More on stochastic intensities (8/6/94)

(i) Suppose that we have some (reversible) Markov process X, and we use non-negative functions f_i of X to provide stochastic intensities of independent Poisson point-processes. So, more precisely, if $\tilde{N}^1, \dots, \tilde{N}^k$ are standard Poisson counting processes, we let $N_t^i \equiv \tilde{N}^i\left(\int_0^t f_i(X_s)\,ds\right)$. If we define

$$Y_i \equiv \int_{-\infty}^0 e^{\theta_i t}\, dN_t^i$$

what can we say about the first two moments of the Y_i? We have

$$EY_i = E\int_{-\infty}^0 e^{\theta_i t} f_i(X_t)\,dt = E\int_0^\infty e^{-\theta_i t} f_i(X_t)\,dt \qquad \text{(by reversibility)}$$

$$= R_{\theta_i} f_i(X_0).$$

(Actually, reversibility isn't really needed). Further, for i and j distinct,

$$E\, Y_i Y_j = E\int_{-\infty}^0 du \int_{-\infty}^0 dv\; e^{\theta_i u + \theta_j v} f_i(X_u) f_j(X_v)$$

$$= E\int_0^\infty du \int_0^\infty dv\; e^{-\theta_i u - \theta_j v} f_i(X_u) f_j(X_v)$$

$$= E\int_0^\infty du\; e^{-(\theta_i+\theta_j)u} f_i(X_u)\, R_{\theta_j} f_j(X_u) \quad + \quad \text{other term}$$

$$= R_{\theta_i+\theta_j}\left(f_i R_{\theta_j} f_j + f_j R_{\theta_i} f_i\right)(X_0).$$

If we further assume the existence of a stationary distribution π, and that $X_0 \sim \pi$, we get

$$\boxed{\begin{aligned} EY_i &= \frac{1}{\theta_i}(\pi, f_i) \\ EY_iY_j &= \frac{1}{\theta_i+\theta_j}\left(\pi, f_i R_{\theta_j} f_j + f_j R_{\theta_i} f_i\right)\end{aligned}}$$

(ii) Let's suppose as a special example that the underlying Mkv pr is an OU process in d dimensions,

$$dX_t = dW_t - \beta X_t\, dt$$

$$= a_j \left(\mu_i^T Q_i \mu_i + \frac{\mathrm{tr}\, Q_i}{2\beta} \right) + \frac{2}{\beta}\, b_j^T Q_i \mu_i + \frac{\mu_i^T Q_i \mu_i}{\theta_j + 2\beta}\, \frac{\mathrm{tr}\, Q_j}{2\beta}$$

$$+ \frac{1}{4\beta^2} \left(\mathrm{tr}\, Q_i \,\mathrm{tr}\, Q_j + 2\,\mathrm{tr}\, Q_i Q_j \right) (\theta_j + 2\beta)^{-1}$$

$$= \frac{1}{\theta_j} \left(\alpha_j + \frac{t_j}{\theta_j + 2\beta} \right) \left(\alpha_i + \frac{t_i}{2\beta} \right) + \frac{2}{\beta(\theta_j + \beta)}\, \mu_j^T \theta_j Q_i \mu_i + \frac{\alpha_i t_j}{2\beta(\theta_j + 2\beta)}$$

$$+ \frac{1}{4\beta^2} \left[t_i t_j + 2\,\mathrm{tr}\, Q_i Q_j \right] (\theta_j + 2\beta)^{-1}$$

Where $\alpha_i \equiv \mu_i^T Q_i \mu_i$, $t_i \equiv \mathrm{tr}\,(Q_i)$

If all the θ_j are equal to θ, when we calculate $(\pi, f_i R_\theta f_j + f_j R_\theta f_i)$ we get

$$\frac{1}{\theta} \left\{ 2\alpha_i \alpha_j + \frac{t_i t_j}{\beta(\theta + 2\beta)} + \frac{\alpha_i t_j + \alpha_j t_i}{2\beta(\theta + 2\beta)} (\theta + 4\beta) \right\} + \frac{4}{\beta}\, \frac{\mu_j^T Q_j Q_i \mu_i}{\theta + \beta} + \frac{\alpha_i t_j + \alpha_j t_i}{2\beta(\theta + 2\beta)} + \frac{1}{2\beta^2} \left(\frac{t_i t_j + 2\,\mathrm{tr}\, Q_i Q_j}{\theta + 2\beta} \right)$$

whose invariant law is $N(0,(2\beta)^{-1}I)$, and let's take $f_i(x) = (x-\mu_i)^T Q_i (x-\mu_i)$ for some non-negative definite Q_i, $\mu_i \in \mathbb{R}^d$.

Then we have

$$(\pi, f_i) = (2\beta)^{-1} \operatorname{tr} Q_i + \mu_i^T Q_i \mu_i$$

and

$$P_t f(x) = (\mu - x e^{-\beta t})^T Q (\mu - x e^{-\beta t}) + \operatorname{tr}(Q) \frac{1-e^{-2\beta t}}{2\beta},$$

dropping the subscripts for now. Hence

$$R_\lambda f(x) = \frac{1}{\lambda}\mu^T Q \mu - 2x^T Q\mu \cdot \frac{1}{\lambda+\beta} + x^T Q x \frac{1}{\lambda+2\beta} + \operatorname{tr}(Q)\left\{\frac{1}{\lambda} - \frac{1}{\lambda+2\beta}\right\}\cdot\frac{1}{2\beta}$$

$$= \frac{\mu^T Q\mu}{\lambda} - \frac{2x^T Q\mu}{\lambda+\beta} + \frac{x^T Q x}{\lambda+2\beta} + \frac{\operatorname{tr}(Q)}{\lambda(\lambda+2\beta)}$$

$$= \left\{\frac{\mu^T Q \mu}{\lambda} + \frac{\operatorname{tr}(Q)}{\lambda(\lambda+2\beta)}\right\} - \frac{2\mu^T Q x}{\lambda+\beta} + \frac{x^T Q x}{\lambda+2\beta}$$

$$\equiv a - 2b^T x + \frac{x^T Q x}{\lambda+2\beta}, \text{ for short.}$$

Hence

$$(\pi, f_i R_{\theta_j} f_j) = \left(\pi, (x-\mu_i)^T Q_i (x-\mu_i)\left\{a_j - 2b_j^T x + \frac{x^T Q_j x}{\theta_j + 2\beta}\right\}\right)$$

$$= \frac{a_j}{2\beta}\operatorname{tr}(Q_i) + \frac{4}{2\beta}\operatorname{tr}(Q_i \mu_i b_j^T) + \mu_i^T Q_i \mu_i \left(a_j + \frac{\operatorname{tr}(Q_j)}{(\theta_j+2\beta)\,2\beta}\right) + \text{last term}$$

$$= \frac{a_j \operatorname{tr}(Q_i)}{2\beta} + \frac{2}{\beta} b_j^T Q_i \mu_i + \mu_i^T Q_i \mu_i \left(a_j + \frac{\operatorname{tr}(Q_j)}{2\beta(\theta_j+2\beta)}\right)$$

$$+ 3\sum Q_i(r,r) Q_j(r,r)\, (2\beta)^{-2}(\theta_j+2\beta)^{-1}$$

$$+ (2\beta)^{-2} \sum_{r \neq p}\sum \left[Q_i(r,r)Q_j(p,p) + 2Q_i(r,p)Q_j(r,p)\right](\theta_j+2\beta)^{-1}.$$

Making the simplifying assumption

$$Q_i = q_i I \quad \text{for all } i$$

we find that the expressions above reduce somewhat; we obtain

Should be $\quad (c_i \equiv |\mu_i|^2)$

$$q_i q_j \left[\left\{ \frac{d}{2\beta} \left(|\mu_i|^2 + |\mu_j|^2 \right) + c_i c_j \right\} \left(\frac{1}{\theta_i} + \frac{1}{\theta_j} \right) + \frac{d^2}{2\beta} \left(\frac{1}{\theta_j(\theta_j+2\beta)} + \frac{1}{\theta_i(\theta_i+2\beta)} \right) + \frac{d(d+2)}{2\beta^2} + \frac{2}{\beta} \mu_i^T \mu_j \left(\frac{1}{\theta_i+\beta} + \frac{1}{\theta_j+\beta} \right) \right]$$

$$\frac{a_j q_i}{2\beta} d + \frac{2q_i}{\beta} b_j^T \mu_i + q_i |\mu_i|^2 \left(a_j + \frac{q_j d}{2\beta/(\theta_j + 2\beta)} \right)$$

$$+ \frac{3d}{4\beta^2} \frac{q_i q_j}{\theta_j + 2\beta} + \frac{d(d-1)}{4\beta^2} \frac{q_i q_j}{\theta_j + 2\beta} ,$$

with

$$a_j = \frac{q_j |\mu_j|^2}{\theta_j} + \frac{q_j d}{\theta_j(\theta_j + 2\beta)} , \quad b_j = \frac{q_j \mu_j}{\theta_j + \beta}$$

After further calculations, I get that

$$(\pi, f_i R_{\theta_j} f_j + f_j R_{\theta_i} f_i) = q_i q_j \Bigg[\Big\{ \frac{d}{2\beta} \left(|\mu_i|^2 + |\mu_j|^2 \right) + c_i c_j \Big\} \left(\frac{1}{\theta_i} + \frac{1}{\theta_j} \right) + \frac{d^2}{2\beta} \left\{ \frac{1}{\theta_j(2\beta + \theta_j)} + \frac{1}{\theta_i(2\beta + \theta_i)} \right\}$$

$$+ \frac{d(d+2)}{2\beta^2(\theta_j + 2\beta)} + \frac{2}{\beta} \mu_i^T \mu_j \left(\frac{1}{\theta_i + \beta} + \frac{1}{\theta_j + \beta} \right) \Bigg]$$

Taking $\theta_i = \theta$ for all i simplifies this to

$$q_i q_j \left[\frac{d}{\beta\theta} \left(|\mu_i^2| + |\mu_j|^2 \right) + \frac{d^2}{\beta\theta(2\beta + \theta)} + \frac{d(d+2)}{(\theta + 2\beta) 2\beta^2} + \frac{4}{\beta(\theta + \beta)} \mu_i^T \mu_j \right]$$

$$+ 2|\mu_i|^2 |\mu_j|^2 / \theta$$

(iii) Suppose that the Markov process $X_t \equiv (X_1(t), \dots, X_k(t))^T$ solves

$$\boxed{dX_j = \sigma_j \sqrt{X_j}\, dW_t^j + (\alpha_j + (AX)_j)\, dt} \qquad (j = 1, \dots, k)$$

where $\alpha_j > 0$ for all j, the W^j are independent BMs, A is a matrix for which $a_{ij} \geq 0$ $(i \neq j)$ $a_{ii} < 0$ for all i, and A is irreducible. Suppose also that $f_i(x) = x_i$.
Then easily

$$\boxed{(\pi, x) = -A^{-1} \alpha}$$

[If $\gamma = \max_i(-a_{ii})$, then $A + \gamma I$ is a non-negative matrix with largest e-value $\lambda > 0$ and corresponding e-vector $u \gg 0$. Thus if $U \equiv \mathrm{diag}(u_i)$

$Q \equiv U^{-1}(A + \gamma - \lambda) U$ is a Q-matrix, and if $\lambda < \gamma$ then spectrum of $A \subset \{z : \mathrm{Re}\, z < 0\}$, so A^{-1} exists, and, indeed, $-A^{-1}$ contains only non-negative entries.

Now

$$R_{\theta_j} f_j = g_j$$

where

$$\theta_j g_j(x) - \tfrac{1}{2}\sigma_i^2 x_i D_i^2 g_j(x) - (\alpha + Ax)_i D_i g_j(x) = x_j$$

so if we try

$$\boxed{g_j(x) = a_j + b_j \cdot x,}$$

we find a solution

$$\theta_j a_j = \alpha \cdot b_j \quad , \quad \theta_j b_j - A^T b_j = e_j$$

so

$$\boxed{\begin{aligned} b_j &= (\theta_j - A^T)^{-1} e_j \\ a_j &= \frac{1}{\theta_j}\left((\theta_j - A)^{-1} \alpha \right)_j \end{aligned}}$$

So in order to calculate $(\pi, f_i R_{\theta_j} f_j + f_j R_{\theta_i} f_i)$, we shall need to have some expression for

$$m_{ij} \equiv (\pi, x_i x_j).$$

Using $f(x) = x_j^2$, we find

$$0 = (\pi, \mathcal{G}f) = \left(\pi, \tfrac{1}{2}\sigma_j^2 \cdot 2x_j + 2(\alpha_j + (Ax)_j) x_j\right)$$

$$\therefore \quad 2\sum_\ell a_{j\ell}\, m_{j\ell} = -(\pi, x_j)(\sigma_j^2 + 2\alpha_j).$$

Using $f(x) = x_i x_j$ gives (for $i \neq j$, of course)

$$0 = (\pi, (\alpha_j + (Ax)_j) x_i + (\alpha_i + (Ax)_i) x_j)$$

$$\therefore \quad \sum_\ell (a_{j\ell}\, m_{\ell i} + a_{i\ell}\, m_{\ell j}) = -(\pi, \alpha_j x_i) - (\pi, \alpha_i x_j)$$

$$\therefore \quad \boxed{AM + MA^T = \alpha (A^{-1}\alpha)^T + (A^{-1}\alpha)\alpha^T - \Delta}$$

where $\Delta = -\mathrm{diag}\left(\sigma_j^2 (A^{-1}\alpha)_j\right)$. Hence a few calculations give

$$\boxed{\mathrm{Cov}_\pi(X, X) = \int_0^\infty e^{tA} \Delta e^{tA^T} dt.}$$

(See also p 23)

Our particular interest is in

$$(\pi,\ f_i R_{\theta_j} f_j + f_j R_{\theta_i} f_i) = (\pi,\ x_i\,(a_j + b_j\cdot x)) + \text{similar}$$

$$= (\pi,\ x_i\,(a_j - b_j^T A^{-1}\alpha + b_j\cdot(x + A^{-1}\alpha)))\ + \text{similar}$$

$$= (\pi, x_i)\ \frac{1}{\theta_j}\,(-e_j^T A^{-1}\alpha) + \sum_\ell b_{j\ell}\, V_{i\ell} \qquad + \text{similar},$$

after some calculations. Here, $\boxed{V \equiv \text{cov}_\pi(X,X) = \int_0^\infty e^{tA}\,\Delta\, e^{tA^T}\,dt.}$

$(AV + VA^T = \Delta)$

This reduces to

$$(-e_i^T A^{-1}\alpha)(-e_j^T A^{-1}\alpha)\left(\frac{1}{\theta_i} + \frac{1}{\theta_j}\right) + e_i^T\, V(\theta_j - A^T)^{-1} e_j + e_j^T\, V(\theta_i - A^T)^{-1} e_i$$

Hence finally

$$\boxed{\text{cov}(Y_i, Y_j) = \frac{1}{\theta_i + \theta_j}\left\{ e_i^T\, V(\theta_j - A^T)^{-1} e_j + e_j^T\, V(\theta_i - A^T)^{-1} e_i \right\}}$$

$$\boxed{E Y_i = \frac{1}{\theta_i}\,(-e_i^T A^{-1}\alpha)}$$

Suppose we required $\theta_i = \theta\ \forall i$, and that we sought symmetric A to solve this. If we also demanded that $\Delta = -\gamma I$, we'd have that $V = -\frac{1}{2}\gamma A^{-1}$, and

$$\text{cov}(Y,Y) = -\frac{\gamma}{\cancel{2}\theta}\ A^{-1}(\theta - A)^{-1}$$

Could a general covariance matrix be represented this way with A a Q-matrix? No – if so, we could rotate LHS and this would simply rotate the solution A (assumed unique) – but a rotated Q-matrix RAR^T needn't be a Q-matrix.

Recovering utility from a single path (21/6/94)

(i) Suppose we've got a complete Brownian market

$$dS_t^i = S_t^i \Big[\sum_{j=1}^d \sigma_{ij}\, dW_t^j + \mu^i dt \Big] \qquad (i=1,\dots,d)$$

$$= S_t^i \Big[\sum_{j=1}^d \sigma_{ij}\, d\tilde{W}_t^j + r\,dt \Big] \qquad \text{in risk-neutral prob}^{ty}\ \tilde{P}$$

where we'll assume σ, μ, r are fairly general adapted processes, σ always invertible.

Phil Dybvig asks "Suppose we see the wealth process $(X_t^*)_{0\le t\le T}$ of an investor investing so as to max $E\,U(X_T)$ – can we (from that one sample path) decide what U is?"

General theory says

$$X_T^* = I(\lambda \zeta_T)$$

where $\zeta_T = \exp(-\int_0^T r_u du)\, Z_T$, and $Z_T = (d\tilde{P}/dP)\big|_{\mathcal{F}_T}$, and λ is chosen to match up

$$X_0 = E\big(\zeta_T I(\lambda\zeta_T)\big).$$

Also,

$$e^{-rt} X_t^* = \tilde{E}_t\Big[e^{-rT} I(\lambda\zeta_T)\Big] \qquad (0\le t\le T).$$

(ii) Special case of σ, μ, r all constant.

In this case,

$$Z_t = \exp\Big[\, b\cdot W_t - \tfrac{1}{2}|b|^2 t \,\Big] \qquad (b \equiv \sigma^{-1}(r\mathbf{1} - \mu))$$

$$= \exp\Big[\, b\cdot \tilde{W}_t + \tfrac{1}{2}|b|^2 t\,\Big]$$

So if $c \equiv |b|$, $\beta_t = b\cdot\tilde{W}_t / c$ is a standard $\tilde{P}$-Brownian motion, we have

$$e^{-rt} X_t^* = e^{-rT}\, \tilde{E}_t\, f(\beta_T)$$

where $f(x) \equiv I\left(\lambda e^{-rT} \exp\left(cx + \frac{1}{2}c^2 T\right)\right)$. Hence

$$e^{-rt} X_t^* = e^{-rT} P_{T-t} f(\beta_t)$$

where $(P_t)_{t \geq 0}$ denotes the Brownian transition semigroup:

$$(P_t h)(x) \equiv \int_{-\infty}^{\infty} e^{-y^2/2t} \frac{dy}{\sqrt{2\pi t}} h(x+y).$$

We thus have by Itô's formula or whatever that

$$e^{-rt} X_t^* = X_0 + \int_0^t e^{-rT} (P_{T-s} f)'(\beta_s)\, d\beta_s .$$

Now it is known (see, for example, Revuz + Yor, p 136, ex. 2.17) that if H is a continuous adapted process then for any t

$$\frac{1}{\beta_{t+h} - \beta_t} \int_t^{t+h} H_s\, d\beta_s \xrightarrow[(h \downarrow 0)]{p} H_t$$

Hence, knowing X^* and β as we assume, we can deduce for any t

$$(P_{T-t} f)(\beta_t) \quad , (P_{T-t} f)'(\beta_t)$$

and by a similar argument we can deduce $\frac{\partial^n}{\partial x^n}(P_{T-t} f)(\beta_t)$ for any n.

Now since

$$(P_t h)(x) = \int e^{-(x-y)^2/2t} \frac{dy}{\sqrt{2\pi t}} h(y)$$

$$= e^{-x^2/2t} \int e^{xy/t} \frac{dy}{\sqrt{2\pi t}} e^{-y^2/2t} h(y)$$

if we know $\left(\frac{\partial^n}{\partial x^n} P_t h\right)(x)$ for some fixed x, t, for every n, then we can deduce $\frac{\partial^n}{\partial x^n}\left(\int e^{xy/t} \frac{dy}{\sqrt{2\pi t}} e^{-y^2/2t} h(y)\right)$ for that fixed t, x, for every n. Since

$$z \mapsto \int e^{zy/t}\, e^{-y^2/2t} \frac{dy}{\sqrt{2\pi t}}\, h(y)$$

is analytic (at least on its domain of finiteness) if we know all of its derivatives at a point in its domain of definition then we know it everywhere. Hence we could know the function

$$z \mapsto \int e^{zy/(T-t) - y^2/2(T-t)} \frac{f(y)\,dy}{\sqrt{2\pi(T-t)}}$$

from which we could deduce f.

(iii) In general, if $dX_t = \sigma(X_t)\, dB_t$ is a one-dimensional diffusion, and we see

$$(D^n P_{T-t} f)(X_t) \qquad \text{for all } n, \text{ for some } 0<t<T,$$

can we deduce what f is?

If $\sigma(x) = a\, I_{(-\infty,0]}(x) + b\, I_{(0,\infty)}(x)$, $0<a<b$,

then $(P_t f)$ will not be smooth for typical smooth f ...

Foreign exchange and term structure (13/7/94)

1) Imagine a (risk-neutral) world, where assets are traded; $(S(t))_{t\geq 0}$ will be a generic traded asset price process, which will be a P-martingale. Let S_j be the price (in terms of some global commodity) of rolled over investment in country j, and let $(r_j(t))_{t\geq 0}$ be the riskless interest rate in country j, $R_j(t) \equiv \int_0^t r_j(s)\,ds$. If P_j is the/an EMM as perceived in country j, $dP_j/dP \equiv \zeta_j$, then

$$\frac{S}{S_j} \text{ is a } P_j\text{-mg} \quad \therefore \quad \frac{S}{S_j}\zeta_j \text{ is a } P\text{-mg}$$

and so $\zeta_j = S_j N_j$, where N_j is a P-mg orthogonal to each P-mg S.

Now let

$$Y_{ij}(t) \equiv \text{price at time } t \text{ of } 1 \text{ unit of currency } j \text{ in terms of currency } i.$$

We have

$$\frac{S}{S_j} e^{R_j} \cdot Y_{ij} = \text{value of } S \text{ in terms of currency } i = \frac{S}{S_i} e^{R_i}$$

from which

$$\boxed{Y_{ij} = \frac{S_j\, e^{-R_j}}{S_i\, e^{-R_i}}}.$$

2) Can we get anywhere further if we now assume MARKETS COMPLETE? Then

$$\frac{dP_j}{dP} = S_j(t)/S_j(0) \qquad \text{on } \mathcal{F}_t$$

(a) A simple model would take (in $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$)

$$dr_i = \sigma_i \sqrt{r_i}\, dW_i + (\alpha_i - \beta_i r_i)\,dt$$

with the W_i indept P-Brownian motions, and now

$$dS_i/S_i = -\sigma_i\gamma_i \sqrt{r_i}\, dW_i,$$

say. This gives $S_i(t) = S_i(0)\exp\left[-\gamma_i(r_i(t) - r_i(0)) + \gamma_i\alpha_i t - \int_0^t \left(\tfrac{1}{2}\gamma_i^2\sigma_i + \gamma_i\beta_i\right) r_i(s)\,ds\right]$

so that

$$S_j(t)\, e^{-R_j(t)} = S_j(0) \exp\left[-\gamma_j (r_j(t) - r_j(0)) + \alpha_j \gamma_j t - \left(\tfrac{1}{2}\gamma_j^2 \sigma_j^2 + \gamma_j \beta_j + 1\right) R_j(t) \right].$$

Changing measure to P_i alters the SDE for r_i by $\beta_i \mapsto \beta_i + \sigma_i^2 \gamma_i$, but doesn't change the other r_j. This allows us to find expressions for derivatives etc.

(b) Try an FX option. In country i, a call option on currency j with maturity T will have fair price at time $t < T$ of

$$E_i\left[(Y_{ij}(T) - K)^+ e^{-R_i(T) + R_i(t)} \,\middle|\, \mathcal{F}_t \right]$$

$$= e^{R_i(t)} E\left[S_i(T)/S_i(t) \cdot e^{-R_i(T)} (Y_{ij}(T) - K)^+ \,\middle|\, \mathcal{F}_t \right]$$

$$= \left\{ e^{R_i(t)}/S_i(t) \right\} E\left[\left(e^{-R_j(T)} S_j(T) - K S_i(T) e^{-R_i(T)} \right)^+ \,\middle|\, \mathcal{F}_t \right].$$

If we had the r_i indept*, and the S_i, we could use

$$E(X-Y)^+ = \iint F(dx)\, G(dy)\, (x-y)^+ = \int G(dy) \int_y^\infty F(dx) \int_y^x dt = \int G(t) \bar{F}(t)\, dt$$

to evaluate if we knew enough about the laws of the component parts.

(c) As a trivial calculation, a fair price to pay in country i for 1 unit of currency j at time T would be

$$E_i\left[Y_{ij}(T) \cdot e^{-R_i(T)} \,\middle|\, \mathcal{F}_0 \right] = E_i\left[\frac{S_j(T)}{S_i(T)} e^{-R_j(T)} \,\middle|\, \mathcal{F}_0 \right]$$

$$= \frac{S_j(0)}{S_i(0)} E_j\left[e^{-R_j(T)} \,\middle|\, \mathcal{F}_0 \right]$$

$$= Y_{ij}(0)\, P_j(0, T)$$

which is obvious in other ways!!

* Clearly a totally unrealistic assumption; interest rate movements in different countries are typically highly correlated.

If $S_t \equiv t\bar{X}_t \equiv \int_0^t X_u \, du$, we find that

$$\boxed{V_t = \delta I + \int_0^t \left(X_u - \frac{S_u}{\varepsilon + u} \right) \left(X_u - \frac{S_u}{\varepsilon + u} \right)^T du}$$

$$\begin{aligned}
\hat{B}_t V_t &= \int_0^t dW_s \, X_s^T + a t \bar{X}_t^T + B \int_0^t X_s X_s^T ds - X_t \bar{X}_t^T + \frac{\varepsilon}{\varepsilon + t} X_t \bar{X}_t^T \\
&= \int_0^t dW_s \, X_s^T + B \int_0^t X_s X_s^T ds - \left(B \int_0^t X_s ds + W_t \right) \bar{X}_t^T + \frac{\varepsilon}{\varepsilon + t} X_t \bar{X}_t^T \\
&= B \int_0^t (X_s - \bar{X}_s)(X_s - \bar{X}_s)^T ds + \int_0^t dW_s \, X_s^T - W_t \bar{X}_t^T + \frac{\varepsilon}{\varepsilon + t} X_t \bar{X}_t^T
\end{aligned}$$

If B is an $n \times n$ matrix, $\beta \equiv \sup \{ \mathrm{Re}\, \lambda : \lambda \in Sp(B) \}$, $\tilde{\beta} = \sup \{ \lambda : \lambda \in Sp(\frac{1}{2}(B + B^T)) \}$, then it is not hard to prove that $\tilde{\beta} \geq \beta$. In general, the inequality is strict, as we see by taking

$$B = \begin{pmatrix} a & \theta \\ -\theta & b \end{pmatrix}, \qquad \theta^2 > \left(\frac{a+b}{2} \right)^2.$$

Bayesian estimation in linear diffusions (25/7/94)

(i) If we take

$$dX_t = (a + BX_t)\,dt + dW_t,$$

where a and B are random variables with prior density $\propto \exp(-\frac{1}{2}\varepsilon|a|^2 - \frac{1}{2}\delta\, tr(BB^T))$ then the likelihood conditional on $\mathcal{Y}_t$ is

$$\exp\left\{-\tfrac{1}{2}\varepsilon|a|^2 - \tfrac{1}{2}\delta\, tr(BB^T) + \int_0^t (a + BX_u)\,dX_u - \tfrac{1}{2}\int_0^t |a+BX_u|^2\,du\right\},$$

with respect to $da \times dB \times d\mathbb{W}$.

Now the posterior means $\hat{a}_t$, $\hat{B}_t$ are given by

$$\boxed{\begin{aligned} \hat{a}_t &= \frac{1}{\varepsilon+t}\left(X_t - \int_0^t \hat{B}_t X_u\,du\right) \equiv \frac{X_t - t\hat{B}_t\bar{X}_t}{\varepsilon+t} \\ \hat{B}_t &= \left\{\int_0^t dX_s\, X_s^T - \frac{t}{\varepsilon+t} X_t \bar{X}_t^T\right\} V_t^{-1} \\ V_t &\equiv \int_0^t (X_u - \bar{X}_u)(X_u - \bar{X}_u)^T du + \frac{\varepsilon t}{\varepsilon+t}\bar{X}_t\bar{X}_t^T + \delta I \end{aligned}}$$

Writing $\alpha \equiv a - \hat{a}_t$, $\beta \equiv B - \hat{B}_t$, we get after some calculations that the posterior likelihood for (a, B) is proportional to

$$\boxed{-\tfrac{1}{2}(t+\varepsilon)\,|\alpha|^2 - \alpha^T\beta\, t\bar{X} - \tfrac{1}{2}\delta\, tr(\beta\beta^T) - \tfrac{1}{2}\int_0^t |\beta X_u|^2\,du}$$

(ii) Can we compute the conditional variance from this? The above expression is

$$-\tfrac{1}{2}(t+\varepsilon)\,\alpha_i\alpha_i - \alpha_i\beta_{ij}w_j - \tfrac{1}{2}\beta_{ij}M_{jk}\beta_{ik}$$

where $w_j \equiv t\bar{X}^j$ and $M_{jk} \equiv \int_0^t X_u^j X_u^k\,du + \delta . I_{\{i=j\}}$ and, if we think of a big composite vector $\xi \equiv (\alpha_1,\dots,\alpha_d, \beta_{11}, \beta_{12},\dots,\beta_{1d},\beta_{21},\dots,\beta_{dd})^T$ then the quadratic form is $\xi^T J \xi$, where

$$J = \left(\begin{array}{cccc|cccc} t+\varepsilon & & & & W^T & & & \\ & t+\varepsilon & & & & W^T & & \\ & & \ddots & & & & \ddots & \\ & & & t+\varepsilon & & & & W^T \\ \hline W & & & & M & & & \\ & W & & & & M & & \\ & & \ddots & & & & \ddots & \\ & & & W & & & & M \end{array}\right) \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and we can express the inverse of J as

$$J^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)BD^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

and

$$(A - BD^{-1}C)^{-1} = \left\{ (t+\varepsilon)I - W^T M^{-1} W \right\}^{-1}$$

$$(D - CA^{-1}B)^{-1} = \begin{pmatrix} M - \frac{WW^T}{t+\varepsilon} & & \\ & \ddots & \\ & & M - \frac{WW^T}{t+\varepsilon} \end{pmatrix}^{-1} = \begin{pmatrix} V_t & & \\ & \ddots & \\ & & V_t \end{pmatrix}^{-1}.$$

(iii) <u>An instructive example.</u> Take one dimension, $b > 0$, and

$$dX_t = (a + bX_t)\,dt + dW_t$$

so that
$$e^{-bt} X_t = a\,\frac{1 - e^{-bt}}{b} + \int_0^t e^{-bu}\,dW_u \;\rightarrow\; \frac{a}{b} + \int_0^\infty e^{-bu}\,dW_u .$$

Thus, although $\hat{a}_t$ <u>does</u> converge a.s., the limit is $E(a \mid \mathcal{F}_\infty) \neq a$!!?

We can develop this a bit more:

$$\frac{d}{dt}\left(e^{-bt} \int_0^t X_s\,ds\right) = -be^{-bt} \int_0^t X_s\,ds + e^{-bt} X_t = e^{-bt}(at + W_t)$$

so
$$e^{-bt} \int_0^t X_s\,ds = \frac{a}{b^2} e^{-bt}(e^{bt} - 1 - bt) - \frac{e^{-bt}}{b} W_t + b^{-1} \int_0^t e^{-bs}\,dW_s$$

$\frac{1}{b}\tilde{X}_\infty$

$$e^{-bt}\int_0^t X_s\,ds \;\to\; \cancel{\frac{a}{b^2}} - b\int_0^\infty e^{-bu}\,dW_u$$

$$= b^{-1}\left(\tilde{X}_t - e^{-bt} W_t - at\, e^{-bt}\right)$$

$$\cancel{= -e^{-bt} W_t - b\tilde{X}_t + \frac{a}{b^2} e^{-bt}(e^{bt} - 1 - bt) + a(1 - e^{-bt})}$$

where $\tilde{X}_t \equiv e^{-bt} X_t$. Similarly,

$$b \cdot \int_0^t X_s^2\, ds = \int_0^t X_s \left(dX_s - a\, ds - dW_s\right)$$

$$= \tfrac{1}{2}(X_t^2 - t) - a\int_0^t X_s\, ds - \int_0^t X_s\, dW_s$$

so that

$$e^{-2bt}\int_0^t X_s^2\, ds = \frac{1}{2b}\left(e^{-2bt} X_t^2 - t e^{-2bt}\right) - \frac{a e^{-bt}}{b} \cdot e^{-bt}\int_0^t X_s\, ds - \frac{e^{-2bt}}{b}\int_0^t X_s\, dW_s$$

$$\to \frac{1}{2b}\left(\lim e^{-bt} X_t\right)^2 \qquad (t \to \infty).$$

Hence we conclude that

$$\boxed{e^{-2bt} V_t \xrightarrow{a.s.} \frac{1}{2b}\left(\lim_{t\to\infty} e^{-bt} X_t\right)^2}$$

and now it is easy to deduce that

$$\boxed{\hat{b}_t \xrightarrow{a.s.} b} \qquad (t \to \infty).$$

(iv) To learn more about this, let's consider the difference between the true values and the estimates;

$$\hat{a}_t - a = \frac{X_t - t\hat{B}_t \bar{X}_t}{t + \varepsilon} - a = \frac{at + W_t}{t + \varepsilon} - \frac{(\hat{B}_t - B)t\bar{X}_t}{t + \varepsilon} - a$$

so

$$\boxed{\begin{aligned} (\hat{a}_t - a) &= -\frac{\varepsilon a}{t+\varepsilon} + \frac{W_t}{t+\varepsilon} - (\hat{B}_t - B)\bar{X}_t \cdot \frac{t}{t+\varepsilon}, \\ (\hat{B}_t - B)V_t &= \frac{\varepsilon}{\varepsilon + t}\, t\, a\bar{X}_t^T + \int_0^t dW_s\, X_s^T - \delta B - \frac{W_t}{\varepsilon + t}\left(\int_0^t X_s\, ds\right)^T. \end{aligned}}$$

For the 1-dimensional example with $b > 0$, we have

$$e^{-bt}\int_0^t X_s\, dW_s = e^{-bt}\, \tilde{W}(A_t) \qquad \left(A_t \equiv \int_0^t X_s^2\, ds\right)$$

$$= e^{-bt}\sqrt{A_t}\;\; \tilde{W}(A_t)/\sqrt{A_t}.$$

Hence clearly for all large enough t

$$\left| e^{-bt} \int_0^t X_s \, dW_s \right| \leq c' \sqrt{\log\log A_t} \leq c \sqrt{\log t}$$

for some suitable constants c', c. From this

$$\left| (\hat{b}_t - b) V_t e^{-bt} \right| \leq c (1 + \sqrt{\log t}) \quad \text{for large enough } t.$$

It is now easy to see that not only is $\hat{b}_t \xrightarrow{a.s.} b$ but also $\hat{a}_t \to a$ a.s.

(v) For the multidimensional case, it's not so easy to describe the behaviour of X, but we can similarly analyse $(\hat{B}_t - B) V_t$

$$= (\varepsilon a - W_t) \frac{\int_0^t X_u \, du}{t+\varepsilon} + \int_0^t dW_s \cdot X_s^T - \delta B .$$

If we consider the ij^{th} element of this, and set $A_t = \int_0^t (X_s^j)^2 \, ds$, then for all large enough t,

$$\left| \int_0^t X_s^j \, dW_s^i \right| \leq (3 A_t \log\log A_t)^{\frac{1}{2}}$$

and the first term can be estimated similarly: for large enough t,

$$\left| \frac{\varepsilon a^i - W_t^i}{\varepsilon + t} \int_0^t X_u^j \, du \right| \leq (3 t \log\log t)^{\frac{1}{2}} \sqrt{t A_t} \,/ t$$

$$= (3 A_t \log\log t)^{\frac{1}{2}}$$

using Cauchy-Schwarz on the integral.

(vi) Just out of interest, let's do the analysis of the Bayesian problem as a filtering problem. So if $\mathcal{X}_t \equiv \sigma (X_u ; u \leq t)$, and $\hat{\varphi}_t$ denotes the $\mathcal{X}$-optional projection of an adapted process φ, we have

$$\begin{cases} X_t = \hat{a}_t \cdot t + \hat{B}_t \left(\int_0^t X_u \, du \right) + \hat{W}_t \\ \int_0^t dX_s \, X_s^T = \hat{a}_t \left(\int_0^t X_s \, ds \right)^T + \hat{B}_t \left(\int_0^t X_s X_s^T \, ds \right) + \hat{\varphi}_t , \end{cases}$$

where $\varphi_t \equiv \int_0^t dW_s\, X_s^T$. Hence easily

$$\boxed{\hat{W}_t = \varepsilon \hat{a}_t \,, \qquad \hat{\varphi}_t = \delta \hat{B}_t \,.}$$

The fundamental "innovations" martingale is $dX_t - (\hat{a}_t + \hat{B}_t X_t)\,dt$. Using the fact that

$$V_t = \delta I + \int_0^t \left(X_u - \frac{S_u}{\varepsilon + u}\right)\left(X_u - \frac{S_u}{\varepsilon + u}\right)^T du$$

(where $S_t \equiv \int_0^t X_s\,ds$), a little Itô calculus gives us

$$\begin{aligned} d\hat{B}_t &= \left\{ dX_t - (\hat{a}_t + \hat{B}_t X_t)\,dt \right\} \left(X_t - \frac{S_t}{\varepsilon+t}\right)^T V_t^{-1} \\ &\equiv dM_t \left(X_t - \frac{S_t}{\varepsilon+t}\right)^T V_t^{-1} \end{aligned}$$

and also

$$\begin{aligned} \varepsilon\, d\hat{a}_t = d\hat{W}_t &= \frac{\varepsilon}{\varepsilon+t}\left(dM_t - d\hat{B}_t \,.\, S_t \right) \\ &= \frac{\varepsilon}{\varepsilon+t}\, dM_t \left(1 - \left(X_t - \frac{S_t}{\varepsilon+t}\right)^T V_t^{-1} S_t \right) \end{aligned}$$

Effect of big investor on prices (2/9/94)

(i) Consider a world where $r \equiv 0$, all semimartingales are continuous, and there are only two assets, cash and shares. There are K shares in total, and at time t, the big investor holds ξ_t in cash, η_t in shares. The price of a share at time t is

$$p_t = f(\eta_t, S_t)$$

where f is some suitably smooth function, increasing in first argument, and S is some semimartingale.

Wealth equation for Big is

$$\boxed{w_t = w_0 + \int_0^t \eta_u \, dp_u = \eta_t p_t + \xi_t}$$

as usual, so

$$d\xi_t = - p_t \, d\eta_t - d\langle \eta, p \rangle_t .$$

(Note that η will have to be a semimartingale; arbitrary integrands don't really make physical sense.)

(ii) Suppose that Big wants to invest in such a way as to ensure

$$\max E \, U(w_T).$$

If $(\eta_t)_{t\geq 0}$ were the optimal investment strategy, and this were perturbed to $(\eta_t + \varepsilon H_t)_{t \geq 0}$, then the change in dp_t is to (first order in ε)

$$\varepsilon \, d\left(H_t \, f_1(\eta_t, S_t) \right).$$

Hence to first order in ε, the change in wealth is

$$\varepsilon \int_0^T H_u \, dp_u + \varepsilon \int_0^T \eta_u \, d\left(H_u f_1(\eta_u, S_u)\right).$$

It's more helpful to write the perturbation in terms of

$$Y_u \equiv H_u \, f_1(\eta_u, S_u),$$

because then the first-order condition says

$$\boxed{0 = E\, U'(w_T) \left\{ \int_0^T Y_u \frac{dp_u}{f_1(\eta_u, S_u)} + \int_0^T \eta_u \, dY_u \right\}}$$

for any reasonably well-behaved perturbation Y. Let's define P^* by

$$\frac{dP^*}{dP} = U'(w_T) / E\, U'(w_T) .$$

By taking $Y = (\sigma, \tau] I_A$, where $0 \le \sigma < \tau \le T$ are stopping times, $A \in \mathcal{F}_\sigma$, we conclude that

$$M_t \equiv \eta_t - \int_0^t \frac{dp_u}{f_1(\eta_u, S_u)} \quad \underline{\text{is a}} \ P^* \underline{\text{-martingale}} .$$

Now taking the boxed equation at the top, integrating by parts in the second term and using the fact that M is a P^*-martingale gives us

$$0 = E^* \left[Y_T \eta_T - Y_0 \eta_0 - \int_0^T d\langle Y, \eta \rangle_u \right] .$$

Now by taking Y to be an FV process ($Y_0 = 0$ is an obvious constraint) we deduce that

$$\boxed{\eta_T = 0 \text{, whence } \eta \text{ must be FV !!?}}$$

(iii) But now we see why this is <u>completely wrong</u>! If at time $T-$ we hold ξ_T in cash and η_T in shares, if we were to increase our holding of shares to $y > \eta_T$, then the nominal value of the portfolio is now

$$y\, f(y, S_T) + \xi_T - \int_{\eta_T}^{y} f(x, S_T)\, dx$$

$$= \xi_T + \eta_T f(\eta_T, S_T) + \int_{\eta_T}^{y} f'(x, S_T)\, x \, dx$$

– which can be arbitrarily big for large y !!

(iv) So we should instead be working with the liquidation value of terminal wealth? A holding of η_t shares is worth, when liquidated,

$$F(\eta_t, S_t) = \int_0^{\eta_t} f(x, S_t)\, dx$$

so if v_t denotes the liquidation value of the portfolio at time t,

$$\boxed{v_t = F(\eta_t, S_t) + \xi_t \equiv w_t - \varphi(\eta_t, S_t)}$$

where $\varphi(x, s) = -F(x, s) + x f(x, s)$. So

$$dv_t = \eta_t\, dp_t - d\varphi(\eta_t, S_t)$$

$$= \eta_t \Big(f_1(\eta_t, S_t)\, d\eta_t + f_2(\eta_t, S_t)\, dS_t + \tfrac{1}{2}\big(f_{11}\, d\langle\eta\rangle + 2 f_{12}\, d\langle S, \eta\rangle + f_{22}\, d\langle S\rangle \big)\Big)$$

$$- \varphi_1\, d\eta - \varphi_2\, dS - \tfrac{1}{2}\big(\varphi_{11}\, d\langle\eta\rangle + 2\varphi_{12}\, d\langle S, \eta\rangle + \varphi_{22}\, d\langle S\rangle \big)$$

$$= (\eta f_2 - \varphi_2)\, dS + \tfrac{1}{2}\big(-f_1\, d\langle\eta\rangle + (\eta f_{22} - \varphi_{22})\, d\langle S\rangle \big)$$

$$= F_2\, dS + \tfrac{1}{2} F_{22}\, d\langle S\rangle - \tfrac{1}{2} f_1\, d\langle\eta\rangle,$$

so

$$\boxed{dv_t = F_2\, dS + \tfrac{1}{2} F_{22}\, d\langle S\rangle - \tfrac{1}{2} F_{11}\, d\langle\eta\rangle.}$$

Thus the liquidation value of terminal wealth is

$$v_T = v_0 + \int_0^T \eta_u\, dp_u - \varphi(\eta_T, S_T) + \varphi(\eta_0, S_0).$$

We can repeat the perturbation analysis of part (ii) to obtain

$$0 = E^*\left[\int_0^T Y_u \frac{dp_u}{f_1(\eta_u, S_u)} + \int_0^T \eta_u\, dY_u - \frac{\varphi_1(\eta_T, S_T)\, Y_T}{f_1(\eta_T, S_T)} \right],$$

which simplifies to

$$0 = E^* \left[\int_0^T Y_u \frac{dp_u}{f_1(\eta_u, S_u)} + \int_0^T \eta_u \, dY_u - \eta_T Y_T \right].$$

As before, taking $Y = I_F (\sigma, \tau]$ leads us to conclude

$$\boxed{\eta_T = 0, \quad M_t \equiv \eta_t - \int_0^t \frac{dp_u}{f_1(\eta_u, S_u)} \quad \text{is a } P^*\text{-martingale}}$$

where $dP^*/dP \propto U'(v_T)$, and also that η should be FV!

To say that M is a P^*-mg amounts to saying

$$\boxed{f_2(\eta_t, S_t)\, dS_t + \tfrac{1}{2} f_{22}(\eta_t, S_t)\, d\langle S\rangle_t \quad \text{is a } P^*\text{-mg}}$$

(v) Let's now consider a special example. Suppose that apart from Big, there are N ~~investors~~ all with utility $U(x,y) = \log x + c \log y$, where x is the cash held, y is the number of shares. Initially, agent j has (x_j, y_j) as endowment. If an agent maximises utility subject to the price constraint $x + py = x_j + p y_j$, then the optimum will be at

$$x_j^* = \frac{x_j + p y_j}{1+c}, \quad y_j^* = \frac{c}{p} x_j^*.$$

The price at which markets clear is then easily computed to be

$$\boxed{p = c \sum_1^N x_j \Big/ \sum_1^N y_j}$$

So in this example, if we now imagine that the total cash available to the small agents is S_t, a continuous semimartingale, then

$$\boxed{p_t \equiv f(\eta_t, S_t) = c\, S_t / (K - \eta_t).}$$

This is particularly simple; we shall have that S ~~must~~ be a P^*-mg, so if

$$dS_t = S_t \, \sigma_t (dW_t + \mu_t \, dt)$$

we have to have

$$\boxed{U'(v_T) \propto \exp\left(- \int_0^T \mu_s \, dW_s - \tfrac{1}{2} \int_0^T \mu_s^2 \, ds \right).}$$

So let's simplify even more, and assume that $\boxed{\sigma, \mu \text{ are constant, and that } U = \log.}$

Thus $S_t = S_0 \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t + \sigma\mu t\right)$, and we need

$$U'(v_T) = U'(w_T) = \frac{1}{v_T} \propto \exp\left(-\mu W_T - \frac{1}{2}\mu^2 T\right),$$

which is to say

$$\boxed{w_T = w_0 + \int_0^T \eta_t \, dp_t = \lambda e^{\mu W_T + \frac{1}{2}\mu^2 T}}$$

We also know that

$$dp_t = p_t \left\{ \sigma(dW_t + \mu dt) + \frac{d\eta_t}{K - \eta_t} \right\}$$

so if $d\eta_t = \dot{\eta}_t dt$, we have

$$dp_t = \sigma p_t \left[dW_t + \theta_t dt \right], \qquad \theta_t \equiv \mu + \dot{\eta}_t / \sigma(K - \eta_t)$$

If we were to change measure to $\tilde{P}$, in such a way that p becomes a loc. martingale, we use density $(d\tilde{P}/dP)|_{\mathcal{F}_t} \equiv \rho_t = \exp\left[-\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du\right]$, and then

$$w_t = \tilde{E}\left[w_T | \mathcal{F}_t \right] = E(\rho_T w_T | \mathcal{F}_t) / \rho_t$$

$$= \frac{1}{\rho_t} E_t \, \lambda \exp\left(\int_0^T (\mu - \theta_u) dW_u - \frac{1}{2}\int_0^T \theta_u^2 du + \frac{1}{2}\mu^2 T \right),$$

implying

$$dw_t = w_t \, \mu (dW_t + \theta dt) \qquad \text{if } \theta \text{ were constant, say.}$$

(vi) This appears to violate the theory in general, --- ??! The problem appears to be that the equation for the cash value, w_t, of the portfolio is not correct; the nominal value of the portfolio changes when you alter η. It's the liquidation value which doesn't change. So I think we must have $v_t = \xi_t + F(\eta_t, S_t)$ satisfies

$$\boxed{dv_t = F_2(\eta_t, S_t) dS_t + \frac{1}{2} F_{22}(\eta_t, S_t) d\langle S \rangle_t,}$$

implying

$$d\xi + f \, d\eta + \frac{1}{2} f_1 d\langle\eta\rangle + f_2 \, d\langle\eta, S\rangle = 0.$$

Stochastic intensities : log Gaussian models (16/9/94)

(i) Let's suppose that we have $\lambda_i(t) = \exp(x_i(t))$, where

$$dx_i(t) = \sigma_i \, dB_i(t) + (\alpha_i - \beta_i x_i(t))\, dt$$

and the Brownian motions B_i are in general correlated; $dB_i dB_j = \rho_{ij}\, dt$. What can we say of the covariance of $\Lambda_i(t) \equiv \int_0^t \lambda_i(s)\,ds$ for different i, j, t?

We have

$$x_i(t) = e^{-\beta_i t} x_i(0) + (1 - e^{-\beta_i t})\, \alpha_i/\beta_i + e^{-\beta_i t} \int_0^t e^{\beta_i s}\, dW_i(s)\, \sigma_i$$

so

$$E\, x_i(t) = e^{-\beta_i t} x_i(0) + (1 - e^{-\beta_i t})\alpha_i/\beta_i, \quad \mathrm{var}(x_i(t)) = \sigma_i^2 (1 - e^{-2\beta_i t})/2\beta_i$$
$$\equiv m_i(t) \qquad\qquad\qquad\qquad \equiv v_i(t)$$

and for $0 \le s \le t$,

$$E\left[x_i(s)\, x_j(t)\right] = m_i(s)\, m_j(t) + \sigma_i \sigma_j\, e^{-\beta_j t - \beta_i s} \rho_{ij} \left(e^{(\beta_i + \beta_j)s} - 1\right)/(\beta_i + \beta_j).$$

Thus

$$E\, \Lambda_i(t) \Lambda_j(t) = \int_0^t ds \int_s^t du\; E\left[\lambda_i(s)\lambda_j(u) + \lambda_j(s)\lambda_i(u)\right]$$

$$= \int_0^t ds \int_s^t du \Big[\exp\{m_i(s) + m_j(u) + \tfrac{1}{2}\mathrm{var}(x_i(s) + x_j(u))\}$$
$$+ \text{ similar term} \Big]$$

$$= \int_0^t ds \int_s^t du\; \exp\left[m_i(s) + m_j(u) + \tfrac{1}{2}\{v_i(s) + v_j(u) + 2\,\mathrm{cov}(x_i(s), x_j(u))\}\right]$$
$$+ \text{ other term}$$

(ii) In the special case where $\beta_i = 0$ for all i, $m_i(t) = x_i(0) + \alpha_i t$, $v_i(t) = \sigma_i^2 t$ and $\mathrm{cov}(x_i(s), x_j(t)) = \sigma_i \sigma_j \rho_{ij} (s \wedge t)$, so we obtain more simply

$$\boxed{E\, \Lambda_i(t) = e^{x_i(0)}\, \frac{e^{(\alpha_i + \frac{1}{2}\sigma_i^2)t} - 1}{\alpha_i + \frac{1}{2}\sigma_i^2}}$$

and

If the change in price of asset i over $[0,t]$ is $\qquad Y_t^i = \sum_{r=1}^{N_t^i} X_r^i$, where the X_r^i are IID, then

$$E\,Y_t^i = (E X_i^i) \cdot E N_t^i \equiv \mu_i E \Lambda_i(t)$$

and

$$E\left[Y_t^i Y_t^j\right] = E\left[\left\{ \sum_{r=1}^{N_t^i}(X_r^i - \mu_i) + \mu_i N_t^i \right\}\left\{ \sum_{s=1}^{N_t^j}(X_s^j - \mu_j) + \mu_j N_t^j \right\}\right]$$

$$= \delta_{ij}\; v_i \; E N_t^i \; + \; \mu_i \mu_j \; E(N_t^i N_t^j) \qquad \left(v_i \equiv \operatorname{var}(X_1^i)\right)$$

so

$$\operatorname{cov}\left(Y_t^i, Y_t^j\right) = \delta_{ij}\; v_i \; E\Lambda_i(t) \; + \; \mu_i \mu_j \; \operatorname{cov}\left(N_t^i, N_t^j\right)$$

$$= \delta_{ij}\left\{v_i + \mu_i^2\right\} E\,\Lambda_i(t) \; + \; \mu_i \mu_j \; \operatorname{cov}\left(\Lambda_i(t), \Lambda_j(t)\right)$$

$$\boxed{\begin{aligned}\operatorname{cov}\left(\Lambda_i(t), \Lambda_j(t)\right) = {} & \frac{e^{kt} - e^{(c_i+c_j)t}}{(k-c_i)(k-c_j)} \\ & + (k-c_i-c_j)\left\{ \frac{e^{kt}}{k(k-c_i)(k-c_j)} - \frac{k\, e^{(c_i+c_j)t}}{c_i c_j (k-c_i)(k-c_j)} - \frac{1}{k c_i c_j} \right. \\ & \left. \qquad + \frac{1}{c_i c_j}\left\{ \frac{e^{c_i t}}{k-c_i} + \frac{e^{c_j t}}{k-c_j} \right\} \right\}\end{aligned}}$$

where $c_i = \alpha_i + \frac{1}{2}\sigma_i^2$, $k = c_i + c_j + \rho_{ij}\sigma_i\sigma_j$, and assuming $x_i(0) = x_j(0) = 0$

For the special case $i=j$, we get (with $x_i(0)=0$)

$$\boxed{\operatorname{var} \Lambda_i(t) = \frac{2\, e^{(2c_i+\sigma_i^2)t}}{(2c_i+\sigma_i^2)(c_i+\sigma_i^2)} - \frac{e^{2c_i t}}{c_i^2} + \frac{2\sigma_i^2\, e^{c_i t}}{c_i^2(c_i+\sigma_i^2)} - \frac{\sigma_i^2}{c_i^2(2c_i+\sigma_i^2)}}$$

This gives us that for large t,

$$\operatorname{var} \Lambda_i(t) \sim \frac{2\, e^{(2c_i+\sigma_i^2)t}}{(2c_i+\sigma_i^2)(c_i+\sigma_i^2)}$$

and hence that

$$\boxed{\operatorname{corr}\left(\Lambda_i(t), \Lambda_j(t)\right) \underset{(t\to\infty)}{\longrightarrow} 0} \quad \text{unless } \sigma_i = \sigma_j \text{ and } \rho_{ij} = 1.$$

As $t \downarrow 0$, $\boxed{\operatorname{corr}\left(\Lambda_i(t), \Lambda_j(t)\right) \longrightarrow \rho_{ij}}$

Stochastic intensities: the CBI model again (9/10/94)

Let's look again at the model considered on pages 3–5:

$$dX_i(t) = \sigma_i \sqrt{X_i(t)}\, dW^i_t + (\alpha_i + (AX)_i)\, dt$$

where the W^i are independent, $\alpha_i > 0$, $a_{ij} \geq 0$ $\forall i \neq j$. Assume that all e-values of A have strictly negative real part (this implies – assuming irreducibility – that there is a positive diagonal U s.t. $U^{-1}AU$ is a transient Q-matrix, and all entries of $-A^{-1}$ are therefore positive). Define the mean of the invariant dist^{n} to be μ:

$$\mu = -A^{-1}\alpha.$$

As was calculated on p 4, the covariance of the invariant dist^{n} is

$$\mathrm{cov}_\pi(X,X) = + \int_0^\infty e^{tA} \Delta e^{tA^T} dt \quad , \qquad \Delta \equiv \mathrm{diag}(\sigma_j^2 \mu_j).$$

(i) We observe that

$$X(t) = e^{tA} X(0) + e^{tA} \int_0^t e^{-sA} \sigma \sqrt{X(s)}\, dW_s + (e^{tA} - I) A^{-1} \alpha,$$

so

$$\boxed{E\left[X(t) \mid X(0)\right] = e^{tA}(X(0) - \mu) + \mu,}$$

and

$$\mathrm{cov}(X_i(t), X_j(t)) \equiv c_{ij}(t)$$

$$= E \int_0^t (e^{(t-s)A})_{ik}\, \sigma_k^2 X_k(s)\, (e^{(t-s)A})_{jk}\, ds$$

$$= \int_0^t (e^{(t-s)A})_{ik}\, \sigma_k^2 \left\{ \mu_k + (e^{sA}(X(0) - \mu))_k \right\} (e^{(t-s)A})_{jk}\, ds$$

$$= \left(\int_0^t e^{uA} \Delta(t-u) e^{uA^T} du \right)_{ij}$$

where $\Delta(s) \equiv \mathrm{diag}\left\{ \sigma_k^2 (\mu_k + (e^{sA}(X(0) - \mu))_k) \right\}$. This explains the earlier limiting form of the covariance, but it seems that no simplification will be possible in general.

We can similarly compute things concerning $\int_0^t X(u)\,du$; in particular, it is not

Hard to calculate

$$\int_0^t \{X_j(s) - E X_j(s)\} ds = \int_0^t \left((e^{(t-u)A} - I) A^{-1} \right)_{jk} \sigma_k \sqrt{X_k(u)} \, dW_u^k ,$$

from which

$$\boxed{\mathrm{Cov}\left(\int_0^t X_j(s) ds, \int_0^t X_i(s) ds \right) = \left(\int_0^t (e^{sA} - I) A^{-1} \Delta(t-s) \left((e^{sA} - I) A^{-1} \right)^T ds \right)_{ij}}$$

Notice that if we start with $X(0) = \mu$, then the covariance is non-decreasing with t.

(ii) Will this model allow sufficiently general correlation? Note that if one component of the motion gets big, then <u>all</u> components of the motion tend to get pulled up, since $a_{ij} \geq 0 \ \forall i \neq j$. This makes it extremely unlikely that any such model will do.

Note that

$$p_t^0(x,y) = \frac{1}{2}\left[p_{\frac{t}{4}}\left(\frac{x}{2}, \frac{y}{2}\right) - p_{\frac{t}{4}}\left(\frac{x}{2}, \frac{-y}{2}\right) \right] \qquad (x, y \in [0, 2\pi])$$

Brownian motions on the unit circle avoiding each other (11/10/94)

I'd like to find the transition density for the motion of n independent Brownian motions on the unit circle $\cong [0,2\pi)$ killed when any two meet. The first case to understand is that where there are only two particles.

(i) From p 47 in Crank, if we take BM in $[0,2\pi]$ killed at each end, the transition density is (for $0 \le x,y \le 2\pi$)

$$\boxed{\begin{aligned} p_t^0(x,y) &= \frac{1}{\pi}\sum_{n\ge 1} \sin\left(\tfrac{nx}{2}\right)\sin\left(\tfrac{ny}{2}\right) e^{-n^2t/8} \\ &= \frac{1}{4\pi}\sum_{n\in\mathbb{Z}} e^{-n^2t/8}\, e^{iny/2}\left\{e^{-inx/2} - e^{inx/2}\right\}\end{aligned}}$$

after some routine manipulations. As for the situation with reflection at the ends of the interval, we obtain the transition density

$$\boxed{p_t(x,y) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-n^2t/2}\, e^{in(x-y)}.} \qquad (x,y\in\mathbb{R})$$

(ii) Now let's write $q_t(x_1,x_2\,;\,y_1,y_2)$ for the transition density of the pair of particles with killing when they ever meet. We understand the parametrisation $x_1 \in [0,2\pi)$, $x_2 - x_1 \in (0,2\pi)$, likewise for y. For the transition $(x_1,x_2)\mapsto(y_1,y_2)$ to happen without contact, we can alternatively say

(a) the centre of the arc (x_1,x_2) must move to the centre of the arc (y_1,y_2);

(b) the separation (x_2-x_1) must move to (y_2-y_1) without ever leaving $(0,2\pi)$.

Now the centre of the arc is a BM on the circle running at half speed, and the separation is a BM running at double speed, so

$$\boxed{q_t(x_1,x_2\,;\,y_1,y_2) = p_{\frac{t}{2}}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) p^0_{2t}\left(x_2-x_1, y_2-y_1\right).}$$

If we now think of y_1, y_2 as being fixed and develop this expression as a Fourier series in x_1, x_2, using the expressions for p, p^0, we shall have after a few calculations that

- Note that

$$q_t(x_1, x_2; y, y+2\pi) = - q_t(x_2, x_1; y, y+2\pi) \quad \text{from the Fourier representation,}$$

so it must be zero!

- If we solve heat equation in $[-2\pi, 2\pi]$ with $\delta_0 - \delta_{2\pi}$ as the condition at $t=0$, then we get the solution

$$g(t,x) = \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \exp\left\{ i(k+\tfrac{1}{2})x - (k+\tfrac{1}{2})^2 t/2 \right\}$$

so we have an expression

$$q_t(\underline{x}, \underline{y}) = \frac{1}{2} \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=1}^{m} p_t(x_j, y_{\sigma(j)}) + \frac{1}{2} \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=1}^{m} g(t, x_j - y_{\sigma(j)})$$

$$q_t(x_1, x_2; y_1, y_2) = \frac{1}{2}\, p\left(\frac{t}{2}; \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \cdot \left\{ p\left(\frac{t}{2}; \frac{x_2-x_1}{2}, \frac{y_2-y_1}{2}\right) - p\left(\frac{t}{2}; \frac{x_2-x_1}{2}, \frac{y_1-y_2}{2}\right)\right\}$$

(iii) And now we see (eventually!) how to generalise this to m points;

$$q(t; x_1, \dots, x_m; y_1, \dots, y_m) = \frac{1}{2}(2\pi)^{-m} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \sum_{n_j}{}' \exp\left\{ \frac{i}{2} \sum_{j=1}^{m} n_j (x_{\sigma(j)} - y_j) - \frac{t}{8} \sum_{j=1}^{m} n_j^2 \right\} \quad ??$$

(Σ' denotes summation over m-tuples of either odd nos or even nos.)

where $0 \le x_1, y_1 < 2\pi$, $0 < x_2 - x_1 < x_3 - x_1 < \dots < x_m - x_1 < 2\pi$, $0 < y_2 - y_1 < \dots < 2\pi$.

This function satisfies the heat equation (thinking of the y's as fixed, say) and vanishes if any two of the x_i come together. Moreover, as $t \downarrow 0$, we get the point mass at $y_1, \dots, y_m$. Also, as $x_m \uparrow x_1 + 2\pi$, it all vanishes.

Stochastic intensities: any better with squared OU? (14/10/94)

(i) We'll try to model an intensity process $\lambda_i(t) = x_i(t)^2$, where x_i is $\mathbb{R}$-valued,

$$d x_i(t) = dM_i(t) - \beta_i x_i(t)\,dt$$

with $dM_i\, dM_j = a_{ij}\, dt$. We have

$$x_i(t) = e^{-\beta_i t}\left[x_i(0) + \int_0^t e^{\beta_i s}\, dM_i(s) \right]$$

and

$$\lambda_i(t) = e^{-2\beta_i t}\left[\lambda_i(0) + \frac{a_{ii}}{2\beta_i}\left(e^{2\beta_i t} - 1\right) + \int_0^t 2\, x_i(s)\, e^{2\beta_i s}\, dM_i(s) \right],$$

so that

$$\boxed{\lambda_i(t) - E\,\lambda_i(t) = e^{-2\beta_i t} \int_0^t 2\, x_i(s)\, e^{2\beta_i s}\, dM_i(s).}$$

So if $\Lambda_i(t) \equiv \int_0^t \lambda_i(s)\,ds$, we have

$$\boxed{\Lambda_i(t) - E\,\Lambda_i(t) = \int_0^t \beta_i^{-1}\, x_i(u) \left\{1 - e^{-2\beta_i(t-u)}\right\} dM_i(u).}$$

Hence

$$\mathrm{cov}\left(\Lambda_i(t), \Lambda_j(t)\right) = \frac{a_{ij}}{\beta_i \beta_j}\, E \int_0^t x_i(s)\, x_j(s) \left\{1 - e^{-2\beta_i(t-s)}\right\}\left\{1 - e^{-2\beta_j(t-s)}\right\} ds.$$

But

$$E\, x_i(s)\, x_j(s) = E\, x_i(s)\; E\, x_j(s) + \int_0^s e^{-(\beta_i+\beta_j)u}\, a_{ij}\, du$$

$$= x_i(0)\, x_j(0)\, e^{-(\beta_i+\beta_j)s} + a_{ij}\, \frac{1 - e^{-(\beta_i+\beta_j)s}}{\beta_i + \beta_j},$$

so that (with $\theta \equiv a_{ij}$, $a \equiv \beta_i$, $b \equiv \beta_j$)

$$\boxed{\begin{aligned} \mathrm{cov}\left(\Lambda_i(t), \Lambda_j(t)\right) &= \frac{\theta^2}{ab(a+b)}\left\{ t - \frac{1-e^{-2at}}{2a} - \frac{1-e^{-2bt}}{2b} + \frac{1-e^{-2(a+b)t}}{2a+2b} \right\} \\ &\quad + \frac{\theta}{ab}\left(x_i(0)\, x_j(0) - \frac{\theta}{a+b}\right)\left\{ \frac{1-e^{-2(a+b)t}}{a+b} + \frac{e^{-2bt} - e^{-2at}}{b-a} \right\} \end{aligned}}$$

Notice that for very large t this is like $\dfrac{\theta^2 t}{ab(a+b)}$, which is always positive

(ii) Let's just look at a single process for now, and consider how we would filter λ from the observations of the counting process. We have that if $\lambda(\cdot)$ were a deterministic intensity, the law of the counting process with that intensity has density

$$\prod_{\tau_j \le t} \lambda(\tau_j) \; . \exp\left(-\int_0^t (\lambda_u - 1)\, du\right)$$

(on $\mathcal{G}_t$) wrt the law of the standard Poisson process. Here, τ_j are the jump times. Thus

$$E\left[f(\tau_1\wedge t, \tau_2 \wedge t, \dots, \tau_n \wedge t)\, g(\lambda_t) \right]$$

$$= E\left[g(\lambda_t)\, e^{-\int_0^t (\lambda_u - 1)\,du} \; \underset{\substack{\uparrow \\ \text{standard Poisson law.}}}{\tilde{E}} \left(\prod_{\tau_j \le t} \lambda(\tau_j) . f(\tau_1 \wedge t, \dots, \tau_n \wedge t) \right) \right],$$

for any bdd Borel g, f. So if $(\mathcal{G}_t)$ is the filtration of the counting process

$$E\left[g(\lambda_t) \,|\, \mathcal{G}_t \right] = E\left[g(\lambda_t) \prod_{\tau_j \le t} \lambda(\tau_j)\, e^{-\int_0^t \lambda_u du} \right] \Big/ E\left[\prod_{\tau_j \le t} \lambda(\tau_j)\, e^{-\int_0^t \lambda_u du} \right]$$

where we insert the realised τ_j into the expectations.

So our goal is to understand the law of $\lambda_t = x_t^2$ when we reweight by

$$\prod_{j=1}^n \lambda(\tau_j) \; . \; e^{-\int_0^t x_u^2\, du}$$

where $\tau_1 < \dots < \tau_n < t$ is some fixed sequence. Well, first, we must understand what happens to the process x, $dx_t = \sigma dW_t - \beta x_t dt$, $x_0 = 0$, when we reweight by $\exp(-\int_0^t x_u^2 du)$. It remains a Gaussian process, and after some calculations of a well-known type, we compute the SDE for x under the reweighted law to be

$$\boxed{dx_u = \sigma d\tilde{W}_u - \rho \frac{(\beta+\rho)e^{\rho(t-u)} - (\rho-\beta)e^{-\rho(t-u)}}{(\rho+\beta)e^{\rho(t-u)} + (\rho-\beta)e^{-\rho(t-u)}}\, x_u \, du}\, , \qquad \rho \equiv \sqrt{\beta^2 + 2\sigma^2}$$

So we have a Gaussian process, and we are going to reweight the law with $\prod_1^n x(\tau_j)^2$ — how does this affect $E\, x(t)^2$??

If we write $\rho_{st} \equiv E\, x_s x_t$, then the Gauss-Markov relation is that for

$s \le t \le u$ we shall have $\rho_{us}\,\rho_{tt} = \rho_{ut}\,\rho_{us}$, and now if we want to calculate $E\left[\prod_{i}^{n} x(t_i)^2\right]$ we could approach this recursively. We know that

$$\mathcal{L}(x(t_n) \mid \mathcal{F}(t_{n-1})) = N\left(\frac{\rho_{t_{n-1}t_n}}{\rho_{t_{n-1}t_{n-1}}}\, x_{t_{n-1}},\ \frac{\rho_{t_nt_n}\,\rho_{t_{n-1}t_{n-1}} - \rho_{t_{n-1}t_n}^2}{\rho_{t_{n-1}t_{n-1}}} \right)$$

so by conditioning back to t_{n-1}, we'd get

$$E \prod_{1}^{n} x(t_i)^2 = a\, E\Big(\prod_{1}^{n-1} x(t_i)^2\Big) + b\, E\Big(\prod_{1}^{n-2} x(t_i)^2 \cdot x(t_{n-1})^4\Big)$$

for certain constants a, b. So to be able to build up recursively, we'd need to carry round the memory of $E\left[\prod_{1}^{n-1} x(t_i)^2 \cdot x(t_n)^{2r}\right]$ for all r ... So this is completely impractical as a filter. And it would probably be numerically unstable in any case!

(iii) We could also set up filtering equations for $\exp(-\alpha\lambda_t)$; we start from

$$d(e^{-\alpha\lambda_t}) = e^{-\alpha\lambda_t}\left(-\alpha\, d\lambda_t + \tfrac{1}{2}\alpha^2\, d\langle\lambda\rangle_t\right)$$

$$\doteq e^{-\alpha\lambda_t}\left\{-\alpha(\sigma^2 - 2\beta\lambda_t) + 2\sigma^2\alpha^2\lambda_t\right\} dt,$$

so, if $(\mathcal{G}_t)$ is the filtration of N, and $\zeta(\alpha,t) \equiv E[\exp(-\alpha\lambda_t) \mid \mathcal{G}_t]$, we have

$$\zeta(\alpha,t) - \zeta(\alpha,0) + \int_0^t \left[\alpha\sigma^2\zeta(\alpha,s) + 2(\beta\alpha + \sigma^2\alpha^2)\frac{\partial\zeta}{\partial\alpha}(\alpha,s)\right] ds = \int_0^t \varphi(\alpha,s)\, d\tilde{N}_s$$

for some $\mathcal{G}$-previsible integrand φ, where $d\tilde{N}_s = dN_s - \hat{\lambda}_s\, ds$.

Now we do the old trick of filtering theory – we do an Itô development of $N_t \exp(-\alpha\lambda_t)$ and then project onto $(\mathcal{G}_t)$, and compare with what we get if we Itô-expand $N_t\,\zeta(\alpha,t)$; the bottom line is the relation

$$\boxed{E\left[\lambda_t e^{-\alpha\lambda_t} \mid \mathcal{G}_t\right] = \hat{\lambda}_t\left\{E\left[e^{-\alpha\lambda_t} \mid \mathcal{G}_t\right] + \varphi(\alpha,t)\right\}.}$$

It appears impossible to use this to find either $\hat{\lambda}_t$ or φ.

Big trader: another try! (20/10/94)

(i) Returning to the situation introduced on p. 16, we again work with the liquidation value of the portfolio, where the liquidation value of η shares would be

$$F(\eta, S) = \int_0^\eta f(x, S)\, dx$$

at a time when the stochastic variable took value S. The liquidation value at time t, v_t, is expressed as

$$v_t = \xi_t + F(\eta_t, S_t).$$

I claim that the self-financing condition must be

$$\boxed{dv_t = F_2(\eta_t, S_t)\, dS_t + \tfrac{1}{2} F_{22}(\eta_t, S_t)\, d\langle S\rangle_t}$$

It's clear that this is true if η is held constant, and if we hold η constant until time t, then increase it rapidly from $a \equiv \eta_{t-}$ to b, the change in liquidation value of the holding of shares will be $F(b, S_t) - F(a, S_t)$, which is exactly balanced by the loss of cash needed to pay for it. We still have that the nominal wealth process w_t is related by

$$w_t = \xi_t + \eta_t f(\eta_t, S_t) = v_t + \varphi(\eta_t, S_t),$$

but really it's v which is to be considered as fundamental.

(ii) Our big trader might be trying to max $E\, U(v_T)$, so if η were his optimal portfolio choice, and he perturbs to $\eta + \varepsilon H$, then the usual first-order argument gives

$$0 = E\left[U'(v_T) \int_0^T H_u \left\{ f_2(\eta_u, S_u)\, dS_u + \tfrac{1}{2} f_{22}(\eta_u, S_u)\, d\langle S\rangle_u \right\} \right]$$

so if we take $dP^*/dP = c \cdot U'(v_T)$, we have the conclusion that

$$\boxed{dS_t + \frac{f_{22}}{2f_2}(\eta_t, S_t)\, d\langle S\rangle_t \quad \text{must be a } P^*\text{-martingale}}$$

(iii) In general it's going to be difficult to make much of this, but if we treat the special case

$$\boxed{\frac{f_{22}}{2f_2}(\eta, S) = g_0(S) \quad \text{indep of } \eta}$$

we can do more, because then f must have the form

$$f(\eta, S) = a(\eta)\, g(S) + c(\eta)$$

and the change-of-measure martingale may be expressed without reference to η.
Let's suppose that

$$dS_t = S_t\, \sigma_t\, (dW_t + \mu_t\, dt),$$

so that $\quad dW_t + \mu_t\, dt + \sigma_t S_t\, g_0(S_t)\, dt$ is a P^*-mg, and hence

$$Z_t = \exp\left[-\int_0^t (\mu_u + \sigma_u S_u g_0(S_u))\, dW_u - \tfrac{1}{2}\int_0^t (\mu_u + \sigma_u S_u g_0(S_u))^2 du \right]$$

is the change-of-measure martingale, and

$$v_T = I(\lambda Z_T)$$

is now a known r.v.. However, most importantly, $F(\eta, S) = A(\eta)\, g(S) + C(\eta)$
$(A(\eta) \equiv \int_0^\eta a(x)dx, \ldots)$ so that

$$\frac{F_{22}(\eta,S)}{F_2(\eta,S)} = \frac{g''}{g'}(S) = \frac{f_{22}(\eta,S)}{f_2(\eta,S)}$$

and particularly

$$\boxed{v_t \text{ is a } P^*\text{- martingale}}$$

(iv) Let's now understand the nice example where

$$f(\eta, S) = \frac{cS}{K-\eta}, \qquad F(\eta,S) = cS \log\left(\frac{K}{K-\eta}\right)$$

and $U'(x) = x^{-R}$, σ, μ are constant. The change of measure is

$$Z_T = \exp(-\mu W_T - \tfrac{1}{2}\mu^2 T),$$

and we have

$$v_t = v_0 \exp\left[\frac{\mu}{R} \tilde{W}_t - \frac{1}{2} \left(\frac{\mu}{R}\right)^2 t \right] \qquad \tilde{W}_t \equiv W_t + \mu t$$

with $\quad v_T = I(\lambda Z_T), \qquad \lambda = v_0^{-R} \exp\{ \mu^2 T (1-R)/2R \}$

Now

$$dv = F_2(\eta, S)\, dS$$
$$= c \log\left(\frac{K}{K-\eta}\right) \cdot \sigma S \, d\tilde{W}$$

and $\quad dv = \frac{\mu}{R} v \, d\tilde{W} \qquad$ from which we deduce that

$$\boxed{\begin{aligned} \log \frac{K}{K-\eta_t} &= \frac{\mu}{c\sigma R} \frac{v_t}{S_t} \\ &= \frac{\mu}{c\sigma R} v_0 \exp\left[\left(\frac{\mu}{R} - \sigma\right) \tilde{W}_t + \frac{1}{2}\left(\sigma^2 - \frac{\mu^2}{R^2}\right) t \right] \end{aligned}}$$

This determines the optimal investment for the big investor, and, as a consequence, it gives us the price process too; the price at time t is

$$\boxed{f(\eta_t, S_t) = \frac{cS_t}{K} \exp\left\{ \frac{\mu v_0}{c\sigma R} \exp\left[\left(\frac{\mu}{R} - \sigma\right) \tilde{W}_t + \frac{1}{2}\left(\sigma^2 - \left(\frac{\mu}{R}\right)^2\right) t \right] \right\}}$$

(V) Now let's assume that Big is try to maximise his integrated expected discounted utility from consumption;

$$\max \; E \int_0^T e^{-\rho t} U(c_t)\, dt$$

where the liquidation-value process v must now satisfy

$$dv_t = F_2(\eta_t, S_t)\, dS_t + \frac{1}{2} F_{22}(\eta_t, S_t)\, d\langle S \rangle_t - c_t\, dt.$$

As before, we assume that F_{22}/f_2 is a function only of S. We have as usual the constraint

$$v_0 = E^* \int_0^T c_t \, dt$$

So by taking a Lagrangian form of the problem we obtain the problem

$$\max \; E\left[\int_0^T e^{-\rho t} U(c_t)\, dt - \lambda \int_0^T Z_t \, c_t \, dt \right]$$

where Z is the change-of-measure martingale making $dS + \frac{f_{22}}{2f_2}(\eta, S)\, d\langle S\rangle$ into a martingale. The usual first-order condition yields

$$c_t^* = I\left(e^{\rho t} \lambda Z_t\right),$$

and

$$v_t + \int_0^t c_u \, du = E^*\left(\int_0^T c_u \, du \,\Big|\, \mathcal{F}_t \right).$$

Once again, we could solve the problem where $f(\eta, S) = cS/(K-\eta)$, in which case $Z_t = \exp(-\mu W_t - \frac{1}{2}\mu^2 t) = \exp(-\mu \tilde{W}_t + \frac{1}{2}\mu^2 t)$, and some standard calculations give us

$$c_t^* = \lambda^{-1/R} \exp\left[\frac{\mu}{R}\tilde{W}_t - \frac{\mu^2 t}{2R} - \frac{\rho t}{R} \right]$$

and

$$dv_t = v_t \frac{\mu \, d\tilde{W}_t}{R} - c_t^* \, dt,$$

with

$$v_t = \lambda^{-1/R} \exp\left[\frac{\mu}{R}\tilde{W}_t - \frac{1}{2}\left(\frac{\mu}{R}\right)^2 t \right] \frac{e^{\theta T} - e^{\theta t}}{\theta}$$

where $\theta \equiv \mu^2(1-R)/2R^2 - \rho/R$. Now in this case we know that

$$dv_t = c \log\left(\frac{K}{K-\eta_t}\right) \sigma S_t \, d\tilde{W}_t - c_t^* \, dt$$

so that

$$\boxed{c\sigma S_t \log\left(\frac{K}{K-\eta_t}\right) = \frac{\mu}{R} v_t ,}$$

exactly as before! In fact, the process v differs from what it was when we maximised expected terminal utility, but as we let $T \to \infty$, we get exactly the same. Thus the same (rather unrealistic?) price process for the shares will result

Robust variance estimation (23/10/94)

(i) Let's suppose that X_i are zero-mean Gaussian variables, $E X_i X_j = \sigma_{ij}$, and with constants $a_i < b_i$ fixed we define

$$Y_i \equiv (X_i \vee a_i) \wedge b_i = X_i - (X_i - b_i)^+ + (a_i - X_i)^+.$$

Then we have

$$\boxed{E Y_i = b\,\bar\Phi\left(\frac{b}{\sigma}\right) + \frac{\sigma}{\sqrt{2\pi}}\left[e^{-a^2/2\sigma^2} - e^{-b^2/2\sigma^2}\right] + a\,\bar\Phi\left(\frac{-a}{\sigma}\right)}$$

However the second moments of Y look absolutely frightful.

(ii) Another approach might be as follows. Suppose we have n-vector r.v.s, $X_1, X_2, \ldots$ which are IID $N(\mu, V)$. If μ were not too far from 0, we could form

$$\frac{1}{N}\sum_1^N X_j \exp(-\varepsilon |X_j|^2/2)$$

which has expected value

$$(I+\varepsilon V)^{-1}\mu \,.\, \det(I+\varepsilon V)^{-\frac{1}{2}} \,.\, e^{-\frac{1}{2}\varepsilon\,\mu^T (I+\varepsilon V)^{-1}\mu}$$

But better is to observe that $E \exp(-\frac{1}{2}\varepsilon |X|^2) = \det(I+\varepsilon V)^{-1/2}\, e^{-\frac{1}{2}\mu^T (I+\varepsilon V)^{-1}\varepsilon\mu}$

So if we were to reweight the density of X by the function

$$\exp(-\tfrac{1}{2}\varepsilon |X|^2) \quad \det(I+\varepsilon V)^{\frac{1}{2}} \quad \exp\left(\tfrac{\varepsilon}{2}\mu^T (I+\varepsilon V)^{-1}\varepsilon\mu\right)$$

we get a new density with mean $\tilde\mu \equiv (I+\varepsilon V)^{-1}\mu$, covariance $\tilde V \equiv (V^{-1}+\varepsilon I)^{-1}$.

In this case, we think that

$$\boxed{\hat\mu \equiv \frac{\sum_1^N X_j \exp(-\frac{1}{2}\varepsilon |X_j|^2)}{\sum_1^N \exp(-\frac{1}{2}\varepsilon |X_j|^2)}}$$

will estimate $\tilde\mu$ (it will certainly converge to it a.s. as $N\to\infty$) and

$$\boxed{\frac{\sum_1^N (X_j - \hat\mu)(X_j - \hat\mu)^T \exp(-\frac{1}{2}\varepsilon |X_j|^2)}{\sum_1^N \exp(-\frac{1}{2}\varepsilon |X_j|^2)} \equiv \hat V}$$

will estimate $\tilde V$. Then $(\hat V^{-1} - \varepsilon I)^{-1}$ should estimate V, $(I-\varepsilon\hat V)^{-1}\hat\mu$ should estimate μ.

In equilibrium, $E\left[Z^i(\alpha)\right] = a_{ii} / 2\alpha\beta_i$

Stochastic intensities and GMM (23/10/94)

(i) We are going to stick with the model of p 27, and consider the random variables

$$Z_i(\alpha) \equiv \int_{-\infty}^{0} e^{\alpha t}\, dN^i(t) \overset{\mathcal{D}}{=} \int_0^\infty e^{-\alpha t}\, dN^i(t)$$

by reversibility of the OU process. We have

$$E\, Z_i(\alpha) = E \int_0^\infty e^{-\alpha t}\, x_i(t)^2\, dt$$

$$= \int_0^\infty e^{-\alpha t} \left[x_i(0)^2\, e^{-2\beta_i t} + a_{ii}\,(1-e^{-2\beta_i t})/2\beta_i \right] dt$$

$$\therefore \quad \boxed{E\, Z_i(\alpha) = \frac{x_i(0)^2\,\alpha + a_{ii}}{\alpha\,(\alpha + 2\beta_i)}}$$

Now we could try to work out the parameters β_i, $x_i(0)$, a_{ij}, $i,j = 1,\dots,n$ if we knew the values of $Z_i(\alpha)$ for some finite set of values $\alpha_1,\dots,\alpha_K$, and $i=1,\dots,n$; we would "best fit" $\alpha^{-1}(\alpha+2\beta_i)^{-1}\{\alpha\, x_i(0)^2 + a_{ii}\}$ to the observed $Z_i(\alpha)$ ... but how should we best fit?

The obvious thing is use the covariance matrix of $Z_i(\alpha_j)$ to tell us how good a fit is. Of course, this will depend on the unknowns also, but we should be able to get away with using the values computed at the previous update.

(ii) For the covariance matrix, we need to compute (among other things)

$$E\left[Z_i(\alpha)\, Z_i(\gamma)\right] = E\left[\int_0^\infty e^{-(\alpha+\gamma)t}\lambda_i(t)\,dt + \int_0^\infty e^{-\alpha t}\lambda_i(t)dt \int_0^\infty e^{-\gamma t}\lambda_i(t)dt\right]$$

so that

$$\mathrm{cov}(Z(\alpha), Z(\gamma)) = \mathrm{cov}\left(\hat\lambda(\alpha), \hat\lambda(\gamma)\right) + E \int_0^\infty e^{-(\alpha+\gamma)t}\lambda(t)\,dt,$$

dropping the subscript i for the time being. Of course, $\hat\lambda(\alpha) \equiv \int_0^\infty e^{-\alpha t}\lambda(t)dt$.

Now

$$\hat\lambda(\alpha) - E\,\hat\lambda(\alpha) = \int_0^\infty \frac{2x_u}{\alpha+2\beta}\, e^{-\alpha u}\, dM_u,$$

from which we conclude that

In equilibrium, $E(x_i x_j) = a_{ij} / (\beta_i + \beta_j)$

$$\boxed{\operatorname{cov}(\hat\lambda(\alpha),\hat\lambda(\gamma)) = \frac{4\sigma^2}{(\alpha+2\beta)(\gamma+2\beta)} \quad \frac{(\alpha+\gamma)x_0^2+\sigma^2}{(\alpha+\gamma)(\alpha+\gamma+2\beta)}}$$

where $\sigma^2 = a_{ii}$. So if we assemble, we get

$$\boxed{\operatorname{cov}(Z(\alpha), Z(\gamma)) = \left\{1+\frac{4\sigma^2}{(\alpha+2\beta)(\gamma+2\beta)}\right\}\cdot\frac{(\alpha+\gamma)x_0^2+\sigma^2}{(\alpha+\gamma)(\alpha+\gamma+2\beta)}}$$

(iii) The other thing we'll need for the covariance matrix will be (for i and j distinct) expressions of the form

$$\operatorname{cov}(Z_i(\alpha), Z_j(\gamma)) = \operatorname{cov}(\hat\lambda_i(\alpha), \hat\lambda_j(\gamma))$$

$$= E\int_0^\infty \frac{2x_i(t)}{\alpha+2\beta_i} e^{-\alpha t}\, dM_i(t)\cdot\int_0^\infty \frac{2x_j(t)e^{-\gamma t}}{\gamma+2\beta_j}\, dM_j(t)$$

$$= \frac{4a_{ij}}{(\alpha+2\beta_i)(\gamma+2\beta_j)} E\int_0^\infty e^{-(\alpha+\gamma)t} x_i(t)x_j(t)\,dt$$

$$= \frac{4a_{ij}}{(\alpha+2\beta_i)(\gamma+2\beta_j)}\int_0^\infty e^{-(\alpha+\gamma)t}\left\{x_i(0)x_j(0)e^{-(\beta_i+\beta_j)t} + a_{ij}\frac{1-e^{-(\beta_i+\beta_j)t}}{\beta_i+\beta_j}\right\}dt$$

so

$$\boxed{\operatorname{cov}(Z_i(\alpha),Z_j(\gamma)) = \frac{4a_{ij}}{(\alpha+2\beta_i)(\gamma+2\beta_j)}\left[\frac{x_i(0)x_j(0)}{\alpha+\gamma+\beta_i+\beta_j} + \frac{a_{ij}}{(\alpha+\gamma)(\alpha+\gamma+\beta_i+\beta_j)}\right]},$$

Consistent with the calculation of $\operatorname{cov}(\hat\lambda(\alpha),\hat\lambda(\gamma))$ at top of page.

[See reverse of p.21 for connection between this and covariance of log-price changes]

(iv) Notice that using only $Z_i(\alpha)$ will not tell us anything about correlation between the different intensity processes. How would we go about estimating this correlation? One possibility is the following. We can compute at any time the matrix

$$\xi_t \equiv \int_{-\infty}^{t} \theta\, e^{\theta(u-t)}\, Z_u^i(a)\, Z_u^j(b)\, du - \left(\int_{-\infty}^{t} \theta e^{\theta(u-t)} Z_u^i(a)\,du\right)\left(\int_{-\infty}^{t} \theta e^{\theta(u-t)} Z_u^j(b)\,du\right)$$

which will undoubtedly be non-negative-definite. Now if θ is small, we will find that the value of $x(0)$ is essentially irrelevant; so let's assume that the process x is in equilibrium. Notice also that

$$\int_{-\infty}^{0} \theta e^{\theta u}\, Z_u^i(a)\,du = \frac{\theta}{\theta-a}\left\{Z_0^i(a) - Z_0^i(\theta)\right\}$$

and in equilibrium

$$E\, Z_0^i(a) = \frac{a_{ii}}{2a\beta_i}, \quad \mathrm{cov}\left(Z_0^i(a), Z_0^j(b)\right) = \frac{4\, a_{ij}^2}{(a+b)(\beta_i+\beta_j)\,(a+2\beta_i)(\,b+2\,\beta_j\,)}$$

for $i \neq j$, and for $i=j$ we get

$$\mathrm{cov.}\left(Z_0^i(a), Z_0^i(b)\right) = \left\{1 + \frac{4a_{ii}}{(a+2\beta_i)(b+2\beta_i)}\right\} \cdot \frac{a_{ii}}{2\beta_i\,(a+b)}$$

so generally

$$\boxed{\mathrm{cov}\left(Z_0^i(a), Z_0^j(b)\right) = \left\{\delta_{ij} + \frac{4\,a_{ij}}{(a+2\beta_i)(b+2\beta_j)}\right\} \frac{a_{ij}}{(\beta_i+\beta_j)\,(a+b)}}$$

from which

$$\boxed{E\left[Z_0^i(a) Z_0^j(b)\right] = \left\{\delta_{ij} + \frac{4a_{ij}}{(a+2\beta_i)(b+2\beta_j)}\right\} \frac{a_{ij}}{(\beta_i+\beta_j)(a+b)} + \frac{a_{ii}\, a_{jj}}{4ab\,\beta_i\beta_j}},$$

We can thus simplify

$$\boxed{E\,\xi_0 = \mathrm{cov}\left(Z_0^i(a), Z_0^j(b)\right) - \frac{\theta^2}{(\theta-a)(\theta-b)}\,\mathrm{cov}\left(Z_0^i(a) - Z_0^i(\theta),\, Z_0^j(b) - Z_0^j(\theta)\right).}$$

(v) Computing $E(\xi_0)$ needs to be done more cunningly. Firstly, if $\bar{Z}_t^i(a) \equiv Z_t^i(a) - E(Z_t^i(a))$, notice that we can obtain $E\,\xi_0$ as

$$\lambda_0^i = \frac{a_{ii}}{2\beta_i} + \int_{-\infty}^{0} 2x_u^i \, e^{2\beta_i u} \, dM_u^i$$

so $\operatorname{cov}(\lambda_0^i, \lambda_0^j) = \frac{4a_{ij}}{2(\beta_i+\beta_j)} E(x_0^i x_0^j)$

$$= \frac{2a_{ij}^2}{(\beta_i+\beta_j)^2}$$

$$\therefore E\left[\lambda_0^i \lambda_j^0\right] = \frac{2a_{ij}^2}{(\beta_i+\beta_j)^2} + \frac{a_{ii}\,a_{jj}}{4\beta_i\beta_j}$$

$$E\xi_0 = E\left[\int_{-\infty}^0 \theta e^{\theta u} Z_u^i(a) Z_u^j(b)\,du - \int_{-\infty}^0 \theta e^{\theta u} Z_u^i(a)\,du \int_{-\infty}^0 \theta e^{\theta s} Z_s^j(b)\,ds\right]$$

(1) $$= \mathrm{cov}\left(Z_0^i(a), Z_0^j(b)\right) - \int_{-\infty}^0 \theta e^{\theta s}\,ds \int_s^0 du\, \theta e^{\theta u}\, \mathrm{cov}\left(Z_s^j(b), Z_u^i(a)\right) - \text{other term}.$$

Now for $s < u < 0$ we have

$$\mathrm{cov}\left(Z_s^j(b), Z_u^i(a)\right) = E\left(Z_s^j(b) - \frac{a_{jj}}{2\beta_j b}\right) Z_u^i(a)$$

$$\delta \equiv u - s > 0$$

(2) $$= e^{-a\delta}\, \mathrm{cov}\left(Z_s^j(b), Z_s^i(a)\right) + E\left[\left(Z_s^j(b) - \frac{a_{jj}}{2b\beta_j}\right) \int_s^u e^{a(t-u)}\, dN_t^i\right].$$

Now only the final term requires further clarification. Taking $s=0$, wlog, we obtain

$$E\left[\left(Z_0^j(b) - \frac{a_{jj}}{2b\beta_j}\right) \int_0^\delta e^{a(t-\delta)} \lambda_t^i\, dt\right]$$

$$= e^{-a\delta} E\left[\frac{\lambda_0^j - \frac{a_{jj}}{2\beta_j}}{b+2\beta_j} \cdot \int_0^\delta e^{at}\left(e^{-2\beta_i t}\lambda_0^i + \frac{a_{ii}}{2\beta_i}\left(1 - e^{-2\beta_i t}\right)\right) dt\right]$$

$$= \frac{e^{-a\delta}}{b+2\beta_j} \cdot \frac{e^{(a-2\beta_i)\delta} - 1}{a - 2\beta_i}\, \mathrm{cov}\left(\lambda_0^j, \lambda_0^i\right)$$

(3) $$= 2\left(\frac{a_{ij}}{\beta_i + \beta_j}\right)^2 \frac{e^{-2\beta_i\delta} - e^{-a\delta}}{a - 2\beta_i} \cdot \frac{1}{b+2\beta_j}.$$

If we abbreviate $\mathrm{cov}\left(Z_0^i(a), Z_0^j(b)\right) \equiv R$, we shall obtain from (1), (2) and (3)

$$E\xi_0 = R - \frac{\theta R}{2(\theta+a)} - \frac{\theta R}{2(\theta+b)} - \theta\left(\frac{a_{ij}}{\beta_i+\beta_j}\right)^2 \left\{\frac{1}{(\theta+a)(\theta+2\beta_i)(b+2\beta_j)} + \frac{1}{(\theta+b)(\theta+2\beta_j)(a+2\beta_i)}\right\}$$

which gets summarised as

$$\boxed{E\xi_0 = \frac{\theta(a+b)+2ab}{2(\theta+a)(\theta+b)}\, R \; - \theta\left(\frac{a_{ij}}{\beta_i+\beta_j}\right)^2 \left\{ \frac{1}{(\theta+a)(\theta+2\beta_i)(b+2\beta_j)} + \frac{1}{(\theta+b)(\theta+2\beta_j)(a+2\beta_i)} \right\}}$$

with

$$R = \left\{ \delta_{ij} + \frac{4a_{ij}}{(a+2\beta_i)(b+2\beta_j)} \right\} \frac{a_{ij}}{(\beta_i+\beta_j)(a+b)} .$$

<u>American put with exponential expiry</u> (24/11/94)

The first idea is that we try to find

$$V(x) \equiv \sup_{0\le\tau\le T} E^*\left(e^{-r\tau} f(S_\tau) \mid S_0 = e^x\right)$$

$$\equiv \sup_{0\le\tau\le T} E^*\left(e^{-r\tau} g(X_\tau) \mid X_0 = x\right) \qquad (T \sim \exp(\lambda))$$

where $X_t \equiv \log S_t = \sigma B_t + (r - \frac{1}{2}\sigma^2)t$ under P^*. Abbreviate $\mu \equiv r - \sigma^2/2$.

(i) Suppose we decide to use the strategy of stopping at the time $H \equiv \inf\{t>0;\ X_t \le a\}$, where a is arbitrary, or at T, whichever is sooner. Then we have the value for this strategy is

$$\tilde{V}(x) = E^x\left(e^{-rH} : H<T\right) g(a) + E^x\left(\int_0^H \lambda e^{-\lambda t - rt} g(X_t)\,dt\right)$$

$$= \lambda R_{\lambda+r} g(x) + E^x\left(e^{-rH-\lambda H}\right)\left\{g(a) - \lambda R_{\lambda+r} g(a)\right\},$$

and some routine calculations give us

$$\lambda R_\lambda g(x) = \int_{-\infty}^{\infty} \frac{\lambda}{\sqrt{\mu^2+2\lambda\sigma^2}}\, e^{\mu(y-x)/\sigma^2 - \theta|x-y|}\, g(y)\,dy$$

$$\left(\theta \equiv \sqrt{\mu^2+2\lambda\sigma^2}/\sigma^2\right)$$

and

$$E^x e^{-\lambda H} = \exp\left[-(x-a)\left(\frac{\mu}{\sigma^2}+\theta\right)\right].$$

Hence

$$\boxed{\tilde{V}(x) = \int_a^\infty \frac{\lambda e^{\mu(y-x)/\sigma^2}}{(\mu^2+2\sigma^2(\lambda+r))^{1/2}}\left[e^{-\theta'|y-x|} - e^{-\theta'(x+y-2a)}\right] g(y)\,dy + g(a)\exp\left\{-(x-a)\left(\frac{\mu}{\sigma^2}+\theta'\right)\right\}.}$$

$$\left(\theta' \equiv \sqrt{\mu^2+2\sigma^2(\lambda+r)}/\sigma^2\right)$$

(ii) Differentiating at $x=a$ yields

$$\tilde{V}'(a) = \int_a^\infty \frac{\lambda\, g(y)\, dy}{(\mu^2 + 2(\lambda+r)\sigma^2)^{1/2}} \cdot 2\theta' \cdot \exp\left(-(\theta' - \mu/\sigma^2)(y-a)\right)$$

$$- \left(\mu/\sigma^2 + \theta'\right) g(a)$$

$$\therefore \boxed{\tilde{V}'(a) = \frac{2\lambda}{\sigma^2} \int_a^\infty g(y)\, e^{-(\theta' - \mu/\sigma^2)(y-a)}\, dy - \left(\theta' + \mu/\sigma^2\right) g(a)}$$

(iii) In the special case where $g(y) = (K - e^y)^+$ we get (with $k \equiv \log K$)

$$\tilde{V}'(a) = \frac{2\lambda}{\sigma^2} \left\{ K \frac{1 - e^{-\beta(k-a)}}{\beta} - e^a \frac{e^{(1-\beta)(k-a)} - 1}{1-\beta} \right\} - \left(\beta + 2\mu/\sigma^2\right)\left(K - e^a\right)$$

with $\beta \equiv \theta' - \mu/\sigma^2 > 0$. The smooth pasting condition would be $\tilde{V}'(a) = -e^a$, which has no explicit solution. Notice that the solution would have the form

$$e^a = \rho K,$$

where

$$\frac{2\lambda}{\sigma^2} \left[\frac{1-\rho^\beta}{\beta} - \rho \frac{\rho^{\beta-1} - 1}{1-\beta} \right] - \left(\beta + \frac{2\mu}{\sigma^2}\right)(1-\rho) = -\rho .$$

This has exactly one solution in $(0,1)$.

(iv) If we have a Markov process with generator $\mathcal{G}$ which is self-adjoint wrt m, and now we consider the problem of optimal stopping where we stop at the smaller of τ (first entry time to the stopping region) and T ($\sim \exp(\lambda)$), then in the continuation region, if V is the value function, we shall have to have

$$(\lambda - \mathcal{G}) V = \lambda g$$

and $V \geq g$ everywhere (g is the reward function for stopping.) We could now try to rephrase this as the problem of

$$\min \quad \frac{1}{2}\int \lambda (V-g)^2 \, dm \quad - \frac{1}{2}\int V \mathcal{G} V \, dm$$

subject to $V \geq g$ everywhere. A formal perturbation argument shows that a minimising V would satisfy the equation $(\lambda - \mathcal{G})V = \lambda g$ where $V > g$. (If we were simply to receive nothing if we hadn't stopped when T occurred, then we'd get analogously $(\lambda - \mathcal{G})V = 0$ when $V > g$, so we'd solve

$$\min \quad \frac{1}{2}\int \lambda V^2 dm - \frac{1}{2}\int V \mathcal{G} V \, dm \qquad \text{subj to } V \geq g. \;)$$

While true, it's not clear that this is going to be any easier than the straightforward American put by finite-difference.

Fourier methods? Minimisation problem for fixed maturity? Minimising over a parametric family of stopping rules? (Check out Myneni's survey)

Share prices with dividends (6/12/94)

Suppose a cash flow of $\delta_u S_u \, du$ is generated as a dividend from a share whose price at time t is S_t, and $R_t \equiv \int_0^t r_u \, du$ is the integral of the spot-rate process. We then have for $s < t$

$$\tilde{S}_s \equiv e^{-R_s} S_s = E_s^*\left[\int_s^t e^{-R_u} \delta_u S_u \, du + e^{-R_t} S_t \right]$$

$$= E_s^*\left[\int_s^t \delta_u \tilde{S}_u \, du + \tilde{S}_t \right]$$

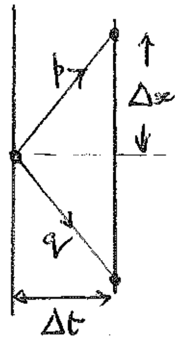
If we set $\boxed{Y_t = e^{\int_0^t \delta_u du} \tilde{S}_u}$, then $d\tilde{S} = \tilde{S}\{ -\delta \, dt + dY/Y \}$, so

$$E_s^*\left[\int_s^t \tilde{S}_u \frac{dY_u}{Y_u} \right] = 0$$

implying that $\boxed{Y \text{ is a } P^*\text{-martingale}}$

Binomial & trinomial lattice models (6/12/94)

This must be in the literature somewhere, but in a rather confused form! The idea is to approximate the log price process $X_t = \sigma B_t + (r - \sigma^2/2) t$ by a random walk on a lattice whose spatial step is Δx, and whose time step is Δt. Either the rw step ± 1, (binomial case) or it steps $-1, 0,$ or 1 (trinomial). How should one pick the parameters to match up?



1) Binomial case We can either match the first 2 moments of the log price, or the first 2 moments of the price.

(i) With the first two moments of log price firstly. If we set $\mu \equiv r - \sigma^2/2$, we get

$$\left.\begin{aligned} E\,\Delta X &= \Delta x\,(p-q) = \mu\,\Delta t \\ E[(\Delta X)^2] &= (\Delta x)^2 = \sigma^2 \Delta t + \mu^2 (\Delta t)^2 \end{aligned}\right\}$$

from which

$$\left(\frac{\mu}{p-q}\right)^2 \Delta t = \sigma^2 + \mu^2 \Delta t$$

so

$$\frac{\mu^2 \Delta t}{(p-q)^2} \cdot 4pq = \sigma^2$$

Thus once we have selected Δt, the values of p and Δx are determined via

$$\boxed{\begin{aligned} p &= \frac{1}{2} \pm \frac{1}{2\sqrt{\alpha}} \\ \Delta x &= |\mu\,\Delta t\,\sqrt{\alpha}\,| \end{aligned}} \qquad \left(\alpha \equiv 1 + \frac{\sigma^2}{\mu^2 \Delta t}\right)$$

(we choose the sign for the expression for p according to the sign of μ).

(ii) If we want to match the first two moments of $S_{\Delta t} = \exp(X_{\Delta t})$, we find the equations are

$$\begin{cases} p\,e^{\Delta x} + q\,e^{-\Delta x} = e^{r\Delta t} \\ p\,e^{2\Delta x} + q\,e^{-2\Delta x} = e^{(2r+\sigma^2)\Delta t} \end{cases}$$

If we make a choice of Δt, then the variable $y = e^{\Delta x}$ solves the quadratic

$$y^2 - y\left(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\right) + 1 = 0$$

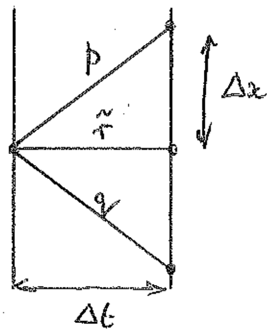
which is to say that y is the unique root in $(1,\infty)$, equal to

$$\boxed{\tfrac{1}{2}\left[e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t} + \left\{ \left(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\right)^2 - 4 \right\}^{\frac{1}{2}} \right].}$$

[This is what is in the footnote on p 222 of edition 4 of Hull's book.]

2) <u>Trinomial case</u> This time, we may pick Δx and Δt, and select the probabilities $p, q, \tilde{r}$.

(i) Matching the first two moments of log price change,



$$\begin{cases} (p-q)\,\Delta x = \mu \Delta t \\ (p+q)(\Delta x)^2 = \mu^2(\Delta t)^2 + \sigma^2 \Delta t \end{cases}$$

Thus

$$\boxed{\begin{aligned} p &= \tfrac{1}{2}(c^2+b+c) \\ q &= \tfrac{1}{2}(c^2+b-c) \end{aligned}} \qquad \begin{aligned} c &\equiv \mu\Delta t/\Delta x \\ b &\equiv \sigma^2\Delta t/(\Delta x)^2 \end{aligned}$$

For these to define genuine probabilities, we must have the inequalities

$$\boxed{|c| < c^2 + b < 1.}$$

There are various ways we could go about choosing Δt, Δx. For example, if we select $\rho \equiv \Delta t/\Delta x < |\mu|^{-1}$ then the above inequalities translate to

$$\rho|\mu| < \rho^2\mu^2 + \sigma^2\Delta x \cdot \rho^2 < 1$$

which can always be satisfied for <u>some</u> choice of Δx. Or we could choose Δt, whereupon the inequalities become

$$|\mu|\Delta t\,\Delta x < \mu^2\Delta t^2 + \sigma^2\Delta t < \Delta x^2$$

which can always be solved. Or we could choose Δx, and find that Δt must lie in the non-empty interval

$$\left(\left(2|\mu|\Delta x - \sigma^2\right)^+, \ \frac{(\sigma^4 + 4\mu^2\Delta x^2)^{\frac{1}{2}} - \sigma^2}{2\mu^2} \right).$$

Set $e^{rh} \equiv \theta$, $e^{\sigma^2 h} \equiv \rho$, so one inequality says

$$e^{\delta} < \frac{\theta(\theta\rho - 1)}{\theta - 1}$$

and the other

$$e^{\delta} > \tfrac{1}{2}\left(\theta^{-1} + \theta\rho + \{(\theta^{-1}+\theta\rho)^2 - 4\}^{\frac{1}{2}} \right)$$

We check

$$\left[\frac{\theta(\theta\rho-1)}{\theta-1}\right]^2 - \frac{\theta(\theta\rho-1)}{\theta-1}\left(\theta^{-1}+\theta\rho\right) + 1 > 0,$$

which it is; since clearly $\xi > 1$, we must therefore have $\xi > e^{\delta^*}$.

(ii) We could also match the first two moments of $S(\Delta t)$, and then

$$\begin{cases} p(e^{\Delta x}-1)+q(e^{-\Delta x}-1) = e^{r\Delta t}-1 \\ p(e^{2\Delta x}-1) + q(e^{-2\Delta x}-1) = e^{(2r+\sigma^2)\Delta t}-1 \end{cases}$$

Algebra gives us (with $\Delta t \equiv h$, $\Delta x \equiv \delta$)

$$q = \frac{e^{(2r+\sigma^2)h} - e^{rh} - e^{rh+\delta} + e^{\delta}}{(e^{2\delta}-1)(e^{\delta}-1)} \cdot e^{2\delta}$$

$$p = \frac{\left(e^{(2r+\sigma^2)h+\delta} - e^{rh+\delta} - e^{rh} + 1\right)}{(e^{2\delta}-1)(e^{\delta}-1)} > 0,$$

both of which are positive, provided

$$e^{\delta} < e^{rh}\left(e^{(r+\sigma^2)h}-1\right)/\left(e^{rh}-1\right) \equiv \xi$$

So for h fixed, p and q are both >0 provided δ is not too big. What is the condition for $p+q<1$? This reduces after a little algebra to

$$e^{2\delta} - e^{\delta}\left(e^{-rh} + e^{(r+\sigma^2)h}\right) + 1 > 0$$

so the condition $p+q<1$ will be satisfied provided δ is at least δ^*,

$$e^{\delta^*} = \frac{1}{2}\left[e^{-rh} + e^{(r+\sigma^2)h} + \left\{\left(e^{-rh} + e^{(r+\sigma^2)h}\right)^2 - 4\right\}^{\frac{1}{2}}\right]$$

which appeared before in the binomial analysis, not surprisingly. The interval for δ will be non-empty iff

$$\xi^2 - \xi\left(e^{-rh} + e^{(r+\sigma^2)h}\right) + 1 > 0$$

which, it may be confirmed, certainly holds: we may therefore pick any $\delta \equiv \Delta x$ in the interval $(\delta^*, \log \xi)$.

How in this instance could we choose the good values of δ, h? We are particularly concerned with construction of a look-up table, and wish to keep values of δ, h which are going to be valid for a range of values $r \in [\underline{r}, \bar{r}]$, $\sigma \in [\underline{\sigma}, \bar{\sigma}]$.

Taking the inequality guaranteeing positivity of p, q, we shall have

$$\frac{e^{(r+\sigma^2)h}-1}{1-e^{-rh}} > \frac{r+\sigma^2}{r} = 1+\frac{\sigma^2}{r} \geq 1+\underline{\sigma}^2/\bar{r}$$

So if we pick δ so that

$$\boxed{\exp(\delta) \leq 1+\underline{\sigma}^2/\bar{r}}$$

then whatever values of σ, r in the given interval, we shall have $p>0$, $q>0$.
How about $p+q<1$? The inequality now is

$$e^{\delta}+e^{-\delta} > e^{-rh}+e^{(\sigma^2+r)h}$$

so if we choose h so small that

$$\boxed{e^{\delta}+e^{-\delta} > e^{-\bar{r}h}+e^{(\bar{\sigma}^2+\bar{r})h}}$$

then throughout the range of values of r, σ, we are guaranteed $p+q<1$.

Initial term-structure of volatility in time-dependent CIR models (8/12/94)

(i) I'll use the notation of "Which model for ... ". Simon Babbs points out that the volatility of f_{tT} $(T>t)$ must inevitably be decreasing in T for a time-dependent Vasicek model; this we see from the fact that

$$f_{tT} = r_t \; B_T(t,T) + A_T(t,T)$$

so that the volatility of f_{tT} is $\sigma_t \; B_T(t,T)$, and from the expression (2.9) in that paper, we see that (assuming $\beta > 0$ always) this has to be a decreasing function of T

(ii) <u>Is the same true of the time-dependent CIR?</u> Here, the volatility of f_{tT} is $\sigma_t^{-1} \, 2\sqrt{r_t} \; B_T(t,T)$, where the function B is now the function introduced in the appendix, solving the Riccati equation (A.4). If we were given a function $T \mapsto \rho(T)^{-2}$, when could we be sure that there existed a time-dependent CIR model with $B_T(t,T) = \rho(T)^{-2}$? If we consult the equation (A.14), we see we need to be able to find non-negative $\beta(\cdot)$ and $\sigma^2(\cdot)$ such that

$$\frac{\partial^2}{\partial T^2}\left(-2\log\rho\right) = -\beta' - \tfrac{1}{2}\beta^2 - \sigma^2 + \tfrac{1}{2}\left[\frac{\partial}{\partial T}\left(\log\rho^{-2}\right)\right]^2$$

or again

$$-\frac{2\rho''}{\rho} = -\beta' - \tfrac{1}{2}\beta^2 - \sigma^2$$

so if we express β as $\beta = 2\varphi'/\varphi$, we find that the equation for φ is

$$\varphi'' = \frac{\varphi}{2}\left(\frac{2\rho''}{\rho} - \sigma^2\right).$$

If we can find a solution φ to this which is <u>always positive</u>, then we have a solution β, σ^2 which give the required initial term-structure of vol. If ρ were <u>convex</u>, this can certainly be done, but what then about A? Can we be sure that the function a is non-negative??

(iii) Let's also investigate some <u>necessary</u> conditions. From the interpretation $-\log P(t,T) = z_t \, B(t,T) + A(t,T)$, it is clear that $T \mapsto A(t,T)$, $B(t,T)$ must be increasing functions. So let's suppose we are given certain yield and volatility curves

$$\begin{cases} Y(T) = z_t\, B(t,T) + A(t,T) & Y'(t) = r_t = \tfrac{1}{2}\sigma_t^2 z_t \\ V(T) = \sqrt{2z_t}\; B'(t,T) & V(t) = \sigma_t \sqrt{r_t} \end{cases}$$

Now look at $Y'(T)/V(T) = \{ z_t\, B'(t,T) + A'(t,T) \} / \sqrt{2z_t}\; B'(t,T)$

$$= \sqrt{\frac{z_t}{2}} + \frac{1}{\sqrt{2z_t}} \cdot \frac{A'(t,T)}{B'(t,T)}$$

I claim that this is non-decreasing in T, because, from (A.10),

$$\frac{A'(t,T)}{B'(t,T)} = \frac{\int_t^T a_u\; B'(u,T)\,du}{B'(t,T)} = \int_t^T a_u\; \frac{\xi_u}{\xi_t} \left\{ \frac{\psi(t,T)}{\psi(u,T)} \right\}^2 du\,.$$

We have a, ξ both non-negative, so provided for fixed $t < u$ we have that $\psi(t,T)/\psi(u,T)$ is increasing with T, we've all we need. But from (A.11) we have

$$\frac{\partial^2}{\partial t\,\partial T} \log \psi(t,T) < 0 \quad \therefore \quad \frac{\partial}{\partial T} \log \psi(t,T) - \frac{\partial}{\partial T} \log \psi(u,T) > 0\,. \quad \text{Done.}$$

So the nice interpretation is that for these time-inhomogeneous CIR models

$$\boxed{\frac{f_{tT}}{\mathrm{vol}(f_{tT})} \text{ is non-decreasing in } T, \text{ for each } t.}$$

(iv) And another thing worth recording. In the time-inhomogeneous versions of V and CIR, we have an expression

$$f_{tT} = r_t\, B'(t,T) + A'(t,T)\,, \qquad \sigma_{tT}^2 = B'(t,T)^2 \; \frac{d\langle r\rangle_t}{dt}$$

and hence

$$\boxed{\frac{f_{tT}}{f_{tt}} - \frac{\sigma_{tT}}{\sigma_{tt}} = \frac{A'(t,T)}{r_t} \geq 0}$$

(This is certainly clear for CIR, but less so for V, because it's not automatically true that $A(t,\cdot)$ should be increasing..!)

<u>How long does BM spend outside an interval?</u> (10/1/95)

Take $X_t = B_t + ct$ where the drift c is strictly positive, fix $a, b > 0$ and let

$$A_t^+ \equiv \int_0^t I_{\{X_u > b\}}\, du, \qquad A_t^- = \int_0^t I_{\{X_u < -a\}}\, du.$$

The aim is to calculate

$$E \exp\left(-\alpha A_T^- - \beta A_T^+\right) \qquad \text{where } T \sim \exp(\lambda), \text{ indep of } X.$$

(i) Let's collect a few facts about the excursion law of X away from 0.

$$\text{Rate of excns reaching } b>0 = \lim \frac{1}{2\varepsilon} P^{2\varepsilon}(\text{hit } b \text{ before } 0) = \frac{c}{1-e^{-2cb}}$$

$$\text{Rate of excns reaching } -a<0 = \frac{c}{e^{2ca}-1} = -c + \frac{c}{1-e^{-2ca}}.$$

Since the excursion looks like a diffusion with generator $\mathcal{G} = \frac{1}{2}D^2 + c\coth cx\, D$ if conditioned to hit $b>0$ before 0, we know that it's the law of the mod of a 3d BM with drift c; hence from Cameron Martin, we conclude that

$$E_c \exp(-\lambda H_b) = E_0\left[\exp-(\lambda+\tfrac{1}{2}c^2)H_b\right] / E_0 \exp\left(-\tfrac{1}{2}c^2 H_b\right)$$

$$= \frac{\tilde\theta}{c} \cdot \frac{\sinh cb}{\sinh \tilde\theta b} \qquad \left[\tilde\theta \equiv \sqrt{2\lambda + c^2}\right]$$

Hence

Rate of excursions with max $< b$ which get killed is

$$\int_0^b d\left(\frac{-c}{1-e^{-2cx}}\right)\left[1 - \left(\frac{\tilde\theta}{c}\,\frac{\sinh cx}{\sinh\tilde\theta x}\right)^2\right]$$

$$= \int_0^b \frac{c^2}{2}\,\frac{dx}{\sinh^2 cx}\left[1 - \left(\frac{\tilde\theta}{c}\cdot\frac{\sinh cx}{\sinh\tilde\theta x}\right)^2\right]$$

$$= \frac{1}{2}\left[\tilde\theta\coth\tilde\theta b - c\coth cb\right].$$

(ii) We shall have

$$E\left[e^{-\alpha A_\tau^- - \beta A_\tau^+}\right] = (k_+ + k_-)/(K_+ + K_-)$$

where K_+ = rate of upward excursions which contain β or λ mark

k_+ = ——————————————— a λ mark, and no β-mark before the first λ mark,

where also k_-, K_- are similarly defined. We shall therefore need to compute

P^b (β or λ marked before hit 0), P^b (λ-marked before hit 0, and before β-marked).

Rate of upward excursions from b which are β or λ marked (from previous page)

$$= \tfrac{1}{2}(\theta^* + c), \qquad \theta^* \equiv \sqrt{2\lambda + 2\beta + c^2}$$ [DON'T FORGET THE INFINITE EXCURSION]

Rate of downward excursions from b which hit zero and are λ-marked before that

$$= \frac{c}{e^{2cb}-1}\cdot\left\{1 - \frac{\tilde\theta}{c}\,\frac{\sinh cb}{\sinh\tilde\theta b}\right\}$$

Rate of excursions down from b which get λ-marked or hit 0

$$= \tfrac{1}{2}\left[\tilde\theta\coth\tilde\theta b - c\coth cb\right] + \frac{c}{e^{2cb}-1} = \tfrac{1}{2}\left[\tilde\theta\coth\tilde\theta b - c\right],$$

Thus

$$\boxed{P^b\left[\text{hit } 0 \text{ before } \beta \text{ or } \lambda \text{ marked}\right] = \frac{\tilde\theta e^{-cb}/\sinh\tilde\theta b}{\theta^* + \tilde\theta\coth\tilde\theta b}}$$

Rate of excursions down which get λ-marked before hitting 0.

$$= \tfrac{1}{2}\left(\tilde\theta\coth\tilde\theta b - c\coth cb\right) + \frac{c}{e^{2cb}-1}\left(1 - \frac{\tilde\theta}{c}\,\frac{\sinh cb}{\sinh\tilde\theta b}\right)$$

$$= \tfrac{1}{2}\left[\tilde\theta\coth\tilde\theta b - c - \frac{\tilde\theta e^{-cb}}{\sinh\tilde\theta b}\right],$$

so

$$\boxed{P^b\left[\lambda \text{ mark before hit } 0, \text{ and before } \beta\text{-mark}\right] = \frac{\frac{\lambda}{\lambda+\beta}(\theta^* + c) + \tilde\theta\coth\tilde\theta b - c - \frac{\tilde\theta e^{-cb}}{\sinh\tilde\theta b}}{\theta^* + \tilde\theta\coth\tilde\theta b}}$$

So now we can assemble all this.

$$2K_+ = 2\times\text{rate of upward excursions containing a } \beta \text{ or } \lambda \text{ mark}$$

$$= \tilde{\theta}\coth\tilde{\theta}b - c\coth cb + \frac{2c}{1-e^{-2cb}}\left(1 - \frac{\tilde{\theta}}{c}\cdot\frac{\sinh cb}{\sinh\tilde{\theta}b}\right)$$

$$+ \frac{\tilde{\theta}e^{cb}}{\sinh\tilde{\theta}b} P^b(\text{for } \lambda \text{ mark before hit } 0)$$

$$= \tilde{\theta}\coth\tilde{\theta}b + c - \frac{\tilde{\theta}e^{cb}}{\sinh\tilde{\theta}b} P^b(\text{hit } 0 \text{ before } \beta \text{ or } \lambda \text{ mark})$$

$$= \tilde{\theta}\coth\tilde{\theta}b + c - \frac{\tilde{\theta}^2/\sinh^2\tilde{\theta}b}{\theta^* + \tilde{\theta}\coth\tilde{\theta}b}$$

$$2k_+ = 2\times\text{rate of upward excursions containing a } \lambda\text{-mark before } \beta\text{-mark}$$

$$= \tilde{\theta}\coth\tilde{\theta}b - c\coth cb + \frac{2c}{1-e^{-2cb}}\left[1 - \frac{\tilde{\theta}}{c}\,\frac{\sinh cb}{\sinh\tilde{\theta}b}\right]$$

$$+ \frac{\tilde{\theta}e^{cb}}{\sinh\tilde{\theta}b} P^b(\lambda\text{-mark before } \beta \text{ mark, and before hit } 0)$$

$$= \tilde{\theta}\coth\tilde{\theta}b + c - \frac{\tilde{\theta}e^{cb}}{\sinh\tilde{\theta}b} P^b(\text{no } \lambda\text{-mark before } \beta\text{-mark or hit } 0)$$

$$= \tilde{\theta}\coth\tilde{\theta}b + c - \frac{\tilde{\theta}e^{cb}}{\sinh\tilde{\theta}b}\;\frac{\beta(\beta+\lambda)^{-1}(\theta^*+c) + \tilde{\theta}e^{-cb}/\sinh\tilde{\theta}b}{\theta^* + \tilde{\theta}\coth\tilde{\theta}b}$$

Likewise,

$$2K_- = \tilde{\theta}\coth\tilde{\theta}a - c - \frac{\tilde{\theta}^2/\sinh^2\tilde{\theta}a}{\theta' + \tilde{\theta}\coth\tilde{\theta}a} \qquad \left[\theta' = \sqrt{2\lambda+2\alpha+c^2}\right]$$

$$2k_- = \tilde{\theta}\coth\tilde{\theta}a - c - \frac{\tilde{\theta}e^{-ca}}{\sinh\tilde{\theta}a}\;\frac{\alpha(\alpha+\lambda)^{-1}(\theta'-c) + \tilde{\theta}e^{ca}/\sinh\tilde{\theta}a}{\theta' + \tilde{\theta}\coth\tilde{\theta}a}$$

(iii) In the special case where $c = 0$, $\alpha = \beta$, we write $\theta = \sqrt{2\lambda}$, $\theta^* = \sqrt{2\lambda + 2\alpha}$ and obtain

$$K_+ = \theta \, \frac{\theta^* \cosh \theta b + \theta \sinh \theta b}{\theta^* \sinh \theta b + \theta \cosh \theta b}$$

$$k_+ = \theta \, \frac{\theta^* \cosh \theta b + \theta \sinh \theta b - 2\alpha/\theta^*}{\theta^* \sinh \theta b + \theta \cosh \theta b}$$

with similar expressions for K_-, k_-. If we also assume $a = b$, the expression for $E\, e^{-\alpha A_T^- - \beta A_T^+}$ reduces even further to

$$E\, e^{-\alpha (A_T^- + A_T^+)} = \frac{\theta^* \cosh \theta b + \theta \sinh \theta b - 2\alpha/\theta^*}{\theta^* \sinh \theta b + \theta \cosh \theta b} .$$

If we take the special case $b \to 0$, $a \to \infty$, we get $K_+ \to \theta^*$, $k_+ \to \theta^* - 2\alpha/\theta^*$, $K_- \to \theta$, $k_- \to \theta$, so we get

$$E\, e^{-\alpha A_T^+} = \frac{\theta^* - 2\alpha/\theta^* + \theta}{\theta + \theta^*} = \sqrt{\frac{\lambda}{\lambda + \alpha}} .$$

(ii) Since $P[S_t - J_t > a] = P[\tau_a < t]$, we deduce that

$$\int_0^\infty \lambda e^{-\lambda t} P[S_t - J_t > a]\, dt \equiv P[S_T - J_T > a] \qquad \text{if } T \sim \exp(\lambda) \text{ indep of } B$$

$$= \operatorname{sech}^2(\theta a/2)$$

so this determines the law of $S_T - J_T$. But if we observe from Brownian scaling that

$$S_t - J_t \overset{\mathcal{D}}{=} \sqrt{t}\,(S_1 - J_1)$$

then we can link the moments;

$$E\left[(S_T - J_T)^n\right] = \int_0^\infty \lambda e^{-\lambda t}\, t^{n/2}\, E\left[(S_1 - J_1)^n\right] dt$$

$$= \lambda^{-n/2}\, E\left[(S_1 - J_1)^n\right]\, \Gamma\left(\tfrac{n}{2} + 1\right).$$

Since the LHS is easy to evaluate numerically, we can from this rapidly compute as many moments of $S_1 - J_1$ as we choose.

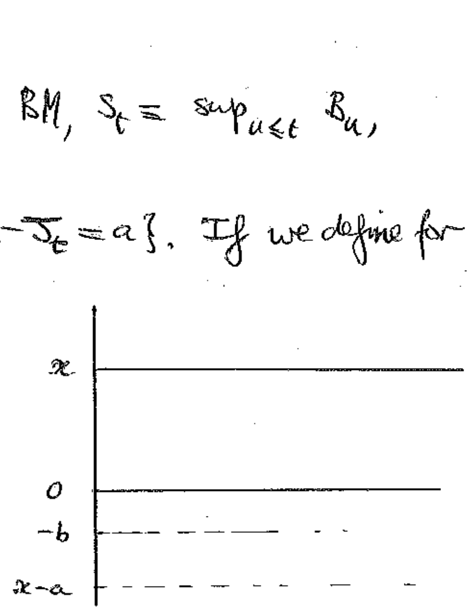
(iii) We could also get the law of $S_T - J_T$ from the Wiener-Hopf factorisation of $\{B_u : u \le t\}$. This would perhaps be the best way to tackle the case of drifting Brownian motion.

Distribution of $S_t - J_t$ (15/1/95)

Charlie Freifeld asks this. If B is BM, $S_t \equiv \sup_{u \le t} B_u$, $J_t \equiv \inf_{u \le t} B_u$, what is the law of $S_t - J_t$?

(i) One method is to go for the law of $\tau_a \equiv \inf\{t : S_t - J_t = a\}$. If we define for $0 \le x \le a$

$$\varphi(x) = E\left[e^{-\lambda \tau_a} \,\middle|\, \text{start with range} = x,\ \text{BM at } 0\right],$$



what is φ?

It's easy to prove that, with $\theta \equiv \sqrt{2\lambda}$,

$$E^y\left[e^{-\lambda H_x} ;\ H_x < H_{-b}\right] = \frac{\sinh \theta(y+b)}{\sinh \theta(x+b)} \qquad b > 0, \quad -b < y < x$$

so that

$$E^0\left[e^{-\lambda H_x} : -\min_{0 \le u \le H_x} B_u \in db\right] / db = \frac{\theta \sinh \theta x}{\sinh^2 \theta(x+b)}.$$

Hence

$$\varphi(x) = \int_{b=0}^{a-x} \frac{\theta \sinh \theta x}{\sinh^2 \theta(x+b)} \varphi(x+b)\, dx + \frac{\sinh \theta x}{\sinh \theta a},$$

the decomposition arising from considering the two possibilities that B reaches x before $x-a$, or alternatively that B reaches $x-a$ before x. Setting $f(x) \equiv \varphi(x)/\sinh \theta x$, we quickly derive an ODE for f:

$$f'(x) = -\theta f(x) / \sinh \theta x,$$

solved by

$$f(x) = \text{const.} \coth(\theta x/2).$$

With the boundary condition $\varphi(a) = 1$, this determines $\varphi(0)$;

$$\boxed{E\left[e^{-\lambda \tau_a}\right] = \operatorname{sech}^2(\theta a/2).}$$

So $\tau_a \overset{\mathcal{D}}{=} T_1 + T_2$,

T_j are IID, with law of exit time of BM from $[-a/2, a/2]$

Questions, conjectures, remarks.

(1) The diffusions

$$dx = \sigma x^{\gamma} dW + (\alpha - \beta x)\, dt$$

are considered in modelling term-structure. If $\gamma < \frac{1}{2}$, there's no pathwise uniqueness, but is there uniqueness in law if $\alpha > 0$?

Chan Karolyi Longstaff + Sanders look at these models + crudely assume that

$x_{t+1} - x_t - \alpha - \beta x_t \sim N(0, \sigma^2 x_t^{2\gamma})$ ← but how good is this?

2) If B is a $BM(\mathbb{R}^d)$, $d \geq 2$, $B_0 = 0$, and A is polar for B, is it the case that A is polar for $\int_0^t B_s\, ds$? (Not in general the same polar-sets, because if we take Γ to be $(W_t)_{0 \leq t \leq 1}$ to be a piece of BM in $\mathbb{R}^3$, then it will a.s. miss any C^1 curve (eg $\int_0^{\cdot} B_s\, ds$) but may with pos prob^y intersect $(B_t)_{0 \leq t \leq 1}$ since $BM(\mathbb{R}^3)$ has double points ⇝ so Γ is polar for $\varphi_t \equiv \int_0^t B_s\, ds$, but not for B. But could we have all B-polar sets are φ-polar?)

3) How about some analysis of a world where 'true' price is only imperfectly observed? Does this save us from infinite trading?

4) Extend the high/low estimators of volatility to help in the case of correlations?