^x(\text{reach unit circle before } 0) \leqslant \text{const}\ x^{2\beta/\pi}$

for x very close to 0. SO THE BOUNDARY VALUES are HÖLDER NEAR 0, BUT NOT LIPSCHITZ.

(ii) <u>Constructing a function which 'straightens out' the boundary condition</u>

The idea is that the analytic function

$$f(z) = \int_{-\infty}^{\infty} h(x) \frac{dx}{\pi} \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\}$$

satisfies the boundary conditions

$$0 = \sin\theta(x) \frac{\partial h}{\partial x}(x) + \cos\theta(x) \frac{\partial h}{\partial y}(x)$$

$$= \sin\theta(x) \frac{\partial h}{\partial x}(x) + \cos\theta(x) \frac{\partial g}{\partial x}(x) \qquad (f = g + ih)$$

by Cauchy-Riemann, where $\theta(\cdot)$ is the direction-of-reflection function, taking values between $-\pi/2$ and $\pi/2$. If we could find some φ analytic in $\mathbb{H}$ s.t. $\varphi(x) = |\varphi(x)| \{\cos\theta(x) - i\sin\theta(x)\}$, then $\mathrm{Re}(\varphi f') = 0$ on $\mathbb{R}$. We build such a φ as follows.

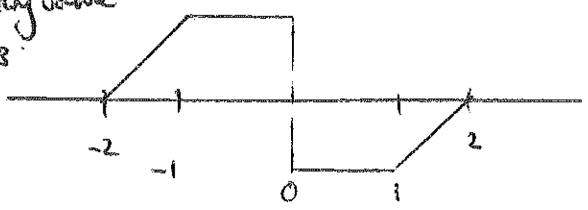
Set

$$f_2(z) \equiv \int_{\mathbb{R}\setminus(-2,2)} \frac{dx}{\pi} \theta(x) \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\}$$

$$f_3(z) = \int_{-2}^{-1} \frac{dx}{\pi} (x+2) \frac{1}{x-z} + \int_{-1}^{0} \frac{dx}{\pi} \frac{1}{x-z}$$

$$- \int_{0}^{1} \frac{dx}{\pi} \frac{1}{x-z} + \int_{1}^{2} (x-2) \frac{dx}{\pi} \frac{1}{x-z}.$$

[Boundary value of $\mathrm{Im} f_3$



$$\int_{\alpha}^{\beta} \frac{dx}{\pi(x-z)} = \frac{1}{\pi}\left[\tfrac{1}{2}\log|x-z| + i\tan^{-1}\left(\frac{x-a}{b}\right) \right]_{\alpha}^{\beta}$$

$$= \frac{1}{\pi}\left[\tfrac{1}{2}\log|x-z| + i\frac{\pi}{2} + i\arg(x-z) \right]_{\alpha}^{\beta}$$

The function f_3 can be evaluated completely explicitly*:

$$\pi f_3'(z) = 2 + (2+z)\{\log(-1-z) - \log(-2-z)\} + 2\log(-z) - \log(-1-z) - \log(1-z)$$
$$+ (z-2)\{\log(2-z) - \log(1-z)\}.$$

Notice that $z f_3(z) \to 0$ as $|z| \to \infty$;
f_3 remains bounded near $2, -2, 1, -1$;
$f_3(z) - \frac{2}{\pi}\log|z|$ remains bounded near 0.

Now choose some $\beta \in (0, \pi/2)$ and set

$$\boxed{\varphi(z) \equiv \exp\{-f_2(z) - \beta f_3(z)\}}$$

This function φ will straighten out the boundary values: $Re(\varphi f') = 0$ on $\mathbb{R}$.

(iii) Rephrasing in terms of BM in the strip $\mathbb{R} \times [0,1]$ with skew reflection on the edges, we can see that the criterion for hitting the ends of R cannot be the same as for the diffusion analogue with this example.

Suppose
$$dX_t = dB_t - e^{-X_t}\{dL_t^0 - dL_t^1\}$$

where L_t^i is local time at i of the vertical component. The diffusion analogue is got by replacing dL_t^i by dt, so in this case the diffusion analogue is recurrent. But if $V_t \equiv L_t^0 - L_t^1$, $Z_t = e^{X_t}$, we get

$$dZ_t = Z_t(dB_t + dt) - dV_t$$

whence
$$Z_t = e^{B_t + t/2}\left\{Z_0 - \int_0^t e^{-B_s - s/2}\, dV_s\right\}$$

which will hit zero in finite time $\therefore$ X reaches $-\infty$ in finite time.

[*Note that $\int_\alpha^\beta \frac{dx}{x-z} = \log(\beta - z) - \log(\alpha - z)$, $z \in \mathbb{H}$.]

<u>Diffusion of shape</u> (13/2/89)

Take $N+1$ independent Brownian motions $X_1, \dots, X_{N+1}$ in $\mathbb{R}^N$. <u>Is the subspace spanned by</u> $X_1, \dots, X_{N+1}$ <u>ever of dimension</u> $< N$?

(i) Let M be the matrix $(X_1 \dots X_{N+1})$, $A \equiv MM^T$. We shall prove that a.s. $\det A > 0$ for all time, and hence the answer to the question posed above is, "No."

$$d(\det A) = \det A \; a^{rs} \, da_{sr} + \tfrac{1}{2} \frac{\partial}{\partial a_{ij}} (a^{rs} \det A) \; d\langle a_{sr}, a_{ij} \rangle$$

$$= \det A \left\{ a^{rs} \, da_{sr} + \tfrac{1}{2} \, d\langle a_{ij}, a_{sr} \rangle \left(a^{rs} a^{ji} - a^{ri} a^{js} \right) \right\}$$

Now

$$da_{rs} = d(X_p^r X_p^s) = X_p^r \, dX_p^s + X_p^s \, dX_p^r + \delta_{rs} (N+1) \, dt,$$

so

$$da_{ij} \, da_{rs} = \left(X_p^r \, dX_p^s + X_p^s \, dX_p^r \right) \left(X_q^i \, dX_q^j + X_q^j \, dX_q^i \right)$$

$$= \{ a_{ri} \delta_{js} + a_{rj} \delta_{is} + a_{si} \delta_{jr} + a_{sj} \delta_{ir} \} \, dt$$

Hence

$$\frac{d(\det A)}{\det A} = a^{rs} \left(X_p^r \, dX_p^s + X_p^s \, dX_p^r \right) + (N+1) \, a^{rr} \, dt$$

$$+ \tfrac{1}{2} dt \left(a^{rs} a^{ji} - a^{ri} a^{js} \right) \left(a_{ri} \delta_{js} + a_{rj} \delta_{is} + a_{si} \delta_{jr} + a_{sj} \delta_{ir} \right)$$

$$= a^{rs} \left(X_p^r \, dX_p^s + X_p^s \, dX_p^r \right) + 2 a^{rr} \, dt$$

The quadratic variation of this semimartingale is

$$a^{rs} a^{ij} \, d\langle a_{ij}, a_{rs} \rangle = 4 a^{jj} \, dt$$

Thus $\boxed{d(\det A) = \det A \; \{ 2 \sqrt{\operatorname{tr} A^{-1}} \, dW_t + 2 \operatorname{tr} A^{-1} \, dt \}}$.

In particular, $\boxed{d(\log \det A) = 2 \sqrt{\operatorname{tr} A^{-1}} \, dW_t}$

So $\log \det A$ is a continuous local martingale $\therefore P(\det A > 0 \; \forall t) = 1$.

(ii) The importance of this is that one may effectively <u>ignore</u> degenerate shapes when N+1 Brownian motions diffuse in $\mathbb{R}^N$. Indeed, if Σ_N^{N+1} is the space of shapes of N+1 points in N-space, we have that

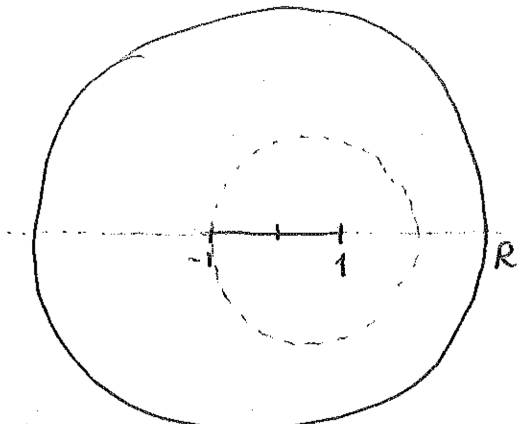
$$\boxed{(X_1, \dots, X_{N+1}) \in \Sigma_N^{N+1} \times SO(N) \times (0,\infty) \times \mathbb{R}^N \qquad \text{for all } t}$$

in an obvious sense. Why? Suppose that after centering we had a configuration $X_1 - \bar{X}, \dots, X_{N+1} - \bar{X}$ (which might as well be supposed to be rescaled). Now consider all the configurations which are $SO(N)$-equivalent to this one: I claim that this set is in 1-1 correspondence with $SO(N)$. If not, there would be $R \in SO(N)$ st.

$$R(X_1 - \bar{X}, \dots, X_{N+1} - \bar{X}) = (X_1 - \bar{X}, \dots, X_{N+1} - \bar{X}), \quad R \neq I;$$

so if $\mathcal{U} = \text{lin}(X_1 - \bar{X}, \dots, X_{N+1} - \bar{X})$, we have that R acts as I on $\mathcal{U}$, and, easily, R fixes $\mathcal{U}^\perp$ and is a rotation on $\mathcal{U}^\perp$. But $\mathcal{U}$ can be of codimension at worst 1 in view of the result above, and $SO(1) = \{1\}$ – that is, $R = I$, a contradiction.

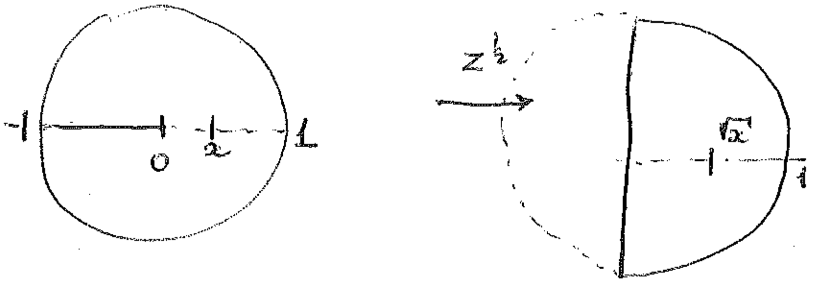
[All of this is in a paper by M-F Bru, CRAS]



More on the skew-reflecting BM problem (22/2/89)

If we had absorption in $[-1,1]$, and had some bdd bdy values on $|z| = R$, would the solution of the Dirichlet problem be Lipschitz in this domain? Alas not, it seems.

Consider what happens in the disc radius 2 centred on 1 $(R > 3)$. This is essentially the same as



and for small x, this is not Lipschitz. Indeed, the boundary integral representation gives

$$\text{Re}\left[\int_{-\pi}^{\pi} \frac{dt}{2\pi}\, h(t)\, \frac{e^{it} + \sqrt{x}}{e^{it} - \sqrt{x}}\right],$$

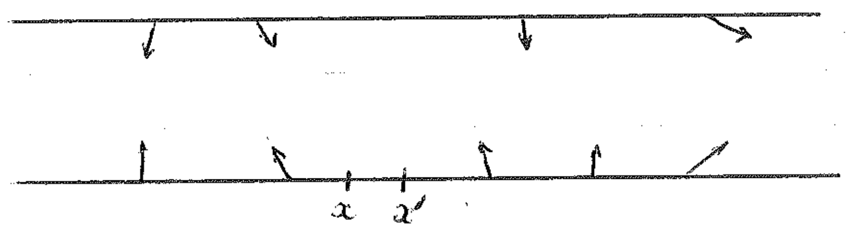
where h, originally defined on $(-\frac{\pi}{2}, \frac{\pi}{2})$, is extended antisymmetrically to the other half of the circle (method of images)

This integral now splits into two terms, like

$$\mathrm{Re} \int_0^{\pi/2} \frac{dt}{\pi}\, h(t) \left[\frac{e^{it}+\sqrt{x}}{e^{it}-\sqrt{x}} - \frac{e^{i(\pi-t)}+\sqrt{x}}{e^{i(\pi-t)}-\sqrt{x}} \right] = \mathrm{Re} \int_0^{\frac{\pi}{2}} \frac{dt}{\pi}\, h(t)\, \frac{4\sqrt{x}\, e^{it}}{e^{2it}-x}$$

$$\sim \sqrt{x}\ \text{const} \quad \text{as } x \downarrow 0 .$$

<u>BM in the strip with skew-reflection, 24/2/89.</u>



Let's take BM in the strip $0 \le \mathrm{Im}(z) \le 1$, with the Y-component simply reflecting BM in $[0,1]$, and the X-component satisfying

$$X_t = x + B_t + \int_0^t c_0(X_s)\, dL_s^0 + \int_0^t c_1(X_s)\, dL_s^1 .$$

Let's suppose further that c_0, c_1 are Lipschitz, constant K, and suppose we have another solution

$$X_t' = x' - B_t + \int_0^t c_0(X_s')\, dL_s^0 + \int_0^t c_1(X_s')\, dL_s^1$$

where $x' > x$. Let $T \equiv \inf\{u : X_u' - X_u \le 0\}$, $\Delta_t = X_t' - X_t$, $L_t \equiv L_t^0 + L_t^1$. Then the process

$$\boxed{V_t = x' - x - 2B_t + K \int_0^t V_s\, dL_s}$$

provides a bound for Δ on $[0,T]$. Indeed, on $[0,T]$,

$$V_t - \Delta_t = K \int_0^t V_s\, dL_s - \left[\int_0^t \{c_0(X_s') - c_0(X_s)\}\, dL_s^0 + \int_0^t \{c_1(X_s') - c_1(X_s)\}\, dL_s^1 \right]$$

$$\ge K \int_0^t (V_s - \Delta_s)\, dL_s$$

so that

$$V_t - \Delta_t = \eta_t + K \int_0^t (V_s - \Delta_s)\, dL_s , \qquad 0 \le t \le T ,$$

where η_t is non-negative, FV, $\eta_0 = 0$.

$$\therefore \quad d\left((V_t - \Delta_t) e^{-KL_t} \right) = e^{-KL_t} \{ d\eta_t \}$$

$$\therefore \quad (V_t - \Delta_t) e^{-KL_t} = \int_0^t e^{-KL_s} d\eta_s = \eta_t e^{-KL_t} + \int_0^t K e^{-KL_s} \eta_s\, dL_s$$

which is non-negative, so $\boxed{V_t \geq \Delta_t \quad \text{for } t \leq T}$. Hence

$$P(X, X' \text{ never couple}) \leq P(V \text{ never reaches } 0),$$

and we can express V explicitly:

$$V_t = e^{KL_t}\left\{ x' - x - 2\int_0^t e^{-KL_s}\,dB_s \right\}$$

whence

$$P(V \text{ never reaches } 0) = P\left\{ \sup_t \int_0^t e^{-KL_s}\,dB_s < \tfrac{1}{2}(x'-x) \right\}$$

$$= P\left\{ \sup_t \tilde{B}\left(\int_0^t e^{-2KL_s}\,ds\right) < \tfrac{1}{2}(x'-x) \right\}$$

$$= P\left[\sup\left\{ \tilde{B}_s : s \leq \int_0^\infty e^{-2KL_u}\,du \right\} < \tfrac{1}{2}(x'-x) \right]$$

$$= P\left[\sup\left\{ \tilde{B}_s : s \leq 1 \right\} < \tfrac{1}{2}(x'-x) \Big/ \left(\int_0^\infty e^{-2KL_u}\,du\right)^{\frac{1}{2}} \right]$$

$$= E\int_0^{\delta/Y} 2e^{-t^2/2}\,\frac{dt}{\sqrt{2\pi}}, \qquad \text{where } \delta \equiv \frac{x'-x}{2},\ Y \equiv \left(\int_0^\infty e^{-2KL_u}\,du\right)^{\frac{1}{2}}$$

$$= \int_0^\delta E\left[\frac{e^{-v^2/2Y^2}}{Y}\right]\frac{dv}{\sqrt{2\pi}}\cdot 2$$

$$\leq \text{const}\,.\,\delta$$

iff $E(1/Y) < \infty$. To estimate $1/Y$, let $\tau \equiv \inf\{u : |B_u| = a\}$, and observe that

$$Y^2 \geq \tau e^{-2KL_\tau}$$

We can by the usual techniques get the joint LT of τ, L_τ:

$$E\left[e^{-\alpha L_\tau - \lambda\tau}\right] = \left\{\cosh\theta a + \frac{\alpha\sinh\theta a}{\theta}\right\}^{-1} \qquad \theta \equiv \sqrt{2\lambda}$$

so that
$$E\left[\frac{e^{-\alpha L_\tau}}{\tau}\right] = \int_0^\infty d\lambda\left\{\cosh a\sqrt{2\lambda} + \alpha\frac{\sinh a\sqrt{2\lambda}}{\sqrt{2\lambda}}\right\}^{-1}$$

and therefore

$$E\left[\frac{e^{2KL_\tau}}{\tau}\right] = \int_0^\infty d\lambda\left\{\cosh a\sqrt{2\lambda} - 2Ka\,\frac{\sinh a\sqrt{2\lambda}}{a\sqrt{2\lambda}}\right\}^{-1}$$

$$< \infty \quad \text{provided } 2Ka < 1.$$

Thus $1/Y \in L^2$, and there is some constant depending on K such that $P(X, X' \text{ don't meet}) \le P(V \text{ never reaches } 0) \le \text{const } (x'-x)$.

<u>Optimal control problem of Beneš, Karatzas</u> (10/7/89).

Suppose given some continuous local martingale M, and define

$$\sigma(x) = \begin{cases} 1 & , x>0 \\ 0 & , x=0 \\ -1 & , x<0. \end{cases}$$

We consider the SDE

$$\boxed{Y_t = y + M_t - \int_0^t \sigma(Y_s)\,ds.}$$

(i) <u>Does pathwise uniqueness hold for this?</u> (ii) <u>Does there exist a strong solution?</u>

(i) Pathwise uniqueness is quite easy. If Y, Y' are two solutions, we have that

$$0 \le (Y_t - Y'_t)^2 = 2\int_0^t (Y_s - Y'_s)\{\sigma(Y'_s) - \sigma(Y_s)\}\,ds$$

and the integrand is always ≤ 0. Note that continuity of M is not needed for this result.

(ii) To get existence of a strong solution, we can always build approximations by solving

$$Y_t = y + M_t - \int_0^t \varphi(Y_s)\,ds,$$

where $\varphi : \mathbb{R} \to [-1,1]$ is increasing, C^∞, $\varphi(x) = 1$ for $x \ge \delta$, $= -1$ for $x \le -\delta$. Suppose now that ψ is another such smooth function, and that

$$\psi(x) = \varphi(x) \quad \text{for all } x \notin (-\varepsilon, \eta).$$

Let Z be the corresponding solution with initial point y, so that

$$Z_t - Y_t = \int_0^t \{\varphi(Y_s) - \psi(Z_s)\}\,ds$$

Then $\boxed{\text{a.s. for all } t \quad Z_t - Y_t \in [-\theta_1, \theta_2]}$ where $\theta_1 = \inf\{\lambda : \varphi(y) \ge \psi(y-\lambda)\ \forall y\}$, $\theta_2 \equiv \inf\{\lambda : \psi(\lambda+y) \ge \varphi(y)\ \forall y\}$.

The proof of this is entirely elementary. In this case, we certainly have $\theta_1, \theta_2 \leq c \cdot \eta$.

One way to build approximations is to fix inc C^∞ φ, $\varphi(x) = -1 \; \forall x \leq 0$, $\varphi(x) = 1 \quad \forall x \geq 1$, and use $\varphi_n(x) \equiv \varphi(nx)$. In this case, comparing φ_n and φ_m, $n > m$, we see that $\theta_2 = 0$, $\theta_1 = \frac{1}{m} - \frac{1}{n}$, so that

$$-\left(\tfrac{1}{m} - \tfrac{1}{n}\right) \leq Y_t^n - Y_t^m \leq 0$$

for all t, where Y^n is solution arising from φ_n. Thus we certainly get monotone uniform almost sure convergence to some continuous limit process Y; and, equally, it is clear that whatever approximation regime one uses will produce the same limit process Y — but does Y solve the SDE we started with?

<u>ASSUMPTION</u>: $dt \ll d\langle M \rangle_t$. Under this assumption, the limit process Y does indeed solve the SDE of interest.

This is because

$$\int_0^t \varphi_n(Y_s^n)\, dt = \int_0^t \varphi_n(Y_s^n) I_{\{Y_s \neq 0\}}\, ds + \int_0^t \varphi_n(Y_s^n) I_{\{Y_s = 0\}}\, ds$$

$$\to \int_0^t \sigma(Y_s)\, ds + \text{some Lipschitz cts process which grows only when } Y = 0.$$

Let this latter process be denoted by A.

Then
$$Y_t = y + M_t - \int_0^t \sigma(Y_s)\, ds - A_t.$$

But it's a well-known fact that $\int_0^t I_{\{Y_s = 0\}}\, d\langle Y \rangle_s = 0 = \int_0^t I_{\{Y_s = 0\}}\, d\langle M \rangle_s \geq \int_0^t I_{\{Y_s = 0\}}\, ds$ so therefore $A \equiv 0$.

(iii) Notice that the argument of (ii) above only uses smoothness of φ, ψ to ensure existence of solutions. Thus if $\tilde{Y}$ were a solution of our original SDE, <u>it must be that</u> $\tilde{Y} = Y$, the process constructed by approximation.

(iv) <u>Some</u> sample-path properties must be used, because if we had $M_t = t/2$, the SDE has no solution.

[Karatzas knows all this!]

Hölder continuity of boundary function implies Hölder continuity of analytic extension (26/7/88)

Suppose that we take $\theta: \mathbb{R} \to \mathbb{R}$ with the property that for some $\alpha < 1$, some $C > 0$,

$$|\theta(x) - \theta(y)| \leq C|x-y|^\alpha \qquad \forall x, y \in \mathbb{R}$$

[NB – must have $\alpha < 1$; $\alpha = 1$ is quite different.]

Then if

$$f(z) \equiv \int_{-\infty}^{\infty} \frac{dx}{\pi}\, \theta(x) \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\},$$

it follows that f is again Hölder(α).

Proof We estimate $|f(a+ib) - f(a'+ib')| \leq |f(a+ib) - f(a'+ib)| + |f(a'+ib) - f(a'+ib')|$ and take the two pieces separately.

(i) For the second, assume $a = 0$ wlog, and consider

$$|f(a'+ib) - f(a'+ib')| = \left| \int \frac{\theta(x)}{\pi} dx \left(\frac{x+ib}{x^2+b^2} - \frac{x+ib'}{x^2+ib'^2} \right) \right|$$

$$\leq \left| \int x\, \theta(x) \frac{dx}{\pi} \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+b'^2} \right) \right| + \left| \int (\theta(x) - \theta(0)) \frac{dx}{\pi} \left\{ \frac{b}{x^2+b^2} - \frac{b'}{x^2+b'^2} \right\} \right|$$

Now the second term contributes

$$\left| \int (\theta(bx) - \theta(b'x)) \frac{dx}{\pi} \frac{1}{1+x^2} \right| \leq C\, |b-b'|^\alpha \int \frac{|x|^\alpha\, dx}{\pi(1+x^2)}$$

$$\leq C_1\, |b-b'|^\alpha,$$

and the first is estimated by (with $b' \equiv b + \varepsilon$)

$$\left| \int \frac{|x|^{\alpha+1} dx}{\pi} \left\{ \frac{2b\varepsilon + \varepsilon^2}{(x^2+b^2)(x^2+(b+\varepsilon)^2)} \right\} \right|$$

$$\cancel{\leq \varepsilon^2 \int \frac{|x|^{\alpha+1}\, dx}{(x^2+b^2)(x^2+(b+\varepsilon)^2)} + 2b\varepsilon \int \frac{|x|^{\alpha+1}\, dx}{(x^2+b^2)(x^2+(b+\varepsilon)^2)}}$$

$b \equiv \beta\varepsilon$

$$= \left| \int \frac{|x|^{\alpha+1}\, dx}{\pi} \cdot \frac{2\beta+1}{(x^2+\beta^2)(x^2+(1+\beta)^2)} \right| \cdot \varepsilon^\alpha$$

$$\leq C_2\, \varepsilon^\alpha$$

(ii) Now let's look at the first piece of the estimate, assuming wlog that $a'=0$, $a>0$. We have

$$\left|\int \theta(x)\frac{dx}{\pi}\left\{\frac{x-a+ib}{(x-a)^2+b^2}-\frac{x+ib}{x^2+b^2}\right\}\right|$$

$$\leq \left|\int \theta(x)\frac{dx}{\pi}\left(\frac{x-a}{(x-a)^2+b^2}-\frac{x}{x^2+b^2}\right)\right|+\left|\int \frac{\theta(x+a)-\theta(x)}{\pi}\frac{b}{x^2+b^2}\,dx\right|$$

and the second piece is $\leq C\,a^\alpha$. The first piece is

$$\left|\int \frac{\theta(x)-\theta(0)}{\pi}\,dx\left\{\frac{x-a}{(x-a)^2+b^2}-\frac{x}{x^2+b^2}\right\}\right| \qquad (b=a\beta)$$

$$=\left|\int \frac{\theta(ax)-\theta(0)}{\pi}\left\{\frac{x-1}{(x-1)^2+\beta^2}-\frac{x}{x^2+\beta^2}\right\}dx\right|.$$

We split the integral at $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$ and estimate the pieces.

$$\left|\int_{3/2}^{\infty}\frac{\theta(ax)-\theta(0)}{\pi}\,dx\left\{\frac{x-1}{(x-1)^2+\beta^2}-\frac{x}{x^2+\beta^2}\right\}\right|$$

$$\leq C\,a^\alpha\int_{3/2}^{\infty}|x|^\alpha\frac{dx}{\pi}\;\frac{\beta^2+x(x-1)}{(x^2+\beta^2)((x-1)^2+\beta^2)}$$

$$\leq C_3\,a^\alpha.$$

Next, $\displaystyle\left|\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\theta(ax)-\theta(0)}{\pi}\,dx\,\frac{x}{x^2+\beta^2}\right|\leq C a^\alpha\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{|x|^{\alpha+1}}{x^2+\beta^2}\,dx\leq C_4\,a^\alpha$

and $\displaystyle\left|\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\theta(ax)-\theta(0)}{\pi}\,dx\,\frac{x-1}{(x-1)^2+\beta^2}\right|\leq C a^\alpha\int_{-\frac{1}{2}}^{\frac{1}{2}}|x|^\alpha\,dx\,\frac{|x-1|}{(x-1)^2+\beta^2}\leq C_5\,a^\alpha.$

The integral from $\frac{1}{2}$ to $\frac{3}{2}$ is handled similarly.

There is also a local form of this result. Dynkin gives a reference:
F.D. Gakhov "Kraevye zadachi (Boundary value problems)" Izd. Fiz.-Mat. Lit. Moscow '63.

Examination of the above proof shows that if θ is locally Hölder(α), and satisfies a growth condition $|\theta(x)|\leq C(1+|x|^\alpha)$, then f is locally Hölder(α).

[The local version is trivial; if θ is locally Hölder α, then write $\theta=\theta_0+\theta_1$ where θ_0 is globally Hölder α, θ_1 is zero in some interval $(-N,N)$, say. Then the extension of θ_1 is analytic in $(-N,N)$!]

Existence of an equivalent martingale measure in a continuous setting (24/7/89.)

Dalang, Morton + Willinger take an adapted process $(X_n)_{n\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$ and show that there is a probability $Q \sim P$ under which X is a martingale iff the condition

$$\text{" } H \text{ previsible }, \; H \Delta X \geq 0 \Rightarrow H \Delta X = 0 \text{ "}$$

holds.

The analogue of this for a continuous semimartingale $X_t = M_t + V_t$ in continuous time is the condition

$$\boxed{\text{(a)} \quad H \text{ previsible, } \int H\,dX \text{ increasing} \Rightarrow \int H\,dX = 0.}$$

I claim that this is equivalent to

$$\boxed{\text{(b)} \quad |dV| \ll d[M].}$$

Proof (i) Suppose (a), and yet (b) fails. Then there is a previsible process $C \geq 0$ such that

$$\int C\,|dV| \text{ is non-null, yet } \int C^2\,d[M] \text{ is null.}$$

Thus, in particular, $\int C\,dM = 0$. If now $\sigma_s\,dV_s = |dV_s|$, we can take $H \equiv \sigma C$, and see

$$\int H\,dX = \int H\,dV = \int C\,|dV| \text{ which is increasing, non-null } \times.$$

(ii) If now (b) holds, and $\int H\,dX$ is increasing, then $\int H\,dM = \int H\,dX - \int H\,dV$ is a f.v. continuous local martingale, therefore null; in particular,

$$\int H^2\,d[M] = 0$$

and so by hypothesis $\underline{\int H^2\,|dV| = 0, \text{ so } \int H\,dV = 0}$, and $\int H\,dX = 0$.

It is easy to see that the condition

$$\boxed{\text{(c)} \quad \exists\, Q \sim P \text{ s.t. } X \text{ is a } Q\text{-martingale}}$$

implies (a) and (b), for if (c) holds, and $(dQ/dP)\big|_{\mathcal{F}_t} = Z_t$ is the Girsanov martingale, $dY_t = Z_t^{-1}\,dZ_t$, we have that, under Q,

$$N_t \equiv M_t - [M,Y]_t \text{ is a local martingale.}$$

Thus $V_t = X_t - M_t = X_t - N_t - [M,Y]_t$, and so $V + [M,Y]$ is a continuous f.v. local martingale, therefore null, so $V = -[M,Y]$. If there were some previsible C s.t. $\int C_s |dV_s| > 0$, then we would have by the Kunita-Watanabe inequalities

$$0 < \int_0^t |C_s||dV_s| \leq \left(\int_0^t C_s^2 \, d[M]_s\right)^{\frac{1}{2}} [Y]_t^{1/2}$$

implying $\int_0^t C_s^2 \, d[M]_s > 0$, whence (b).

It is also easy to see that (c) is not equivalent to (b), because if one considers $(X_t)_{0\leq t\leq 1}$ to be Brownian bridge $X_t = B_t - tB_1$,

then X is a continuous semimartingale:

$$X_t + \int_0^t \frac{X_s}{1-s}\, ds \equiv W_t \quad \text{is a Brownian motion} \quad (0 \leq t \leq 1)$$

and $\int_0^t \frac{X_s}{1-s}\, ds$ is of integrable variation. However, any equivalent measure Q which transformed X into a continuous local martingale would preserve the quadratic variation, t, of X; thus X would be a Brownian motion under Q, with value 0 at $t=1$ Q-a.s.. This is impossible!

Nonetheless, if (b) holds, and $dV = c \, d[M]$, then the solution Z to the SDE

$$dZ = -cZ \, dM$$

is a true martingale if the Novikov criterion is satisfied:

$$\boxed{(d) \quad \mathbb{E} \exp\left\{ \tfrac{1}{2} \int_0^t C_s^2 \, d[M]_s \right\} < \infty \quad \text{for all } t.}$$

In this case, we use Z to define the equivalent martingale measure.
So we have

$$(c) \Rightarrow (a) \Leftrightarrow (b), \qquad ((b)+(d)) \Rightarrow (c).$$

<u>Semimartingale property of fractional Brownian motion</u> (25/7/89)

Consider the process

$$X_t \equiv \int_0^t (t-s)^{H-\frac{1}{2}}\, dB_s$$

where $H>0$ is self-similarity parameter: $(X_{ct})_{t\geq 0} \overset{\mathcal{D}}{=} (c^H X_t)_{t\geq 0}$.

$$\boxed{\text{Conjecture: except for } H=\tfrac{1}{2},\ X \text{ is not a semimartingale}}$$

1) Consider

$$\mathrm{Var}_n(r) \equiv \sum_{j=1}^{2^n} |X(j2^{-n}) - X((j-1)2^{-n})|^r \overset{\mathcal{D}}{=} 2^{-nrH} \sum_{j=1}^{2^n} |X_j - X_{j-1}|^r$$

$$\equiv 2^{-nrH}\, V^r_{2^n}, \quad \text{say}$$

Now $X_j - X_{j-1}$ is zero-mean Gaussian, with variance

$$\int_{j-1}^{j} (j-u)^{2\gamma}\, du + \int_0^{j-1} \left((j-u)^\gamma - (j-1-u)^\gamma\right)^2 du \qquad [\gamma = H - \tfrac{1}{2}]$$

$$= \frac{1}{2H} + \int_0^{j-1} \left((v+1)^\gamma - v^\gamma\right)^2 dv$$

$$= c + \int_1^{j-1} v^{2\gamma} \left((1+\tfrac{1}{v})^\gamma - 1\right)^2 dv$$

$$\leq \begin{cases} c\,(1 + j^{2\gamma-1}) & \text{if } \gamma \neq \tfrac{1}{2} \\ c\,(1+\log j) & \text{if } \gamma = \tfrac{1}{2}. \end{cases}$$

2) Now we consider the order-2 variation.

<u>Case 1</u>: $\gamma > \frac{1}{2}$ (i.e. $H>1$). We have

$$E[\mathrm{Var}_n(2)] = 2^{-2nH}\, E[V^2_{2^n}] \sim (2^{-n})^{2H} \cdot (2^n)^{2\gamma} = 2^{-n} \to 0.$$

<u>Case 2</u>: $\gamma = \frac{1}{2}$, $H=1$: We get

$$E[\mathrm{Var}_n(2)] \sim 2^{-2n} \cdot n2^n \longrightarrow 0.$$

<u>Case 3</u>: $\gamma < \frac{1}{2}$. We now have

$$E\left[\mathrm{Var}_n(2)\right] \sim (2^{-n})^{2H} \cdot 2^n \quad \begin{array}{ll} \to 0 & \text{if } H > \frac{1}{2} \\ \to \infty & \text{if } H < \frac{1}{2}. \end{array}$$

Thus for all $H > \frac{1}{2}$, the quadratic variation of X (if it is a semimartingale) will be 0, and thus X would have to be a f.v. process.

3) Now consider the order-1 variation.

Case 1 : $H > 1$. Now we have

$$E\left[\mathrm{Var}_n(1)\right] \sim (2^{-n})^{H} \quad (2^n)^{(\gamma - \frac{1}{2})+1} \quad = 1$$

Case 2: $H \leq 1$. This time we get

$$E\left[\mathrm{Var}_n(1)\right] \sim (2^{-n})^{H} \; 2^n \quad \to \infty \qquad \text{if } H < 1$$

$$\sim 2^{-n} \cdot 2^n \sqrt{n} \quad \to \infty \qquad \text{if } H = 1.$$

4) Next we shall get the exploding behaviour, using the following pretty result.

Lemma (A.P. Selby)

Suppose Y_n take values 0 or 1, Y_n is $\mathcal{F}_n$-measurable, and for some $\alpha > 0$, for all n, $E(Y_{n+1} | \mathcal{F}_n) \geq \alpha$. Then on some augmentation of the probability space there can be constructed i.i.d. r.v.'s W_n s.t. $W_n \sim B(1, \alpha)$ and $Y_n \geq W_n$ a.s.

We know that if $\tilde{Y}_n \equiv X_n - X_{n-1}$, then $\tilde{Y}_{n+1}$ has a Gaussian law conditional on $\mathcal{F}_n$, and indeed

$$\tilde{Y}_{n+1} | \mathcal{F}_n \sim N\left(\int_0^n \left((n+1-s)^{\gamma} - (n-s)^{\gamma}\right) dB_s \, , \; \frac{1}{2H} \right).$$

Hence for each $a > 0$,

$$P\left(|\tilde{Y}_{n+1}| > a \,/\, \mathcal{F}_n \right) \geq \theta_a$$

where θ_a depends on a. By Selby's lemma, therefore, for any $a > 0$

$$\liminf \frac{1}{n} \sum_1^n |X_j - X_{j-1}|^r \geq a^r \theta_a \quad \text{a.s.}$$

Hence immediately for $H < \frac{1}{2}$, $2^{-2nH} V^2_{2^n} \to +\infty$ a.s., and so $\mathrm{Var}_n(2) \to +\infty$ in probability. But the limit of $\mathrm{Var}_n(2)$ is $\langle X \rangle_1$, assuming X were a semimartingale. Thus the quadratic variation process of X

explodes for $H<\frac{1}{2}$, Therefore for $H<\frac{1}{2}$, X is not a semimartingale

Next, for $\frac{1}{2}<H<1$, we know that the q.v. of X is zero, so X can only be a process of finite variation. But exactly the same argument shows that the variation of X explodes. Thus for $H\in(\frac{1}{2},1)$, X is not a semimartingale.

5) Rather to my surprise, for $H>1$ the process X is C^1 !! This may well follow from general results on Gaussian processes (though X is not stationary) but we can differentiate directly to obtain an expression for the derivative.

By parts,

$$X_t = \int_0^t (t-s)^\gamma \, dB_s = \int_0^t B_u (t-u)^{\gamma-1} \gamma \, du,$$

so that

$$X_{t+h}-X_t = \int_t^{t+h} (t+h-u)^{\gamma-1} \gamma B_u \, du + \int_0^t B_u \, du \, \gamma \left[(t+h-u)^{\gamma-1} - (t-u)^{\gamma-1}\right]$$

$$= \int_t^{t+h} (t+h-u)^{\gamma-1} \gamma (B_u - B_t) \, du + \int_0^t (B_u - B_t) \, \gamma\left[(t+h-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] du$$

$$+ B_t \left\{ h^\gamma + (t+h)^\gamma - h^\gamma - t^\gamma \right\}$$

$$= \int_0^h \gamma v^{\gamma-1} (B_{t+h-v} - B_t) \, dv + \int_0^t (B_{t-v} - B_t) \, \gamma \left\{ (v+h)^{\gamma-1} - v^{\gamma-1} \right\} dv$$

$$+ B_t \left\{ (t+h)^\gamma - t^\gamma \right\}$$

Dividing by h and letting $h\downarrow 0$, noticing that for $\delta < \varepsilon \equiv \gamma - \frac{1}{2} \equiv H-1$

$$\int_0^h \gamma v^{\gamma-1} (B_{t+h-v} - B_t) \, dv \le C_\delta \int_0^h \gamma v^{\gamma-1} (h-v)^{\frac{1}{2}-\delta} \, dv$$

$$= \text{const.}\, h^{\gamma+\frac{1}{2}-\delta} = c.h^{1+\varepsilon-\delta} = o(h),$$

we deduce that

$$\boxed{\lim_{h\downarrow 0} \frac{1}{h} (X_{t+h} - X_t) = \int_0^t (B_{t-v} - B_t) \, \gamma(\gamma-1) v^{\gamma-2} \, dv + \gamma t^{\gamma-1} B_t}$$

The left-derivative is handled similarly, and takes the same value.

6) So this only leaves the case $H=1$ to decide.

Growth of analytic functions in $\mathbb{H}$ once again. (26/7/89)

1) From the item on p. 38 we know that if f is the analytic extension of a Lipschitz boundary function which takes values in $[-\pi/2, \pi/2]$, then $|f(z)| \leq C_1 + \log(1+|z|^2)$. The constant C_1 is a universal constant, so what we have is that

$$\sup_{f \in \mathcal{F}} |f(z)| \lesssim C_1 + \log(1+|z|^2).$$

where $\mathcal{F}$ is the class of such f. This leaves open the (very desirable) possibility that for each $f \in \mathcal{F}$,

$$|f(z)| \leq C_f + \tfrac{1}{2}\log(1+|z|^2).$$

Here we build an example of such an f which shows that the factor $\frac{1}{2}$ is not achievable. The b.v.'s of f are not Lipschitz, but this is irrelevant – we shall bust the bound along $\mathbb{R}+i$, and failure of Lipschitz only screws things up near $\mathbb{R}$.

2) Fix $C \gg 2$, and define for $x \in \mathbb{R}$

$$f_0(x) = \frac{\pi}{2} \sum_{k \geq 0} I_{(C^{2k}, C^{2k+1}]}(x) - \frac{\pi}{2}\left\{\sum_{k \geq 0} I_{(C^{2k+1}, C^{2k+2}]}(x) + I_{(-\infty,0)}(x)\right\},$$

and set

$$f(z) \equiv \int_{-\infty}^{\infty} f_0(x)\, \frac{dx}{\pi} \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\},$$

so that for $a>0$

$$\mathrm{Re}\, f(a+i) = \int_{-\infty}^{\infty} f_0(x)\, \frac{dx}{\pi} \left\{ \frac{x-a}{1+(x-a)^2} - \frac{x}{1+x^2} \right\} \equiv \int_{-\infty}^{\infty} f_0(x) \frac{dx}{\pi}\, k_a(x),$$

say. Notice that $k_a(x) \leq 0$ iff $|x - a/2| \leq \sqrt{\frac{a^2}{4}+1}$. We shall always suppose $a>0$, and, indeed, we shall take $a = C^{2k+1}$ straight away. There are three important pieces to the integral for $\mathrm{Re}\, f(a+i)$:

(i) $$\int_{-\infty}^{0} f_0(x) \frac{dx}{\pi} k_a(x) = -\frac{1}{4} \log(1+a^2);$$

(ii) $$\int_{C^{2k}}^{C^{2k+1}} f_0(x) \frac{dx}{\pi} k_a(x) = -\frac{1}{4}\left\{\log(1+a^2) - \log(1+C^{4k}) + \log\left(1+(a-C^{2k})^2\right)\right\};$$

(iii) $$\int_{C^{2k+1}}^{C^{2k+2}} f_0(x) \frac{dx}{\pi} k_a(x) = -\frac{1}{4}\left\{\log(1+a^2) + \log\{1+(C^{2k+2}-a)^2\} - \log(1+C^{4k+4})\right\}.$$

Added together, these make up something $-\frac{3}{4}\log(1+a^2) + O(1)$.

The remaining pieces are negligible:

(iv) $$\left|\int_{c^{2k+2}}^{\infty} f_0(x)\,\frac{dx}{\pi}\,k_a(x)\right| \leq \frac{1}{4}\left\{\log\left(1+c^{4k+4}\right) - \log\left(1+\left(c^{2k+2}-c^{2k+1}\right)^2\right)\right\} = O(1)$$

(v) $$\int_0^{c^{2k}} f_0(x)\,\frac{dx}{\pi}\,k_a(x) = \frac{1}{4}\sum_{j=0}^{k-1}\left\{g(c^{2j+1}) - g(c^{2j}) - g(c^{2j+2}) + g(c^{2j+1})\right\}$$

where $g(x) \equiv \log(1+x^2) - \log\left(1+(x-a)^2\right)$.

Now $$g(c^\ell) = \log(1+c^{2\ell}) - \log\left(1+\left(c^{2k+1}-c^\ell\right)^2\right)$$
$$= \log(1+c^{-2\ell}) - (4k+2-2\ell)\log c - \log\left\{\left(1-c^{-2k-1+\ell}\right)^2 + c^{-4k-2}\right],$$

which shows that this last contribution is also $O(1)$. $\square$

It's easy to think of other oscillatory examples where the change-over points b_k grow very rapidly (e.g. $b_k = \exp(k^2)$ seems good) which would get arbitrarily close to the growth rate $\log(1+|z|^2)$, so that really looks like the best we can hope for in general.

<u>Pathwise uniqueness for a stochastic differential equation</u> (27/7/89).

In their treatment of a control problem of Beneš, Ikeda + Watanabe consider the one-dimensional SDE

$$dX_t = 2\sqrt{X_t^+}\,dB_t + \left(1-2\sqrt{X_t^+}\right)dt,$$

and claim that it has pathwise uniqueness. Their proof is wrong, at the point where they apply Itô's formula to the non-C^2 scale function $s(x) = \exp(2\sqrt{x})$.

Nonetheless, pathwise uniqueness does hold for this SDE. The construction they give of maximal and minimal solutions $\overline{X}_t \geq \underline{X}_t$ works OK, though justification of continuity of these processes <u>is</u> needed.

Now fix $\lambda > 0$, let $\alpha, \beta = 1 \pm \sqrt{1+2\lambda}$, and let

$$f(x) \equiv \frac{e^{\alpha x}}{\alpha} - \frac{e^{\beta x}}{\beta} = e^x\left\{\frac{e^{x\sqrt{1+2\lambda}}}{\alpha} - \frac{e^{-x\sqrt{1+2\lambda}}}{\beta}\right\}.$$

This function is increasing in $\mathbb{R}^+$, $f'(0) = 0$, and f solves $\frac{1}{2}f'' - f' - \lambda f = 0$.

The aim is to apply Itô's formula to $f(\sqrt{x})$ – but is this C^2? Notice that

$$f(y) = \sum_{n\geq 0}\left(\frac{(\alpha y)^n}{n!\,\alpha} - \frac{(\beta y)^n}{n!\,\beta}\right)$$

$$= \sum_{n\neq 1}\left(\frac{(\alpha y)^n}{\alpha} - \frac{(\beta y)^n}{\beta}\right)\frac{1}{n!}$$

since $f'(0)=0$. The only term in the expansion of $f(\sqrt{x})$ which could possibly give trouble therefore is the $x^{3/2}$ term – but this is a convex function, to which Itô's formula can be applied. So we get

$$d\, f(\sqrt{X_t}) = \frac{1}{2\sqrt{X_t}} f'(\sqrt{X_t})\, dX_t + \frac{1}{2}\left[\frac{f''(\sqrt{X_t})}{4X_t} - \frac{f'(\sqrt{X_t})}{4X_t^{3/2}}\right] 4X_t\, dt$$

$$= f'(\sqrt{X_t})\, dB_t + \lambda f(X_t)\, dt.$$

Thus $f(\sqrt{X_t})\, e^{-\lambda t}$ is a martingale for each of the solutions $\bar{X}$, $\underline{X}$, and hence $\bar{X} = \underline{X}$ a.s.

Constructing an equivalent martingale measure in discrete time. (8/8/89)

The problem reduces to this: suppose given a prob'y measure μ on $\mathbb{R}^d$ such that

" if $\mu(\{x \mid a^Tx \geq 0\}) = 1$, then $\mu(\{x \mid a^Tx = 0\}) = 1$."

Find a measure $\tilde{\mu}$ equivalent to μ such that $\int x\, \tilde{\mu}(dx) = 0$. Of course, there are many such measures $\tilde{\mu}$; all we want to do is find one, in a fairly natural way.

We lose no generality by assuming

(*) $\quad \forall\, a \in \mathbb{R}^d, \quad \mu(\{x \mid a^Tx > 0\}) > 0.$

(i) Let us firstly assume that the moment generating function

$$M(\theta) \equiv \int e^{\theta x}\, \mu(dx), \qquad \theta \in \mathbb{R}^d$$

is everywhere finite-valued; non-negativity and strict convexity follow easily, using the assumption (*).

One can show also that for some $\lambda > 0$, $\inf_{u \in S^{n-1}} M(\lambda u) > 1$, from which it follows that $M(\theta) \to \infty$ as $|\theta| \to \infty$. [The assumption (*) is used again here, because one knows that for any $u \in S^{n-1}$

$$M(\lambda u) \equiv E\, e^{\lambda u.X} \to \infty \quad \text{as } \lambda \to \infty. \quad]$$

Continuity of M, strict convexity of M, and $M \to \infty$ imply that M has a unique minimum, θ^*, say, where

$$0 = \nabla M(\theta^*) = \int x\, e^{\theta^* x} \mu(dx).$$

Renormalising $e^{\theta^* x}\mu(dx)$ to be a probability solves the problem.

(ii) The rationale behind (i) is that one is <u>minimising the information gain</u> $\int f \log f \, d\mu$ (or whatever this expression is called).

If one tries to

$$\min \int f \log f \, d\mu \qquad \text{subj to } \int f\, d\mu = 1, \quad \int x f \, d\mu = 0,$$

then the Lagrangian is

$$\int f \log f \, d\mu + c\left(1 - \int f \, d\mu\right) - a^T \int x f \, d\mu,$$

which is (naively) minimised at $f(x) = \exp(c - 1 + a \cdot x)$. This suggests the approach via the moment generating function

(iii) In the case where M is not finite-valued, replace $\mu(dx)$ by const. $e^{-|x|^2}\mu(dx) \equiv \nu(dx)$ – because any measure equivalent to μ is certainly equivalent to ν, and the mgf for ν certainly exists.

<u>Last exit times for one-dimensional diffusions</u> 10/8/89.

Let's consider a diff^n on $(0,1]$ in natural scale, speed measure m, instantaneously reflecting at 1, 0 approached but never reached (thus $\int_{0+} x\, m(dx) = +\infty$.)

If now $\sigma_x \equiv \sup\{t > 0 : X_t = x\}$, we know that

$$\boxed{\begin{aligned} & E^x\left[e^{-\lambda \sigma_a}\right] = r_\lambda(x,a)/2a \\ \text{and} \quad & \lim_{\lambda \downarrow 0} r_\lambda(x,a) = 2(x \wedge a) \end{aligned}}$$

(see Thm VI.35.5 in RW)

(see V.50.10 in RW).

Let's find out the first two moments of σ_a.

From the resolvent equation,

$$-\frac{\partial}{\partial\lambda} r_\lambda(x,y) = \int r_\lambda(x,z)\, r_\lambda(z,y)\, m(dz)$$

$$\to 4\int (x\wedge z)(y\wedge z)\; m(dz) \qquad \text{as } \lambda\downarrow 0$$

Differentiating once more gives

$$\frac{\partial^2}{\partial\lambda^2} r_\lambda(x,y) = \int r_\lambda(x,z)\, m(dz) \int r_\lambda(z,u)\, r_\lambda(u,y)\, m(du) \quad + \text{other term}$$

$$= 2\iint m(dz)\, m(du)\; r_\lambda(x,z)\, r_\lambda(z,u)\, r_\lambda(u,y)$$

by symmetry of $r_\lambda(\cdot,\cdot)$. Hence

$$\boxed{\begin{aligned} E^1[\sigma_a] &= \frac{2}{a}\int_0^1 z\,(a\wedge z)\; m(dz) \\ E^1[\sigma_a^2] &= \frac{8}{a}\int_0^1\int_0^1 m(du)\, m(dz)\; z\,(z\wedge u)(u\wedge a) \end{aligned}}$$

These could easily be infinite!

and take $c(x) \equiv E^1 H_x = 2\int_x^1 m((v,1])\,dv$

$$V(x) = \mathrm{var}^1(H_x) = 8\int_x^1 (v-x)\,\bar{m}(v)^2\,dv$$

where $\bar{m}(v) \equiv m((v,1])$.

(i) What limit laws can one obtain for H_x as $x \downarrow 0$?
When the tail σ-field is non-trivial, equivalently, $V(0+) < \infty$, we can say $H_x - c(x) \xrightarrow{a.s.}$ to some non-degenerate limit.
If $V(0+) = +\infty$, then we can consider

$$(H_x - c(x))\,V(x)^{-\frac{1}{2}}$$

and look for a CLT-type result. If $a_n \downarrow 0$, $Y_n \equiv H_{a_n} - H_{a_{n-1}}$, then the Y_n are indept r.v.'s, so some classical version of the CLT should do it for us

(ii) I have been able to build an example where $V(x)/c(x)^2$ gets unbounded as $x \downarrow 0$. Is there a general result to this effect? Or do I need a better example?

(iii) What about limit behaviour for σ_x as $x \downarrow 0$? We have once again that $Z_n \equiv \sigma_{a_n} - \sigma_{a_{n-1}}$ are independent r.v.'s, but now they need not be integrable.
If one were in a situation where the limit laws of H_x and σ_x were similar, one might then be able to deduce something about X_t ...

10. Suppose $X, X' \geq 0$ are i.i.d. and one knows the law of $X - X'$. Can one deduce the law of X?
More generally, what symmetric laws can be represented in the form $X - X'$??
[See Problems Corner of IMS Bulletin

11. Let $f: \mathbb{R}^+ \to \mathbb{R}$, and let $\tau = \inf\{u : B_u > f(u)\}$. It's a hard problem to calculate the law of τ (not soluble in closed form) but if we are given the law of τ, can we deduce f?

12. Integral representation for BM stopped at a previsible time? Stricker, Stochastics 6 73–77 has this (early attempt: Engelbert + Hess Stochastics 4 121–142)

7. <u>A control problem with partial observation</u> ("Degenerate diffusion arising in a control problem with partial observations" by Beneš, Karatzas + Rishel.)

They consider minimising

$$E\left[\int_0^\infty e^{-\alpha s} Y_s^2 \, ds\right]$$

where $Y_t = y + W_t + z\int_0^t u_s\,ds$,

and u is bdd between $[-1,1]$, Z is indept of W, and unknown, $P(Z=1)=p=1-P(Z=-1)$. They get a bang-bang optimal control u^*, which has to be adapted to a filtration <u>bigger</u> than $\mathcal{Y}$! What is going on?!

8. <u>Another control problem of Beneš-Karatzas.</u>

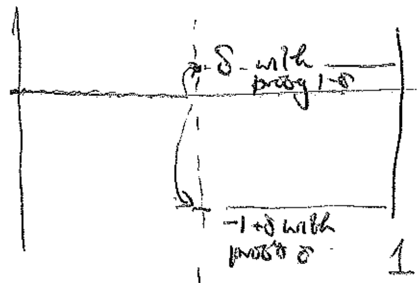
This time, they take a process

$$X_t^u = x + Z_t + \int_0^t u_s\,ds$$

where Z is a given continuous local mg (or perhaps even a semimg) and u is previsible control, $|u|\leq 1$. The aim is to minimise $E\,\varphi(X_T^u)$, where φ is C^2 (say), even, incr in $\mathbb{R}^+$.

Their conjecture is that optimal control (in mg case?) should be based on the predicted miss $R_t^u \equiv X_t^u + E(Z_T - Z_t \mid \mathcal{F}_t)$, and should always be of opposite sign. The example of Gareth Roberts below shows that this can't be correct:

If Z were this mg, it's going to be good to move up a bit before $t=\frac{1}{2}$.



So what form should the optimal control take?

9. <u>Diffusion hitting time limit laws.</u>

Suppose one has a diffusion in natural scale in $(0,1]$, where 1 is reflecting, and m is the speed measure. Let

$$H_x \equiv \inf\{t;\ X_t = x\}, \quad \sigma_x \equiv \sup\{t;\ X_t = x\}$$

Thus exploiting the fact that $\forall v$, $\int p(v,\lambda)d\lambda = 1$, we get that

$$\int_0^\infty z^{1+\theta\eta-2\theta}\, h(z^\theta)\, e^{-\alpha z}\, dz = \frac{\theta c}{\alpha} \exp\left[(1+c)\gamma\left(\frac{\alpha}{c}\right)^\theta\right]$$

and the RHS is not a CM function of α.
Thus such a model is impossible.

(b) Trying to model $\log \lambda_n$ as a r.w. in such a way that the conditional law of $\lambda_n / \mathcal{Y}_n$ is gamma is also futile.

(c) I tried a cts time model where $Y_t \equiv N_t - \int_0^t \lambda_u\, du$ is a martingale, and λ_u is a positive diffusion, $\lambda_u \equiv s(X_u)$.
The conditional law of $X_t / \mathcal{Y}_t$ is a mess; given that there are jumps at times $0 < \tau_1 < \dots < \tau_n < t$, the law of X_0 has density

$$c \exp\{-\int_0^t s(X_u)du\} \prod_1^n s(X_{\tau_k})$$

w.r.t. the law of X unconditioned, and this involves us in finding expressions for

$$E\left\{\varphi(X_t) \exp[-\int_0^t s(X_u)\, du] \prod_1^n s(X_{\tau_k})\right\}$$

for various φ.

6/. A question posed by Mike Harrison. Let D be a polygonal region in the plane, and suppose that we have BM with drift μ (constant) in this region, with reflection on the boundary in direction $a(\cdot)$. What is the invariant law?
This looks quite hard. One thing one can show easily is that

$$0 = \int_D \pi(dz)\, \mathcal{G}f(z) + \int_{\partial D} a(z)\cdot\nabla f(z)\, \underbrace{\gamma(dz)}_{\text{invariant law on } \partial D}$$

But does this help?
Can we get the Poisson kernel for drifting BM in D?
Any other approach?

4/. In the work on diffusion of triangular shapes, there's rotational symmetry of the shape diffusion about the N. Pole. To what does this group action correspond in terms of the vertices?

/. Can one build some nice filtering model where an underlying parameter process λ_n is observed only through $Y_n \sim P(\lambda_n)$? We particularly need that there's a simple updating as in Kalman filter.

(a) One model I've tried is this. Want (λ_n) to be Mkv pr with transition density $p(\cdot,\cdot)$, and want

(i) $Y_n \sim P(\lambda_n)$ given $Y_0, \dots, Y_{n-1}, \lambda_0, \dots, \lambda_n$;

(ii) given $\mathcal{Y}_n = \sigma(Y_0, \dots, Y_n)$, $\lambda_n \sim \Gamma(a_n(Y), b_n(Y))$;

(iii) given $\mathcal{Y}_n$, $\lambda_{n+1} \sim \Gamma(\tilde{a}_{n+1}(Y), \tilde{b}_{n+1}(Y))$.

Now if we want the simple updating

$$a_{n+1} b_{n+1} = (c\, a_n b_n + Y_{n+1}) / (1+c)$$

for the conditional mean $a_n b_n$ of λ_n, one very natural way is to have

$$b_n = \frac{1}{1+c} \,\forall n, \quad a_{n+1} = \theta a_n + Y_{n+1}, \quad \theta \equiv c/(1+c).$$

[This follows from straightforward calculations, using the fact that $\lambda \sim \Gamma(a,b)$ and $Y \sim P(\lambda)$ $\Rightarrow$ $(\lambda / Y = k) \sim \Gamma(a+k, b/(1+b))$]

The natural way to begin is $\lambda_0 \sim \Gamma(\eta, 1/c)$ for some $\eta > 0$. I then worked out that $\forall k$, $\forall \lambda$,

$$\int_0^\infty e^{-(1+c)x} x^{\eta-1} p(x,\lambda)\, x^k \,dx = \frac{\Gamma(k+\eta)}{\Gamma(\theta k + \theta\eta)} \left(\frac{\lambda}{c}\right)^{\theta(\eta+k)} e^{-c\lambda} (1+c)^{-k-\eta}.$$

This rearranges to

$$\int_0^\infty v^k \,dv\; e^{-\kappa(1+c)v} v^{\eta-1} p(\kappa v, \lambda) = \frac{\Gamma(k+\eta)}{\Gamma(\theta k + \theta\eta)} e^{-c\lambda} \qquad (\kappa \equiv \lambda^\theta \gamma)$$

with $\gamma = c^{-\theta}(1+c)^{-1}$. So, fixing λ, we ask whether the RHS is a moment sequence. If it were the moment sequence of some f^n h, then

$$p(v,\lambda) = h\left(\frac{v}{\lambda^\theta \gamma}\right) \exp\left[-c\lambda + (1+c)v\right] \left(\frac{v}{\lambda^\theta \gamma}\right)^{1-\eta}.$$

PROBLEMS, SILLY EXAMPLES, ETC...

1/ Does every Dirichlet process on $\mathbb{R}$ with cts sample paths have a local time? This seems a hard question. Various ideas occur for constructing bad examples.

$$dX_t^n = dB_t - n \sin nX_t^n \, dt$$

looks like it should concentrate around the points $2\pi k/n$, $k \in \mathbb{Z}$. Are the laws of these things tight? If so, is (some) limit law the law of a nice Dirichlet process? If so, does this break the conjecture?

The scale function f_n satisfies

$$e^{-2} \le f_n'(x) = \exp(2\cos nx) \le e^2$$

which is very well behaved, so that $Y^n \equiv f_n(X^n)$ is a diffusion in natural scale with a C^∞ variance bounded between e^{-4} and e^4. The laws of $\{Y^n\}$ are tight, but can one deduce anything about limits...? (3/8/87)

[I seem to recall a paper of Stroock + Varadhan in a Berkeley Symposium looked at something like this...]

2/ Consider one-dimensional SDE $dX_t = \sigma(t, X_t)\, dB_t$, where σ is continuous in x uniformly in t, and

$$1 \ge \sigma(t,x) \ge b(x) \qquad \forall x, t,$$

where

$$\int_{-n}^{n} b(x)^{-1} dx < \infty \qquad \text{for all } n.$$

Does uniqueness in law hold? (I think this conjecture is due to Martin)

3/ Suppose X is a Dirichlet process with cts paths, $\varepsilon \equiv \lambda^{-1} > 0$. Let

$$X_t^\varepsilon \equiv E\left[\int_0^\infty \lambda e^{-\lambda u} X_{t+u}\, du \;\middle|\; \mathcal{F}_t\right].$$

Now $e^{-\lambda t} X_t^\varepsilon$ is the optional projection of the decr. process $\int_t^\infty \lambda e^{-\lambda u} X_u \, du$, and is therefore a supermg (assume wlog that X is bounded).

Thus X^ε is a semimartingale, and $X_t^\varepsilon \to X_t$ as $\varepsilon \downarrow 0$.

Does this help in the analysis of Dirichlet processes?