Joint law of the biggest jump of subordinator, and the process (26/1/95)

Suppose that Y is a subordinator, $\eta_t = \sup_{s\le t} \Delta Y_s$. What is the law of (η_t, Y_t)?

If $E\exp(-\lambda Y_t) = \exp\left[-t\left(c\lambda + \int_0^\infty (1-e^{-\lambda x})\,\mu(dx)\right)\right]$, then we have

$$P[\eta_t \le a] = \exp\{-t\bar\mu(a)\} \quad \therefore \quad P[\eta_t \in da] = t\, e^{-t\bar\mu(a)}\,\mu(da),$$

and so

$$\boxed{E\, e^{-\alpha\eta_t - \lambda Y_t} = \int_0^\infty t\,\mu(dx)\, e^{-t\bar\mu(x)} \exp\left[-\alpha x - \lambda x - t\left(c\lambda + \int_0^x (1-e^{-\lambda v})\,\mu(dv)\right)\right]}$$

Interest in this is sparked by a result of Bertoin & Doney, that if $X_t = Y_t^+ - Y_t^-$ is the difference of two independent driftless subordinators, then 0 is regular for $\mathbb{R}^+$ iff

$$\int_{0+} G_-(dy)\,\bar\mu_+(y) = +\infty$$

where $G_-(dy) = \int_0^\infty P(Y_t^- \in dy)\,dt$ is the Green function of Y^- (Compare with my 1984 AIHP paper, where the same test would arise, but using the law of the minimum up to an independent exponential time....)

Gaussian measure on $L^2(\mathbb{R};\mathbb{C})$

Noah Linden is interested in a Gaussian measure μ on $L^2(\mathbb{R};\mathbb{C}) \equiv \{f_1 + if_2 : f_1, f_2 \in L^2(\mathbb{R})\}$ which should be described formally by the property

$$\int \mu(d\rho)\,\exp\{\langle f,\rho\rangle + \langle \rho, g\rangle\} = \exp(\langle f, g\rangle) \qquad \forall f, g \in L^2(\mathbb{R},\mathbb{C}).$$

It's impossible to make μ live on the topological dual $(= L^2(\mathbb{R},\mathbb{C}))$ of $L^2(\mathbb{R};\mathbb{C})$, since we have a concrete representation of this Gaussian measure. Indeed, if X_1 and X_2 are independent BMs, it's not difficult to prove that

$$E\exp\left[\frac{1}{\sqrt2}\int (f_1 - if_2,\, dX_1 + i\,dX_2) + \frac{1}{\sqrt2}\int (g_1 + ig_2,\, dX_1 - i\,dX_2)\right]$$

$$= \exp\langle f, g\rangle \equiv \exp\left(\int (f_1 - if_2)(g_1 + ig_2)\,dx\right).$$

So the Gaussian linear functional exists, but is not continuous...

Some thoughts on recursive utility (30/1/95)

(i) Duffie and Epstein consider a set-up where there is an agent who consumes according to $(c_t)_{0 \le t \le T}$ and who obtains residual utility V_t from the consumption over $[t,T]$, where V satisfies

$$(*) \qquad V_t = E_t\left[\int_t^T f(c_s, V_s)\, ds \right],$$

and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is concave (strictly) and strictly increasing in both arguments. The agent's aim is to

$$\max \; E \int_0^T f(c_s, V_s)\, ds \qquad \text{subj to} \quad E \int_0^T \xi_s c_s\, ds = w_0$$

$$\text{and} \quad V_t = E_t\left[\int_t^T f(c_s, V_s)\, ds \right].$$

(ii) If $f_*(y,z) \equiv \sup\{ f(c,v) - cy - vz \; : \; c \ge 0, v \in \mathbb{R} \}$, then f_* is the concave conjugate of f, and the sup is attained at a unique (c,v), because of strict concavity of f. Given V satisfying $(*)$, choose processes (y_s), (z_s) so that

$$f(c_s, V_s) = f_*(y_s, z_s) + c_s y_s + V_s z_s$$

and note that

$$V_t + \int_0^t \left(f_*(y_s, z_s) + c_s y_s + V_s z_s \right) ds = M_t, \quad \text{a martingale}$$

$$\equiv V_t + \int_0^t (\gamma_s + V_s z_s)\, ds, \qquad \text{say}$$

Hence $e^{Z_t} V_t + \int_0^t e^{Z_s} \gamma_s\, ds$ is a martingale $(Z_s \equiv \int_0^s z_u\, du)$, and $V_T = 0$, so we get the representation

$$\boxed{\begin{gathered} e^{Z_t} V_t = E_t\left(\int_t^T \gamma_u e^{Z_u}\, du \right) \\ \gamma_t \equiv f_*(y_t, z_t) + c_t y_t . \end{gathered}}$$

(iii) Suppose now that we had found an optimal solution c^*, V^*, with corresponding multiplier processes y^*, z^*. Suppose that (c, V) were any feasible (consumption, residual utility) pair. Then

$$V_t = E_t \int_t^T f(c_s, V_s)\, ds$$

$$\leq E_t \Big[\int_t^T \big(f_*(y_s^*, z_s^*) + y_s^* c_s + z_s^* V_s \big)\, ds \Big]$$

$$\equiv E_t \int_t^T (\alpha_s + z_s^* V_s)\, ds$$

and now feed this back into itself;

$$V_t \leq E_t \int_t^T \Big(\alpha_s + z_s^* \int_s^T (\alpha_u + z_u^* V_u)\, du \Big)\, ds$$

$$= E_t \int_t^T \Big(\alpha_s + \alpha_s Z_{ts}^* + \big(\textstyle\int_t^s z_u^*\, du\big)\, z_s^* V_s \Big)\, ds$$

$$Z_{ts}^* = \int_s^t z_u^*\, du$$

so we prove inductively that

$$V_t \leq E_t \int_t^T \Big\{ \alpha_s \Big(\sum_{r=0}^{N-1} \frac{1}{r!} (Z_{ts}^*)^r \Big) + \frac{1}{(N-1)!} (Z_{ts}^*)^{N-1} z_s^* V_s \Big\}\, ds$$

leading to

$$V_t \leq E_t \Big[\int_t^T \alpha_s \exp(Z_{ts}^*)\, ds \Big],$$

or again

$$\boxed{e^{Z_t^*} V_t \leq E_t \Big[\int_t^T \big(f_*(y_s^*, z_s^*) + y_s^* c_s \big) e^{Z_s^*}\, ds \Big]}$$

(iv) If $V_t = M_t + \int_0^t \alpha_s\, ds$ is a quite general semimartingale, $V_T = 0$, and if $f(\cdot, V)$ is onto $\mathbb{R}$ for each V, we can choose a consumption plan (c_t) to make V the residual utility, just by choosing

$$f(c_t, V_t) = -\alpha_t .$$

Arbitrage from fractional Brownian motion? (5/2/95)

(i) The fractional Brownian motion (X_t) is a zero-mean Gaussian process defined by

(1) $$X_t \equiv \int_{-\infty}^{t} (t-s)^{H-\frac12}\, dW_s - \int_{-\infty}^{0} (-s)^{H-\frac12}\, dW_s \qquad (t\in\mathbb{R})$$

where the parameter H is in $(0,1)$. Since $H=\frac12$ is the Brownian case, let's exclude it from all further consideration.

In view of this representation, it's tempting to conjecture that for $t>0$

(2) $$E\left[X_t \mid X_u : u\le 0\right] = \int_{-\infty}^{0} (t-s)^{H-\frac12}\, dW_s - \int_{-\infty}^{0} (-s)^{H-\frac12}\, dW_s .$$

(If we took the c/e given $(W_u, u\le 0)$, this would yield the RHS, but it's not immediately obvious that these two σ-fields are the same.)

(ii) The case $H\equiv \frac12-\varepsilon \in (0,\frac12)$. In this instance, it turns out that for $a>0$,

(3) $$E\left[X_a \mid X_u : u\le 0\right] = \int_{-\infty}^{0} \frac{|t/a|^{\varepsilon-1}}{a+|t|}\, X_t \,\frac{dt}{\Gamma(\varepsilon)\,\Gamma(1-\varepsilon)}$$

(4) $$= \int_{-\infty}^{0}\left[(a-s)^{H-\frac12} - (-s)^{H-\frac12}\right] dW_s .$$

How do we see this? If we define a measure m on $\mathbb{R}^+$ by

$$\frac{m(dt)}{dt} = \frac{(t/a)^{\varepsilon-1}}{(a+t)\ \Gamma(\varepsilon)\ \Gamma(1-\varepsilon)}$$

then m is a probability measure, and for $s>0$

$$\int_0^\infty m(dt)\big((s-t)^+\big)^{-\varepsilon} \equiv \int_0^\infty m(dt)\big((s-t)^+\big)^{H-\frac12}$$

(5) $$= \int_0^s (s-t)^{-\varepsilon}\, m(dt)$$

(6) $$= (a+s)^{-\varepsilon},$$

as we shall presently show. This then establishes (3)=(4) and that's enough.
To prove the claim that (5)=(6), Laplace transform both sides. The LT of (5) is

$$\int_0^\infty e^{-\lambda s} \int_0^s (s-t)^{-\varepsilon} m(dt) = \int_0^\infty m(dt)\, e^{-\lambda t} \int_0^\infty e^{-\lambda u}\, u^{-\varepsilon} du$$

$$= \int_0^\infty m(dt)\, e^{-\lambda t}\ \Gamma(1+\varepsilon)\ \lambda^{-1+\varepsilon}$$

and the LT of (6) gives

$$\int_0^\infty e^{-\lambda s}\, ds \int_0^\infty e^{-(a+s)v}\ v^{\varepsilon-1} \frac{dv}{\Gamma(\varepsilon)}$$

$$= \int_0^\infty e^{-av} \frac{v^{\varepsilon-1}}{\lambda+v} \frac{dv}{\Gamma(\varepsilon)} = \int_0^\infty e^{-\lambda u} \frac{(u/a)^{\varepsilon-1}}{a+u} \frac{du}{\Gamma(\varepsilon)}\ \lambda^{\varepsilon-1}$$

and that's it.

(iii) <u>The case $H = \frac{1}{2}+\delta \in (\frac{1}{2}, 1)$</u>. By extension of what happened for $0<H<\frac{1}{2}$, we'd guess that the mixing measure we take ought to be

$$(7)\quad m(dt) = \frac{(t/a)^{-\delta-1}}{(a+t)} \frac{-\delta}{\Gamma(1+\delta)\Gamma(1-\delta)}\ dt\ .$$

Note that this measure is negative, and not of finite mass. We need to check that for all $s>0$

$$(8)\quad \int_0^\infty m(dt) \left[((s-t)^+)^\delta - s^\delta \right] = (a+s)^\delta - s^\delta$$

$$= \int_0^a \delta\, (u+s)^{\delta-1}\, du$$

$$= \int_0^a \delta \left(\int_0^\infty e^{-(u+s)x}\ x^{-\delta} \frac{dx}{\Gamma(1-\delta)} \right) du$$

We can express the LHS more helpfully as

$$(9)\quad -\int_0^\infty m(dt) \int_{(s-t)^+}^s \delta u^{\delta-1}\, du = -\int_0^s \delta u^{\delta-1}\, du\ \bar{m}(s-u)\ .$$

Forming the LT of the LHS of (8) therefore yields

$$-\int_0^\infty e^{-\lambda s}\,\bar{m}(s)\,ds \quad \delta\,\Gamma(\delta)\,\lambda^{-\delta} = -\Gamma(1+\delta)\,\lambda^{-\delta}\int_0^\infty e^{-\lambda s}\,\bar{m}(s)\,ds.$$

Taking the LT of the RHS of (8) gives

$$\int_0^a \delta\,du \int_0^\infty \frac{1}{\lambda+x}\, e^{-ux}\, \frac{x^{-\delta}\,dx}{\Gamma(1-\delta)}$$

$$= \int_0^\infty \frac{\delta x^{-\delta}\,dx}{\Gamma(1-\delta)} \cdot \frac{1}{\lambda+x} \cdot \frac{1-e^{-ax}}{x}$$

$x = \lambda t/a$

$$= \int_0^\infty \frac{\delta (t/a)^{-\delta-1}}{\Gamma(1-\delta)}\, \frac{dt}{a+t}\, \frac{1-e^{-\lambda t}}{\lambda}\,.\ \lambda^{-\delta}$$

and that's all we need.

So in either case, $\forall a>0$

$$\boxed{E\left[X_a \mid X_s : s\leq 0\right] = \int_{-\infty}^0 \left[(a-s)^{H-\frac12} - (-s)^{H-\frac12}\right] dW_s}$$

(iv) Define $\mathcal{G}_n \equiv \sigma(X_u : u \leq -2^{-n})$, $n\in\mathbb{Z}$, and let

$$Y_n \equiv \{X(-2^{-n}) - X(-2^{-n+1})\}\, 2^{nH}.$$

Then the process $(Y_n)_{n\in\mathbb{Z}}$ is stationary, and

$$Y_n - E(Y_n \mid \mathcal{G}_{n-1}) = 2^{nH}\int_{-2^{-n+1}}^{-2^{-n}} (-2^{-n}-s)^{H-\frac12}\, dW_s$$

which has variance 1. If we set $\eta_n \equiv E(Y_n \mid \mathcal{G}_{n-1})$, then we also have that the process $\left(\begin{pmatrix} Y_n \\ \eta_n\end{pmatrix}\right)_{n\in\mathbb{Z}}$ is stationary.

Abbreviate $-2^{-n} \equiv a_n$. Notice that the process $\binom{Y_n}{\eta_n}$ is ergodic, since any r.v. in $\bigcap_n \sigma\left(\binom{Y_k}{\eta_k}; k\leq n\right)$ is in the tail σ-field of W and therefore is trivial.

Making arbitrage <u>is</u> possible, but requires a more careful analysis. This is the topic of a short paper.

$$dX_{ir}\, dX_{jr} = \delta_{ij}\,(n - n\varepsilon)\,dt$$

Random ellipsoids again (15/2/95)

This is pretty similar to the calculations in M-F Bru's work on Wishart processes, and to the calculations of earlier work with David, so I'll just outline what happens. The differences are enough to be worth taking time over.

(1) We'll take a matrix $X \equiv (X_{ij})_{i=1,\dots,p,\ j=1,\dots,n}$ of diffusion processes.

$$\boxed{dX_{ir} = dM_{ir} - \beta X_{ir}\,dt} \qquad \boxed{\text{Assume } p \geq n}$$

where $dM_{ir}\,dM_{js} = \delta_{ij}(\delta_{rs} - \varepsilon)\,dt$, and ε is either 0 or $1/n$. The $\varepsilon = 0$ case is clear, the interpretation for $\varepsilon = 1/n$ follows from considering the columns of X to be IID OU processes centred on the sample mean, so that always $X\mathbf{1} = 0$. We now define the rnds matrix $(p \times p)$

$$\Sigma(t) \equiv X(t)\,X(t)^T$$

which, as previously, we express as $\Sigma = R\Lambda R^T$, $\partial R = R\,\partial A$, leading to

$$\partial\Lambda = \Lambda\,\partial A - \partial A.\Lambda + R^T\partial\Sigma\,R.$$

We have

$$d\Sigma_{ij} = X_{ir}\,dM_{jr} + X_{jr}\,dM_{ir} - 2\beta\,\Sigma_{ij}\,dt + \delta_{ij}(n - n\varepsilon)\,dt,$$

which yields

$$d\Sigma_{ij}\,d\Sigma_{k\ell} = (\delta_{j\ell}\Sigma_{ik} + \delta_{jk}\Sigma_{i\ell} + \delta_{ik}\Sigma_{j\ell} + \delta_{i\ell}\Sigma_{jk})\,dt.$$

If we consider the martingale part of $d\lambda_i$, we get

$$R_{qi}\,d\Sigma_{qs}\,dR_{si} + FV$$

and thus the quadratic covariations are

$$d\lambda_i\,d\lambda_j = R_{qi}R_{si}\,R_{rj}\,R_{tj}\,d\Sigma_{qs}\,d\Sigma_{rt}$$

$$= 4\delta_{ij}\,\Lambda_{ij}\,dt,$$

so we see the beginnings of the SDE for the λ's;

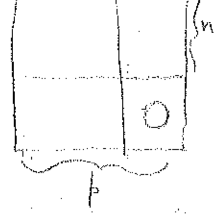
$$d\lambda_i = 2\sqrt{\lambda_i}\,dW_i + FV.$$

We may next analyse the martingale part of a_{ij}, and we find

$$(\lambda_j-\lambda_i)\,\partial a_{ij} = (R^T \partial\Sigma R)_{ij} \quad \text{so} \quad da_{ij}\, da_{rs}\,(\lambda_j-\lambda_i)(\lambda_s-\lambda_r)$$

$$= R_{ki}\, d\Sigma_{kl}\, R_{lj}\; R_{nr}\, d\Sigma_{mq}\, R_{qs}$$

$$= \delta_{ir}\Lambda_{js} + \delta_{is}\Lambda_{jr} + \delta_{jr}\Lambda_{is} + \delta_{js}\Lambda_{ir}$$

$d\Lambda =$ 

$$da_{ij}\, d\lambda_k = R_{rk}\,(da_{ij}\, d\Sigma_{rs})\,R_{sk}$$

$$= (\lambda_j-\lambda_i)^{-1}\, R_{rk}R_{sk}\; R_{qi}R_{mj}\; d\Sigma_{rs}\, d\Sigma_{qm}$$

$$= (\lambda_j-\lambda_i)^{-1}\left[2\delta_{ik}\,\Lambda_{kj} + 2\delta_{jk}\,\Lambda_{ki}\right] \quad = 0$$

If $\lambda_i \neq 0$, we can usefully rewrite

$$d\lambda_i = 2\sqrt{\lambda_i}\, dW_i - 2\beta\lambda_i\, dt + 2\lambda_i\left(\sum_{\substack{k\neq i\\ k=1}}^{n} \frac{1}{\lambda_i-\lambda_k}\right) dt + (\underset{-n}{p} - n\varepsilon + 1)\, dt$$

after some calculations that for $i, j \leq \min\{p, n\}$ or even provided either i or $j \leq n$

$$da_{ij}\ da_{ij} = -da_{ij}\ da_{ji} = \frac{\lambda_j + \lambda_i}{(\lambda_j - \lambda_i)^2}, \quad da_{ij}\, da_{rs} = 0$$

if $\{i,j\} \neq \{r,s\}$.

Further calculations give

$$da_{ij}\ d\lambda_k = 0 \qquad \text{for } i \neq j, \text{ all } k.$$

From here, we can obtain that

$$R^T \partial\Sigma\, R = R^T d\Sigma\, R + \tfrac{1}{2}\left[\partial A^T R^T d\Sigma\, R + R^T d\Sigma\, R\, dA\right]$$

$$\therefore\ (R^T\partial\Sigma R)_{ij} = (R^T d\Sigma\, R)_{ij} + \delta_{ij}\left(\sum_{q \neq i} \frac{\lambda_i + \lambda_q}{\lambda_i - \lambda_q}\right) dt$$

$$= \left\{R^T(X\,dM^T + dM\,X^T)R - 2\beta\Lambda\, dt + (A - n\varepsilon)dt\right\}_{ij}$$

$$+ \delta_{ij}\left(\sum_{\substack{q \neq i \\ q=1}}^{c} \frac{\lambda_i + \lambda_q}{\lambda_i - \lambda_q}\right) dt$$

$\left(\begin{array}{l} c = p \text{ if } \lambda_i > 0 \\ = n \text{ if } \lambda_i = 0 \end{array}\right)$

NB if $\lambda_i > 0$, sum is over all $q \neq i$, else if $\lambda_i = 0$, it's over all $q (\neq i)$ s.t. $\lambda_q > 0$.

So we shall have

$$\boxed{da_{ij} = \frac{\sqrt{\lambda_i + \lambda_j}}{|\lambda_i - \lambda_j|}\, dW_{ij} \qquad (i \neq j)}$$

$(\lambda_i > 0 \text{ or } \lambda_j > 0)$

where W_{ij} are independent Brownian motions indept of W_i, and $dW_{ij} = -dW_{ji}$

$$\boxed{d\lambda_i = 2\sqrt{\lambda_i}\, dW_i - 2\beta\lambda_i\, dt + n(1-\varepsilon)\, dt + \sum_{\substack{k \neq i \\ k=1}}^{p} \frac{\lambda_k + \lambda_i}{\lambda_i - \lambda_k}\, dt} \quad (\lambda_i \neq 0)$$

[our conventions are: if $\varepsilon = 0$, there are $n (\leq p)$ e-values which are generically positive, when $\varepsilon = n^{-1}$ there are $(n-1)$ generically positive evalues. We have $a_{ij} \equiv 0$ if λ_i and λ_j are both identically zero, otherwise a_{ij} moves in accordance with the rules above.]

$$\sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} \frac{\lambda_i}{\lambda_i - \lambda_j} = \frac{1}{2} n(n-1)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\lambda_i}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)} = 0$$

i, j, k distinct.

Writing the equation for the λ_i, $i \le n$, (where now we assume $\varepsilon = 0$ for definiteness) in another form

$$d\lambda_i = 2\sqrt{\lambda_i}\, dW_i - 2\beta\lambda_i\, dt + dt + 2\lambda_i \sum_{\substack{k=1\\ k\ne i}}^{n} \frac{1}{\lambda_i-\lambda_k}\, dt + (p-n)\,dt,$$

we see that it *will* depend on $p \ge n$. This agrees with Bru's result.

(ii) Can we obtain this diffusion by change of measure, or h-transforming, from the simpler SDE

$$(*) \qquad d\lambda_i = 2\sqrt{\lambda_i}\, dW_i - 2\beta\lambda_i\, dt + \kappa\, dt \qquad (\kappa \equiv p-n+1)\ ?$$

Well, the change-of-measure local martingale is

$$\exp\Big[\underbrace{\sum_i \int_0^t \sqrt{\lambda_i} \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{1}{\lambda_i-\lambda_j}\, dW_i} - \frac{1}{2}\sum_i \int_0^t \lambda_i \Big(\sum_{\substack{j=1\\ j\ne i}}^{n} \frac{1}{\lambda_i-\lambda_j}\Big)^2 ds\Big]$$

$$\frac{1}{2}\sum\sum_{i\ne j} \int_0^t \frac{1}{\lambda_i-\lambda_j}\left(d\lambda_i + 2\beta\lambda_i\, ds - \kappa\, ds\right)$$

$$= \frac{1}{4}\sum\sum_{i\ne j}\int_0^t \frac{d(\lambda_i-\lambda_j)}{\lambda_i-\lambda_j} + \frac{\beta}{2}\, n(n-1)t$$

$$= \frac{1}{2}\sum\sum_{i<j}\left\{\log(\lambda_i-\lambda_j) + \frac{1}{2}\int_0^t \frac{d\langle\lambda_i-\lambda_j\rangle}{(\lambda_i-\lambda_j)^2}\right\} + \beta n(n-1)t \qquad + \text{const}$$

(from $\lambda_i(0)$),

$$= \frac{1}{2}\sum\sum_{i<j}\log(\lambda_i-\lambda_j) + \sum\sum_{i<j}\int_0^t \frac{\lambda_i+\lambda_j}{(\lambda_i-\lambda_j)^2}\, ds + \beta n(n-1)t + \text{const}$$

On the other hand, the FV part in the exponential is

$$-\frac{1}{2}\sum_i \int_0^t \lambda_i \sum\sum_{j,k\ne i} \frac{1}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\, ds = -\frac{1}{2}\int_0^t \sum\sum_{i\ne j} \frac{\lambda_i}{(\lambda_i-\lambda_j)^2}\, ds$$

$$= -\sum\sum_{i<j}\int_0^t \frac{\lambda_i+\lambda_j}{(\lambda_i-\lambda_j)^2}\, \frac{ds}{2}$$

which very disappointingly does not cancel the integral term from the first bit. If it did, we'd be dealing with a nice h-transform.

(iii) Can show

$$\boxed{\exp(\beta n(n-1)t) \prod\prod_{i<j}(\lambda_j-\lambda_i) \equiv h(t,\lambda) \text{ is harmonic.}} \quad (\text{for } (*))$$

OU bridges (17/2/95)

(i) What's the covariance structure of an OU bridge on $[0,T]$? If the OU process is $dX = d\tilde{W} - \beta X\,dt$, we have $(X_0 \equiv x)$

$$e^{\beta t} X_t = x + \int_0^t e^{\beta s} d\tilde{W}_s = x + W\left(\frac{e^{2\beta t}-1}{2\beta}\right)$$

and therefore if we condition on $X(T)=y$, this is the same as conditioning on $W(\tau) = e^{\beta T} y - x$ $\quad (\tau \equiv (e^{2\beta T}-1)/2\beta)$. Conditioning W in this way produces a familiar Brownian bridge:

$$W(s) = \frac{s}{\tau}(e^{\beta T} y - x) + \left(1-\frac{s}{\tau}\right) W'\left(\frac{s\tau}{\tau - s}\right)$$

and hence

$$e^{\beta t} X_t = x + \frac{e^{2\beta t}-1}{2\beta\tau}(e^{\beta T}y - x) + \left(1 - \frac{e^{2\beta t}-1}{2\beta\tau}\right) W'\left(\frac{(e^{2\beta t}-1)(e^{2\beta T}-1)}{2\beta\,(e^{2\beta T}-e^{2\beta t})}\right),$$

or again

$$X_t = x e^{-\beta t} + \frac{\sinh\beta t}{\beta\tau}(e^{\beta T}y - x) + \left(1 - \frac{e^{2\beta t}-1}{2\beta\tau}\right) e^{-\beta t}\, W'\left(\frac{\sinh\beta t\,\sinh\beta T}{\beta\sinh\beta(T-t)}\right).$$

Hence

$$\boxed{E[X_t \mid X_0 = x, X_T = y] = x e^{-\beta t} + \frac{\sinh\beta t}{\sinh\beta T}(y - x e^{-\beta T})}$$

and after a few calculations

$$\boxed{X_t - E(X_t \mid X_0, X_T) = \frac{\sinh\beta(T-t)}{\sinh\beta T}\, W'\left(\frac{\sinh\beta t\,\sinh\beta T}{\beta\sinh\beta(T-t)}\right)}$$

(ii) The special case $x=y$ is of some interest, as this is the OU bridge "wrapped around on itself". Here

$$E(X_t \mid X_0 = X_T = x) = x\,\frac{\cosh\beta(t-T/2)}{\cosh\beta T/2}$$

We can specify the covariance structure of the OU bridge; for $0 \le s \le t \le T$,

$$\boxed{\operatorname{cov}(X_s, X_t \mid X_0, X_T) = \frac{\sinh\beta s\,\sinh\beta(T-t)}{\beta\sinh\beta T}}$$

Notice that if we take BM killed when it exits $[0,T]$, then the λ-resolvent density is exactly twice this ($\lambda \equiv \beta^2/2$) [see, for example, equation (38) in Chan - Dean - Jansons - Rogers].

Now it's of interest to mix the starting = finishing point of this OU bridge using a $N(0, \sigma^2)$ where σ^2 is chosen suitably (which means that $\sigma^2 = (2\beta)^{-1} \coth(\beta T/2)$, in fact!). If we do this, the covariance structure becomes (for $0 \le s \le t \le T$)

$$\boxed{\begin{aligned} E(X_s X_t) &= \sigma^2 \frac{\cosh \beta(s-T/2) \cosh \beta(t-T/2)}{\cosh^2 \beta T/2} + \frac{\sinh \beta s \sinh \beta(T-t)}{\beta \sinh \beta T} \\ &= \frac{\cosh \beta(s-t+T/2)}{2\beta \sinh \beta T/2} \end{aligned}}$$

once the dust settles.

(iii) Now here's another thing we can do. Let's identify $0, T \in [0,T]$ to make a circle of circumference T, and run a Brownian motion on it, but killing it at a constant rate $\lambda \equiv \beta^2/2$. What is the Green function for this process?

Rate of killed excursions from $0 = \beta \coth \beta T - \frac{1}{T} + \frac{1}{T}\left(1 - \frac{\beta T}{\sinh \beta T}\right)$

$$= \beta \tanh \beta T/2$$

and for $0 \le x \le T$

$$P^x[\text{reach } \{0,T\} \text{ without killing}] = \frac{\cosh \beta(x-T/2)}{\cosh \beta T/2}$$

so that for $0 \le x \le y \le T$

$$\boxed{g(x,y) = \frac{\cosh \beta(y-x-T/2)}{\beta \sinh(\beta T/2)}},$$

which is exactly twice the covariance structure of the wrapped OU process! So the covariance of the wrapped OU process IS the Green function of a symmetric Markov process.

(iv) What is the SDE for an OU bridge? Working firstly with an OU bridge from 0 to 0, we have that

$$\frac{\sinh\beta T}{\sinh\beta(T-t)}\, X_t = W\left(\frac{\sinh\beta t\,\sinh\beta T}{\beta\,\sinh\beta(T-t)}\right)$$

$$= W\left(\int_0^t \left(\frac{\sinh\beta T}{\sinh\beta(T-s)}\right)^2 ds\right)$$

so that

$$d\left(\frac{\sinh\beta T}{\sinh\beta(T-t)}\, X_t\right) = \frac{\sinh\beta T}{\sinh\beta(T-t)}\, dW_t$$

leading to

$$dX_t = dW_t - \beta X_t \coth\beta(T-t)\, dt.$$

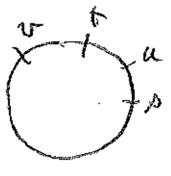
If we condition it to go to some point a from a, we get

$$\boxed{dX_t = dW_t - \beta\left\{X_t\cosh\beta(T-t) - a\right\}\frac{dt}{\sinh\beta(T-t)}}\,.$$

(v) What stationary zero-mean Gaussian processes are there indexed by the circle $[0,T]$ which have the Markov random field property?

For $0 \le s \le u \le t \le v \le T$, we want

$$E\left[X_u \mid X_s, X_t\right] = E\left[X_u \mid X_r,\ r \notin [s,t]\right],$$



and the LHS is

$$\begin{pmatrix}\rho_{us} & \rho_{ut}\end{pmatrix}\begin{pmatrix}\rho_{ss} & \rho_{st}\\ \rho_{st} & \rho_{tt}\end{pmatrix}^{-1}\begin{pmatrix}X_s\\ X_t\end{pmatrix} = \frac{\begin{pmatrix}\rho_{us} & \rho_{ut}\end{pmatrix}}{\rho_{ss}\rho_{tt}-\rho_{st}^2}\begin{pmatrix}\rho_{tt} & -\rho_{st}\\ -\rho_{st} & \rho_{ss}\end{pmatrix}\begin{pmatrix}X_s\\ X_t\end{pmatrix}$$

and we shall have to have

$$\rho_{uv} = E(X_u X_v) = \frac{\begin{pmatrix}\rho_{us} & \rho_{ut}\end{pmatrix}}{\rho_{ss}\rho_{tt}-\rho_{st}^2}\begin{pmatrix}\rho_{tt} & -\rho_{st}\\ -\rho_{st} & \rho_{ss}\end{pmatrix}\begin{pmatrix}\rho_{sv}\\ \rho_{tv}\end{pmatrix}.$$

Let's now write $f(t) = \rho_{0t}$, assuming stationarity. We also have obviously that $f(T-t) = f(t)$. Assume wlog that $f(0) = 1$, if it helps.

Our condition for Markov random field unpacks to

$$f(v-u)\left[1-f(t-s)^2\right] = f(v)\{f(a) - f(t)f(t-u)\} + f(v-t)\{f(t-u) - f(u)f(t)\}.$$

If we write $t = u+a$, $v = t+b$, we get

$$f(a+b)\{1-f(t)^2\} = f(u+a+b)\{f(u) - f(u+a)f(a)\}$$
$$+ f(b)\{f(a) - f(u)f(u+a)\}$$
$$= -f(u+a)\left(f(a)f(u+a+b) + f(b)f(u)\right)$$
$$+ f(u)f(u+a+b) + f(b)f(a)$$
$$\equiv -f(u+a)\,g(u,a,b) + g(a,u,b), \quad \text{say.}$$

Hence

$$\{f(a+b) - f(u+b)\}\{1-f(u+a)^2\} = \{g(a,u,b) - g(u,a,b)\}\{1+f(u+a)\}$$

which gives

$$\{f(a+b) - f(u+b)\}\{1-f(u+a)\} = g(a,u,b) - g(u,a,b)$$
$$= \{f(u) - f(a)\}\{f(u+a+b) - f(b)\},$$

whence

$$\frac{f(u+a+b) - f(b)}{f(a+b) - f(u+b)} = \frac{f(u+a) - 1}{f(a) - f(u)}, \quad \underline{\text{same for all } b}$$

Hence we learn that

$$\frac{f(x+\delta) - f(x-\delta)}{f(x+\varepsilon) - f(x-\varepsilon)} = k(\varepsilon, \delta), \quad \text{same for all } x.$$

Now apply this with $\delta = 2\varepsilon$, $x = n\varepsilon$. If $f(n\varepsilon) \equiv y_n$, we shall have

$$y_{n+2} - y_{n-2} = k(y_{n+1} - y_{n-1})$$

so solving the auxiliary polynomial $t^4 - kt^3 + kt - 1 \equiv (t-1)(t+1)(t^2 - kt + 1)$ gives roots $1, -1, e^{\alpha}, e^{-\alpha}$. The general solution is therefore

$$f(n\varepsilon) = A + B e^{n\varepsilon\alpha(\varepsilon)} + C e^{-n\varepsilon\alpha(\varepsilon)}$$

(the root -1 is discarded by considering what happens when we halve ε.)

Hence $\boxed{f(t) = A + B\cosh\beta(t - T/2)}$ <u>is most general covariance structure!</u>

(vi) Jim Pitman says that if you take BM on a circle, and let τ denote the first time it completes an excursion from its starting point right round the circle, then the local time process $\{ L(\tau, x) \,;\, 0 \le x \le T \}$ is stationary (where T is the circumference of the circle).

To see this, let's consider any $V : \mathbb{R} \to \mathbb{R}^+$ which is periodic with period T, let B denote the Brownian motion, and consider

$$\varphi(x) \equiv E^x \exp - \int_0^\tau V(B_s)\, ds \qquad (-T \le x \le T).$$

Introduce the increasing (decreasing) solution ψ_+ (ψ_-) to

$$\tfrac{1}{2} \psi'' = V \psi \quad , \qquad \psi(0) = 1$$

and observe that because of periodicity of V, $\psi_\pm(T+x) = \psi_\pm(T)\, \psi_\pm(x)$.

Now φ solves $\tfrac{1}{2}\varphi'' = V\varphi$ with boundary conditions $\varphi(T) = \varphi(-T) = 1$, so we can express it simply as

$$\varphi(x) = \frac{\{\psi_-(-T) - \psi_-(T)\}\psi_+(x) + \{\psi_+(T) - \psi_+(-T)\}\psi_-(x)}{\psi_+(T)\psi_-(-T) - \psi_+(-T)\psi_-(T)},$$

so that

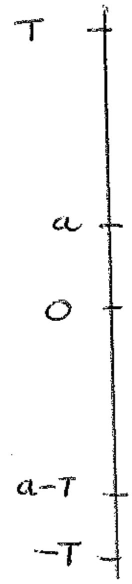
$$\varphi(0) = \frac{\psi_+(T) - \psi_+(-T) + \psi_-(-T) - \psi_-(T)}{\psi_+(T)\psi_-(-T) - \psi_+(-T)\psi_-(T)}.$$

If we abbreviate $\psi_+(T) = \rho_+$, $\psi_-(-T) = \rho_-$, we shall have

$$\boxed{\varphi(0) = (\rho_+ + \rho_-)/(1 + \rho_+\rho_-) .}$$

If we replace V by $\tilde{V}$, $\tilde{V}(x) \equiv V(x+a)$, then we replace $\psi_\pm$ by $\tilde{\psi}_\pm(x) \equiv \psi_\pm(x+a)/\psi_\pm(a)$ and see that $\tilde{\psi}_\pm(T) = \psi_\pm(T)$. This gives us immediately that $\tilde{\varphi}(0) = \varphi(0)$, establishing stationarity.

(vii) Could it be that $(L(\tau, x))_{0 \le x \le T}$ is the sum of squares of two IID Gaussian processes on the circle? In view of the stationarity of the local time process, the first thing to consider would be stationary Gaussian processes on the circle, and possibly also Markovian random fields.... but a little excursion theory shows this has to be wrong!



Why? Fix $a \in (0,T)$, and think about the contribution made to L_a by an excursion from 0 which gets to a, but returns to 0 before it goes on to T. This contribution is

$$\exp\left(\frac{1}{2a} + \frac{1}{2(T-a)}\right) = \exp\left(\frac{T}{2a(T-a)}\right)$$

in law.

The same is true of the law of the contribution collected on the first excursion which gets through all the way to $\{T, -T\}$, so if $L_0 = x$, there will be a $P\left(x\left(\frac{1}{2a} - \frac{1}{2T}\right)\right)$ number of visits to a before the last excursion, a $P\left(x\left(\frac{1}{2(T-a)} - \frac{1}{2T}\right)\right)$ number of visits to $T+a$ ~~before~~ before the last excursion, so in total a $P(\rho x)$ number of visits, with

$$\rho \equiv \frac{a^2 + (T-a)^2}{2a(T-a)T}.$$

Clearly $L_0 \sim \exp(1/T)$, so sticking it all together gives

$$\boxed{E\ \exp\{-\alpha L_a - \beta L_0\} = \{1 + (\alpha+\beta)T + 2\alpha\beta\, a(T-a)\}^{-1}}$$

If $L_x \overset{\mathcal{D}}{=} \xi_x^2 + \tilde{\xi}_x^2$, where $\xi, \tilde{\xi}$ are IID Gaussian random fields, then

$$E\ \exp(-\alpha \xi_a^2 - \beta \xi_0^2) = \{1 + (\alpha+\beta)T + 2\alpha\beta\, a(T-a)\}^{-1/2},$$

from which

$$\begin{pmatrix} \xi_a \\ \xi_0 \end{pmatrix} \sim N\left(0, \begin{pmatrix} \frac{T}{2} & \frac{\sqrt{a^2+(a-T)^2}}{2} \\ \frac{\sqrt{a^2+(a-T)^2}}{2} & \frac{T}{2} \end{pmatrix}\right).$$

<u>This covariance structure is not one of the stationary Gauss-Markov covariance structures!</u>

Moreover when I checked it out using Mathematica, the joint law of (L_0, L_a, L_b) is <u>different</u> for the two — $(\tilde{\xi}_0^2 + \xi_0^2,\ \tilde{\xi}_a^2 + \xi_a^2,\ \tilde{\xi}_b^2 + \xi_b^2) \overset{\mathcal{D}}{\neq} (L_0, L_a, L_b)$!!

For $U(x) = x^{1-R}/(1-R)$ we have

$$\sup E\left[U(W_T) \mid W_0 = w_0\right] = U(w_0) \cdot \exp\left[\left(r + \frac{\tilde{\mu}^2}{2R}\right)(1-R)T\right]$$

so for a well-posed problem we shall have to have

$$\delta \geq \left(r + \frac{\tilde{\mu}^2}{2R}\right)(1-R)$$

An optimal investment problem with habit formation (1/3/95)

Let's consider an agent whose initial wealth is w_0 who can invest in a riskless asset (rate of return $= r$, const) or a risky asset ($dS_t/S_t = \sigma\, dB_t + \mu dt$) and whose aim is to

$$\max \quad E\int_0^\infty e^{-\delta t}\, U(Y_t)\, dt$$

where $\quad Y_t = Y_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} dC_s \quad$, and C_t is total consumption by time t. For a well-posed problem, must assume $\boxed{\delta > (1-R)(r + \tilde{\mu}^2/2R)}$ if we assume $\boxed{U(y) = y^{1-R}/(1-R)}$ for some $0<R<1$.

We'll analyse this from two quite different standpoints, the first a Markov decision problem / value function approach, and the second by spotting the correct Lagrange multiplier process. In fact, we give two essentially equivalent Lagrangian approaches.

(i) Let $V(w,y)$ be the max of $E\left[\int_0^\infty e^{-\delta t}\, U(Y_t)\, dt \;\middle|\; Y_0 = y,\, W_0 = w\right]$. We have certain evident properties;

$$V(aw, ay) = a^{1-R}\, V(w,y) \quad ; \quad V \text{ is increasing in both arguments;}$$

$$V \text{ is concave;} \quad V(0,y) = \int_0^\infty e^{-\delta t} \left(e^{-\alpha t} y\right)^{1-R} \frac{dt}{1-R} \equiv k \equiv \left((1-R)(\delta + \alpha - \alpha R)\right)^{-1}.$$

Using the scaling, let's write $f(x) \equiv V(x,1)$, so $V(w,y) = y^{1-R} f(w/y)$.

Since

$$\begin{cases} dW_t = \theta_t \sigma (dB_t + \tilde{\mu}\, dt) + r W_t dt - dC_t \\ dY_t = -\alpha Y_t\, dt + dC_t \end{cases} \qquad \tilde{\mu} \equiv \sigma^{-1}(\mu - r)$$

we can derive the Bellman equation for the optimal control problem;

$$\boxed{\begin{gathered} \sup_\theta \; -\delta V + (\theta\sigma\tilde{\mu} + rw) V_w + \tfrac{1}{2}\theta^2\sigma^2 V_{ww} - \alpha y V_y + \frac{y^{1-R}}{1-R} \le 0 \\ V_w \ge V_y \,. \end{gathered}}$$

with consumption only where $V_w = V_y$. Optimal θ is $\theta^* = -\frac{\tilde{\mu}}{\sigma} V_w / V_{ww}$ so

$$\boxed{\sigma\theta^* = -\tilde{\mu}\, y \frac{f'(w/y)}{f''(w/y)} = -\tilde{\mu} w \cdot \frac{f'(w/y)}{(w/y)\, f''(w/y)}}\,.$$

We can rework the HJB in terms of f to give

$$-\frac{1}{2}\tilde{\mu}^2 \frac{f'^2}{f''} + (r+\alpha)x f' - (\delta+\alpha-\alpha R)f + \frac{1}{1-R} = 0$$

and if we now write $f(x) = k + h(x)$, we see that $h(x)=0$, and h solves

$$\boxed{-\frac{1}{2}\tilde{\mu}^2 \frac{h'^2}{h''} + (r+\alpha)x h' - (\delta+\alpha-\alpha R)h = 0.}$$

One solution to this is to take $h(x) = c x^\gamma$ for some $c, \gamma > 0$. We find that γ must solve the quadratic

$$(r+\alpha)\gamma^2 - \gamma\left(\tfrac{1}{2}\tilde{\mu}^2 + r+\alpha+\delta+\alpha-\alpha R\right) + \delta+\alpha-\alpha R = 0$$

which has two real roots, one >1, one <1. Since f must be concave, it is only the root $\gamma<1$ that can possibly interest us, and this would give as a candidate for V to be the function

$$(w,y) \longmapsto k.y^{1-R} + c\, w^\gamma y^{1-R-\gamma} \equiv \varphi(w,y)$$

The feasibility constraint $\delta > (1-R)(r + \tilde{\mu}^2/2R)$ translates into an inequality for γ, namely $\gamma > 1-R$, so the above function can only be valid for $(w/y) \le a$, for some a.

If we restrict (w,y) to $w \le ay$, then φ is non-decreasing in y iff

$$a^\gamma \le \frac{k\,(1-R)}{c\,(\gamma-1+R)}$$

and is concave in y iff

$$a^\gamma \le \frac{k\,(1-R)}{c(\gamma-1+R)} \; \frac{R}{R+\gamma}.$$

For concavity of φ as a whole, also need $\varphi_{ww}\varphi_{yy} - \varphi_{wy}^2 \ge 0$, which becomes

$$a^\gamma \le \frac{k\,(1-R)}{c(\gamma-1+R)} \cdot (1-\gamma)$$

which is the tightest bound of all. We also have the constraint on the gradients, This also arises by "smooth pasting", because $f(x) = k + c\,x^{\gamma} \qquad (x \le a)$

$$= f(a) \left(\frac{1+x}{1+a}\right)^{1-R} \qquad (x \ge a)$$

and the condition that f has no discontinuity of gradient at a is exactly $a^{-1}\gamma c a^{\gamma} = k(1-R) - ca^{\gamma}(\gamma - 1 + R)$. The other condition is needed to guarantee that we can actually consume when $X = a$ with no loss.

that $\varphi_W \geq \varphi_Y$ except on the boundary where $w = ay$. This translates into

$$\gamma c\, x^{\gamma-1} \;\geq\; k(1-R) - c(\gamma-1+R)\,x^{\gamma} \qquad \forall x \leq a$$

with equality for $x = a$. One way to think of this is that once a is chosen, c is determined by this equation; and we shall want c to be as big as possible subject to the earlier constraints, on the previous page. This means we seek c, a such that

$$\left.\begin{aligned} a^{-1}\,\gamma\, c a^{\gamma} &= k(1-R) - c a^{\gamma}(\gamma-1+R) \\ c a^{\gamma} &= \frac{k(1-R)}{\gamma-1+R}(1-\gamma) \end{aligned}\right\}$$

implying

$$\boxed{a = (1-\gamma)/(\gamma-1+R).}$$

If we now consider the SDE satisfied by $X_t \equiv W_t/Y_t$, we find that, since $\theta^* = \tilde{\mu}\,\sigma^{-1} w/(1-\gamma)$, we have

$$\boxed{dX_t = \frac{\tilde{\mu}}{1-\gamma} X_t\, dB_t + \left(r + \frac{\tilde{\mu}^2}{1-\gamma} + \alpha\right) X_t\, dt - (1+X_t)\frac{dc_t}{Y_t}}$$

So we now have a candidate for the value function:

$$\boxed{\begin{aligned} V(w,y) = y^{1-R} f(w/y), \qquad f(x) &= k + c x^{\gamma} && (x \leq a) \\ &= f(a)\left(\frac{1+x}{1+a}\right)^{1-R} && (x \geq a) \end{aligned}}$$

It can be verified (at some length) that

$$\begin{cases} (V_W - V_Y)(w,y) \geq 0, & \text{with equality iff } w \geq ay \\ \sup_{\theta}\; -\delta V + (\theta\sigma\tilde{\mu} + rw) V_W + \tfrac{1}{2}\theta^2\sigma^2 V_{WW} - \alpha y V_Y + \dfrac{y^{1-R}}{1-R} \leq 0 \end{cases}$$

with equality iff $w \leq ay$. This confirms the value function.

We can give a nice sample-path description of the optimal solution:

If $x_t \equiv \log X_0 + \frac{\tilde{\mu}}{1-\gamma} B_t + \left[r + \alpha + \frac{\tilde{\mu}^2}{2(1-\gamma)^2}(1-2\gamma) \right] t$

$$\eta_t \equiv \sup_{s \le t} x_s$$

and finally $L_t = (\eta_t - \log a)^+$, then

$$X_t = \exp(x_t - L_t), \quad Y_t = \exp\left[-\alpha t + \frac{a}{1+a} L_t \right].$$

(ii) Now here's the Lagrange multiplier approach.

The objective function is

$$\int_0^\infty e^{-\delta t} Y_t^{1-R} \frac{dt}{1-R} = \int_0^\infty e^{-(\delta + \alpha - \alpha R)t} \left(e^{\alpha t} Y_t \right)^{1-R} \frac{dt}{1-R}$$

$$= \int_0^\infty e^{-(\delta+\alpha-\alpha R)t} \left\{ Y_0^{1-R} + \int_0^t (1-R)(e^{\alpha s} Y_s)^{-R} \cdot e^{\alpha s} dC_s \right\} \frac{dt}{1-R}$$

$$= k Y_0^{1-R} + \int_0^\infty (e^{\alpha s} Y_s)^{-R} e^{\alpha s} \, e^{-(\delta+\alpha-\alpha R)s} \frac{dC_s}{\delta + \alpha - \alpha R}$$

$$= k Y_0^{1-R} + \int_0^\infty e^{-\delta s} Y_s^{-R} \frac{dC_s}{\delta + \alpha - \alpha R}.$$

We want to maximise the mean of this subject to the constraint

$$E \int_0^\infty \xi_s \, dC_s = w_0$$

where $\xi_t = \exp[-rt - \tilde{\mu} B_t - \frac{1}{2}\tilde{\mu}^2 t]$ is the usual state-price density.

Rephrasing the problem in Lagrangian form:

$$\max E\left[\int_0^\infty e^{-\delta s} Y_s^{-R} \frac{dC_s}{\delta+\alpha-\alpha R} - \int_0^\infty \lambda \xi_s \frac{dC_s}{\delta+\alpha-\alpha R} \right]$$

is very suggestive; if we work in terms of the increasing process $A_t \equiv e^{\alpha t} Y_t$, we get

$$\max E \int_0^\infty e^{-bs} A_s^{-R} \, dA_s - \int_0^\infty \lambda \xi_s e^{-\alpha s} dA_s, \qquad [b \equiv \delta + \alpha - \alpha R]$$

Now we can go for the first-order conditions; if we perturb optimal A to $A+\eta$, the resultant change in the payoff is

$$E\left[\int_0^\infty \{e^{-bs}A_s^{-R} - \lambda\xi_s e^{-\alpha s}\}\,d\eta_s \quad - R\int_0^\infty e^{-bs}A_s^{-R-1}\,\eta_s\,dA_s\right] \leq 0$$

$$= E\int_0^\infty \left\{ e^{-bs}A_s^{-R} - \lambda\xi_s e^{-\alpha s} - \int_s^\infty Re^{-bt}A_t^{-R-1}\,dA_t \right\} d\eta_s$$

$$= E\int_0^\infty \left(\int_s^\infty be^{-bt}A_t^{-R}\,dt - \lambda\xi_s e^{-\alpha s}\right) d\eta_s$$

so we shall seek to choose increasing A so that always

$$\boxed{E_s\int_s^\infty e^{-bt}A_t^{-R}\,dt \leq \lambda b^{-1}\xi_s e^{-\alpha s},}$$

with equality on supp(dC). To learn one way to do this, we have that $\xi_s e^{-\alpha s} \equiv v_t$ is a potential, with increasing process

$$\int_0^t (\alpha+r)v_s\,ds$$

and so

$$\xi_t e^{-\alpha t} = E_t\left[\int_t^\infty (\alpha+r)v_s\,ds\right]$$

and so if we took

$$e^{-bt}A_t^{-R} = \inf_{s\leq t}(\alpha+r)\lambda b^{-1}\xi_s e^{-\alpha s},$$

we've got the inequality in the box. Now there won't be equality on supp(dc), but what we can say is that for some constant b^* we have at times τ when A moves that

$$E_\tau\int_\tau^\infty e^{-bt}A_t^{-R}\,dt = b^*\xi_\tau e^{-\alpha\tau} = b^* E_\tau\left[\int_\tau^\infty (\alpha+r)\xi_s e^{-\alpha s}\,ds\right]$$

and at all other times there's the inequality $\leq$. However, this requires very strongly the special properties of this situation (independent increments)

Let's just see another "Lagrangian" approach which yields the same conclusion. We are trying to

$$\max E \int_0^\infty e^{-\delta t} \, Y_t^{1-R} \frac{dt}{1-R} = \max E \int_0^\infty e^{-bt} A_t^{1-R} \frac{dt}{1-R} \qquad (b \equiv \delta + \alpha - \alpha R)$$

and if $U(x) = x^{1-R}/(1-R)$, then $U'(x) = x^{-R}$, $I(x) = x^{-1/R}$, A is increasing and if $\lambda_t \equiv U'(A_t)$, we get that the payoff is

$$\leq E \int_0^\infty e^{-bt} \left\{ \tilde{U}(\lambda_t) + \lambda_t A_t \right\} dt \qquad \tilde{U}(\lambda) = \sup_x \left\{ U(x) - \lambda x \right\}$$

So we'll try to choose a multiplier process λ which is decreasing, and develop the linear term;

$$E \int_0^\infty e^{-bt} \lambda_t A_t \, dt = Y_0 E \int_0^\infty e^{-bt} \lambda_t \, dt + E \int_0^\infty e^{\alpha s} \left(\int_s^\infty \lambda_t e^{-bt} \, dt \right) dC_s$$

and the last term is also

$$E \int_0^\infty \underbrace{e^{\alpha s} \; E_s \left[\int_s^\infty e^{-bt} \lambda_t \, dt \right]} dC_s \, .$$

If we could guarantee $\leq K \xi_s$, then we can get an upper bound on the payoff ... so this is essentially the same problem as before.

Note If we set $f(t,x) \equiv E\,(x e^{B_t - t/2} - 1)^+$, then the BS option price can be expressed simply in terms of this;

$$C(S_0, K, t, r, \sigma) = K e^{-rt} f(\sigma^2 t, \; S_0 e^{rt}/K)$$

We can define the inverse function τ by $f(\tau(v,x), x) = v$ $\quad (0 < v < x)$ and use the pde $\frac{1}{2}x^2 f_{xx} = f_t$ satisfied by f to find a pde satisfied by τ;

$$\tau_x (\tau_{xv} \tau_v - \tau_x \tau_{vv}) + \tau_v (\tau_x \tau_{xv} - \tau_{xx} \tau_v) = 2 (\tau_v / \tau)^2 .$$

But is this any good?

Implied volatility in the Hobson-Rogers model. (6/3/95)

(i) Take the special case where $X_t \equiv \log S_t$ satisfies (under EMM P^*)

$$dX_t = r\,dt + \sigma(R_t)\,dB_t - \tfrac{1}{2}\sigma(R_t)^2 dt$$

where $\sigma(R) \equiv a\sqrt{1+\varepsilon^2 R^2}$, $R_t \equiv \int_{-\infty}^{t} \lambda e^{\lambda(s-t)}(X_t - X_s)\,ds \equiv X_t - Z_t$, and so $dZ = \lambda(X-Z)\,dt$. If we now set

$$V(\tau, x, z, K, r) \equiv E\left[(\exp(X_\tau) - K)^+ e^{-r\tau} \,\middle|\, X_0 = x,\ Z_0 = z \right]$$

and

$$C(\tau, x, \sigma, K, r) \equiv E\left(\exp(x + \sigma B_\tau - \sigma^2\tau/2 + r\tau) - K \right)^+ e^{-r\tau},$$

then the implied volatility process $\hat\sigma_t$ is defined via

$$(*) \quad \boxed{V(\tau, X_t, Z_t, K, r) = C(\tau, X_t, \hat\sigma_t, K, r).} \qquad (\tau \equiv T-t)$$

We have

$$\left.\begin{aligned} -rV - V_\tau + \tfrac{1}{2}a^2(1+\varepsilon^2(x-z)^2)(V_{xx} - V_x) + rV_x + \lambda(x-z)V_z &= 0 \\ -rC - C_\tau + \tfrac{1}{2}\sigma^2(C_{xx} - C_x) + rC_x &= 0 \end{aligned}\right\}.$$

If $d\hat\sigma_t \equiv \psi_t\,dB_t + \alpha_t\,dt$, we deduce that (dropping explicit dependence on K, r)

$$\boxed{\begin{aligned} \psi_t &= \sigma(R_t)\left\{ V_x(\tau, X_t, Z_t) - C_x(\tau, X_t, \hat\sigma_t) \right\} / C_\sigma(\tau, X_t, \hat\sigma_t) \\ \alpha_t &= \left[\tfrac{1}{2}(\sigma(R_t)^2 - \hat\sigma_t^2)(C_x - C_{xx}) - \tfrac{1}{2}\psi_t^2 C_{\sigma\sigma} - \psi_t\sigma(R_t)C_{\sigma x} \right] / C_\sigma \end{aligned}}$$

(ii) Differentiating both sides of $(*)$ w.r.t. K gives us $\dfrac{\partial V}{\partial K} = \dfrac{\partial\hat\sigma}{\partial K}\cdot C_\sigma + \dfrac{\partial C}{\partial K}$, and as

$$\frac{\partial V}{\partial K} = -e^{-r\tau} P\left[X_\tau > \log K \,\middle|\, X_0 = x,\ Z_0 = z \right]$$

we get

$$\boxed{\frac{\partial\hat\sigma}{\partial K} > 0 \iff P\left[\hat\sigma B_\tau + (r - \hat\sigma^2/2)\tau > \log K - x\right] > P\left[X_\tau > \log K - x \,\middle|\, X_0 = 0,\ Z_0 = z\right]}$$

Any better?

Long rate $= \dfrac{bk}{a^2}\left(\left(b^2 + \dfrac{4a^2\beta}{2\gamma+\beta}\right)^{1/2} - b\right)$

Yet another model for TSIR? (8/3/95)

What if we let the spot rate follow

$$dr_t = \sigma\sqrt{r_t}\,dW_t + \beta(\mu_t - r_t)\,dt$$

where μ_t is an independent CIR process, $d\mu_t = a\sqrt{\mu_t}\,dW_t + b(k-\mu_t)\,dt$?
We'll have the bond prices

$$P(t,T) = E_t \exp\{-r_t B(\tau) - A(t,T)\}$$

where $B(\tau) = (\sinh\gamma\tau)\{\gamma\cosh\gamma\tau + \tfrac{1}{2}\beta\sinh\gamma\tau\}^{-1}$ as usual $(2\gamma = \sqrt{\beta^2+2\sigma^2})$,
and where

$$A(t,T) = \int_t^T \beta\mu_s\, B(T-s)\,ds\,.$$

We therefore need to calculate $E_t \exp\{-\int_t^T \beta\mu_s B(T-s)\,ds\} = \exp\left[-\varphi(T-t)\mu_t - c(T-t)\right]$
and routine calculations give

$$\boxed{\tfrac{1}{2}a^2\varphi^2 + \dot\varphi + b\varphi = \beta B\,, \qquad \dot c = bk\varphi\,, \qquad \varphi(0) = c(0) = 0.}$$

With the usual Ricatti substitution $\varphi = 2a^{-2}\dot\psi/\psi$, we get

$$\frac{2}{a^2}\frac{\ddot\psi}{\psi} + \frac{2b}{a^2}\frac{\dot\psi}{\psi} = \beta B$$

or again

$$\ddot\psi + b\dot\psi = \frac{a^2\beta}{2}B\psi$$

or again

$$\boxed{\frac{d^2}{dt^2}\left[e^{bt/2}\psi(t)\right] = \left(\frac{a^2\beta}{2}B(t) + \frac{b^2}{4}\right)e^{bt/2}\psi(t).}$$

This has the form $$f''(x) = \frac{a_1e^x + b_1}{a_2e^x + b_2}f(x) \qquad (x \equiv 2\gamma t)$$

which we can reduce by the substitution $y = e^x$ to $(f(x) = g(e^x))$

$$(a_2y + b_2)\{y^2g''(y) + y\,g'(y)\} = (a_1y + b_1)\,g(y)$$

Explicit solutions impossible in general!

> **NB** This is just a special case of the class of models in Duffie + Kan!!

Trading on a fractional BM with transactions costs (15/3/95)

(i) Let's suppose that we have a fractional BM $(X_t)_{t\in\mathbb{R}}$, $H \neq \frac{1}{2}$, and we pay a fixed proportion ε of any bet in transactions costs, and we choose to bet at intervals of δ.

Set $\varphi(s) \equiv ((s+\delta)_+)^{H-\frac{1}{2}} - (s_+)^{H-\frac{1}{2}}$, $\Delta_n X \equiv X(n\delta) - X(n\delta-\delta)$ so that

$$\xi_n \equiv E\left[\Delta_n X \mid \mathcal{F}_{(n-1)\delta}\right] = \int_{-\infty}^{(n-1)\delta} \varphi(n\delta-\delta-s)\, dW_s$$

and

$$\Delta_n X - \xi_n = \int_{(n-1)\delta}^{n\delta} \varphi(n\delta-\delta-s)\, dW_s .$$

(ii) Let's also suppose our strategy is to bet $\theta_n = \xi_n I_{\{|\xi_n|>\varepsilon\}}$ on $\Delta_n X$, so that the gain from N trades is

$$G_N = \sum_{j=1}^{N} \{\theta_j \Delta_j X - \varepsilon|\theta_j|\}.$$

By the ergodic theorem, $\frac{1}{N} G_N \xrightarrow[L^1]{a.s.} E(\theta_1 \xi_1 - \varepsilon|\theta_1|) = E\left[(\xi_1^2 - \varepsilon|\xi_1|)^+\right] > 0$, which looks good, but let's be more precise.

Observe

$$\xi_n \sim N(0, k\delta^{2H}), \qquad k \equiv \int_0^\infty \{(1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\}^2\, ds$$

so

$$\boxed{E(\xi_1^2 - \varepsilon|\xi_1|)^+ = 2\, k\delta^{2H}\, \bar{\Phi}(\varepsilon/\delta^H\sqrt{k}).}$$

Let's abbreviate $E(\xi_1^2 - \varepsilon|\xi_1|)^+$ to μ, and see what we can say about the second moment of the gains process,

$$\begin{aligned} E(G_N - N\mu)^2 &= N\, E(\theta_j \Delta_j X - \varepsilon|\theta_j| - \mu)^2 \\ &\quad + 2\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} E(\theta_j\Delta_j X - \varepsilon|\theta_j| - \mu)(\theta_k\Delta_k X - \varepsilon|\theta_k| - \mu) \\ &= N\, E(\theta_j\Delta_j X - \varepsilon|\theta_j| - \mu)^2 \\ &\quad + 2\sum_{r=1}^{N-1}(N-r)\, E(\theta_0\Delta_0 X - \varepsilon|\theta_0| - \mu)(\theta_r\Delta_r X - \varepsilon|\theta_r| - \mu) \end{aligned}$$

So the most interesting part is

$$\begin{aligned} &E(\theta_0\Delta_0 X - \varepsilon|\theta_0| - \mu)(\theta_r\Delta_r X - \varepsilon|\theta_r| - \mu) \\ &= E(\theta_0\Delta_0 X - \varepsilon|\theta_0| - \mu)\left((\xi_r^2 - \varepsilon|\xi_r|)^+ - \mu\right). \end{aligned}$$

Now we can write $\Delta_0 X = \xi_0 + \int_{-\delta}^{0} |s|^{H-\frac{1}{2}} dW_s \equiv \xi_0 + \eta_1$ say, to reduce one of the terms in the sum to

$$E\left\{(\xi_0^2 - \varepsilon|\xi_0|)^+ - \mu\right\}\left\{(\xi_r^2 - \varepsilon|\xi_r|)^+ - \mu\right\} + E\left[\theta_0 \eta \left\{(\xi_r^2 - \varepsilon|\xi_r|)^+ - \mu\right\}\right]$$

Explicit computations are probably going to be too fearsome, but we can use Gebelein's inequality (with a multivariate generalization) to estimate; if (X, Y) are jointly Gaussian vectors, and $|\mathrm{corr}(X^i, Y^j)| \leq \rho$ for all i, j, then for any $\varphi(X), \psi(Y) \in L^2$ we shall have

$$|\mathrm{corr}(\varphi(X), \psi(Y))| \leq \rho.$$

$(\gamma = H - \frac{1}{2})$

Now
$$\mathrm{cov}(\xi_0, \xi_r) = \delta^{2H} \int_0^\infty ((1+s)^\gamma - s^\gamma)((1+s+r)^\gamma - (s+r)^\gamma)\, ds$$

<u>In the case $\gamma < 0$</u>, we can easily show that

$$r^{1-\gamma} \mathrm{cov}(\xi_0, \xi_r) \uparrow \delta^{2H} \gamma \int_0^\infty ((1+s)^\gamma - s^\gamma)\, ds.$$

<u>For the case $\gamma > 0$</u>, we observe that

$$\frac{(1+t)^\gamma - t^\gamma}{(t+\frac{1}{2})^{\gamma-1}} \asymp 1$$

(where $f(t) \asymp g(t)$ iff $\frac{f(t)}{g(t)} + \frac{g(t)}{f(t)}$ bdd)

so the asymptotics of $\mathrm{cov}(\xi_0, \xi_r)$ is same as

$$\int_0^\infty (1+s)^{\gamma-1} (1+s+r)^{\gamma-1}\, ds = c \int_0^\infty ds \int_0^\infty x^{-\gamma} e^{-x-sx}\, dx \int_0^\infty y^{-\gamma} e^{-y(1+s+r)}\, dy$$

$$= c \int_0^\infty y^{-\gamma} e^{-(1+r)y} \left(\int_0^\infty \frac{x^{-\gamma}\, dx\, e^{-x}}{x+y} \right) dy$$

Now $\displaystyle\int_0^\infty \frac{x^{-\gamma} e^{-x} dx}{x+y} \asymp \int_0^1 \frac{x^{-\gamma}}{x+y}\, dy = \int_0^{1/y} (ty)^{-\gamma} \frac{dt}{1+t} \sim y^{-\gamma} \int_0^\infty \frac{t^{-\gamma} dt}{1+t}$,

so the whole expression is

$$\asymp \int_0^\infty y^{-2\gamma} e^{-(1+r)y}\, dy \sim \text{const}\; r^{2\gamma - 1}.$$

From this follows

$$\boxed{\begin{array}{lll} \text{for } \gamma > 0, & E(G_N - N\mu)^2 & \sim N^{1+2\gamma} \\ \text{for } \gamma < 0 & E(G_N - N\mu)^2 & \sim N \end{array}}$$

But, as Stewart points out – one doesn't <u>have</u> to liquidate the position every day – one may buy, + hold for a long time...

Some simple-minded ideas about interest rates + FX (23/3/95)

(i) Suppose we have N countries. In country j, there is a production process which produces good j. Assume that all consumption, trading, etc in country j is in terms of good j, and that a unit of good j invested in the production process at time 0 has become $S_j(t)$ units of good j by time t, where S_j is a continuous semimartingale. We will also postulate the existence of a bond $B_j(t)$ in each country, which will be in zero net supply.

In country j there is a single agent with utility U_j, who aims to maximise the expected utility of his wealth at time T (wealth measured in terms of good j). At any time t, one may trade 1 unit of good j for $Y_{ij}(t)$ units of good i.

Assume that the utilities U_j and production semimartingales S_j are given; can one then decide from market clearing what the B_j and Y_{ij} are?

(ii) Let's have

$$Z_j(t) = \left.\frac{dP_j^*}{dP}\right|_{\mathcal{F}_t},$$

where we have that P_j^* is a country-j risk-neutral measure, and let's write

$$\zeta_j(t) \equiv Z_j(t)/B_j(t)$$

for the country-j deflator. Thus $S_j(t).\zeta_j(t)$ will be a P-martingale. But agent j could also invest in country k; if he holds 1 unit of k-production, this is worth $Y_{jk}(t)\, S_k(t)$ in j-good at time t. Hence

$$\zeta_j(t)\, Y_{jk}(t)\, S_k(t) \quad \text{is a } P\text{-martingale.}$$

A sufficient condition for this would be

$$\boxed{Y_{jk}(t) = c.\, \zeta_k(t)/\zeta_j(t)},$$

In a complete market, this is also necessary, so from now on, let's assume that this is what is happening; the exchange rate Y_{jk} is given by $c_{jk}\, \zeta_k/\zeta_j$.

(iii) Now let's suppose that we seek some equilibrium where the market clearing conditions are trivially satisfied by the following prescription:

Agent j holds 1 share of j-production and nothing else.

Thus agent j's optimal wealth process is just $S_j(\cdot)$, and it follows that

$$U_j'(S_j(T)) \propto \zeta_j(T)$$

by the usual analysis, and also $\zeta_j(t)\, S_j(t)$ is a martingale. So this suggests an approach; represent

$$M_j(t) \equiv E\left(S_j(T) \,.\, U_j'(S_j(T)) \,\middle|\, \mathcal{F}_t \right)$$

$$= \mathcal{E}(N_j)_t$$

(which we can do explicitly since S_j and U_j are known) and now if we have $S_j = \mathcal{E}(\eta_j)$ then

$$\zeta_j(t) = \mathcal{E}(N_j)_t \,/\, S_j(t)$$

$$= \mathcal{E}\left(N_j - \eta_j - [N_j, \eta_j] + [\eta_j] \right)_t .$$

The spot-rate process then satisfies

$$\boxed{\int_0^t r_j(s)\, ds = \text{FV part of } \eta_j + [N_j - \eta_j, \eta_j]_t .}$$

This <u>will</u> be an equilibrium; all of the assets which agent j can trade on become P-martingales when multiplied by the deflator ζ_j.

If we write $d\eta_j = \sigma_{jk}\, dW^k + \mu_j\, dt$, then

$$\boxed{r_j(t) = \mu_j(t) + d[N_j - \eta_j, \eta_j]_t \,/dt .}$$

We have moreover

$$\boxed{\begin{aligned} Y_{kj} = \frac{\zeta_j}{\zeta_k} = \mathcal{E}\Big(& N_j - \eta_j - N_k + \eta_k + [\eta_j, \eta_j - N_j] + [N_k, N_k - \eta_k] \\ & - [N_j - \eta_j, N_k - \eta_k] \Big) \end{aligned}}$$

(iv) <u>Example</u> $dS_j(t) = S_j(t)\left[\sigma_{jk}\, dW^k_t + \mu_j\, dt\right]$, $U_j(x) = x^{1-R_j}/(1-R_j)$

Then $I_j(x) = x^{-1/R_j}$, and we need to consider the mg

$$E\left[\, S_j(T)^{1-R_j} \,\middle|\, \mathcal{F}_t \right] = S_j(0)^{1-R_j} \exp\Big[(1-R_j)\,\sigma^j\cdot W_t - \tfrac{1}{2}(1-R_j)^2 |\sigma^j|^2 t$$

$$- R_j(1-R_j)\tfrac{1}{2}|\sigma^j|^2 T + \mu_j(1-R_j)T \Big]$$

$$= \mathcal{E}\big((1-R_j)\sigma^j\cdot W\big)\; S_j(0)^{1-R_j} \exp\left[-R_j(1-R_j)\tfrac{1}{2}|\sigma^j|^2 T + \mu_j(1-R_j)T \right]$$

We therefore get $N_j = (1-R_j)\sigma^j \cdot W$ and $\eta_j = \sigma^j\cdot W + \mu_j$, so that

$$[N_j, \eta_j]_t = (1-R_j)|\sigma^j|^2 t = (1-R_j)[\eta_j]_t .$$

So for this model,

$$\boxed{r_j(t) = \mu_j - R_j |\sigma^j|^2}$$

and the FX rates are

$$Y_{kj}(t) = \exp\Big[(R_k\sigma^k - R_j\sigma^j)\cdot W_t + \tfrac{1}{2}\left(|\sigma^k|^2R_k^2 - |\sigma^j|^2R_j^2\right)t$$

$$+ (\mu_k - \mu_j)t + \left(|\sigma^j|^2 R_j - |\sigma^k|^2 R_k\right)t \Big] \cdot Y_{kj}(0)$$

$$= \exp\Big[(R_k\sigma^k - R_j\sigma^j)\cdot W_t + \tfrac{1}{2}\left(|R_k\sigma^k|^2 - |R_j\sigma^j|^2\right)t - r_j t + r_k t \Big] \cdot Y_{kj}(0).$$

OK, well, that's not the most exciting example one could hope for!

What if we also consider running consumption? of CAPM? This is real interest rates - what about nominal?

Variance estimation with quanta (25/3/95)

(i) Suppose that the underlying model for log share prices is $X_t = \sigma B_t + ct$, (where B is Brownian motion), but that we only observe $X_t^{(\varepsilon)}$, which is the process X which only changes when it has moved by at least ε. Formally, if $\tau_0 = 0$, $\tau_{n+1} = \inf\{t > \tau_n : |X_t - X(\tau_n)| > \varepsilon\}$, then

$$X_t^{(\varepsilon)} = \sum_{n\geq 0} X(\tau_n)\, I_{\{\tau_n \leq t < \tau_{n+1}\}} .$$

Let $\bar{X}_t^{(\varepsilon)} \equiv \sup\{X_u^{(\varepsilon)} : u \leq t\}$. Trying to mimic the Rogers-Satchell estimator, can we compute $E\,\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)})$?

(ii) Introduce the independent $\exp(\lambda)$ random variable T, and then

$$E\,\bar{X}_T^{(\varepsilon)} = \varepsilon \sum_{n\geq 1} P(\bar{X}_T^{(\varepsilon)} \geq n\varepsilon)$$

$$= \varepsilon \sum_{n\geq 1} E \exp(-\lambda H_{n\varepsilon})$$

$$\gamma = \sqrt{c^2+2\lambda\sigma^2}$$

$$= \varepsilon \sum_{n\geq 1} \exp[-n\varepsilon(\gamma-c)/\sigma^2]$$

and

$$E(\bar{X}_T^{(\varepsilon)} - X_T^{(\varepsilon)}) = \varepsilon \sum_{n\geq 1} \exp[-n\varepsilon(\gamma+c)/\sigma^2] \qquad \text{similarly}$$

Hence
$$E\,\bar{X}_T^{(\varepsilon)}(\bar{X}_T^{(\varepsilon)} - X_T^{(\varepsilon)}) = \varepsilon^2 \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \exp[-(j+k)\varepsilon(\gamma-c)/\sigma^2]\, e^{-2k\varepsilon c/\sigma^2}$$

$$= \varepsilon^2 \sum_{n=2}^{\infty} \exp[-n\varepsilon(\gamma-c)/\sigma^2] \cdot \sum_{j=1}^{n-1} e^{-2j\varepsilon c/\sigma^2}$$

$$= \varepsilon^2 \sum_{n\geq 2} \Big(\sum_{j=1}^{n-1} e^{-2j\varepsilon c/\sigma^2}\Big) \int_0^\infty e^{-\lambda t} \cdot \frac{n\varepsilon}{\sigma} \exp[-(n\varepsilon - ct)^2/2\sigma^2 t] \frac{dt}{\sqrt{2\pi t^3}}$$

whence

$$\boxed{E\,\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) = \varepsilon^2 \sum_{n\geq 2}\Big(\sum_{j=1}^{n-1} e^{-2j\varepsilon c/\sigma^2}\Big) \int_0^t \frac{n\varepsilon}{\sigma} \exp[-(n\varepsilon - cs)^2/2\sigma^2 s] \frac{ds}{\sqrt{2\pi s^3}}}$$

Let's abbreviate $h(t,x) = \dfrac{x}{\sigma} \exp[-(x-ct)^2/2\sigma^2 t]\,(2\pi t^3)^{-1/2}$. Our task is

to get as much detail as we can on

$$\varepsilon^2 \sum_{n\geq 1} \frac{e^{-2\varepsilon c/\sigma^2} - e^{-2n\varepsilon c/\sigma^2}}{1 - e^{-2\varepsilon c/\sigma^2}} \cdot h(t, n\varepsilon)$$

$$= \varepsilon^2 \sum_{n\geq 1} \left\{ \frac{1-e^{-2n\varepsilon c/\sigma^2}}{1-e^{-2\varepsilon c/\sigma^2}} - 1 \right\} h(t, n\varepsilon)$$

$$= \frac{\varepsilon^2}{1-e^{-2\varepsilon c/\sigma^2}} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} h(t, n\varepsilon) - \varepsilon^2 \sum_{n\geq 1} h(t, n\varepsilon),$$

or even on the indefinite integral of this with respect to t. Let's introduce

$$\boxed{H(t,a) \equiv \int_0^t h(s,a)\, ds.}$$

We notice that $a\,H(t,a) \geq 0$, and for $a>0$, $H(t,a) = P[H_a \leq t]$ is decreasing in a. Notice also that $H(t,-a) = -H(t,a)\exp(-2ac/\sigma^2)$, and $H(t,0+) = 1$, $H(t,0-) = -1$, for each $t>0$.

Can prove

$$\frac{\sigma^2}{2c} \int_{-\infty}^{\infty} H(t,x)\, dx = \frac{\sigma^2 t}{2},$$

so

$$\varepsilon^{-1}\left\{ E\left[\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)})\right] - \frac{\sigma^2 t}{2} \right\} = \frac{\varepsilon}{1-e^{-2\varepsilon c/\sigma^2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} H(t,n\varepsilon) - \varepsilon \sum_{n\geq 1} H(t,n\varepsilon) - \frac{\sigma^2 t}{2\varepsilon}$$

$$= \frac{\sigma^2}{2c}\left[\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} H(t,n\varepsilon) - \varepsilon^{-1}\int_{-\infty}^{\infty} H(t,x)\, dx \right]$$

$$+ \left[\frac{\varepsilon}{1-e^{-2\varepsilon c/\sigma^2}} - \frac{\sigma^2}{2c} \right] \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} H(t,n\varepsilon) - \varepsilon \sum_{n\geq 1} H(t,n\varepsilon).$$

Just for concreteness, let's assume $c>0$, and take the three terms in the above expression one by one. Splitting the first $[\cdot\cdot]$ into the positive and negative parts, the positive bit is

$$\varepsilon^{-1} \sum_{n\geq 1} \int_{n\varepsilon-\varepsilon}^{n\varepsilon} \{H(t,n\varepsilon) - H(t,x)\}\, dx$$

$$= \varepsilon^{-1} \sum_{n\geq 1} \int_0^\varepsilon dy \int_0^y H'(t, n\varepsilon - z)\, dz$$

$$= \varepsilon^{-1} \sum_{m\geq 0} \int_0^\varepsilon dy \cdot y H'(t, m\varepsilon + y)$$

$$= \varepsilon \sum_{m\geq 0} \int_0^1 v\, H'(t, m\varepsilon + \varepsilon v)\, dv$$

The claim is that this converges to $-\frac{1}{2} = +\frac{1}{2}\{H(t,\infty) - H(t,0)\}$ as $\varepsilon \downarrow 0$. For this, we need a better understanding of the derivatives of H. Now

$$H(t,a) = P[\bar{X}_t \geq a]$$

so

$$H'(t,x) = -P[\bar{X}_t \in dx]/dx$$

$$= -\int_0^\infty \frac{2(x+y)\, e^{-(x+y)^2/2t}}{\sqrt{2\pi t^3}}\, e^{c(x-y) - c^2t/2}\, dy$$

$$= -\int_x^\infty \frac{2v\, e^{-v^2/2t}}{(2\pi t^3)^{1/2}}\, e^{-cv - c^2t/2 + 2cx}\, dv$$

$$= -\frac{2\, e^{-(x-ct)^2/2t}}{\sqrt{2\pi t}} + 2c e^{2cx}\, \bar{\Phi}\left(\frac{x+ct}{\sqrt{t}}\right)$$

and so

$$H''(t,x) = 2\,\frac{x-2ct}{t}\,\frac{e^{-(x-ct)^2/2t}}{\sqrt{2\pi t}} + 4c^2\, \bar{\Phi}\left(\frac{x+ct}{\sqrt{t}}\right) . e^{2cx}.$$

Thus with $t>0$ fixed, we have a bound

$$|H''(t,x)| \leq A(t)\,(1+x)\, \exp\{-(x-ct)^2/2t\}.$$

Therefore

$$\left| \tfrac{1}{2} + \varepsilon \sum_{m\geq 0} \int_0^1 v\, H'(t, m\varepsilon + \varepsilon v)\, dv \right|$$

$$\leq \varepsilon \sum_{m \geq 0} \int_0^1 v\, |H'(t, m\varepsilon + \varepsilon v) - H'(t, m\varepsilon)|\, dv$$

$$+ \left| \tfrac{1}{2} + \varepsilon \sum_{m\geq 0} \tfrac{1}{2} H'(t, m\varepsilon) \right|$$

$$\to 0.$$

A similar analysis holds for the negative part of $[\cdots]$, showing that it converges to $\frac{1}{2}$. Thus, overall, the first term is $o(1)$.

Now for the second term, $\left[\frac{\varepsilon}{1-e^{-2c\varepsilon/\sigma^2}} - \frac{\sigma^2}{2c}\right] \sum_{n\neq 0} H(t, n\varepsilon)$. The term $[\ \ ]$ is

$$\frac{\sigma^2}{2c}\left[\frac{2c\varepsilon/\sigma^2}{1-e^{-2c\varepsilon/\sigma^2}} - 1\right] = \frac{\sigma^2}{2c}\,\frac{2c\varepsilon/\sigma^2 - 1 + e^{-2c\varepsilon/\sigma^2}}{1-e^{-2c\varepsilon/\sigma^2}} \sim \tfrac{1}{2}\varepsilon,$$

and $\displaystyle \varepsilon \sum_{n \neq 0} H(t, n\varepsilon) \to \int_0^\infty H(t,x)\left(1 - e^{-2cx/\sigma^2}\right) dx = ct.$

The third term,

$$- \varepsilon \sum_{n \geq 1} H(t, n\varepsilon) \to -\int_0^\infty H(t,x)\, dx,$$

so, overall, for $c > 0$,

$$\boxed{\varepsilon^{-1}\left\{E\, \bar{X}_t^{(\varepsilon)}\left(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}\right) - \frac{\sigma^2 t}{2}\right\} \to -\int_0^\infty H(t,x)\, e^{-2cx/\sigma^2}\, dx}$$

This has some expression in terms of Φ, but it's ugly except in the case $c=0$ (which need separate verification, because the above argument required $c \neq 0$). It's still OK for $c=0$, when the result says

$$\boxed{\varepsilon^{-1}\left\{E\, \bar{X}_t^{(\varepsilon)}\left(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}\right) - \frac{\sigma^2 t}{2}\right\} \longrightarrow -\sqrt{\frac{2}{\pi}}\sqrt{t}}$$

So our corrected version of the R-S estimator for this situation is

$$\boxed{t\,\hat{\sigma}^2 = \left\{\bar{X}_t^{(\varepsilon)}\left(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}\right) + \underline{X}_t^{(\varepsilon)}\left(\underline{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}\right)\right\} + 2\varepsilon\sqrt{\frac{2t}{\pi}}}$$

[What are typical values? If we work on daily prices, $t = 1/350$, so $2\sqrt{2t/\pi} = 0.0853$, and if we had $\sigma = 0.2$, which is typical, then $\sigma^2 t = 0.0001143$. So we'd really need ε to be at very worst about 10^{-3}, otherwise the "correction" is swamping the truth!]

[Perhaps we need the next term in the expansion??]

Substitute $\theta = \frac{\sigma}{c} \frac{f'}{f}$ whence eventually

$$f(t) = e^{-\beta t} \left\{ \gamma \cosh \gamma t + \left(\beta + \frac{c\rho}{\sigma}\right) \sinh \gamma t \right\}$$

An asymmetric information problem (28/3/95)

This is a model proposed by Stewart Hodges. The log price process is

$$\boxed{dY_t = \sigma dB_t + (a + c\eta_t)dt, \qquad d\eta_t = dB'_t - \beta\eta_t dt}$$

and the uninformed traders only know Y, but the informed trader is also seeing the OU process η. Stewart is interested to know what the informed trader with CRRA felicity function would do under transactions costs. Another interesting question is to do with the estimate $\hat{\eta}_t \equiv E[\eta_t \mid \mathcal{Y}_t]$ which the uninformed trader filters from the observation filtration $\mathcal{Y}_t \equiv \sigma(Y_u : u \leq t)$. Assume that $dB_t\, dB'_t = \rho\, dt$. Yet another interesting question is what the optimal investment + consumption behaviour would be for the two traders!

(i) We have

$$dY_t = \sigma d\tilde{B}_t + (a + c\hat{\eta}_t)\, dt, \qquad d\hat{\eta}_t = dM_t - \beta\hat{\eta}_t dt,$$

where $\tilde{B}$, M are two continuous $\mathcal{Y}$-martingales, $\tilde{B}$ is Brownian motion. Now

$$\begin{cases} d(\eta Y) = \sigma\eta\, dB + Y dB' + \eta(a+c\eta)dt - \beta Y\eta\, dt + \sigma\rho\, dt \\ d(\hat{\eta} Y) = \sigma\hat{\eta}\, d\tilde{B} + Y dM + \hat{\eta}(a+c\hat{\eta})dt - \beta Y\hat{\eta}\, dt + \sigma\, d\tilde{B}\, dM \end{cases}$$

and taking the optional projection and comparing gives

$$\boxed{\sigma\frac{d\tilde{B}\,dM}{dt} = \sigma\rho + c(\widehat{\eta^2} - \hat{\eta}^2)}$$

Since all processes are Gaussian, we have $v_t \equiv (\eta^2)^\wedge_t - \hat{\eta}_t^2$ is <u>deterministic</u>. Represent $dM_t = \theta_t\, d\tilde{B}_t$, and observe that

$$E[(\eta_t^2)^\wedge] = E\eta_t^2 = (1 - e^{-2\beta t})/2\beta$$

so that if $g(t) = E[\hat{\eta}_t^2]$, we have $\dot{g}(t) = -2\beta g(t) + \theta_t^2$; hence we obtain the pair of equations

$$\begin{cases} \dot{g} = -2\beta g + \theta^2 \\ \theta = \rho + \frac{c}{\sigma}\cdot\left\{\frac{1-e^{-2\beta t}}{2\beta} - g\right\}. \end{cases}$$

Hence

$$\dot{\theta} = \frac{c}{\sigma}(e^{-2\beta t} - \dot{g}) = \frac{c}{\sigma}(e^{-2\beta t} + 2\beta g - \theta^2) = -\frac{c}{\sigma}\theta^2 + \frac{c}{\sigma} + 2\beta\rho - 2\beta\theta$$

which is a Ricatti equation, with boundary condition $\theta_0 = \rho$. The solution is

In the special case $\rho = \pm 1$, we get $\theta = \pm 1$.

$$U(x) = \frac{x^{1-R}}{1-R} \qquad (R>0,\ R \neq 1) \Rightarrow \quad \tilde{U}(\lambda) = \frac{R}{1-R}\ \lambda^{(R-1)/R}$$

$$U(x) = \log x \qquad \Rightarrow \quad \tilde{U}(\lambda) = -\log \lambda - 1$$

$$\boxed{\theta_t = \frac{\gamma\rho + (\frac{c}{\sigma} + \beta\rho)\tanh \gamma t}{\gamma + (\beta + \frac{c\rho}{\sigma})\tanh \gamma t}} . \qquad \gamma^2 \equiv \beta^2 + \frac{c}{\sigma}\left(\frac{c}{\sigma} + 2\beta\rho\right)$$

Now the filtering equation for $\hat{\eta}$; we get $d\hat{\eta}_t = \sigma^{-1}\theta_t(dY_t - a\,dt) - (\beta + \sigma^{-1}c\theta_t)\hat{\eta}_t\,dt$

$$\Rightarrow d[f_t e^{\beta t}\hat{\eta}_t] = f_t e^{\beta t}\cdot\frac{\theta_t}{\sigma}(dY_t - a\,dt) = cf_t' e^{\beta t}(dY_t - a\,dt)$$

(ii) Now how would the informed agent behave to max $E\int_0^\infty e^{-\delta t}U(C_t)\,dt$? Suppose M_t = total cash moved from bank to share by time t, L_t = total cash moved from share to cash by time t, with proportional losses μ, λ resp. If x_t = monetary value of bank account, y_t = monetary value of holdings in shares. Thus

$$\begin{cases} dx_t = rx_t\,dt - dM_t + (1-\lambda)\,dL_t - C_t\,dt \\ dy_t = y_t(\sigma\,dB_t + b(\eta_t)\,dt) + (1-\mu)\,dM_t - dL_t \end{cases} \qquad b(\eta) \equiv a + \frac{\sigma^2}{2} + c\eta$$

Define

$$V(x,y,\eta) \equiv \sup E\left[\int_0^\infty e^{-\delta t}U(C_t)\,dt \,\Big|\, x_0 = x,\ y_0 = y,\ \eta_0 = \eta\right].$$

Then the HJB equations reduce to

$$\boxed{\tilde{U}(V_x) - \delta V + rxV_x + yb(\eta)V_y - \beta\eta V_\eta + \tfrac{1}{2}\sigma^2y^2V_{yy} + \sigma y\rho V_{y\eta} + \tfrac{1}{2}V_{\eta\eta} = 0}$$

where $\tilde{U}(\lambda) = \sup\{U(x) - \lambda x\}$. Also $-V_x + (1-\mu)V_y \le 0$, $(1-\lambda)V_x - V_y \le 0$.

(iii) Special case: $U(x) = x^{1-R}/(1-R)$. We have the scaling behaviour

$$V(\kappa x, \kappa y, \eta) = \kappa^{1-R}V(x,y,\eta)$$

So let's define

$$\theta_t \equiv \frac{y_t}{x_t + y_t}, \qquad w_t \equiv x_t + y_t$$

so that

$$\boxed{\begin{aligned} dw_t &= \theta_t w_t[\sigma\,dB_t + \{b(\eta_t) - r\}\,dt] + rw_t\,dt - \mu\,dM_t - \lambda\,dL_t - C_t\,dt \\ d\theta_t &= \theta_t(1-\theta_t)[\sigma\,dB_t + (b(\eta_t) - r)\,dt - \sigma^2\theta_t\,dt] + \frac{\theta_t C_t}{w_t}\,dt \\ &\quad + \frac{1-\mu(1-\theta_t)}{w_t}\,dM_t - \frac{1-\theta_t\lambda}{w_t}\,dL_t \end{aligned}}$$

Compare with Davis & Norman? (which is what happens when there's no η-dependence)

[If we specialise to $U(x) = x^{1-R}/(1-R)$ we obtain $V(w, \eta) = w^{1-R}\, g(\eta)$, where g solves

$$\frac{R}{1-R}\, g^{(R-1)/R} + \big(r(1-R) - \delta\big) g + \frac{(1-R)}{2R\sigma^2} \big\{(b(\eta) - r) g + \sigma\rho g'\big\}^2 g^{-1} - \beta\eta g' + \tfrac{1}{2} g'' = 0 .$$
]

In this special case, we can express the value f^n as $V(x,y,\eta) = w^{1-R} g(\theta,\eta)$, and the HJB story becomes

$$\left[(1-R)g - \theta g_\theta\right]^{(R-1)/R} \frac{R}{1-R} - \delta g + \theta(1-\theta)(b-r-\sigma^2\theta) g_\theta - \beta\eta g_\eta$$

$$+ (1-R)\{\theta b + (1-\theta)r\} g + \tfrac{1}{2}\sigma^2\theta^2(1-\theta)^2 g_{\theta\theta} + \tfrac{1}{2} g_{\eta\eta} - \tfrac{1}{2}R(1-R)\sigma^2\theta^2 g$$

$$+ \sigma\theta(1-\theta)\rho\, g_{\theta\eta} + \rho\theta\sigma(1-R) g_\eta + \sigma^2\theta^2(1-\theta)(1-R) g_\theta = 0.$$

This really looks not a lot better. Suppose instead we write

$$V(x,y,\eta) = x^{1-R}\varphi(u,\eta), \quad u \equiv y/x.$$

Then we obtain

$$\boxed{\begin{aligned}&\frac{R}{1-R}\left((1-R)\varphi - u\varphi_1\right)^{(R-1)/R} - \delta\varphi + r(1-R)\varphi - ru\varphi_1 + u\, b(\eta)\varphi_1\\ &-\beta\eta\varphi_2 + \tfrac{1}{2}\sigma^2u^2\varphi_{11} + \sigma\rho\, u\varphi_{12} + \tfrac{1}{2}\varphi_{22} = 0.\end{aligned}}$$

(iv) Special case: $U(x) = \log x$. In this case, $V(x,y,\eta) = \delta^{-1}\log x + V(1, y/x, \eta)$ and hence

$$\boxed{\begin{aligned}&-1-\log\left[\tfrac{1}{\delta} - uf_1(u,\eta)\right] - \delta f(u,\eta) + r\left\{\tfrac{1}{\delta} - u f_1\right\} + u\, b(\eta) f_1 - \beta\eta f_2\\ &+\tfrac{1}{2}\sigma^2u^2 f_{11} + \rho\sigma u f_{12} + \tfrac{1}{2} f_{22} = 0\end{aligned}}$$

where we write $u \equiv y/x$, $f(u,\eta) \equiv V(1,u,\eta)$.

(v) OK, well, what happens if we don't have transactions costs? The HJB equation for $V(w,\eta)$ (where $w = x+y$ is the wealth) now becomes

$$\tilde{u}(V_w) - \delta V - \frac{\left[(b(\eta)-r)V_w + \sigma\rho V_{w\eta}\right]^2}{2\sigma^2 V_{ww}} + rwV_w - \beta\eta V_\eta + \tfrac{1}{2}V_{\eta\eta} = 0$$

and $\theta^* = -\{(b(\eta)-r)V_w + \sigma\rho V_{w\eta}\}/\sigma^2 V_{ww}$.

Perhaps only the log utility stands much chance here; if $V(w,\eta) = \frac{1}{\delta}\log w + \varphi(\eta)$ then the DE we obtain is

$$\left(\log\delta - 1 + \frac{r}{\delta}\right) + \frac{(b(\eta)-r)^2}{2\sigma^2\delta} - \delta\varphi - \beta\eta\varphi'(\eta) + \tfrac{1}{2}\varphi''(\eta) = 0.$$

We can solve this by

$$\varphi(\eta) = a_0 + a_1\eta + a_2\eta^2$$

where
$$a_0 = \frac{\tilde{a}(\delta+2\beta)+\tilde{c}}{\delta(\delta+2\beta)}, \quad a_1 = \frac{\tilde{b}}{\delta+\beta}, \quad a_2 = \frac{\tilde{c}}{\delta+2\beta}$$

and

$$\tilde{a} = \log\delta - 1 + \frac{r}{\delta} + (a+\tfrac{1}{2}\sigma^2-r)^2/2\sigma^2\delta$$

$$\tilde{b} = c(a+\sigma^2/2 - r)/\sigma^2\delta$$

$$\tilde{c} = c^2/2\sigma^2\delta$$

(vi) Meanwhile, what about the uninformed trader, for whom $Y \equiv \log S$ solves

$$dY = \sigma\, d\tilde{B} + (a + c\hat{\eta})\,dt, \quad d\hat{\eta} = \theta_t\, d\tilde{B} - \beta\hat{\eta}\,dt\ ?$$

Strictly speaking, the value function is now $V(t,w,\eta)$, depending also on t, and HJB is

$$\tilde{u}(V_w) - \delta V - \frac{\left((b(\hat{\eta})-r)V_w + \sigma\theta V_{w\eta}\right)^2}{2\sigma^2 V_{ww}} + rwV_w - \beta\hat{\eta}V_\eta + \tfrac{1}{2}\theta_t^2 V_{\eta\eta} + \dot{V} = 0.$$

But if we let $t\to\infty$, then $\theta_t \to (\gamma\rho + \frac{c}{\sigma} + \beta\rho)/(\gamma+\beta+c\rho/\sigma) \equiv \rho'$, and the HJB equation is very similar to what it is for the informed trader, except the c/e of $V_{\eta\eta}$, and of $V_{w\eta}$. We can, of course, solve exactly as above, by writing $V(w,\eta) = \delta^{-1}\log w + \varphi(\eta)$, with η some other quadratic.

For the case $c=0$,

$$f(a) = \bar{\Phi}\left(\frac{a}{\sqrt{t}}\right), \quad \text{so} \quad f'(a) = -\frac{e^{-a^2/2t}}{\sqrt{2\pi t}}, \quad f''(a) = \frac{a e^{-a^2/2t}}{\sqrt{2\pi t^3}}$$

On the smoothness of Brownian first passage distributions (2/5/95)

Fix $t>0$ and $c>0$, and consider

$$f(a) \equiv \mathbb{P}^c[H_a \le t] = \int_0^t a \exp[-(a-cs)^2/2s] \frac{ds}{\sqrt{2\pi s^3}}.$$

Is this nice + smooth in a, down to and including $a=0$?

(i) Note that

$$e^{-ca} f(a) = \int_0^t a \exp[-a^2/2s - c^2 s/2] \frac{ds}{\sqrt{2\pi s^3}}$$

$$= \frac{\partial}{\partial a}\left\{ -\int_0^t \exp\left[-\frac{a^2}{2s} - \frac{c^2 s}{2}\right] \frac{ds}{\sqrt{2\pi s}} \right\},$$

so this leads us to consider

$$\varphi(\lambda) \equiv \int_0^t \exp\left[-\frac{\lambda}{2s} - \frac{c^2 s}{2}\right] \frac{ds}{\sqrt{2\pi s}}$$

$s \equiv \frac{1}{2u}$

$$= \int_{1/2t}^{\infty} \exp(-\lambda u - c^2/4u) \frac{du}{2\sqrt{\pi u^3}}$$

Hence

$$\boxed{\varphi(0) = \frac{1}{2\sqrt{\pi}} \int_{1/2t}^{\infty} e^{-c^2/4u} \frac{du}{u^{3/2}},}$$

and

$$\boxed{\varphi^{(n)}(\lambda) = (-1)^n \int_{1/2t}^{\infty} u^{n-3/2} e^{-\lambda u - c^2/4u} \frac{du}{2\sqrt{\pi}} \sim \frac{(-1)^n}{2\sqrt{\pi}} \lambda^{-(n-1/2)} \Gamma(n-\tfrac{1}{2})}$$

Note also $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(n-\frac{1}{2}) = (2n-2)!\, 4^{-n+1}\, \Gamma(\frac{1}{2})/(n-1)!$,

so

$$\boxed{\varphi^{(n)}(\lambda) \sim \frac{(-1)^n}{2} \lambda^{-(n-1/2)} \frac{(2n-2)!}{(n-1)!} 4^{-n+1}} \quad \text{as } \lambda \downarrow 0$$

(ii) However, this doesn't really help... the way round is trivial, because

$$\mathbb{P}^c[H_a > t] = \int_t^{\infty} a\, e^{-(a-cs)^2/2s} \frac{ds}{\sqrt{2\pi s^3}},$$

A quick way to see what the Euler-Maclaurin expansion is runs as follows, at least on $[0,1]$.

$$f(1)-f(0) - \sum_{j=1}^{n} \frac{1}{n} f'(j/n) = \sum_{r\geq 1} \frac{c_r}{n^r}\left[f^{(r)}(1) - f^{(r)}(0)\right]$$

$$=\left(e^D - I - \frac{1}{n}\sum_{1}^{n} e^{jD/n} D\right) f(0) \qquad = \sum_{r\geq 1} \frac{c_r}{n^r} D^r (e^D - 1) f(0)$$

So we can do some generating function tale ...

$$e^D - 1 - \frac{D}{n} \frac{e^{(n+1)D/n} - e^{D/n}}{e^{D/n}-1} = \left(I - \frac{D}{n}\cdot \frac{e^{D/n}}{e^{D/n}-1}\right)(e^D - 1),$$

so replacing D/n by θ, we should have

$$1 - \frac{\theta e^\theta}{e^\theta - 1} = \sum_{r\geq 1} c_r \theta^r$$

$$= 1-\theta - \frac{\theta}{e^\theta - 1} = 1-\theta - \sum_{k\geq 0} B_k \frac{\theta^k}{k!}$$ in terms of the Bernoulli numbers B_k

We have $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, and thus

matches what we have found.

So now if $\bar{f}(a) \equiv 1-f(a)$, we get

$$e^{-ca}\bar{f}(a) = \int_t^\infty a\, e^{-a^2/2s\, - c^2s/2} \frac{ds}{\sqrt{2\pi s^3}} = \sum_{n\geq 0} \frac{a^{2n+1}}{n!} \int_t^\infty \left(\frac{-1}{2s}\right)^n e^{-c^2s/2} \frac{ds}{\sqrt{2\pi s^3}}$$

$$\equiv \sum_{n\geq 0} a^{2n+1}\gamma_n, \quad \text{say}$$

where, of course, $\gamma_n \equiv \frac{1}{n!}\int_t^\infty \left(\frac{-1}{2s}\right)^n \exp(-\tfrac{1}{2}c^2 s) \frac{ds}{\sqrt{2\pi s^3}}$. We may now define $\beta_{2n+1} \equiv \gamma_n$, $\beta_{2n}=0$, so that

$$\boxed{\bar{f}(a) = e^{ca}\sum_{k\geq 0}\beta_k a^k = \sum_{r=0}^\infty a^r\left(\sum_{k=0}^r \frac{c^{r-k}}{(r-k)!}\beta_k\right)}$$

Special case: $c=0$. Here we get

$$\gamma_n = \frac{(-2)^{-n}}{n!}\frac{1}{\sqrt{2\pi}} \frac{t^{-n-\frac{1}{2}}}{n+\frac{1}{2}}.$$

(iii)
In general, it's not too hard to see that $\bar{f}^{(n)}(a) \to 0$ as $a\to\infty$, for each $n\geq 1$. Indeed, if $\psi(\lambda) \equiv \int_t^\infty \exp[-\lambda/2s - c^2 s/2]\frac{ds}{\sqrt{2\pi s}}$, then it's easy to see that for each n, there is constant k_n s.t.

$$|\psi^{(n)}(\lambda)| \leq e^{-\lambda/2t}.\,k_n$$

Since $e^{-ca}\bar{f}(a) = \frac{d}{da}\left[-\psi(a^2)\right]$, the n^{th} derivative of the RHS can be bounded by $\exp(-a^2/2t)\times\{\text{polynomial in } a\}$, and this forces $\bar{f}^{(n)}(a)\to 0$ $(a\to\infty)$

(iv) Now it looks like we need some variant of the Euler-McLaurin formula. I intend to prove that there exists a sequence $(c_r)_{r\geq 1}$ of reals such that if f is smooth enough on $\mathbb{R}^+$, and nice enough (in terms of derivatives going to 0 at ∞ fast enough) then $\forall m$

$$\boxed{f(\infty)-f(0) - \varepsilon\sum_{n\geq 1} f'(n\varepsilon) = \sum_{r=1}^m c_r\{f^{(r)}(\infty)-f^{(r)}(0)\}\varepsilon^r + \varepsilon^{m+1}R_{m+1}}$$

where the remainder term R_{m+1} is $O(1)$.
To establish this, let's consider

$a_1 = -\frac{1}{2}$, $a_2 = -\frac{1}{6}$, $a_3 = 0$, $a_4 = \frac{1}{30}$, $a_5 = 0$, $a_6 = -\frac{1}{42}$, ...

One can put the power-series together; if $c_0 = 0$,

$$\sum_{m \geq 0} c_m t^m = 1 + \frac{t e^t}{1 - e^t}$$

$$f(\infty)-f(0)-\varepsilon\sum_{n\geq 1} f'(n\varepsilon)-\sum_{r=1}^{m} c_r\left[f^{(r)}(\infty)-f^{(r)}(0)\right]\varepsilon^r$$

$$=\sum_{n\geq 1}\int_{n\varepsilon-\varepsilon}^{n\varepsilon}\{f'(x)-f'(n\varepsilon)\}\,dx-\sum_{n\geq 1}\sum_{r=1}^{m} c_r\varepsilon^r\{f^{(r)}(n\varepsilon)-f^{(r)}(n\varepsilon-\varepsilon)\}$$

$$=\sum_{n\geq 1}\int_{n\varepsilon-\varepsilon}^{n\varepsilon}\left\{\sum_{k=1}^{m}\frac{(x-n\varepsilon)^k}{k!}f^{(k+1)}(n\varepsilon)+\frac{(x-n\varepsilon)^{m+1}}{(m+1)!}f^{(m+2)}(x_n)\right\}dx$$

$$+\sum_{n\geq 1}\sum_{r=1}^{m} c_r\varepsilon^r\left\{\sum_{j=1}^{m+1-r}\frac{(-\varepsilon)^j}{j!}f^{(r+j)}(n\varepsilon)+\frac{(-\varepsilon)^{m+2-r}}{(m+2-r)!}f^{(m+2)}(y_n)\right\}$$

for certain $y_n, x_n\in(n\varepsilon-\varepsilon, n\varepsilon)$. This now simplifies to

$$\sum_{n\geq 1}\sum_{k=1}^{m}\frac{-(-\varepsilon)^{k+1}}{(k+1)!}f^{(k+1)}(n\varepsilon)+\sum_{n\geq 1}\sum_{\ell=2}^{m+1}\varepsilon^\ell f^{(\ell)}(n\varepsilon)\sum_{r=1}^{\ell-1}c_r\frac{(-1)^{\ell-r}}{(\ell-r)!}$$

$$+\sum_{n\geq 1}\int_{n\varepsilon-\varepsilon}^{n\varepsilon}\frac{(x-n\varepsilon)^{m+1}}{(m+1)!}f^{(m+2)}(x_n)\,dx+\sum_{n\geq 1}\varepsilon^{m+2}f^{(m+2)}(y_n)\sum_{r=1}^{m}c_r\frac{(-1)^{m+2-r}}{(m+2-r)!}$$

If now we have $\boxed{c_r\equiv a_r/r!}$ and forall $k\geq 1$

$$\boxed{1=\sum_{r=1}^{k}a_r\binom{k+1}{r}(-1)^r,}$$

then only the remainder terms are left. If we set $K^{(m)}(n,\varepsilon)$ to be $\sup\{|f^{(m)}(x)| \ :\ n\varepsilon-\varepsilon\leq x\leq n\varepsilon\}$ the remainder term is bounded by

$$\sum_{n\geq 1}\varepsilon^{m+2}K^{(m+2)}(n,\varepsilon)\left\{\frac{1}{(m+2)!}+\sum_{r=1}^{m}|c_r|/(m+2-r)!\right\}$$

$$=\varepsilon^{m+1}\sum_{n\geq 1}\varepsilon\frac{K^{(m+2)}(n,\varepsilon)}{(m+2)!}\left\{1+\sum_{r=1}^{m}|a_r|\binom{m+2}{r}\right\}$$

and provided we have some DRI property of $K^{(m+2)}$, we're OK for an asymptotic expansion!

I've checked this against the Euler-Maclaurin expansion in Olver's book, and it matches exactly.

(v) Let's now return to the earlier analysis of p 29–30, where we obtained

$$E\left[\bar X_t^{(\varepsilon)}(\bar X_t^{(\varepsilon)} - X_t^{(\varepsilon)})\right] = \varepsilon^2 \sum_{n\geq 1}\left\{\frac{1-e^{-2n\varepsilon c/\sigma^2}}{1-e^{-2\varepsilon c/\sigma^2}} - 1\right\} h(n\varepsilon),$$

where

$$h(a) \equiv \int_0^t \frac{a}{\sigma}\exp\left[-\frac{(a-cs)^2}{2\sigma^2 s}\right](2\pi s^3)^{-1/2}\,ds \equiv 1-\tilde h(a).$$

I want to obtain the first few terms of the expansion in ε. We have

$$\frac{\varepsilon}{1-e^{-2\varepsilon c/\sigma^2}}\cdot \varepsilon\sum_{n\geq 1}\left(1-e^{-2n\varepsilon c/\sigma^2}\right)h(n\varepsilon) - \varepsilon\cdot\varepsilon\sum_{n\geq 1} h(n\varepsilon).$$

Now the Bernoulli numbers $(b_n)_{n\geq 0}$ are defined by

$$\frac{t}{e^t-1} = \sum_{n\geq 0} b_n \frac{t^n}{n!}$$

(so $b_0 = 1,\ b_1 = -\frac12,\ b_2 = \frac16,\ b_4 = -\frac1{30},\ b_6 = \frac1{42},\ b_8 = -\frac1{30},\ b_{10} = \frac5{66},\ldots$)

so that

$$\frac{\varepsilon}{1-e^{-2\varepsilon c/\sigma^2}} = \varepsilon + \frac{\varepsilon}{e^{2\varepsilon c/\sigma^2}-1} = \varepsilon + \frac{\sigma^2}{2c}\cdot\left\{b_0 + b_1\theta + b_2\frac{\theta^2}{2!} + b_4\frac{\theta^4}{4!} + \ldots\right\}$$

$$(\theta \equiv 2\varepsilon c/\sigma^2)$$

$$= \frac{\sigma^2}{2c} + \frac12\varepsilon + \frac16\cdot\frac{c}{\sigma^2}\cdot\varepsilon^2 - \frac1{30}\cdot\left(\frac{2c}{\sigma^2}\right)^3\frac{\varepsilon^4}{4!} + \frac1{42}\left(\frac{2c}{\varepsilon^2}\right)^5\frac{\varepsilon^6}{6!}\ \ldots$$

and the E-M expansion gives

$$\varepsilon\sum_{n\geq 1}\left(1-e^{-2n\varepsilon c/\sigma^2}\right)h(n\varepsilon) = \int_0^\infty\left(1-e^{-2cx/\sigma^2}\right)h(x)\,dx - \sum_{r\geq 1} c_r\,\varepsilon^r\left[g^{(r-1)}(\infty) - g^{(r-1)}(0)\right]$$

(where $g(x) \equiv (1-e^{-2cx/\sigma^2})h(x)$;)

$$= ct - \varepsilon^2\frac{c}{6\sigma^2} + \frac{\varepsilon^4}{4!}\cdot\frac1{30}\,g^{(3)}(0) + O(\varepsilon^6)$$

$$= ct - \varepsilon^2\frac{c}{6\sigma^2} + \frac{\varepsilon^4}{4!}\cdot\frac1{30}\cdot\left\{3\cdot\frac{2c}{\sigma^2}h''(0) - 3\left(\frac{2c}{\sigma^2}\right)^2 h'(0) + \left(\frac{2c}{\sigma^2}\right)^3\right\} + O(\varepsilon^6),$$

But $h'(0) = -\frac1\sigma\int_t^\infty e^{-c^2s/2\sigma^2}\frac{ds}{\sqrt{2\pi s^3}}$, $h''(0) = \frac{2c}{\sigma^2}h'(0)$, so we get

$$\varepsilon \sum_{n\geq 1} \left(1-e^{-2n\varepsilon c/\sigma^2}\right) h(n\varepsilon) = ct - \frac{\varepsilon^2}{12}\cdot\frac{2c}{\sigma^2} + \frac{\varepsilon^4}{4!}\cdot\frac{1}{30}\cdot\left(\frac{2c}{\sigma^2}\right)^3 + O(\varepsilon^6)$$

Next, if we write $-2\varepsilon c/\sigma^2 \equiv t$, then

$$\frac{\varepsilon}{1-e^{-2\varepsilon c/\sigma^2}} = \frac{\sigma^2}{2c}\left(\frac{-t}{1-e^t}\right) = \frac{\sigma^2}{2c}\left[1-\frac{1}{2}t+\frac{1}{6}\frac{t^2}{2!} - \frac{1}{30}\frac{t^4}{4!}\right] + O(t^6)$$

$$= \frac{\sigma^2}{2c}\left[1+\varepsilon\frac{c}{\sigma^2} + \frac{\varepsilon^2}{12}\cdot\left(\frac{2c}{\sigma^2}\right)^2 - \frac{\varepsilon^4}{4!}\cdot\frac{1}{30}\cdot\left(\frac{2c}{\sigma^2}\right)^4\right] + O(\varepsilon^6)$$

$$= \frac{\sigma^2}{2c} + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{12}\cdot\left(\frac{2c}{\sigma^2}\right) - \frac{\varepsilon^4}{4!}\cdot\frac{1}{30}\left(\frac{2c}{\sigma^2}\right)^3 + O(\varepsilon^6).$$

Assembling, we have

$$\frac{\varepsilon}{1-e^{-2\varepsilon c/\sigma^2}}\ \varepsilon\sum_{n\geq 1}\left(1-e^{-2n\varepsilon c/\sigma^2}\right) h(n\varepsilon)$$

$$= \frac{\sigma^2 t}{2} + \frac{ct}{2}\varepsilon + \frac{\varepsilon^2}{12}\left(\frac{2c^2t}{\sigma^2}-1\right) - \frac{\varepsilon^3}{24}\cdot\frac{2c}{\sigma^2} - \frac{\varepsilon^4}{4!}\cdot\frac{1}{30}\cdot\left(\frac{2c}{\sigma^2}\right)^2\left[4+\frac{2c^2t}{\sigma^2}\right]$$
$$+O(\varepsilon^5)$$

The other part of the expansion is

$$\varepsilon^2\sum_{n\geq 1} h(n\varepsilon) = \varepsilon\left[\int_0^\infty h(x)\,dx - \sum_{r=1}^{\infty} c_r\left\{h^{(r-1)}(\infty) - h^{(r-1)}(0)\right\}\varepsilon^r\right].$$

Now
$$\int_0^\infty h(x)\,dx = \int_0^\infty da\int_0^t \frac{a}{\sigma}\exp\left[-(a-cs)^2/2\sigma^2 s\right]\frac{ds}{\sqrt{2\pi s^3}}$$

$$= \int_0^t \frac{\sigma\,ds}{\sqrt{2\pi s}}\, e^{-c^2 s/2\sigma^2} + c\int_0^t \bar{\Phi}(-c\sqrt{s}/\sigma)\,ds,$$

$$h(\infty)-h(0) = -1,\quad h'(\infty)-h'(0) = \bar{h}'(0) = \int_t^\infty e^{-c^2 s/2\sigma^2}\frac{ds}{\sigma\sqrt{2\pi s^3}},$$

$h''(\infty) - h''(0) = \bar{h}''(0) = 2c\,\bar{h}'(0)$. Assembling all this, we obtain

$$\boxed{E\,\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) = \frac{\sigma^2 t}{2} + \varepsilon\left\{\frac{ct}{2} - E\bar{X}_t\right\} + \frac{\varepsilon^2}{12}\left(\frac{2c^2t}{\sigma^2}+5\right) - \frac{\varepsilon^3}{12}\left(\frac{c}{\sigma^2}+\bar{h}'(0)\right) - \frac{\varepsilon^4}{4!}\cdot\frac{1}{30}\cdot\left(\frac{2c}{\sigma^2}\right)^2\left[4+\frac{2c^2t}{\sigma^2}\right] + O(\varepsilon^5)}$$

Mathematica check gives (for $c=0$):

$$E\, \bar{X}_t^{(\varepsilon)} (\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)}) = \frac{\sigma^2 t}{2} - \varepsilon \sqrt{\frac{2t}{\pi}} + \frac{5\varepsilon^2}{12} - \frac{\varepsilon^3}{6} (2\pi\sigma^2 t)^{-\frac{1}{2}}$$

$$+ \frac{\varepsilon^4}{360\,\sigma^3} (2\pi t^3)^{-\frac{1}{2}} - \frac{\varepsilon^7}{15120\,\sigma^5} (2\pi t^5)^{-\frac{1}{2}} + O(\varepsilon^8)$$

(vi) We may check this in some formal fashion as follows. Our exp... ...hange (28/5) for $E\,\bar{X}_t^{(\varepsilon)}(\bar{X}_t^{(\varepsilon)} - X_t^{(\varepsilon)})$ is ... we

$$\varepsilon^2 \sum_{n=2}^{\infty} \Big(\sum_{k=1}^{n-1} e^{-kb\varepsilon} \Big) \; \mathbb{P}^c\big[H_{n\varepsilon} \le t \big]$$

$$\simeq \int_0^\infty dx \Big(\int_0^x e^{-by}\,dy \Big)\, \mathbb{P}^c(H_x \le t),$$

so if we set $g(x) \equiv \mathbb{P}^c(H_x \le t)$, and $\varphi(x) \equiv \int_0^x dv \big(\int_0^v e^{-by}\,dy \big)\, g(v)$, we have

$$\varphi'(x) = \frac{1}{b}(1 - e^{-bx})\, g(x), \text{ and so}$$

$$\varepsilon^2 \sum_{n=2}^{[a/\varepsilon]} \Big(\sum_{k=1}^{n-1} e^{-kb\varepsilon} \Big) g(n\varepsilon) = \varepsilon^2 \sum_{n=2}^{[a/\varepsilon]} \frac{e^{-b\varepsilon} - e^{-nb\varepsilon}}{1 - e^{-b\varepsilon}} \cdot \frac{b\,\varphi'(n\varepsilon)}{1 - e^{-nb\varepsilon}}$$

$$= \varepsilon^2 \sum_{n=1}^{[a/\varepsilon]} \Big\{ \frac{b}{1 - e^{-b\varepsilon}}\, \varphi'(n\varepsilon) - \frac{b\,\varphi'(n\varepsilon)}{1 - e^{-nb\varepsilon}} \Big\}$$

$$= \frac{b\varepsilon}{1 - e^{-b\varepsilon}} \cdot \sum_{n=1}^{[a/\varepsilon]} \varphi'(n\varepsilon) - \varepsilon \sum_{n=1}^{[a/\varepsilon]} \varepsilon\, g(n\varepsilon)$$

$$= \frac{b\varepsilon}{1 - e^{-b\varepsilon}} \Big\{ \varphi(a) - \varphi(0) - \sum_{r \ge 1} \varepsilon^r c_r \big(\varphi^{(r)}(a) - \varphi^{(r)}(0) \big) \Big\}$$

$$- \varepsilon \Big\{ G(a) - G(0) - \sum_{r \ge 1} \varepsilon^r c_r \big(g^{(r-1)}(a) - g^{(r-1)}(0) \big) \Big\}$$

using standard E-McL expansion (here, $G = \int g$). Now $\varphi(\infty) = \sigma^2 t/2$, and all derivatives of g, φ vanish at ∞. Also, $\varphi(0) = \varphi'(0) = 0$, so we obtain

$$\frac{b\varepsilon}{1 - e^{-b\varepsilon}} \Big[\frac{\sigma^2 t}{2} + \sum_{r \ge 2} c_r \varepsilon^r \varphi^{(r)}(0) \Big] - \varepsilon \Big[G(\infty) + \sum_{r \ge 1} \varepsilon^r c_r\, g^{(r-1)}(0) \Big]$$

Substituting α^r in place of $\varphi^{(r)}(0)$, β^r in place of $g^{(r-1)}(0)$, we could expand

$$\frac{b\varepsilon}{1 - e^{-b\varepsilon}} \Big\{ \frac{\sigma^2 t}{2} + \frac{\varepsilon\alpha\, e^{\varepsilon\alpha}}{1 - e^{\varepsilon\alpha}} \Big\} - \varepsilon \Big\{ G(\infty) + \frac{\varepsilon\beta\, e^{\varepsilon\beta}}{1 - e^{\varepsilon\beta}} \Big\}$$

in powers of ε using Mathematica, and check it that way.

Modelling tick data: another candidate (13/5/95)

(i) News shocks arrive at the market at rate $\alpha(\cdot)$ (where α is deterministic and known). Each shock imparts a jump $\xi \geq 0$ to $\lambda \equiv (\lambda_1, \dots, \lambda_N)$, which is the vector of intensities with which trades in each of the N shares are happening. While no news shocks are coming in, λ_j decays to 0 at rate β_j. Thus if ν is the Poisson random measure on $\mathbb{R} \times (\mathbb{R}^+)^N$ with intensity $\alpha(t)\,dt\; G(dx)$ governing the news shocks, the intensity of trading in share j at time t is given by

$$\boxed{\lambda_j(t) = \int_{-\infty}^{t} \int_{(\mathbb{R}^+)^N} e^{-\beta_j(t-s)}\, \xi_j\, \nu(ds, d\xi).}$$

Suppose that the jumps in X_j ($\equiv$ log of j^{th} share price) have distribution $F_j(dx \mid \lambda)$ when $\lambda_j = \lambda$, and make the abbreviations

$$a_j(\lambda) \equiv \int x\, F_j(dx \mid \lambda), \qquad b_j(\lambda) \equiv \int x^2\, F_j(dx \mid \lambda).$$

Then

$$M_j(t) \equiv X_j(t) - X_j(0) - \int_0^t a_j(\lambda_j(s))\, \lambda_j(s)\, ds$$

is a martingale, and so

$$\boxed{E\left[X_j(t) - X_j(0)\right] = E \int_0^t a_j(\lambda_j(s))\, \lambda_j(s)\, ds.}$$

We can abbreviate $\int_0^t a_j(\lambda_j(s))\, \lambda_j(s)\, ds \equiv R_j(t)$ and then we get

$$\boxed{\operatorname{Cov}(X_i(t), X_j(t)) = \delta_{ij}\, E\left[M_j(t)^2\right] + \operatorname{cov}(R_i(t), R_j(t)).}$$

Without more explicit assumptions on the F_j, there's not much more we can do.

(ii) For fitting model, we shall probably want F_j not to depend on λ. Note that

$$E_0 \int_0^t \lambda_j(s)\, ds = \lambda_j(0)\, \frac{1 - e^{-\beta_j t}}{\beta_j} + \mu_j \int_0^t \frac{1 - e^{-\beta_j(t-u)}}{\beta_j}\, \alpha(u)\, du$$

where $\mu_j \equiv \int x_j\, G(dx)$ is the mean size of the shock to λ_j. We can also get the covariance between the $R_j(t)$ by expressing

$$\int_0^t \lambda_j(s)ds = \lambda_j(0)\,\frac{1-e^{-\beta_j t}}{\beta_j} + \int_0^t ds \int_0^s\!\!\int \nu(du,d\xi)\,\xi_j\, e^{-\beta_j(s-u)}$$

Centering the second term, we get

$$\int_0^t ds \int_0^s \int \{\nu(du,d\xi) - \alpha(u)du\, G(d\xi)\}\,\xi_j\, e^{-\beta_j(s-u)}$$

$$= \int_0^t \int \frac{\nu(du,d\xi) - \alpha(u)du\, G(d\xi)}{\beta_j}\,(1-e^{-\beta_j(t-u)})\,\xi_j$$

which is independent of $\mathcal{F}_0$. Hence the covariance given $\mathcal{F}_0$ between $\int_0^t \lambda_i(s)ds$ and $\int_0^t \lambda_j(s)\,ds$ will be

$$\int_0^t du\ \alpha(u)\,(1-e^{-\beta_i(t-u)})(1-e^{-\beta_j(t-u)})\,(\beta_i\beta_j)^{-1}\ .\ \int \xi_i\xi_j\, G(d\xi)$$

So once again the curse of all earlier efforts reappears; <u>the covariance cannot be general</u>. But should we demand this? We couldn't in practice get correlation 1 or −1!

More importantly, perhaps, is this. We envisage some random intensity governing jumps with distribution $F(dx|\lambda)$. Then the law of the log share price must obey

$$E\, e^{i\theta X_t} = E \exp\left\{ \int_0^t \lambda_s \int (e^{i\theta x}-1)\, F(dx|\lambda_s)\, ds \right\}$$

– and now how would we choose F so as to obey the risk-neutrality condition

$$\lambda \int (e^x-1)\, F(dx|\lambda) = r \qquad \forall \lambda$$

while at the same time keeping the above reasonably tractable?

<u>Equilibrium derivation of real and nominal rates of interest, and foreign exchange (28/5)</u>

(i) Let's suppose we're in a world generated by N Brownian motions, and we have complete markets. There's initially a single productive asset generating a flow $\delta_t \geq 0$ of dividend, expressed in terms of the only good in this world. The unique agent (think of him as a representative agent) has the goal of maximising

$$E \int_0^\infty U(t, c_t)\, dt$$

where c is the consumption process. The work of Karatzas, Lehoczky + Shreve gives us that

$$U'(t, c_t^*) = \lambda \zeta_t$$

where ζ is the state-price density process, so for an equilibrium we shall have

$$\boxed{U'(t, \delta_t) = \lambda \zeta_t \quad , \quad \pi_t \zeta_t = E_t\left[\int_t^\infty \delta_s \zeta_s\, ds\right]}$$

This is well known. (Here, π_t is the price of the unit share in the productive asset.)

(ii) Let's now introduce money. How? Well, since no economist seems willing or able to <u>define</u> money (though various descriptions are offered), let's take as our definition

$$\boxed{1 \text{ unit of money at time } t = \pi_t \text{ of the good} = \text{value of the productive asset at } t.}$$

Now if (S_t) is any asset, denominated in the good, it follows that $\zeta_t S_t$ is a martingale. So if b_t is the value at time t of one unit invested in the bank, we have

$b_t \pi_t \zeta_t$ <u>is a martingale</u>

and since $b_t = \exp\left(\int_0^t r_u\, du\right)$, where r is the nominal interest rate, and $\pi_t \zeta_t + \int_0^t \delta_s \zeta_s\, ds$ is a martingale, we conclude that

$$\boxed{r_t = \delta_t / \pi_t .}$$

The deflator for assets expressed in cash terms is simply $\pi_t \zeta_t$, and we have an easy expression for the bond prices; in fact, for $0 \leq t \leq T$

$$\boxed{P(t,T) = E_t(\pi_T \zeta_T) / \pi_t \zeta_t .}$$

This expression is even more general than the equilibrium derivation; if we have in general that $(Y_t)_{t\geq 0}$ is the deflator needed to convert assets (denominated in cash) into martingales, then

$$\boxed{P(t,T) = E_t(Y_T)/Y_t .}$$

(ii) If we now had various nations, with deflator $(Y_j(t))_{t\geq 0}$ in nation j, then an asset whose price at time t in nation j's cash is $S_j(t)$ is worth at that time $Y_{ij}(t)\, S_j(t)$ in nation i's cash (this defines Y_{ij}). Now

$Y_j(t)\, S_j(t)$ <u>and</u> $Y_i(t) . Y_{ij}(t)\, S_j(t)$ <u>are both martingales</u>,

and the completeness assumption says that

$$\boxed{Y_{ij}(t) = \frac{Y_j(t)}{Y_i(t)}}$$

so that <u>the exchange rate is the ratio of deflators in the two countries</u>.

(iii) Note that if $U(t,x) = e^{-\rho t} U(x)$, then

$$r_t = \frac{\delta_t\, U'(\delta_t)}{E_t\left[\int_t^\infty e^{-\rho(s-t)}\, \delta_s\, U'(\delta_s)\, ds\right]}$$

so that <u>the rate of interest increases with ρ</u>; the more impatient the agent is, the higher the nominal rate of interest!

If we also assume that (δ_t) is a Markov process with resolvent $(R_\lambda)_{\lambda\geq 0}$ then it's immediate that

$$\pi_t\, \zeta_t = e^{-\rho t} . \lambda . R_\rho\, g(\delta_t) \qquad\qquad g(x) \equiv x\, U'(x)$$

so

$$\boxed{r_t = \frac{g(\delta_t)}{R_\rho\, g(\delta_t)}}$$

(iv) Example: $U'(x) = x^{-R}$, δ is a CIR process, $d\delta_t = \sigma\sqrt{\delta_t}\, dW_t + (\alpha - \beta\delta_t)dt$.
The first thing we need to do to this example is to compute $R_\rho g$, where $g(x) = x^{1-R}$.
Note that $y_t \equiv e^{\beta t}\delta_t$ solves

$$dy_t = \sigma e^{\beta t/2}\sqrt{y_t}\, dW_t + \alpha e^{\beta t} dt$$

so that $y(\tau_t)$ is a BESQ$(4\alpha/\sigma^2)$, where $\tau_t = \frac{1}{\beta}\log(1 + 4\beta t/\sigma^2)$. Thus

$$\int_0^\infty e^{-\rho t} g(\delta_t)\, dt = \int_0^\infty e^{-\rho t} g(e^{-\beta t}\tilde{y}(\tau_t^{-1}))\, dt \qquad \tilde{y}_t \equiv y(\tau_t)$$

$$[t = \tau_s]$$

$$= \int_0^\infty (1 + 4\beta s/\sigma^2)^{-\rho/\beta} g((1 + 4\beta s/\sigma^2)^{-1}\tilde{y}_s)\, ds$$

$$= \int_0^\infty (1 + 4\beta s/\sigma^2)^{R-1-\rho/\beta}\, \tilde{y}_s^{\,1-R}\, ds$$

where we take $g(x) = x^{1-R}$. Now there is an explicit expression for the transition density of $\tilde{y}$; it is

$$p_t(x,y) = \frac{1}{2t}\exp\{-(x+y)/2t\}\left(\frac{y}{x}\right)^{\nu/2} I_\nu\left(\frac{\sqrt{xy}}{t}\right), \qquad \nu \equiv \frac{2\alpha}{\sigma^2} - 1.$$

The function g is integrable with respect to this iff $\boxed{\nu > R-2}$, which we now assume, along with $\nu \geq 0$.

Hence

$$E\left[\tilde{y}_t^{\,1-R} \,\middle|\, \tilde{y}_0 = x\right] = (2t)^{1-R} e^{-x/2t} \sum_{k\geq 0}\left(\frac{x}{2t}\right)^k \frac{\Gamma(k+\nu+2-R)}{\Gamma(k+\nu+1)\cdot k!}$$

We are therefore interested in

$$\int_0^\infty (1 + 4\beta t/\sigma^2)^{R-1-\rho/\beta}(2t)^{1-R} e^{-x/2t}\left(\frac{x}{2t}\right)^k dt$$

$$\left[\frac{x}{2t} = u\right]$$

$$= \int_0^\infty \left(1 + \frac{2\beta x}{\sigma^2 u}\right)^{R-1-\rho/\beta} u^{R-1+k} e^{-u}\cdot x^{1-R}\cdot\frac{x}{2u^2}\, du$$

$$= \int_0^\infty \left(u + \frac{2\beta x}{\sigma^2}\right)^{R-1-\rho/\beta}\cdot u^{\frac{\rho}{\beta}+k-2}\cdot e^{-u}\,\frac{du}{2}\; x^{2-R}$$

Small transactions costs (8/6/95)

(i) One stock, $dS = S(\sigma dB + c\,dt)$, bank account riskless return rate r, and agent with utility U, $U'(x) = x^{-R}$. Optimal proportion of wealth in risky asset is $\theta^* \equiv (c-r)/\sigma^2 R$, but suppose we invest θ_t, which is uniformly within δ of θ^*. Then

$$dw_t = \theta_t w_t \left\{ \sigma dB_t + (c-r)\,dt \right\} + r w_t\,dt$$

$$\therefore \quad w_T = w_0 \exp\left[\int_0^T \sigma \theta_t\,dB_t + \int_0^T \{ r + (c-r)\theta_t - \tfrac{1}{2}\sigma^2\theta_t^2 \}\,dt \right]$$

and

$$U(w_T) = U(w_0)\exp\left[\int_0^T (1-R)\sigma\theta_t\,dB_t - \tfrac{1}{2}(1-R)^2\sigma^2\int_0^T \theta_t^2\,dt \right.$$

$$\left. + (1-R)\int_0^T \left\{ r + (c-r)\theta_t - \tfrac{1}{2}\sigma^2\theta_t^2 R \right\} dt \right]$$

$$= U(w_0)\, \mathcal{E}((1-R)\sigma\theta\cdot B)_T \exp (1-R)\int_0^T \left\{ -\tfrac{1}{2}\sigma^2 R(\theta_t - \theta^*)^2 + r + \frac{\sigma^2 R}{2}\theta^{*2} \right\} dt$$

Hence easily

$$\boxed{\exp\left[-\tfrac{1}{2}\sigma^2 R|1-R|T\,\delta^2\right] EU(w_T^*) \le EU(w_T) \le EU(w_T^*)}$$

Here, w^* is the actual optimal wealth process for this problem. The moral is

<u>keeping within δ</u> <u>costs us</u> $\frac{\sigma^2}{2} R T \delta^2$ <u>of the initial wealth at most</u>.

We expect that the actual loss will be more like $\frac{\sigma^2}{6} R T \delta^2$, since $\theta - \theta^*$ should (under no trading) be approximately uniformly distributed on $[-\delta, \delta]$.

(ii) Now let's look at transactions costs.

$$\left.\begin{aligned} dx_t &= r x_t\,dt - dL_t + (1-\mu)\,dM_t \\ dy_t &= y_t(\sigma dB_t + c\,dt) + (1-\lambda)\,dL_t - dM_t \end{aligned}\right\}$$

We imagine both λ and μ are very small. Then with $\theta_t \equiv y_t/(x_t+y_t)$,

$$d\theta_t = \theta_t(1-\theta_t)\left\{ \sigma dB_t + (c - r - \sigma^2\theta_t^2)\,dt \right\} + \left\{ \theta_t + (1-\lambda)(1-\theta_t) \right\} \frac{dL_t}{x_t+y_t}$$

$$- \left\{ 1-\theta_t + (1-\mu)\theta_t \right\} \frac{dM_t}{x_t+y_t}$$

$$\equiv \theta_t(1-\theta_t)\left\{\sigma\, dB_t + (c-r-\sigma^2\theta_t^2)\,dt\right\} + d\tilde{L}_t - d\tilde{M}_t,$$

Assy. Now we have easily that $\tilde{L}_t \simeq (\sigma\ \theta^*(1-\theta^*))^2 t/4\delta$, as is $\tilde{M}_t$, so the loss in transactions costs is on average

$$E\left[\lambda L_t + \mu M_t\right] \simeq \frac{(\sigma\,\theta^*(1-\theta^*))^2}{4\delta}(\lambda+\mu)\, E\int_0^t w_s\, ds$$

$$\simeq \frac{\{\sigma\,\theta^*(1-\theta^*)\}^2}{4\delta}(\lambda+\mu)\cdot w_0\cdot \int_0^t e^{(r+\theta^*(c-r))s}\, ds$$

Thus our overall loss from transactions costs and poor tracking comes to approximately

$$\frac{\lambda+\mu}{\delta}\cdot\left\{\frac{\sigma\,\theta^*(1-\theta^*)}{2}\right\}^2 \frac{e^{(r+\theta^*(c-r))T}-1}{r+\theta^*(c-r)} + \frac{\sigma^2 RT}{2}\,\delta^2$$

– this is minimised by

$$\delta = k\,(\lambda+\mu)^{1/3}$$

for some suitable constant K, which can be determined from the above.

Real + Nominal rates: a class of examples (14/6/95)

Going back to the Markovian situation on p. 46, it seemed hard to pick δ to make r nice. But perhaps we should consider instead the inverse problem, beginning with nice r, + trying to deduce δ. We have

$$r_t = \frac{g(\delta_t)}{R_\rho g(\delta_t)} \equiv f_0(\delta_t)$$

for short. Assuming f_0 is one-one, we can alternatively express

$$r_t = \varphi(r_t) / V_\rho \varphi(r_t) \equiv \varphi(r_t) / h(r_t),$$

where (V_α) is the resolvent of r, and the change-of-measure martingale is

$$Z_t \equiv \left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t r_u\,du\right) \cdot \pi_t S_t = \exp\left(\int_0^t r_u\,du - \rho t\right) V_\rho \varphi(r_t).$$

Assume that, under $\tilde{P}$, r is a nice one-dimensional diffusion with generator $\tilde{\mathcal{L}}$;

$$dr = \sigma(r)\,d\tilde{W} + \mu(r)\,dt.$$

The change-of-measure back to P is $dZ_t^{-1} = Z_t^{-1}\left(-\frac{h'}{h}(r_t)\,\sigma(r_t)\,d\tilde{W}_t\right)$

$$\Rightarrow \quad dr_t = \sigma(r_t)\left[dW_t - \frac{h'\sigma}{h}(r_t)\,dt\right] + \mu(r_t)\,dt$$

relative to P, so that relative to P, r has generator

$$\mathcal{L} \equiv \tfrac{1}{2}\sigma(r)^2 D^2 + \left\{\mu(r) - \sigma(r)^2 \frac{h'(r)}{h(r)}\right\} D.$$

Now we know that

$$(\rho - \mathcal{L})h \equiv (\rho - \mathcal{L}) V_\rho \varphi = \varphi = r h$$

so h is a solution to

$$0 = \rho h - r h - \tfrac{1}{2}\sigma^2 h'' - \left(\mu - \frac{\sigma^2 h'}{h}\right) h'$$

or again

$$0 = \rho - r - \tfrac{1}{2}\sigma^2\left(\frac{h''}{h} - \left(\frac{h'}{h}\right)^2\right) - \mu\frac{h'}{h} + \tfrac{1}{2}\left(\frac{\sigma h'}{h}\right)^2$$

So if we let $f = \log h$, we have

$$\boxed{0 = \rho - r - \tilde{\mathcal{L}} f(r) + \tfrac{1}{2}\sigma(r)^2 f'(r)^2}$$

This may in general be solved for f, and h can be deduced from this. We may then find

$$r\, h(r) \equiv \varphi(r) = g(\delta).$$

Since $g(x) = x\, u'(x)$ is known, we can deduce what the diffusion δ has to be to make this work. The state-price density is

$$\boxed{\pi_t \zeta_t = V_\rho\, \varphi(r_t) = h(r_t)}$$

so we can convert (monetary) prices into martingales with this.

The boxed equation at the foot of the previous page can be simplified somewhat; we have

$$\tfrac{1}{2}\sigma(r)^2 \left[f''(r) - f'(r)^2 \right] + \mu(r)\, f'(r) + r - \rho = 0,$$

so if we set $h(x) \equiv 1/k(x)$, then $f' = -k'/k$, $f'' = -k''/k + (k'/k)^2$, and we get the 2^{nd}-order linear ODE

$$\boxed{\tfrac{1}{2}\sigma(x)^2\, k''(x) + \mu(x)\, k'(x) + (\rho - x)\, k(x) = 0}$$

Ratcheting of consumption (14/6/95)

(i) Phil Dybvig considers the problem of an agent in a complete market with felicity function U, with state-price density process ζ, who aims to

$$\max \; E \int_0^\infty \rho e^{-\rho t} U(C_t)\, dt$$

where C_t is the rate of consumption at time t, satisfying the budget constraint

$$a = E \int_0^\infty \zeta_t C_t \, dt$$

and the constraint that C is non-decreasing.

(ii) A necessary condition. If we consider the Lagrangian form

$$L(C,\lambda) = E\Big[\int_0^\infty \rho e^{-\rho t} U(C_t)\, dt + \lambda \big(a - \int_0^\infty \zeta_s C_s \, ds\big) \Big]$$

$$= E\Big[U(C_0) - \lambda C_0 \int_0^\infty \zeta_s \, ds + \lambda a$$

$$+ \int_0^\infty e^{-\rho s} U'(C_s)\, dC_s - \lambda \int_0^\infty \big(\int_s^\infty \zeta_t \, dt\big)\, dC_s \Big]$$

by integration by parts, we see that

$$L(C,\lambda) = E \int_0^\infty \Big\{ e^{-\rho s} U'(C_s) - \lambda \, E_s\big(\int_s^\infty \zeta_t \, dt\big) \Big\}\, dC_s \quad + \lambda a$$

$$+ E\Big[U(C_0) - \lambda C_0 \int_0^\infty \zeta_s \, ds \Big],$$

so that a necessary condition for consumption process C^* with multiplier λ to be optimal is that

$$\boxed{e^{-\rho s} U'(C^*_s) \leq \lambda E_s\big(\int_s^\infty \zeta_t \, dt\big) \equiv \lambda \eta_s \quad \text{for short.}}$$

Thus if we take

$$\boxed{C^*_t = \sup_{s \leq t} \; (U')^{-1}(\lambda^* e^{\rho s} \eta_s)}$$

we have a candidate for optimal consumption. We can choose λ^* to satisfy the budget constraint under some mild integrability assumptions

12) The traditional no-arbitrage $\Leftrightarrow$ EMM can't be expected to hold unmodified if there are limits to the volumes you can trade one way or another at any time (eg, quotes with volumes are posted...)

we would know $\int_{-\infty}^{\infty} e^{\theta v - v} E(S_T - e^v)^+ \, dv$, and now we've a chance to use FFT to calculate option prices. Was this well-known already??

11) The Duffie-Kan class of models seems to boil down to $r_t = \beta \cdot X_t$, where

$$dX_t^i = \sqrt{X_t^i}\, dW_t^i + (\mu_i + \sum_j c_{ij} X_t^j)\, dt$$

where we'll need $c_{ij} \geq 0 \ \forall i \neq j$, and $\mu_i > 0 \ \forall i$. In this case, the bond martingale is as usual

$$\exp\left[-\int_0^t r_u\, du - A(T-t) - B(T-t) \cdot X_t\right]$$

from which we obtain

$$\beta + C^T B - \tfrac{1}{2} B^2 - \dot{B} = 0, \qquad \dot{A} = B\mu$$

so that

$$\dot{B}_i + \tfrac{1}{2} B_i^2 = \beta_i + \sum_j c_{ji} B_j \qquad B_i(0) = 0, \quad \dot{B}_i(0) = \beta_i.$$

Volatility of yield on maturity-T bond is $\frac{1}{\tau}\left(\sum B_i(\tau)^2 x_i(t)\right)^{1/2}$ $\quad (\tau \equiv T-t)$ so this is decreasing in τ iff $B_i(\tau)/\tau$ is decreasing. If we write $f_i(\tau) \equiv B_i(\tau)/\tau$, we have

$$\tau \dot{f}_i(\tau) + f_i(\tau) + \tfrac{1}{2} f_i(\tau)^2 \cdot \tau^2 = \beta_i + \sum c_{ji}\, \tau f_j(\tau)$$

$$\therefore \quad \dot{f}_i(\tau) + \frac{f_i(\tau) - \beta_i}{\tau} = -\tfrac{1}{2} \tau f_i(\tau)^2 + \sum c_{ji} f_j(\tau)$$

Letting $\tau \downarrow 0$ gives

$$\boxed{\dot{f}_i(0) = \sum_j c_{ji} \beta_j}$$

So it's clear that by suitably choosing the β's and c's, this expression could be made to be positive

10) David Hobson asks how one would hedge a lookback option in the presence of transactions costs.

11) (9-lives option) You agree to pay 3mo LIBOR every 3 months for next $N/4$ years, except that you may in k periods opt to pay nothing. What's this worth?

and if we set $\rho \equiv \sqrt{\delta^2+2b\delta}$, then evalues of $-V^{-1}Q$ are $\pm\rho$ and 0. We get

$$f(t) = \underbrace{\begin{pmatrix} b & b+\delta+\rho \\ b+\delta+\rho & b \end{pmatrix}}_{M} \begin{pmatrix} e^{\rho t} & \cdot \\ \cdot & e^{-\rho t} \end{pmatrix} M^{-1} f(0)$$

and $f_0(0) = f_a(0)$ by symmetry, $f_a(k) = f_0(-k) = 1$. We get

$$f(t) = \xi(\delta+\rho) \begin{pmatrix} b e^{\rho t} + (b+\delta+\rho) e^{-\rho t} \\ b e^{-\rho t} + (b+\delta+\rho) e^{\rho t} \end{pmatrix}$$

where $\xi^{-1} = (\delta+\rho)(b e^{-\rho k} + (b+\delta+\rho)e^{\rho k})$.

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6) John Toland and Edward Fraenkel are trying to discover (properties of) a function $f: \mathbb{H} \to \mathbb{H}$ which is analytic, periodic in x, $\operatorname{Im} f(n) = 0 \ \forall n \in \mathbb{Z}$, and which satisfies $\frac{1}{2}|f'(z)|^2 = \operatorname{Im} f(z)$ for $z \in \mathbb{R}$. If one considers the restriction of f to $\mathbb{R}$, they know that $g(x) \equiv \operatorname{Im} f(x)$ has the property that $g'(n+) = -1/\sqrt{3} = -g'(n-) \ \forall n \in \mathbb{Z}$. They'd like to know whether g is concave in $(0,1)$

7) Mark Prentiss fits GARCH(1,1) to various spot rates, especially USD/DEM, and DEM/CHF (and says that trades of USD/CHF is usually done via DEM ...) but that his models (based on hourly data) for USD/CHF began to fit poorly in 9/94, and then for USD/DEM went wrong a month or so later.... and he doesn't know why.

8) Artashes Terzian suggests we could extend the approximation used in pricing Asian options to the case of a basket of shares (eg FT 100)

9) If we consider $\{ \int_\varepsilon^1 \theta_{t-\varepsilon}\, dB_t ;\ \theta$ previsible, $E \int_0^{1-\varepsilon} \theta_s^2\, ds < \infty \}$, this is a closed subspace of $L^2_0(\mathcal{B}_1)$, but what can one say about it...??

10) We have for $x>0$, $\theta>1$, $x^\theta = \int_0^\infty \theta(\theta-1) y^{\theta-2}(x-y)^+\, dy$, so if we could calculate $E[S_T^\theta]$ for $\theta>1$ in some stochastic volatility model

Questions and problems.

1) Thomas Bjørk points out that in the situation considered by Stewart and Andrew Carverhill in their paper on which risk premia can arise in an equilibrium setting, the market price of risk $\alpha(t, S_t)$ which arises is a martingale in the objective measure. Does this happen more generally?

2) Phil Dybvig asks about the problem of a foundation, which can close down at a time of its choosing, τ, say, but which attempts to maximise

$$E \int_0^\tau e^{-\delta t} U(C_t)\, dt$$

subject to $dw_t = \theta_t(\sigma_t dB_t + (\mu_t - r_t)dt) + r_t w_t dt - C_t dt$, w_0 given, $w_t \geq 0$ $\forall t$. Special case of μ, σ, r seems to be quite tough to solve, even with U being CRRA. [Abel Cadenillas shows me the paper of Sethi et al, which "solves" this]

3) Phil Dybvig also asks about choosing the cumulative consumption C to maximise $E \int_0^\infty e^{-\delta t} U(Y_t, Z_t)\, dt$, where $dY_t = -\alpha Y_t dt + dC_t$, $dZ_t = -\beta Z_t dt + dC_t$, and U may be nice, eg $U(y,z) = y^a z^b \ldots$

4) Artashes Terzian says they have been trying to price an American put where the price of the share is replaced by a moving average...

5) Take BM in $[-1, 1]$ with reflection at ± 1, and let $\varphi_t \equiv L(t,1) - L(t,-1)$. If $\tau = \inf\{u : |\varphi_u| > K\}$, what's the law of τ? (Claudia says this is a question of a grad student in Oxford, so probably one of the plumbers' protégés.)

Solution isn't too hard. If we have a BM in $[0,a]$ reflected at both ends, and killed at constant rate $\lambda = \frac{1}{2}\theta^2$, then when we time change by $L(t,0) + L(t,a)$ we get a MC on $\{0, a, \delta\}$ with Q-matrix

$$\begin{array}{cc} \begin{pmatrix} -b-\delta & b & \delta \\ b & -b-\delta & \delta \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ a \\ \delta \end{matrix} \end{array} \qquad b \equiv \tfrac{1}{2}\theta \operatorname{cosech} \theta a, \quad \delta = \tfrac{1}{2}\theta \tanh \theta a/2$$

$$b + \delta = \tfrac{1}{2}\theta \coth \theta a$$

If φ_t grows at rate 1 when $X = a$, falls at rate 1 when $X = 0$, and $f(x, \varphi) = P(\varphi \text{ reaches } \pm k \text{ before killing})$, then easily

$$Q f + V \dot{f} = 0 \qquad V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

BS price:

$$C = S_0\, \bar{\Phi}(a - \sigma\sqrt{T}) - Ke^{-rT}\, \bar{\Phi}(a) \qquad a \equiv \frac{\log(K/S_0) - rT + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$\frac{\partial C}{\partial S_0} = \bar{\Phi}(a - \sigma\sqrt{T})$$

$$\frac{\partial C}{\partial \sigma} = S_0 \sqrt{\frac{T}{2\pi}}\, \exp\{-\tfrac{1}{2}(a - \sigma\sqrt{T})^2\}$$

$$\frac{\partial^2 C}{\partial \sigma^2} = \frac{1}{\sigma}(a - \sigma\sqrt{T})\, a\, \frac{\partial C}{\partial \sigma}$$

$$\frac{\partial^2 C}{\partial S_0 \partial \sigma} = \frac{a}{\sigma\sqrt{2\pi}} \exp\{-(a - \sigma\sqrt{T})^2/2\}$$

$$\frac{\partial^2 C}{\partial S_0^2} = \frac{1}{S_0 \sigma\sqrt{T}} \exp(-(a - \sigma\sqrt{T})^2/2) \cdot \frac{1}{\sqrt{2\pi}}$$

$$\frac{\partial C}{\partial T} = \frac{\sigma S_0}{2\sqrt{2\pi T}}\, e^{-(a - \sigma\sqrt{T})^2/2} + rKe^{-rT}\, \bar{\Phi}(a)$$

$I_\nu(x) \equiv \left(\frac{x}{2}\right)^\nu \sum_{k \geq 0} \left(\frac{x}{2}\right)^{2k} / k!\, \Gamma(\nu + k + 1)$ solves $f'' + \frac{1}{x} f' = \left(1 + \frac{\nu^2}{x^2}\right) f$

The transition density of BES$(2\nu + 2)$ is $\qquad [\nu = \frac{d}{2} - 1]$

$$p_t^\nu(x, y) = \frac{y}{t}\, e^{-(x^2 + y^2)/2t} \left(\frac{y}{x}\right)^\nu I_\nu\left(\frac{xy}{t}\right)$$

If $\psi_\nu(\delta, x) \equiv \left(\frac{\delta x}{2}\right)^{-\nu} \Gamma(\nu + 1)\, I_\nu(\delta x)$, then $\psi_\nu(\delta, \rho_t)\, e^{-\delta^2 t/2}$ is a mg (when ρ is the BES$(2\nu + 2)$ process).