Non-negative couplings (21/6/95)

(i) Previously we considered the problem of finding (X, Y) with X and Y both distributed like F, and $Y + a \geq X$, so as to

$$\max E\,\varphi(Y-X) \qquad\qquad (a \geq 0)$$

where φ is some convex decreasing function. I claimed that there was a joint distribution which attained this maximum for any such φ. To see this, let's notice

$$E\,\varphi(Y-X) = E\left[\int_{Y-X}^{\infty} -\varphi'(t)\,dt\right] \qquad (\varphi(\infty)=0 \text{ wlog})$$

$$= \int_{-\infty}^{\infty} -\varphi'(t)\,dt\ P(Y-X \leq t)$$

$$= \int \varphi''(ds) \int_{-\infty}^{s} P(Y-X \leq t)\,dt$$

$$= \int_{-\infty}^{\infty} \varphi''(ds)\ E\,(s-Y+X)^+.$$

Now

$$E\,(s-Y+X)^+ = \int_{x=-\infty}^{\infty} \int_{y=x-a}^{s+x} \left(\int_y^{x+s} dt\right) G(dx,dy)$$

$$= \int_{-\infty}^{\infty} dt\ P[\,Y \leq t,\ X \geq t-s\,],$$

so our problem now is to understand how big $P[Y \leq t,\ X \geq t-s]$ can be.

(ii) Let C denote the shaded triangle

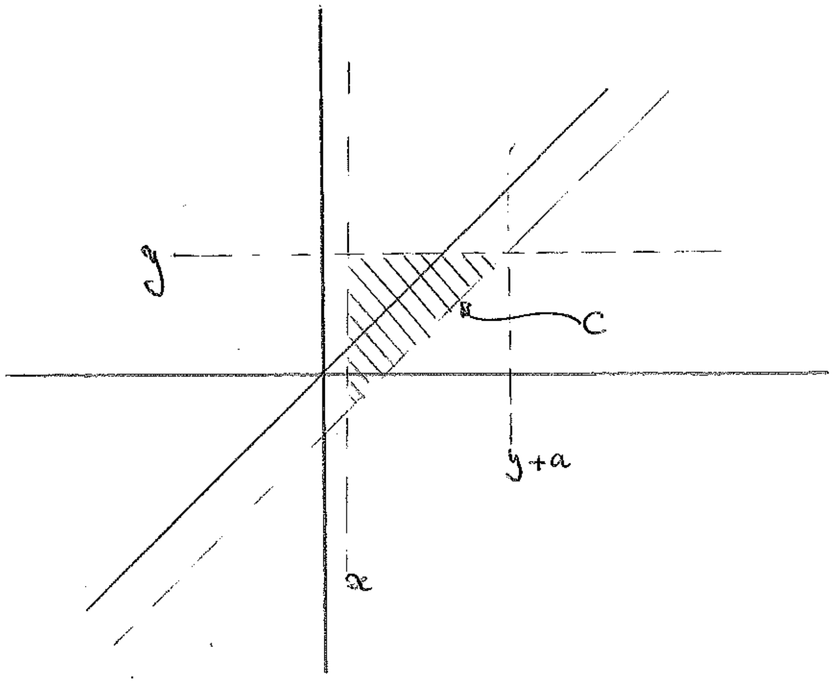
$C = \{(x',y') : x' > x,\ y' \leq y,\ x' \leq y'+a\}$.

Now clearly

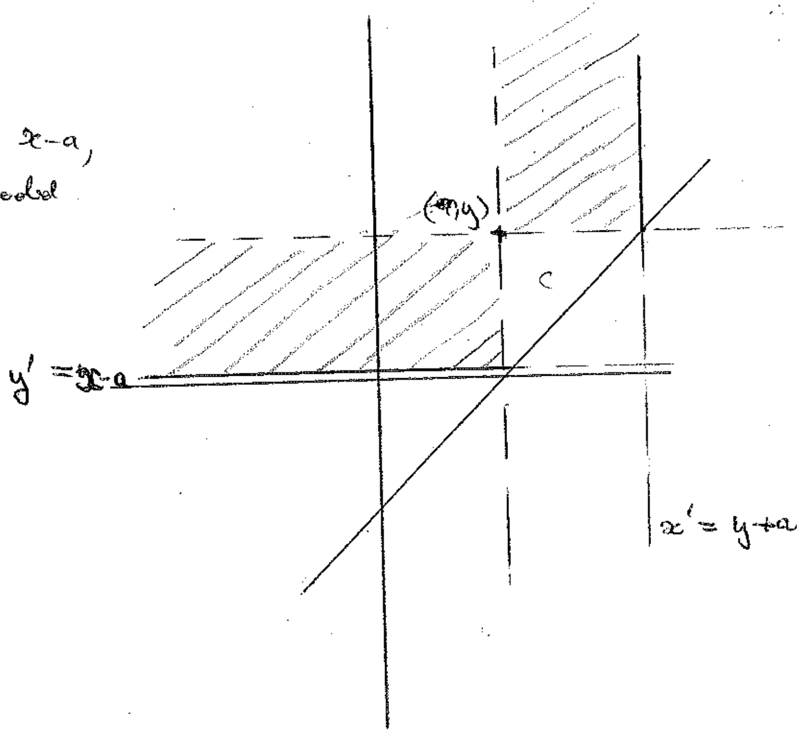
$$G(x,y) + G(C) = F(y),$$

and also

$$G(C) \leq F(y+a) - F(x)$$



If optimal μ charges (x, y), $y > x-a$, then there can be no mass in the shaded region, whence by considering the open triangle C we get



$$F(x'-) - F(x) = F(y-) - F(y')$$

so

$$\boxed{F(y+a-) - F(y-) = F(x) - F(x-a)}$$

from which we deduce the inequality

$$G(x,y) \geq F(x) + F(y) - F(y+a), \qquad \text{for } y \geq x-a$$

Since G is non-decreasing in both arguments, we must also have

$$\boxed{G(x,y) \geq \Gamma_0(x,y) \equiv \sup_{x-a \leq z \leq y} \{ F(x) + F(z) - F(z+a) \}}$$

Accordingly, $G(C) \leq F(y) - \Gamma_0(x,y)$. But Γ_0 may not be increasing in x, so set

$$\boxed{\Gamma(x,y) \equiv \sup_{v \leq x} \Gamma_0(v,y)} \qquad (x \leq y+a)$$

(iii) We must prove that Γ defined above is a valid distribution function. (we assume that this is the definition for $y+a \geq x$; if $x > y+a$, we set $\Gamma(x,y) = \Gamma(y+a, y) = F(y)$.)

Define the non-negative function $\varphi(z) \equiv F(z+a) - F(z)$, so that

$$\Gamma_0(x,y) = F(x) - \inf_{x-a \leq z \leq y} \varphi(z) \qquad (x \leq y+a)$$

Note that as $y \to \infty$, $\Gamma_0(x,y) \to F(x)$, so $\Gamma(x,y) \to F(x)$, since $\Gamma_0(x,y) \leq F(x)$. Also, $\Gamma_0(x,y) \leq F(y)$ for all $x \leq y+a$, with equality when $x = y+a$, so ~~if~~ Γ is a distribution function, it has the correct marginals.

To check the distribution function, take $x < x'$, $y < y'$, and suppose without loss of generality that $x' \leq y+a$. We need to prove that

$$\Gamma(x', y') - \Gamma(x, y') \geq \Gamma(x', y) - \Gamma(x, y).$$

If the RHS is zero, we are done; so suppose that $\Gamma(x', y) > \Gamma(x,y)$. For ease of presentation, assume sups and infs are attained when necessary. Thus we'll assume that there's $x^* \in (x, x']$ for which

$$\Gamma(x', y) = \Gamma_0(x^*, y) = \sup_{x^*-a \leq z \leq y} (-\varphi(z)) + F(x^*)$$

Now as we rise from (x^*, y) to (x^*, y'), the first term here rises by

$$C \equiv \sup_{x^*-a \leq z \leq y'} (-\varphi(z)) - \sup_{x^*-a \leq z \leq y} (-\varphi(z))$$

and evidently

$$c \geq \sup_{\tilde{x}-a \leq z \leq y'} (-\varphi(z)) - \sup_{\tilde{x}-a \leq z \leq y} (-\varphi(z))$$

for any $\tilde{x} \leq x^*$.

Thus for $v \leq x$

$$F(v) + \sup_{v-a \leq z \leq y'} (-\varphi(z))$$

$$\leq c + F(v) + \sup_{v-a \leq z \leq y} (-\varphi(z))$$

whence taking sup over $v \leq x$, we obtain

$$\Gamma(x, y') \leq c + \Gamma(x, y)$$

$$= \Gamma_0(x^*, y') - \Gamma_0(x^*, y) + \Gamma(x, y)$$

$$= \Gamma_0(x^*, y') - \Gamma(x', y) + \Gamma(x, y)$$

$$\leq \Gamma(x', y') - \Gamma(x', y) + \Gamma(x, y),$$

as required.

(iv) To sum up, therefore, we have for any (X, Y) with marginals F and $Y + a \geq X$ and for any $x < y + a$

$$P[X > x, \; Y \leq y] = F(y) - G(x, y)$$

$$\leq F(y) - \Gamma(x, y)$$

and this bound is attained if we take the joint distn to be Γ.

Stochastic investment/Consumption models; some simple examples (27/6/95)

(i) There is some bunch of models (stochastic Ramsey models) which look interesting. The agent is trying to

$$\max E\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \right]$$

where $c_t = f(x_t, k_t) - i_t$ and $dk_t = -\varepsilon k_t\, dt + i_t\, dt$. The interpretations are quite nice; i_t is the rate at which we invest in machinery etc, and this produces an improvement in the level of capital; the amount of good produced at time t is a function of k_t and some underlying stochastic variable x_t (which might denote labour ...)

In the classical Ramsey model, x is deterministic, and in fact $f(x, k) = x\, f(1, k/x)$. The objective is to max $\int_0^\infty e^{-\rho t} U(c_t/x_t)\, dt$, and the interpretation is that x is the size of the population, and we try to maximise utility of per-capita consumption.

(ii) We can get <u>some</u> distance with these. Notice that

$$k_t = e^{-\varepsilon t}\left(k_0 + \int_0^t e^{\varepsilon s} i_s\, ds\right)$$

so the FOC we obtain by perturbing from i^* to $i^* + \eta$ is

$$0 = E \int_0^\infty e^{-\rho t} U'(c_t^*) \left\{ -\eta_t + f_k(x_t, k_t^*) \int_0^t e^{\varepsilon(s-t)} \eta_s\, ds \right\} dt,$$

then at least while i^* and c^* are >0

$$\boxed{U'(c_t^*) = E_t\left[\int_t^\infty e^{-(\rho+\varepsilon)(s-t)}\, U'(c_s^*)\, f_k(x_s, k_s^*)\, ds \right]}$$

Another consequence is that if this holds for <u>all</u> t, and if we define

$$Y_t \equiv \exp(-(\rho+\varepsilon) t)\, U'(c_t^*), \qquad \gamma_s \equiv f_k(x_s, k_s^*)$$

then

$$\boxed{Y_t \exp\left\{ \int_0^t \gamma_s\, ds \right\} \text{ is a martingale.}}$$

(iii) For the deterministic Ramsey model, if we have $\dot{x}_t = \alpha_t x_t$, and

$$C_t + I_t = F(x_t, K_t), \qquad \dot{K}_t = -\varepsilon K_t + I_t$$

then assuming $F(x, K) = x\,F(1, K/x) \equiv x f(K/x)$, and using the abbreviations $C/x \equiv c$, $K/x \equiv k$, $I/x \equiv i$, we end up with

$$\boxed{c_t + i_t = f(k_t), \qquad \dot{k}_t = -\varepsilon k_t + i_t - \alpha_t k_t}$$

and the FOC are (assuming $i_t^* > 0$ for all t)

$$\boxed{U'(c_t^*) = \int_t^\infty e^{-\rho(s-t)} \frac{\beta_s}{\beta_t} U'(c_s^*) f'(k_s^*)\,ds}, \qquad \beta_t = \exp\left(-\int_0^t (\alpha_s + \varepsilon)ds\right)$$

Hence

$$\boxed{U'(c_t^*) \exp\left(-\int_0^t (\rho + \varepsilon + \alpha_s)ds\right) = \text{const.}\ \exp\left(-\int_0^t f'(k_s^*)\,ds\right)}$$

(assuming $\int_0^\infty f'(k_s^*)\,ds = +\infty$).

The special case $\alpha_t \equiv \alpha$ is dealt with in Malliaris & Brock, for example; one solution is to keep k^* constant, $f'(k^*) = \rho + \varepsilon + \alpha$, and likewise $i^* = (\varepsilon + \alpha)k^*$, $c^* = f(k^*) - i^*$, though this is only going to work if $k_0 = k^*$.

Nevertheless, for a given non-constant function α, it is impossible in general to solve the boxed equation for the pair (c^*, k^*) related as in the Ramsey model.

(iv) More generally, if we don't assume $i^* > 0$ always, the FOC say

$$U'(c_t^*) \geq \int_t^\infty \exp\left[-\rho(s-t) - \int_t^s (\varepsilon + \alpha_u)\,du\right] f'(k_s^*)\, U'(c_s^*)\,ds$$

Abbreviate $\int_0^t (\varepsilon + \alpha_u)\,du \equiv \gamma_t$, $\varphi_t \equiv f'(k_t^*)$, $\Phi_t = \int_0^t \varphi_s\,ds$, $y_t \equiv U'(c_t^*) \exp(-\rho t - \gamma_t)$, so the FOC says

$$y_t \geq \int_t^\infty y_s\, \varphi_s\,ds \qquad \text{with equality when } i_t^* > 0.$$

If we insist $i^* \geq 0$, quite reasonably, we shall have $U'(c_t^*) \geq U'(f(k_t^*))$.
During intervals where $i^* > 0$, we shall have

$$y_t \propto \exp(-\Phi_t).$$

But also $e^{\Phi_t} y_t \geq e^{\Phi_t} . e^{-\rho t - \gamma_t} U'(f(k^*_t))$, so if we set

$$\lambda_t \equiv \sup_{u \leq t} \{ \exp(\Phi_u - \rho u - \gamma_u) U'(f(k^*_u)) \}$$

then

$$\boxed{\begin{aligned} y_t &\equiv \exp(-\rho t - \gamma_t) U'(c^*_t) \\ &= \lambda_t \exp(-\Phi_t) \end{aligned}}$$

We would now obtain a coupled pair of DEs which determine the solution:

$$\frac{dk^*_t}{dt} = -(\varepsilon + \alpha_t) k^*_t + f(k^*_t) - I\big(\lambda_t \exp(\rho t + \gamma_t - \Phi_t)\big)$$

$$\frac{d\Phi_t}{dt} = f'(k^*_t)$$

where the λ_t is as above.

(v) Special case: $f(x) = ax + b$. This reduces the problem to a single DE,

$$\frac{dk^*_t}{dt} = -(\varepsilon + \alpha_t) k^*_t + a k^*_t + b - I\left(\lambda_t e^{(\rho - a)t + \gamma_t}\right).$$

Taking $U(\cdot) = \log(\cdot)$ helps a little further;

$$\frac{d}{dt}\left[e^{\gamma_t - at} k^*_t \right] = b e^{\gamma_t - at} - e^{-\rho t}/\lambda_t$$

but still no picnic.

(vi) Well, let's consider the following variant: We allow $i < 0$, but insist on $k_t \geq 0$ $\forall t$. (You may eat the oxen if you must, but when they're gone you starve.) Also for simplicity assume $f(x) = ax + b$.

Then

$$\begin{cases} U'(c^*_t) = \theta \exp\left[\int_0^t (\rho + \varepsilon - a + \alpha_s)\, ds \right] \\ \dot{k}^*_t = \{a - (\varepsilon + \alpha_t)\} k^*_t + b - I\left(\theta \exp \int_0^t (\rho + \varepsilon - a + \alpha_s) ds \right) \end{cases}$$

for some constant θ. So we have

$$\frac{d}{dt}\left\{ \exp\left[\int_0^t (\alpha_s+\varepsilon-a)\,ds\right] k_t^* \right\} = \exp\left(\int_0^t (\alpha_s+\varepsilon-a)\,ds\right)\left\{ b - I\left(\theta e^{\int_0^t (\rho+\varepsilon-a+\alpha_s)\,ds}\right)\right\}$$

$$\Rightarrow k_t^* = e^{-\psi(t)}\left\{ k_0^* + \int_0^t e^{\psi(s)}\left[b - I\left(\theta e^{\psi(s)+\rho s}\right)\right] ds \right\}$$

where $\psi_t \equiv \int_0^t (\alpha_s+\varepsilon-a)\,ds$. Clearly as we increase θ we increase k^*, so presumably one chooses θ so as to make $\inf k_t^* = 0$.

Investment/consumption models with improving technology (9/8/95)

(i) Consider the situation of an investor who can invest in a risky asset and a bond and can also consume. The return of the risky asset depends on the level k_t of capital investment at time t. If x_t is his wealth at time t,

$$\begin{cases} dx_t = (rx_t - c_t - i_t)\,dt + \theta_t\left[\sigma\, dW_t + (\mu(k_t) - r)\,dt\right] \\ dk_t = (i_t - \varepsilon k_t)\,dt \end{cases}$$

The goal is to

$$\max E\int_0^\infty e^{-\rho t} U(c_t)\,dt.$$

(ii) Define
$$V(x,k) \equiv \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \,\middle|\, x_0 = x,\ k_0 = k\right].$$

The HJB equation is

$$U(c) - \rho V + \tfrac{1}{2}\sigma^2\theta^2 V_{xx} + \{rx - i - c + \theta(\mu(k) - r)\} V_x + (i - \varepsilon k) V_k \leq 0$$

with equality when we maximise. Hence we must have

$$\boxed{\begin{aligned} & V_x \geq V_k \quad \text{everywhere, equal if } i \neq 0 \\ & \theta^* = -(\mu(k) - r) V_x / \sigma^2 V_{xx} \\ & U'(c^*) = V_x \end{aligned}}$$

and so $[\tilde{U}(\lambda) \equiv \sup(U(x) - \lambda x)]$ we have

$$\boxed{\tilde{U}(V_x) - \rho V - (\mu(k) - r)^2 V_x^2 / 2\sigma^2 V_{xx} + rxV_x - \varepsilon k V_k = 0.}$$

(iii) Under what circumstances do we get a separable solution $V(x,k) = f(x)g(k)$? Dividing throughout by V, we'd have

$$\frac{\tilde{U}(f'g)}{fg} - \rho - \frac{(\mu(k) - r)^2}{2\sigma^2}\frac{(f')^2}{f f''} + \frac{rxf'}{f} - \varepsilon k \frac{g'}{g} = 0$$

This can only happen if $\tilde{U}(y) = c y^\theta$ for some c, θ, and $f(x) = ax^\alpha$ for

For later: notationally cleaner to take $f(x) = U(x) \equiv \dfrac{x^{1-R}}{1-R}$,

This is apparently a fairly standard piece of ODEs, the Bernoulli method applicable to equations of the form

$$y' + P(x)\,y = Q(x)\,y^{\alpha}.$$

(13/6/2000)

appropriate a, α. If $U(x) = \dfrac{x^{1-R}}{1-R}$ $(R>0,\ R\neq 1)$, then

$$\tilde{u}(\lambda) = \frac{R}{1-R}\,\lambda^{(R-1)/R}, \qquad \alpha = 1-R,$$

Case 1 : $0<R<1$. Since we clearly must have $V>0$, we'll have $g, f>0$ also, so let's take $f(x) = x^{1-R}$, and then the HJB equation collapses to

$$\frac{R}{1-R}\cdot(1-R)^{(R-1)/R}\cdot g^{-1/R} - \rho + \frac{(\mu(k)-r)^2}{2\sigma^2}\,\frac{1-R}{R} + r(1-R) - \varepsilon k\,\frac{g'(k)}{g(k)} = 0$$

or again

$$\boxed{\begin{aligned} \varepsilon k\, g'(k) &= \frac{R}{(1-R)^{1/R}}\, g(k)^{1-\frac{1}{R}} - a(k)\, g(k) \\ a(k) &\equiv -r(1-R) + \rho - \frac{(\mu(k)-r)^2}{2\sigma^2}\cdot\frac{1-R}{R} \end{aligned}}$$

Case 2: $R>1$. This time, we clearly have $V<0$, so we have $f(x) = -x^{1-R}$ and want g to be positive (and decreasing?) The HJB equation reduces this time to

$$\boxed{\varepsilon k\, g'(k) = R(R-1)^{-1/R}\cdot g(k)^{(R-1)/R} - a(k)\, g(k)}$$

where a is as defined above, and now is always positive.

But we may say more about the solutions to these ODEs. Taking Case 1 first, we define

$$h(x) \equiv \exp\left[\int_1^x \frac{a(y)}{\varepsilon y}\, dy\right]$$

and multiply throughout by $h(k)/\varepsilon k$ to obtain

$$\frac{d}{dk}\left[g(k)\,h(k)\right] = \frac{R}{(1-R)^{1/R}}\,\frac{h(k)^{1/R}}{\varepsilon k}\left\{g(k)\,h(k)\right\}^{1-1/R}.$$

If $g(k)\,h(k) \equiv \psi(k)$, this ODE for ψ can now be solved:

$$\boxed{\psi(k)^{1/R} = \psi(1)^{1/R} + \int_1^k (1-R)^{-1/R}\, h(x)^{1/R}\,(\varepsilon x)^{-1}\, dx.}$$

For small x, $h(x) \sim x^{\nu}$, $\nu \equiv a(0)/\epsilon$

Now it turns out that a necessary condition for a well posed problem is that

$$\boxed{a(k) > 0 \quad \text{for all } k}$$

(because if for some k^* this were ≤ 0, we could with a finite initial wealth simply hold k at k^* by investing constantly at rate εk^*, and then by consuming from what remains we may make unbounded payoff.) Among other things, this forces μ to be bounded. If we require g to be positive increasing, we also have to have

$$\boxed{g(k) < a(k)^R \quad R^R/(1-R)}$$

Next, by considering the behaviour of the solution ψ near zero, we see that the only way we can get $g \geq 0$ and bounded near zero, is to have

$$\psi(k)^{1/R} = \int_0^k (1-R)^{-1/R} \; h(x)^{1/R} \; \frac{dx}{\varepsilon x}$$

so that

$$\boxed{g(k) = \frac{1}{(1-R)h(k)} \left\{ \int_0^k h(x)^{1/R} \; \frac{dx}{\varepsilon x} \right\}^R}$$

and as $k \downarrow 0$, $g(k) \to (R/a(0))^R \; (1-R)^{-1}$ by analysing this expression, or by considering the ODE from which g comes.

But is the solution g given by this formula necessarily increasing in k? No. But this is not a problem, because if μ were quite general, it doesn't follow automatically that g should be increasing; even if μ were increasing, this doesn't have to hold.

Case 2 is similar in many respects. Using the same integrating factor h gives

$$\frac{d}{dk}\left[g(k)\, h(k) \right] = \frac{R}{(R-1)^{1/R}} \; \frac{h(k)^{1/R}}{\varepsilon k} \left[g(k)\, h(k) \right]^{1-1/R}$$

so

$$\boxed{\begin{aligned} \psi(k)^{1/R} &= \psi(1)^{1/R} + \int_1^k \frac{1}{(R-1)^{1/R}} \; h(x)^{1/R} \; \frac{dx}{\varepsilon x} \\ &= \psi(0)^{1/R} + \int_0^k \frac{1}{(R-1)^{1/R}} \; h(x)^{1/R} \; \frac{dx}{\varepsilon x} \quad \geq 0 \end{aligned}}$$

What boundary condition at 0 do we have? One policy that the agent could follow is to keep $i \equiv 0$. The effect of this is to reduce to the well-studied problem, and we find that the payoff with initial wealth x is proportional to x^{1-R}. The conclusion is that g cannot be unbounded near zero, so we have as in case 1

$$\boxed{g(k) = h(k)^{-1}\left\{ \int_0^k |R-1|^{-1/R}\, h(x)^{1/R}\, \frac{dx}{\varepsilon x} \right\}^R}$$

(iv) An example. Note that the region where (x, k) lies has to satisfy $V_x \geq V_k$, so for Case 2 this amounts to

$$\boxed{\frac{R-1}{x} \geq -\frac{g'(k)}{g(k)}}$$

which places an upper bound on x whenever $g'(k) < 0$. Let's see what happens if we have $g(k)^{1/R} \equiv \varphi(k)$ is decreasing; what can we say about h and thence a? If $h^{1/R} = v$, we get

$$(\varphi v)(k) = \int_0^k |R-1|^{-1/R}\, v(x)\, \frac{dx}{\varepsilon x},$$

so

$$\varphi'(k)\, v(k) + v'(k)\, \varphi(k) = |R-1|^{-1/R}\, v(k)/\varepsilon k$$

$$\therefore \quad v'(k) = v(k)\left\{ |R-1|^{-1/R} \frac{1}{\varepsilon \varphi(k) k} - \frac{\varphi'(k)}{\varphi(k)} \right\}$$

so

$$\frac{d}{dk} \log \varphi(k) v(k) = \frac{|R-1|^{-1/R}}{\varepsilon k\, \varphi(k)}$$

If we assume $\boxed{\varphi(k) = (\alpha + \beta k)^{-1}}$

then we get

$$\log \varphi(k)\, v(k) = \frac{|R-1|^{-1/R}}{\varepsilon} \{\beta k + \alpha \log k\}$$

So if $|R-1|^{-1/R} \equiv \varepsilon \nu$, we get

$$\varphi(k)\, v(k) = e^{\nu\beta k} \cdot k^{\nu\alpha}$$

$$\therefore \quad \int^k \frac{a(s)}{\varepsilon s}\, ds = R \log v(k) = R\nu(\beta k + \alpha \log k) - R \log \varphi(k)$$

$$\Rightarrow \quad a(k)/\varepsilon k = R\nu\beta + \frac{\alpha R \nu}{k} + \frac{R\beta}{\alpha + \beta k}$$

so that

$$a(k) \equiv \rho + r(R-1) + \frac{R-1}{R} \cdot (\mu(k) - r)^2 / 2\sigma^2$$

$$= R\nu\beta\varepsilon\, k + \alpha R\nu\varepsilon + \frac{R\varepsilon\beta\, k}{\alpha+\beta k}$$

so we must insist that $\alpha R\nu\varepsilon \geq \rho + (R-1)r$ and then

$$\boxed{\mu(k) - r = \left(\frac{2R\sigma^2}{R-1}\right)^{\frac{1}{2}} \left[\alpha R\nu\varepsilon - \rho - r(R-1) + R\varepsilon\beta\left(\nu k + \frac{k}{\alpha+\beta k}\right)\right]^{\frac{1}{2}}}$$

For this choice of μ, the constraint on x takes the form

$$\boxed{x \leq \frac{R-1}{\beta}(\alpha+\beta k)}$$

An especially simple situation arises when $\alpha = 0$, for we get

$$\mu(k) - r = \left(\frac{2R\sigma^2}{R-1}\right)^{\frac{1}{2}} \left[R\varepsilon - \rho - r(R-1) + R\varepsilon\beta\nu\, k\right]^{\frac{1}{2}}$$

$$x \leq (R-1)k$$

(v) This set up is slightly unrealistic, because it assumes that if $\mu(k) < r$ the agent can take a short position in the risky asset, i.e. he can find mugs to put their money in it! More realistic would be to either (i) insist that all his wealth was always invested in the risky asset ("the farm") or else (ii) to put a lower bound on θ ($\theta \geq 0$ would allow the farmer to take some of his farm out of production)

(a) For the first of these, the dynamics are

$$dk = (i - \varepsilon k)\, dt$$

$$dx = (-c - i)\, dt + x\left\{ \sigma dW + \mu(k)\, dt \right\}$$

which then give the following HJB equation for the value function V:

$$\sup_{i\geq 0,\, c\geq 0} \left\{ U(c) - \rho V + \tfrac{1}{2}\sigma^2 x^2 V_{xx} + \{ x\mu(k) - c - i \} V_x + (i - \varepsilon k) V_k \right\} = 0$$

so $V_x \geq V_k$, $U'(c^*) = V_x$ and we get

$$\boxed{\tilde{U}(V_x) - \rho V + \tfrac{1}{2}\sigma^2 x^2 V_{xx} + x\,\mu(k) V_x - \varepsilon k V_k = 0.}$$

Once again, with $U(x) = (1-R)^{-1} x^{1-R}$, $\tilde{U}(v) = R(1-R)^{-1} v^{1-1/R}$, we can go for a separable solution $V(x,k) = U(x)\,g(k)$. This gives us

$$R\, g(k)^{1-1/R} - a(k)\, g(k) - \varepsilon k\, g'(k) = 0 \quad , \quad a(k) \equiv \rho + \frac{\sigma^2 R}{2}(1-R) + (R-1)\mu(k)$$

So if $g(k) = \gamma(k)^R$ we obtain

$$\varepsilon R\, k\, \gamma'(k) = R - a(k)\,\gamma(k)$$

from which (for solution bounded at 0) we get

$$\boxed{\gamma(k) = \eta(k)^{-1} \int_0^k \eta(s) \frac{ds}{\varepsilon s} \quad , \quad \eta(k) \equiv \exp \int_1^k \frac{a(s)}{\varepsilon R s}\, ds}$$

The region where (x,k) lives is where $V_x \geq V_k$, which is to say

$$U'(x) \geq U(x)\, R\gamma'(k)/\gamma(k)$$

Thus $$\boxed{x^{-1} \geq \frac{R}{1-R} \frac{\gamma'(k)}{\gamma(k)}}$$

which needs specific assumptions to go further. For example, if we assume $R>1$ and

$$\gamma(k) = \frac{1}{\alpha + \beta k}$$

we get

$$a(k) = \varepsilon R \left[\frac{\beta k}{\alpha + \beta k} + \frac{\alpha}{\varepsilon} + \frac{\beta k}{\varepsilon} \right]$$

and

$$x \leq \frac{R-1}{\beta R}(\alpha + \beta k)$$

$$c^* \equiv x\, g(k)^{-1/R} = (\alpha + \beta k)\, x$$

(b) If we restrict $\theta \geq 0$, the effect is the same as replacing $\mu(\cdot)$ by $\mu(\cdot) \vee r$.

The 'potential theory' of term structure (22/8/95)

Idea is to represent the deflator ζ_t as $\zeta_t = e^{-\alpha t} R_\alpha g(X_t)$, where X is some Markov process with generator $\mathcal{G}$ and resolvent $(R_\lambda)_{\lambda>0}$. Since

$$\zeta_t + \int_0^t e^{-\alpha s} g(X_s)\, ds \quad \text{is a martingale,}$$

it's easy to conclude that $r_t = g(X_t) / R_\alpha g(X_t)$. In order that this approach can be of some use, must have some examples where bond prices (and prices of other derivatives) work out OK, and must also be able to specify the T-forward measure simply enough to allow various path-dependent derivatives to be priced.

(i) Fix $g>0$, $T>0$ and setting $R_\alpha g \equiv \varphi$ for short, we'll have the density of the T-forward measure wrto reference measure is $R_\alpha g(X_T) / P_T R_\alpha g(X_0)$.

We have

$$\frac{1}{h} \tilde{E}_t\left[f(X_{t+h}) - f(X_t) \right] = \frac{1}{h} E_t\left[\{ f(X_{t+h}) - f(X_t) \} P_{T-t-h}\varphi(X_{t+h}) \right] / P_{T-t}\varphi(X_t)$$

$$= \frac{1}{h} E_t\left[f P_{T-t-h}\varphi(X_{t+h}) - f P_{T-t}\varphi(X_t) \right] / P_{T-t}\varphi(X_t)$$

$$\to \frac{\left(\mathcal{G}(f P_{T-t}\varphi) + f \frac{\partial}{\partial t} P_{T-t}\varphi \right)(X_t)}{P_{T-t}\varphi(X_t)}$$

$$= \frac{\mathcal{G}(f P_{T-t}\varphi) - f\, \mathcal{G} P_{T-t}\varphi}{P_{T-t}\varphi} \, (X_t)$$

which tells us the generator of the process in the T-forward measure. If the Mkv process is a diffusion, $\mathcal{G} \equiv \frac{1}{2} a_{ij} D_i D_j + b_i D_i$, then

$$\boxed{\tilde{\mathcal{G}} = \mathcal{G} + a_{ij}\, D_j (\log P_{T-t}\varphi) \cdot D_i}$$

(ii) Some examples? (a) Try taking the underlying Mkv pr to solve

$$dX = \sigma\, dW - BX\, dt$$

where X is a d-vector pr. This gets solved by

$$X_t = e^{-tB}\left(X_0 + \int_0^t e^{sB} \sigma\, dW_s \right)$$

If we took $g(x) = |x|^2$, we could compute $R_\alpha g$, since

$$E^x X_t = e^{-tB} x \quad , \quad \mathrm{cov}(X_t) = E\left[\int_0^t e^{-sB} \sigma \, dW_s \cdot \left(\int_0^t e^{-sB} \sigma \, dW_s\right)^T \right]$$

$$= \int_0^t e^{-sB} \sigma \sigma^T e^{-sB^T} \, ds$$

which is not in general going to be too simple to work with. If we make the simplifying assumption $\boxed{B = \beta I}$ we get $\mathrm{cov}(X_t) = \sigma\sigma^T (1 - e^{-2\beta t})/2\beta$ and so

$$\begin{cases} E|X_t|^2 = |x|^2 e^{-2\beta t} + \dfrac{1 - e^{-2\beta t}}{2\beta} \, \mathrm{tr}\, \sigma\sigma^T \\ \\ R_\alpha g(x) = \dfrac{1}{\alpha + 2\beta} |x|^2 + \dfrac{\mathrm{tr}\, \sigma\sigma^T}{\alpha(\alpha + 2\beta)} \end{cases}$$

So the spot rate here is

$$\boxed{ r(x) = \frac{\alpha(\alpha + 2\beta) \, |x|^2}{\alpha |x|^2 + \mathrm{tr}\, \sigma\sigma^T} . }$$

One slightly unattractive feature is that r would be <u>bounded</u>. How do the bond prices look?

$$E^x \left[e^{-\alpha t} R_\alpha g(X_t) \right] = E^x \int_t^\infty e^{-\alpha s} |X_s|^2 ds$$

$$= \frac{|x|^2}{\alpha + 2\beta} e^{-(\alpha + 2\beta)t} + \frac{\mathrm{tr}\, \sigma\sigma^T}{2\beta} \left(\frac{e^{-\alpha t}}{\alpha} - \frac{e^{-(\alpha+2\beta)t}}{\alpha + 2\beta} \right)$$

Thus

$$\boxed{ P(0,t) = \frac{e^{-\alpha t} \left\{ \alpha |x|^2 e^{-2\beta t} + \dfrac{\mathrm{tr}\, \sigma\sigma^T}{2\beta} \left((\alpha + 2\beta) - \alpha e^{-2\beta t} \right) \right\}}{\alpha |x|^2 + \mathrm{tr}\, \sigma\sigma^T} }$$

Thus the yield on the t-bond is

$$\alpha + \frac{1}{t} \log \left[\frac{\alpha |x|^2 + \mathrm{tr}\, \sigma\sigma^T}{\alpha |x|^2 + \mathrm{tr}\, \sigma\sigma^T - \alpha \left(|x|^2 - \dfrac{\mathrm{tr}\, \sigma\sigma^T}{2\beta} \right) (1 - e^{-2\beta t})} \right]$$

Writing $y \equiv 2\beta t$, we want to know whether we can have

$$\frac{1-e^{-y}}{y}\left[1-\frac{r-\alpha}{2\beta}(1-e^{-y})\right]^{-1} \quad \text{initially increasing.}$$

Let $\frac{r-\alpha}{2\beta} \equiv c$. Obviously can only have initial increase if $c>0$, so assume $0<c<1$, and ask whether $\frac{y}{1-e^{-y}} - cy$ can be initially decreasing. Then if $\frac{1}{2}<c<1$, it is easy to see that initially this curve <u>is</u> decreasing.

$$= \alpha - \frac{1}{t}\log\left[1 - \frac{\alpha\left(|x|^2 - \operatorname{tr}\sigma\sigma^T/2\beta\right)\left(1-e^{-2\beta t}\right)}{\alpha|x|^2 + \operatorname{tr}\sigma\sigma^T}\right]$$

$$= \alpha - \frac{1}{t}\log\left[1 - \frac{r-\alpha}{2\beta}\left(1-e^{-2\beta t}\right)\right].$$

If we abbreviate $(r-\alpha)/2\beta \equiv y \in (-\alpha/2\beta, 1)$, we have

$$-\frac{1}{t}\log\left[1 - y\left(1-e^{-2\beta t}\right)\right] = \frac{1}{t}\int_0^t \frac{2\beta y e^{-2\beta s}\, ds}{1 - y\left(1-e^{-2\beta s}\right)}$$

$$= \frac{1}{t}\int_0^t \frac{2\beta y}{(1-y)e^{2\beta s} + y}\, ds$$

which is <u>decreasing</u> if $y > 0$, and negative increasing if $y < 0$. Either way, the yield curve is monotone, which is a little disappointing.

How does the term structure of volatility look? For tractability, let's just take $d=1$, in which case $r(\alpha+2\beta)^{-1} = 1 - \sigma^2(\alpha x^2 + \sigma^2)^{-1}$, and

$$\frac{dr}{\alpha+2\beta} = \frac{2\alpha\sigma^2}{(\alpha x^2+\sigma^2)^2}\,\sigma x\, dW + \cdots$$

so the volatility of r is $\quad 2\sqrt{\alpha(\alpha+2\beta)}\; r^{\frac{1}{2}}\left(1 - \frac{r}{\alpha+2\beta}\right)^{3/2}$, so volatility of yield of maturity t is

$$2\sqrt{\alpha(\alpha+2\beta)}\; r^{\frac{1}{2}}\left(1-\frac{r}{\alpha+2\beta}\right)^{3/2} \frac{1-e^{-2\beta t}}{2\beta t}\left\{1 - \frac{r-\alpha}{2\beta}\left(1-e^{-2\beta t}\right)\right\}^{-1}.$$

* As can be shown, this <u>can</u> be increasing then decreasing, though such behaviour is not typical.

(b) With the same diffusion, try $f(x) \equiv R_\alpha g(x) = \cosh \gamma \cdot x$ for some fixed vector γ. We may of course reduce to the scalar case, so let's do so. Now

$$g(x) = (\alpha - \mathcal{G}) f(x) = \alpha\cosh\gamma x - \frac{\sigma^2}{2}\gamma^2\cosh\gamma x + \beta x\,\gamma\sinh\gamma x$$

which is non-negative if $\alpha \geq \gamma^2\sigma^2/2$. Especially interesting is the case of equality, for then

$$g(x) = \gamma\beta\; x\sinh\gamma x$$

and $\boxed{r(x) = \beta\gamma\, x \tanh \gamma x}$

which is OU-like for large x, CIR-like for small x.

More generally we could have $f(x) = \cosh \gamma(x+c)$, and then

$$g(x) = \left(\alpha - \frac{\sigma^2\gamma^2}{2}\right)\cosh \gamma(x+c) + \beta x \gamma \sinh \gamma(x+c).$$

With γ fixed, we pick α to make $\min_x g(x) = 0$, and this yields a spot rate

$$\boxed{r(x) = \alpha - \frac{\sigma^2\gamma^2}{2} + \beta\gamma\, x \tanh \gamma(x+c)}$$

How do bond prices look for this?

$$E^x \cosh \gamma(X_t + c) = \exp\left[\frac{\gamma^2}{2}\cdot\frac{\sigma^2}{2\beta}(1-e^{-2\beta t})\right]. \cosh \gamma(xe^{-\beta t}+c)$$

implying that

$$\boxed{P(0,t) = \frac{\cosh \gamma(xe^{-\beta t}+c)}{\cosh \gamma(x+c)} \exp\left[-\alpha t + \frac{\gamma^2\sigma^2}{2}\frac{1-e^{-2\beta t}}{2\beta}\right]}$$

What can we say about yields?

$$-\frac{1}{t}\log P(0,t) = \frac{1}{t}\int_0^t \beta\gamma x e^{-\beta s}\tanh(\gamma(xe^{-\beta s}+c))\, ds + \alpha - \frac{\gamma^2\sigma^2}{2}\frac{1-e^{-2\beta t}}{2\beta t}$$

This has the property that for $c=0$ we get

$$\frac{1}{t}\int_0^t \beta\gamma\, xe^{-\beta s}\tanh(\gamma xe^{-\beta s})\, ds + \frac{\sigma^2\gamma^2}{2}\left(1 - \frac{1-e^{-2\beta t}}{2\beta t}\right)$$

and the first term is decreasing, the second is increasing. For small t, this is like

$$\beta\gamma x \tanh \gamma x + \left[\frac{\sigma^2\gamma^2}{2} - \frac{\beta\gamma x}{2}\left(\tanh\gamma x + \frac{\gamma x}{\cosh^2\gamma x}\right)\right]\beta t$$

which could be increasing or decreasing, depending on the parameters. The TS of volatility is

$$t \mapsto \frac{\sigma\gamma e^{-\beta t}\tanh \gamma x e^{-\beta t}}{t}$$

, which always decreases.

(iii) Let's observe a generic recipe for making examples. If we can find positive eigenfunctions φ_i of $\mathcal{G}$, $\mathcal{G}\varphi_i = \lambda_i \varphi_i$, then we can form

$$f = \sum_{i=1}^{n} c_i \varphi_i, \quad (\alpha - \mathcal{G}) f = \sum_{i=1}^{n} c_i (\alpha - \lambda_i) \varphi_i,$$

and so provided $\alpha \geq \lambda_i$ for all i we can make an example this way:

$$r(x) = \frac{g(x)}{f(x)} = \frac{\sum (\alpha - \lambda_i) c_i \varphi_i(x)}{\sum c_i \varphi_i(x)}$$

Example (a) above is of this form, though example (b) does not appear to be. If we had the diffusion with generator

$$\tfrac{1}{2}\sigma^2 x^2 D^2 + \mu x D,$$

then $\varphi(x) = x^\theta$ is an eigenf$^{\underline{n}}$ with e-value $\mu\theta + \frac{\sigma^2}{2}\theta(\theta-1)$; this gives lots of examples, but the spot-rate process will always be bounded.

Some brief comments on the Ricardian proposition (4/10/95)

Sargent's book gives a faked account of the Ricardian proposition in § 3.7. Here is a somewhat fuller story for the continuous-time setting.

The government's wealth at time t, y_t, solves

$$dy_t = \{ r_t y_t + \tau_t - g_t \} dt \qquad , \; y_0 = y_T = 0,$$

where τ is the rate of taxation, g is the rate at which govt. spends. The single representative agent of the economy has wealth x_t solving

$$dx_t = -(c_t + \tau_t)\,dt + r_t \theta_t\,dt + \frac{x_t - \theta_t}{\pi_t}\{ d\pi_t + \delta_t\,dt \}$$

where θ_t is wealth held in bonds, π_t is price of 1 share in the sole production process, which generates dividends at rate δ_t. The agent aims to maximise

$$E \int_0^T U(t, c_t)\,dt \qquad \text{subject to } x_0 \text{ given}, \; x_t \geq 0 \;\; \forall t.$$

As usual, $U'(t, c_t^*) = \lambda \zeta_t$ will serve as a state-price density when the agent behaves optimally, and the equilibrium market clearing conditions will be

$$\begin{cases} c_t^* + g_t = \delta_t \\ y_t^* + \theta_t^* = 0 \\ x_t^* + y_t^* = \pi_t = \zeta_t^{-1} E_t\left[\int_t^T \zeta_s \delta_s\,ds \right] \end{cases}$$

Since δ, g are exogenous, we deduce the state price density:

$$\lambda \zeta_t = U'(t, \delta_t - g_t)$$

which determines r and π. What taxation policies could the government pick? Since

$$y_t e^{-\int_0^t r_s ds} = \int_0^t e^{-\int_0^s r_u du} (\tau_s - g_s)\,ds \qquad \left(R_t = \int_0^t r_s ds\right),$$

admissible taxation policies will be any for which

$$\int_0^T e^{-R_t} (\tau_t - g_t)\,dt = 0.$$

Now

$$x_t^* = \zeta_t^{-1} E_t\left[\int_t^T \zeta_s (c_s + \tau_s)\,ds \right]$$

so that

$$x_t^* + y_t^* = E_t\Big[\int_t^T \zeta_s(\delta_s - g_s + \tau_s)\,ds\Big].\zeta_t^{-1} + e^{+R_t}\int_0^t e^{-R_s}(\tau_s - g_s)\,ds$$

$$= E_t\Big[\int_t^T \zeta_s\,(\delta_s - g_s + \tau_s)\,ds\Big]\zeta_t^{-1} + e^{+R_t}\int_t^T e^{-R_s}(g_s - \tau_s)\,ds$$

since τ is admissible and $y_T^* = 0$, and hence

$$x_t^* + y_t^* = \zeta_t^{-1}\; E_t\Big[\int_t^T \zeta_s\,\delta_s\,ds\Big] = \pi_t$$

So for any admissible τ, the productive asset is exactly owned, and taking $y^* + \theta^* = 0$ certainly is one way to ensure that x can be represented as the gains for some portfolio. Uniqueness would be guaranteed if the martingale $E_t^*[\int_0^T \delta_s\,ds]$ is never static.

<u>Ramsey allocations in continuous time</u> (6/10/95)

There's a fascinating paper "Optimal fiscal and monetary policy: some recent results" by V.V. Chari, L.J. Christiano, P.J. Kehoe (J. Money, Credit + Banking 23, 1991, 519–539). They set up several models in discrete time with abysmal notation, and present various results about existence of government fiscal + monetary policies to make certain choices of consumption, labour, ... processes optimal. Let's translate some of the arguments into the continuous-time setting.

(i) Govt. consumption rate $(g_t)_{0\le t\le T}$ is given, and the govt will choose a tax regime $(\tau_t)_{0\le t\le T}$ to be imposed on income (which in this story is synonymous with labour l_t) and also a return process $(\rho_t)_{0\le t\le T}$ on bonds. The govt's wealth at time t, y_t, must solve

(1) $\quad dy_t = y_t\,d\rho_t + (\tau_t\,l_t - g_t)\,dt, \qquad y_T = 0.$

The agent consumes at rate $(c_t)_{0\le t\le T}$, and the value of his holding of bonds at time t, x_t, obeys

(2) $\quad x_t = -c_t\,dt + x_t\,d\rho_t + (1-\tau_t)\,l_t\,dt, \qquad x_0$ given.

His aim (when told what ρ and τ are going to be) is to

$$\max\; E\Big[\int_0^T U(t, c_t, l_t)\,dt\Big].$$

Let's abbreviate $\boxed{c_t - (1-\tau_t)\,\ell_t \equiv \theta_t}$, to denote the agent's net consumption.
Now if

$$\boxed{d\xi_t = \xi_t \, d\rho_t \qquad , \quad \xi_0 = 1,}$$

we shall have

$$d\left(x_t/\xi_t\right) = -\,\theta_t\,\xi_t^{-1}\,dt,$$

so that

$$(3) \qquad \boxed{x_t\,\xi_t^{-1} = x_0 - \int_0^t \theta_s\,\xi_s^{-1}\,ds = \int_t^T \theta_s\,\xi_s^{-1}\,ds}$$

since $x_T = 0$. If we had found optimal x^*, c^*, ℓ^*, and we were to perturb c^* to $c^*+\eta$, where $\int_0^T \eta_s\,\xi_s^{-1}\,ds = 0$, then $(c^*+\eta, \ell^*)$ is a feasible consumption/labour pair, and the FOC says

$$0 = E\left[\int_0^T U_c(t, c_t^*, \ell_t^*)\,\eta_t\,dt\right]$$

(assuming $U_c(t, 0+, \ell) = +\infty$ so that $c_t^* > 0$ for all t). From this it is not hard to deduce that

$$(4) \qquad \boxed{U_c(t, c_t^*, \ell_t^*)\,\xi_t \equiv M_t \quad \text{is a martingale.}}$$

We deduce that

$$U_c(t) \equiv U_c(t, c_t^*, \ell_t^*)$$

$$\begin{aligned}
x_0\,U_c(0) &= E\left[x_0\,M_T\right] \\
&= E\left[M_T \int_0^T \theta_s\,\xi_s^{-1}\,ds\right] \\
&= E \int_0^T \theta_s\,M_s\,\xi_s^{-1}\,ds \\
&= E\left[\int_0^T \theta_s\,U_c(s)\,ds\right].
\end{aligned}$$

There is one last optimality condition, and this obtained by noticing that evidently

$$U(t, c_t^*, \ell_t^*) = \max\left\{ U(t,c,\ell)\ ;\ c - (1-\tau_t)\ell = c_t^* - (1-\tau_t)\,\ell_t^* \right\}$$

from which

$$(5) \qquad \boxed{(1-\tau_t)\,U_c(t) + U_\ell(t) = 0.}$$

so that the budget constraint now reads

$$\boxed{\begin{aligned} x_0\, U_c(0) &= E\left[\int_0^T \theta_t\, U_c(t)\, dt\right] \\ &= E\left[\int_0^T \{U_c(t)\, c_t^* + U_\ell(t)\, \ell_t^*\}\, dt\right]. \end{aligned}} \qquad (B)$$

Condition (B) must be satisfied by an agent who has optimised in the given tax/borrowing environment. For an equilibrium, we want to choose (τ, ρ) to satisfy the market-clearing conditions:

$$\begin{cases} c_t + g_t = \ell_t & \text{(MC1)} \\ x_t + y_t = 0 & \text{(MC2)} \end{cases}$$

In the specification of a Ramsey equilibrium, the government picks a policy $\pi = (\tau, \rho)$ and the agent adopts optimal x^π, c^π, ℓ^π for that environment. Let $\mathcal{A} = \{\text{admissible } \pi\}$, where π is admissible if the market clearing conditions hold (in fact, MC1 $\Rightarrow$ MC2 $\Rightarrow$ MC1 if we have $x_T = y_T = 0$). The optimisation problem is

(I) $\quad \max\left\{ E \int_0^T U(t, c_t^\pi, \ell_t^\pi)\, dt \;;\; \pi \in \mathcal{A} \right\}.$

By what we argued above, if (x^*, c^*, ℓ^*) are the optimal triple for this problem, then (B) and (MC) hold, but the remarkable thing is that the converse is true! That is, if we

(II) $\quad \max\; E \int_0^T U(t, c_t, \ell_t)\, dt$

subject to

(MC1) $\quad c_t + g_t = \ell_t$

(B) $\quad x_0\, U_c(0, c_0, \ell_0) = E \int_0^T \{U_c(t, c_t, \ell_t)\, c_t + U_\ell(t, c_t, \ell_t)\, \ell_t\}\, dt$

then the maximum for problem (II) = max for problem (I), and optimal (x^*, c^*, ℓ^*) for (II) is optimal for (I).

Why? Evidently, max(II) $\geq$ max(I), since we are maximising over a bigger set. What we shall now show is that if we have some (x, c, ℓ)

satisfying (B) & (MC1), together with $x_T = 0$, then there is some choice of π for the government which makes that triple (x, c, ℓ) the same as (x^π, c^π, ℓ^π). This then establishes that the max of the two problems will be same.

Given (c, ℓ) satisfying (B) & (MC1), we firstly find the taxation via (5);

$$1 - \tau_t = \frac{U_\ell}{U_c}(t, c_t, \ell_t)$$

This then determines $\theta = c - (1-\tau)\ell$. Next, from (3) and (4) we had

$$x_t^* \, U_c(t) = \frac{x_t^*}{\xi_t} \cdot M_t = M_t \left(x_0 - \int_0^t \frac{\theta_s}{\xi_s} \, ds \right) \qquad \left[U_c(t) \equiv U_c(t, c_t^*, \ell_t^*) \right]$$

so $$d\left(x_t^* \, U_c(t)\right) \doteq - \theta_t \, U_c(t) \, dt$$

and hence $x_t^* \, U_c(t) + \int_0^t \theta_s U_c(s) \, ds$ is a martingale, and since $x_T^* = 0$, we conclude

(6) $$\boxed{x_t^* \, U_c(t) = E_t \left[\int_t^T \theta_s \, U_c(s) \, ds \right]}$$

This tells us how to define the agent's wealth process in terms of (c, ℓ); we must take

$$x_t \, U_c(t, c_t, \ell_t) = E_t \int_t^T \theta_s \, U_c(s, c_s, \ell_s) \, ds .$$

Note that x so defined vanishes at T, and has value x_0 at 0, in view of (B). Then we use (2) to define the process p, and set $y = -x$. That constructs the policy $\pi = (\tau, p)$ from the given (c, ℓ).

The natural goal is to solve the optimisation problem I, which we now see is equivalent to solving (II). We can absorb the budget constraint via a Lagrange multiplier λ, and we find for each t that we pick (c_t, ℓ_t) to

$$\max \quad U(t, c, \ell) - \lambda c U_c(t, c, \ell) - \lambda \ell \, U_\ell(t, c, \ell)$$

$$\text{subject to} \quad \ell - c = g .$$

Hence

$$\boxed{c_t^* = \bar{c}(t, g_t), \quad \ell_t^* = \bar{\ell}(t, g_t), \quad \tau_t^* = \bar{\tau}(t, g_t).}$$

(17/10/95) (ii) <u>A second model</u> considered in the CCK paper involves capital as well as labour. The (per capita) government spending $(g_t)_{t\geq 0}$ is exogenous, and there be production function F,

(7) $$F(k, \ell, \omega, t) = \ell\, \varphi(k/\ell, \omega, t)$$

which may depend on ω in an adapted fashion. The per capita consumption $(c_t)_{t\geq 0}$ and labour $(\ell_t)_{t\geq 0}$ supplied must satisfy

(8) $$\boxed{-(c_t + g_t + \delta k_t)dt + F(k_t, \ell_t, t)\, dt = dk_t,}$$

where (k_t) is per capita allocation of capital. The govt sets a tax rate (τ_t) on income and a tax rate (θ_t) on capital gains, as well as a bond return process (ρ_t). The marginal price of labour $w_t = F_\ell(k_t, \ell_t, t)$ and the marginal price of capital $r_t = F_k(k_t, \ell_t, t)$ are what the individual agent faces in this optimisation problem, which is simply

$$\max E \int_0^T U(t, c_t, \ell_t)\, dt$$

subject to

(9) $$\boxed{dx_t = -c_t\, dt + (1-\tau_t) w_t \ell_t\, dt + \psi_t\, d\rho_t + (x_t - \psi_t)\underbrace{(1-\theta_t)(r_t dt - \delta dt)}_{d\tilde{r}_t}}$$

and $x_T = 0$. Here, ψ_t is the wealth held in bonds. The government's wealth (y_t) will satisfy

(10) $$\boxed{dy_t = -g_t\, dt + y_t\, d\rho_t + \tau_t w_t \ell_t\, dt + \theta_t (x_t - \psi_t)(r_t dt - \delta dt)}$$

and the market clearing condition is that

(11) $$\boxed{\psi_t + y_t \equiv 0, \qquad k_t = x_t + y_t.}$$

Let's learn some more about the optimal behaviour of the agent. Define

(12) $$\begin{cases} dZ^k_t = Z^k_t\, d\tilde{r}_t, & Z^k_0 = 1, \\ dZ^b_t = Z^b_t\, d\rho_t, & Z^b_0 = 1, \end{cases}$$

the rolling value of wealth invested in capital or bonds. Abbreviate

$$\alpha_t \equiv c_t - (1-\tau_t) w_t \ell_t,$$

and now consider the wealth process denominated in the capital index:

$$d\left(\frac{x_t}{Z^k_t}\right) = \frac{dx_t}{Z^k_t} - \frac{x_t}{Z^k_t}\, d\tilde{r}_t$$

(13) $$= \frac{1}{Z^k_t}\left\{ -\alpha_t\, dt + \psi_t\,(d\rho_t - d\tilde{r}_t) \right\}$$

$$= -\frac{\alpha_t}{Z^k_t}\, dt + \frac{\psi_t}{Z^b_t}\, d\left(Z^b_t / Z^k_t \right)$$

In the bond index, we get similarly

$$d\left(\frac{x_t}{Z^b_t}\right) = \frac{dx_t}{Z^b_t} + \frac{x_t}{Z^b_t}\left\{ -d\rho_t + d\langle\rho\rangle_t \right\} - \frac{dx_t\, d\rho_t}{Z^b_t}$$

(14) $$= -\frac{\alpha_t}{Z^b_t}\, dt + \frac{x_t - \psi_t}{Z^b_t}\, d\left(Z^k_t / Z^b_t \right).$$

Perturb optimal c^* to $c^* + \eta$, where $\int_0^T \eta_t\, (Z^k_t)^{-1}\, dt = 0$, and from (13) we get from FOC

$$0 = E \int_0^T U_c(t, c^*_t, \ell^*_t)\, \eta_t\, dt$$

$$= E \int_0^T Z^k_t\, U_c(t, c^*_t, \ell^*_t)\, \frac{\eta_t}{Z^k_t}\, dt.$$

The conclusion is that

$$Z^k_t\, U_c(t) \equiv Z^k_t\, U_c(t, c^*_t, \ell^*_t) \equiv M^k_t \quad \underline{\text{is a martingale}}.$$

By a similar perturbation argument from (14) we have

$$Z^b_t\, U_c(t) = M^b_t \quad \underline{\text{is a martingale}}.$$

Thus if we develop

$$d\left[x_t\, U_c(t) \right] = d\left(\frac{x_t}{Z^b_t}\, M^b_t \right)$$

$$\doteq M^b_t \left\{ -\frac{\alpha_t}{Z^b_t}\, dt + \frac{x_t - \psi_t}{Z^b_t}\, d\left(\frac{Z^k}{Z^b}\right) \right\} + dM^b \cdot \frac{x_t - \psi_t}{Z^b_t}\, d\left(\frac{Z^k}{Z^b}\right)$$

We now invoke the fact that (Z^k/Z^b) is a martingale in the measure whose density is M^b to argue that the final ~~terms~~ contribute a martingale, so

$$d\left[x_t U_c(t)\right] \doteq M^b_t \left(\frac{-\alpha_t}{Z^b_t}\right) dt$$

so we have the budget constraint

$$\boxed{\begin{aligned} x_0 U_c(0) &= E\int_0^T \alpha_t\, U_c(t)\, dt \\ &= E\left[\int_0^T \{c^*_t U_c(t) + \ell^*_t U_\ell(t)\} dt\right] \end{aligned}} \quad (15)$$

For the converse, suppose we have found (c^*, ℓ^*, k^*) to satisfy the budget condition (15) and also the production equation (8). We want to discover taxes τ, θ and bond returns ρ which make this behaviour optimal for the agent if he faces those government controls. Clearly,

$$w_t = F_\ell(k^*_t, \ell^*_t, t), \qquad r_t = F_k(k^*_t, \ell^*_t, t)$$

determines the wage rate, and gross return on capital. Moreover,

$$(1-\tau_t) w_t\, U_c(t, c^*_t, \ell^*_t) = -U_\ell(t, c^*_t, \ell^*_t)$$

tells us what τ is going to have to be.

When we add (9) to (10) and use the market clearing conditions (11), we recover (8), when we notice that because of the assumed form (7) of the production function

$$F(k,\ell) = kF_k + \ell F_\ell .$$

From (11), we have $y_t = k^*_t - x^*_t = -\psi_t$, and we can obtain x^* easily;

$$\begin{aligned} x^*_t\, U_c(t) &= E_t \int_t^T \{c^*_s - (1-\tau_s) w_s \ell^*_s\} U_c(s)\, ds \\ &= E_t \int_t^T \{c^*_s U_c(s) + \ell^*_s U_\ell(s)\} ds . \end{aligned}$$

The bond return ρ and capital tax θ have to satisfy (9), but there is a degree of indeterminacy here; this isn't surprising, since the govt has three control processes

This is very like a Ramsey model, with x playing the rôle of capital, so it seems sensible to demand also that $x \geq 0$.

$$0 \geq \int_0^T \underbrace{e^{\delta s}\eta_s}_{g(s)} \underbrace{\left(u_c(s)e^{-\delta s} - \int_s^T e^{-\delta t} u_x(t)\,dt\right)}_{f(s)} ds$$

. On the set where $c_s^* \, \ell_s^* > 0$, must have f is , and must be no bigger off that set at its disposal, τ, θ and ρ, via which it tries to induce the agent (with the two controls (c, ℓ) at his disposal) to do wot's best for 'im.

Utility from consumption and possession (18/10/95)

(i) Here is a very simple deterministic model for utility from consumption and possession. The wealth at time t of an agent, x_t, obeys

$$\dot{x}_t = -\delta x_t - c_t + \gamma \ell_t,$$

where c_t is consumption rate, ℓ_t is rate of working, $\gamma, \delta > 0$ constants. The agent is trying to

$$\max \int_0^T U(t, c_t, x_t, \ell_t)\, dt$$

where U is concave, incr in c and x, decr in ℓ. The boundary conditions on x are x_0 given, $x_t \geq 0$. Consumption and labour must be non-negative. Let c^*, ℓ^* denote optimal choices. If $c^*_t > 0$, $\ell^*_t > 0$, we see by perturbing subject to $c - \gamma\ell$ remaining constant that

$$\gamma U_c(t) + U_\ell(t) = 0$$

(where we write $U_c(t) \equiv U_c(t, c^*_t, x^*_t, \ell^*_t)$ for short) If c^*_t or ℓ^*_t is zero, we have the weaker conclusion

$$\gamma U_c(t) + U_\ell(t) \leq 0.$$

Now let's observe that

$$e^{\delta t} x_t = x_0 - \int_0^t (c_s - \gamma \ell_s) e^{\delta s}\, ds,$$

so that if we make a perturbation from c^* to feasible $c^* + \eta$, and ASSUMING $x^*_t > 0$ for all $0 \leq t < T$, we get from FOC

$$0 \geq \int_0^T \left\{ U_c(t)\, \eta_t - U_x(t) \int_0^t e^{-\delta(t-s)} \eta_s\, ds \right\} dt$$

$$\boxed{\therefore \quad e^{-\delta t} U_c(t) \leq \int_t^T e^{-\delta s} U_x(s)\, ds}$$

with equality if $c^*_t > 0$, $\ell^*_t > 0$. If we were to further assume that

$$\lim_{c\downarrow 0} U_c(t,c,x,\ell) = +\infty, \qquad \lim_{\ell\downarrow 0} U_\ell(t,c,x,\ell) = 0$$

this would ensure that $c^*, \ell^* > 0$ and so

$$\gamma U_c(t, c^*, x^*, \ell^*) = - U_\ell(t, c^*, x^*, \ell^*).$$

<u>Example</u> $U(t,c,x,\ell) = (\log c + \beta \log x - \frac{1}{2} a \ell^2)\, e^{-\varepsilon t}$. Here then we obtain

$$\frac{\gamma}{c_t^*} = a \ell_t^*$$

so we only need to consider pairs (c,ℓ) for which $c_t \ell_t = \gamma/a \ \forall t$.
So the dynamics for x become

$$\dot{x} = -\delta x - c + \gamma^2/ac$$

and the optimality condition is

$$\frac{e^{-(\varepsilon+\delta)t}}{c_t} = \int_t^T \frac{\beta}{x_s} e^{-(\varepsilon+\delta)s}\, ds$$

giving

$$\dot{c} + (\varepsilon+\delta)c = c^2 \beta / x$$

So we have the coupled equations to solve:

$$\boxed{\begin{aligned} \dot{x}_t + \delta x_t &= \frac{\gamma^2}{a c_t} - c_t \\ \dot{c}_t + (\varepsilon+\delta) c_t &= \beta c_t^2 / x_t \end{aligned}} \qquad x_t \to 0 \quad (t \uparrow T)$$

If $T = \infty$, the boundary condition's different. It doesn't look like we get any closed-form solution.

<u>Example</u> $\boxed{U(t,c,x,\ell) = -e^{-\rho t} c^{-\alpha} x^{-\beta} (\ell+\varepsilon)^{\theta}}$ $(\alpha, \beta, \varepsilon, \theta, \rho > 0)$

We can equally well use a value function approach, and dig out the HJB equations. So in this example we set

$$c^* = \left(\frac{a}{\alpha}\right)^{(1-\theta)/\nu} \left(\frac{b}{\theta}\right)^{\theta/\nu}, \quad (l^*+\varepsilon) = \left(\frac{a}{\alpha}\right)^{-\alpha/\nu} \left(\frac{b}{\theta}\right)^{(1+\alpha)/\nu}$$ in interior

$$V(x) = \sup \int_0^\infty -e^{-\rho t} c_t^{-\alpha} x_t^{-\beta} (l_t+\varepsilon)^\theta \, dt$$

subject to the dynamics

$$\dot{x}_t = -\delta x_t - c_t + l_t$$

(we shall set $\gamma=1$, wlog). This gives us the HJB equation

$$\sup_{c>0,\, l\geq 0} \Big[-\rho V(x) + (-\delta x - c + l) V'(x) + U(c,x,l) \Big] = 0.$$

The first step is to compute for $a, b > 0$

$$\sup_{c,\, l\geq 0} \Big[-ac + bl - c^{-\alpha} x^{-\beta} (l+\varepsilon)^\theta \Big] \equiv f(a,b,x)$$

$$= x^{-\beta} f(ax^\beta, bx^\beta, 1)$$

so wlog $x=1$, and we shall have

$$\left.\begin{aligned} a &= \alpha c^{-\alpha-1} (l+\varepsilon)^\theta \\ b &= \theta c^{-\alpha} (l+\varepsilon)^{\theta-1} \end{aligned}\right\} \quad \therefore \quad \boxed{\frac{l^*+\varepsilon}{c^*} = \frac{a\theta}{b\alpha}}$$

at least in the interior, when $l>0$ at optimum. This gives us after some calculations

$$\boxed{f(a,b,1) = \nu \left(\frac{a}{\alpha}\right)^{-\alpha/\nu} \left(\frac{b}{\theta}\right)^{\theta/\nu} - b\varepsilon} \qquad [\nu \equiv \theta - \alpha - 1]$$

However, if

$$\left(\frac{a}{\alpha}\right)^{-\alpha} \left(\frac{b}{\theta}\right)^{1+\alpha} \leq \varepsilon^\nu$$

then optimal is to set $l^* = 0$, and

$$c^* = \left(\frac{\varepsilon^\theta \alpha}{a}\right)^{1/(1+\alpha)}.$$

The de to solve when l^* is positive is

$$\boxed{0 = -\rho V(x) - (\varepsilon + \delta x) V'(x) + \nu\, x^{\beta/\nu}\, V'(x)^{(\theta-\alpha)/\nu}\, \alpha^{\alpha/\nu}\, \theta^{-\theta/\nu}}$$

with the following when $l^* = 0$:

$$\boxed{0 = -\rho V(x) - \delta x V'(x) - x^{-\beta/(1+\alpha)} \left(\frac{V'(x)}{\alpha}\right)^{\alpha/(1+\alpha)} \varepsilon^{\theta/(1+\alpha)} (1+\alpha).}$$

Now if we are willing to assume certain relations among the parameters, we may be able to make progress. We seek a solution of the form

$$V'(x) = k x^{\lambda} \qquad \text{for } x > x_c$$
$$= \kappa x^{\mu} \qquad \text{for } x < x_c$$

with positive parameters k, κ, λ, μ. We find

$$\begin{cases} \lambda = -1-\alpha-\beta \\ k\left(\frac{\rho}{\alpha+\beta} - \delta\right) = \left(\frac{k}{\alpha}\right)^{\alpha/(1+\alpha)} \varepsilon^{\theta/(1+\alpha)}(1+\alpha) \\ \mu = -\beta = -1-\rho/\delta \\ \kappa = \left(\frac{\varepsilon}{\nu}\right)^{\nu} \alpha^{-\alpha}\,\theta^{\theta} \end{cases}$$

which imposes the constraint

$$\boxed{\beta = 1+\rho/\delta}$$

and we find that

$$\boxed{x_c = \left(\frac{\theta}{\nu}\right)^{\nu/\beta}}$$

For continuity of V at x_c, we shall require

$$\boxed{\frac{k\, x_c^{1+\lambda}}{1+\lambda} = \frac{\kappa\, x_c^{1+\mu}}{1+\mu}}$$

which imposes one more constraint.

<u>Example</u> If we are willing to place restrictions on the parameters, we could use

$$\boxed{U(c,x,\ell) = -c^{-\alpha} - b x^{-\beta} - \lambda \ell^{\theta},}$$

where $\theta > 1$, $\alpha, \beta > 0$. If we demand $\rho > \delta\alpha$ and

$$\boxed{\beta = \frac{\theta(1+\alpha)}{\theta-1}, \quad b = (\theta-1)\,\lambda^{-1/(\theta-1)}\left(\frac{1+\alpha}{\rho-\delta\alpha}\right)^{\theta(1+\alpha)/(\theta-1)} \left(\frac{\alpha}{\theta}\right)^{\theta/(\theta-1)}}$$

then we can do something. Why? We have

$$\sup_{c,\ell\geq 0}\left[-\rho V(x)-\delta x V'(x) - cV'(x) + \ell V'(x) - c^{-\alpha} - bx^{-\beta} - \lambda \ell^{\theta}\right] = 0$$

which is

$$-bx^{-\beta} - \rho V(x) - \delta x V'(x) - (1+\alpha)\left(\frac{V'(x)}{\alpha}\right)^{\alpha/(1+\alpha)} + (\theta-1)\lambda\left(\frac{V'(x)}{\lambda\theta}\right)^{\theta/(\theta-1)} = 0.$$

Now if we seek a solution

$$\boxed{V'(x) = k\, x^{-1-\alpha}}$$

we can solve with

$$k = \alpha\left(\frac{1+\alpha}{\rho-\delta\alpha}\right)^{1+\alpha},$$

because of the constraints on the parameters β, b. With this setup,

$$\boxed{\begin{aligned} c^* &= c^*(x) = \left(\frac{V'(x)}{\alpha}\right)^{-1/(1+\alpha)} = \frac{\rho-\delta\alpha}{1+\alpha}\, x \\ \ell^* &= \ell^*(x) = \left(\frac{k}{\lambda\theta}\right)^{1/(\theta-1)} x^{-\theta(1+\alpha)/(\theta-1)} \end{aligned}}$$

10/1/96 If the dynamics were $dx_t = -\delta x_t \, dt + l_t \, (\sigma dW_t + \gamma dt)$, with the aim of

$$\max E \left[\int_0^\infty e^{-\rho t} U(x_t, l_t) \, dt \right] \equiv \varphi(x)$$

then the HJB would be

$$\sup_{l \geq 0} \left[-\rho \varphi + \mathcal{G} \varphi + U(x, l) \right] = 0$$

$$= \sup_{l \geq 0} \left[-\rho \varphi(x) + \tfrac{1}{2} \sigma^2 l^2 \varphi''(x) + (\gamma l - \delta x) \varphi'(x) + U(x, l) \right]$$

Assuming $U(x, l) = u(x) - \frac{1}{2} \varepsilon l^2$, we get

$$l^* = \frac{\gamma \varphi'(x)}{\varepsilon - \sigma^2 \varphi''(x)}$$

and

$$\boxed{-\rho \varphi(x) + u(x) - \delta x \varphi'(x) + \frac{\gamma^2 \varphi'(x)^2}{2 \{ \varepsilon - \sigma^2 \varphi''(x) \}} = 0}$$

Now we could fake this by looking for a solution, $\varphi(x) = \lambda \log x$, say. Then we'd have

$$\boxed{u(x) = \rho \lambda \log x + \lambda \delta - \frac{\gamma^2 \lambda^2}{2(\sigma^2 \lambda + \varepsilon x^2)}}$$

which is certainly concave increasing if $3 \rho \sigma^2 > \gamma$ (and even more generally)
The optimal rule is

$$\boxed{l^* = \frac{\gamma x}{\sigma^2 + \varepsilon x^2}}$$

Other choices would be possible, of course.

Wealth-dependent utility: stochastic version (3/11/95)

Suppose the agent's wealth was generating some random return as well as decaying: his wealth equation would then be

$$dx_t = x_t[\sigma\, dW_t - \delta\, dt] + l_t\, dt - c_t\, dt$$

in a time-homogeneous case, with the aim to

$$\max \quad E\left[\int_0^\infty e^{-\rho t}\, U_0(x_t, c_t, l_t)\, dt\right].$$

Approach 1. Since everything is clearly Markovian, the optimal consumption and labour will be functions of x_t. We could more generally define

$$dz_t = z_t\,[\sigma\, dW_t - \delta\, dt], \qquad z_0 = 1$$

so that

$$x_t = z_t\left(x_0 - \int_0^t y_s/z_s\, ds\right),$$

where

$$y_s \equiv c_s - l_s.$$

The usual FOC gives (assuming always optimal consumption is positive, etc...)

$$\boxed{e^{-\rho t}\, U_y(t) = E_t\left[\int_t^\infty e^{-\rho s} z_s\, U_x(s)\, ds\right] / z_t}$$

where

$$U_y(t) \equiv \frac{\partial U}{\partial y}(x_t^*, y_t^*), \qquad U_x(t) \equiv \frac{\partial U}{\partial x}(x_t^*, y_t^*),$$

and where

$$\mathcal{U}(x,y) \equiv \sup\{\, U_0(x,c,l)\ ;\ c, l \geq 0,\ c - l = y\,\}.$$

If we have the change of measure $(dQ/dP)\big|_{\mathcal{F}_t} = \exp(\sigma W_t - \sigma^2 t/2)$, then the FOC says that

$$\boxed{e^{-(\rho+\delta)t}\, U_y(t) = E_t^Q\left[\int_t^\infty e^{-(\rho+\delta)s}\, U_x(s)\, ds\right]}.$$

Now if we express

$$y_t^* = \psi(x_t^*)$$

as will be the case under optimal play, it is clear that under Q the process x^* will solve the SDE

$$dx^*_t = x^*_t\left[\sigma dW^*_t + (\sigma^2-\delta)dt\right] - y_t\, dt$$

$$= x^*_t\left[\sigma dW^*_t + (\sigma^2-\delta)dt\right] - \psi(x^*_t)dt,$$

As the FOC can be re-expressed as saying that

$$f(x) \equiv U_y(x, \psi(x)) \quad \text{and} \quad g(x) \equiv U_x(x, \psi(x))$$

are related by

$$f = R_{\rho+\delta}\, g,$$

or again

$$\boxed{\left(\rho+\delta - \tfrac{1}{2}\sigma^2 x^2 D^2 - (\sigma^2-\delta)xD - \psi(x)D\right)f(x) = g(x)}$$

This is fairly complicated, but maybe we can get somewhere with it.

<u>Approach 2</u> Conventional HJB approach. If we define

$$V(x) \equiv \sup E\left[\int_0^\infty e^{-\rho t}\, U_0(x_t, c_t, \ell_t)\, dt \;\middle|\; x_0 = x\right]$$

then the HJB formulation is

$$\sup_{c,\ell\geq 0}\left[-\rho V(x) + U_0(x,c,\ell) + \tfrac{1}{2}\sigma^2x^2 V''(x) - \delta x V'(x) + (\ell - c)V'(x)\right] = 0$$

Another way to think of the American put (19/11/95)

(i) Let's fix strike 1 and consider an American put on a share paying continuous dividends at rate $\delta > 0$. Then the value of the option when time-to-go is t and the log share price is x is given by

$$\varphi(t,x) = \sup_{0\le\tau\le t} E\left[e^{-r\tau}\left(1 - \exp\{\sigma B_\tau + (r-\delta-\sigma^2/2)\tau + x\}\right)^+ \right]$$

$$= \sup_{0\le\tau\le t} E\left[e^{-r\tau + \gamma B_\tau - \gamma^2\tau/2} \left(1 - e^{\sigma B_\tau + x}\right)^+ \right]$$

where $\gamma \equiv (r-\delta-\sigma^2/2)\sigma^{-1}$.

If we take

$$f(s,x) \equiv \exp\left[\gamma x - (r + \tfrac{\gamma^2}{2})s\right]\left(1 - e^{-\sigma x^-}\right)$$

and expand $f(t, B_t)$, we get

$$d f(t,B_t) \doteq e^{\gamma B_t - (r+\gamma^2/2)t} I_{\{B_t \le 0\}}\left[-(r - \delta e^{-\sigma B_t^-})\,dt + \tfrac{1}{2}\sigma\, d\ell_t\right],$$

where ℓ is loc. time at 0 of B. Thus

$$\varphi(t,x) = (1-e^x)^+ + e^{-\gamma x}\sup_{0\le\tau\le t} E^x\left[\frac{\sigma}{2}\int_0^\tau e^{-(r+\gamma^2/2)s}\,d\ell_s - \int_0^\tau (r - \delta e^{-\sigma B_s^-})\, e^{\gamma B_s - (r+\gamma^2/2)s} I_{(B_s\le 0)}\,ds\right]$$

Accordingly, we should continue if and only if

$$\boxed{\sup_{0\le\tau\le t} E^x\left[\frac{\sigma}{2}\int_0^\tau e^{-\mu s}\,d\ell_s - \int_0^\tau (r-\delta e^{+\sigma B_s})\, e^{\gamma B_s - \mu s} I_{\{B_s\le 0\}}\,ds\right] \ge 0}$$

(ii) We certainly continue if $\left[\mu = r + \tfrac{1}{2}\gamma^2\right]$

$$\sup_{a\in[0,t]} E^x\left[\frac{\sigma}{2}\int_0^a e^{-\mu s}\,d\ell_s - \int_0^a e^{\gamma B_s - \mu s} I_{\{B_s\le 0\}}(r - \delta e^{\sigma B_s})\,ds\right] > 0.$$

With $x > 0$ fixed, we consider

$$\psi_1(t) \equiv \frac{\sigma}{2}\int_0^t e^{-\mu s}\, p_s(x,0)\, ds$$

and $\psi_2(t,\theta) \equiv \int_0^t e^{-\mu s} \left(\int_{-\infty}^0 e^{\theta y} \, p_s(x,y)\, dy \right) ds$

$$= \int_0^t e^{-\mu s} e^{\theta x + \theta^2 s/2} \, \Phi\left(\frac{-x-\theta s}{\sqrt{s}} \right) ds$$

What interests us therefore is

$$\psi_1(t) - r\,\psi_2(t,\gamma) + \delta\,\psi_2(t,\gamma+\sigma).$$

The derivative of this is

$$e^{-\mu s}\left[\frac{\sigma}{2} \frac{e^{-x^2/2s}}{\sqrt{2\pi s}} - r\, e^{\gamma x + \gamma^2 s/2}\, \Phi\left(\frac{-x-\gamma s}{\sqrt{s}}\right) + \delta\, e^{(\gamma+\sigma)x + (\gamma+\sigma)^2 s/2}\, \Phi\left(\frac{-x-(\gamma+\sigma)s}{\sqrt{s}}\right)\right]$$

so for large s this looks like

$$\frac{e^{-\mu s - x^2/2s}}{\sqrt{2\pi s}} \left[\frac{\sigma}{2} - \frac{r s}{x+\gamma s} + \frac{\delta s}{x+(\gamma+\sigma)s} \right].$$

(iii) If we have $\delta > 0$, it's possible that $\delta > r$, and there is some $a > 0$ s.t. $r = \delta e^{-\sigma a}$. Suppose we change the game, and only count the decreases of $-\int_0^t (r - \delta e^{\sigma B_s}) e^{\gamma B_s} I_{\{B_s \le 0\}} ds\, e^{-\mu s}$ until B first rises above $-a$. After that, we shall only allow the increases of $\frac{\sigma}{2}\int e^{-\mu s} d\ell_s - \int_0^t (r - \delta e^{\sigma B_s}) e^{\gamma B_s} I_{(B_s \le 0)} e^{-\mu s} ds$. If for this more favourable game it is advantageous to stop, then certainly it will be advantageous to stop for the original problem.

We have

$$E^x\left[\frac{\sigma}{2}\int_0^t e^{-\mu s} d\ell_s + \int_0^t (\delta e^{\sigma B_s} - r)\, e^{\gamma B_s - \mu s}\, I_{\{-a \le B_s \le 0\}}\, ds \right]$$

$$= \psi_1(t) + \delta\,\tilde{\psi}_2(t,\sigma+\gamma) - r\,\tilde{\psi}_2(t,\gamma)$$

where $\tilde{\psi}_2(t,\rho) \equiv \int_0^t e^{-\mu s}\left(\int_{-a}^0 p_s(x,y)\, e^{\rho y} dy\right) ds$. The expected negative part is

$$\int_0^t e^{-\mu s} ds \int_{-\infty}^{a} (r - \delta e^{\sigma y})\, e^{\gamma y} \left(p_s(x,y) - p_s(-x-2a, y) \right) dy.$$

(iv) We can very crudely estimate the boundary below by comparing with the game in which we can at any time stop, or else continue and receive the overall future maximum of $(1-S_t)^+$, but at the time at which it happens (so, although this max will be bigger than the present value, the time we have to wait for it makes it better to stop now).

If strike is 1, and $\log S_0 = x < 0$, and time-to go is T, the expected reward if we go on to collect max is

$$\int_0^T dt \int_0^\infty dz \int_0^\infty dy\ 2h(t,z)h(T-t,y)\left(1-e^{x-\sigma z}\right) e^{\gamma(y-z)-\gamma^2 T/2}-rt$$

where h is the Brownian first-passage density. This can't be computed in closed form, so it's no better than (iii)

(v) For start values of $\log S$ which are $\gg 0$, we can bound the value by

$$E\left(1-\exp(x+\sigma\xi)\right)^+$$

where $\xi \equiv \inf_{s\le T}\{B_s + (r-\delta-\sigma^2/2)s\sigma^{-1}\}$. A very crude bound is

$$\le P\left[\xi < -x/\sigma\right]$$

$$\le P\left[\inf_{s\le T} B_s < -\frac{x}{\sigma} + \left(r-\delta-\frac{\sigma^2}{2}\right)^-\frac{T}{\sigma}\right]$$

$$= 2\,\Phi\left(\frac{x-(r-\delta-\sigma^2/2)^- T}{\sigma\sqrt{T}}\right)$$

which is at least simple to use!

(vi) In terms of the game at (iv), we'd stop if

$$E\ 1-e^x - e^{-rT^*}\left(1-e^{x-\sigma\xi}\right) > 0$$

where $-\xi$ is the min value on $[0,T]$ of the drifting BM, T^* the time at which it occurs.

Now this is the same condition as

$$E\left(1-e^{-rT^*}\right) \ge e^x\ E\left(1-e^{-rT^*-\sigma\xi}\right)$$

which is implied by

$$\left(E(1-e^{-rT^*})\ge\right)\ \boxed{\frac{1-e^{-rT}}{T}\ ET^* \ge e^x\left(E\,rT^* + E\,\sigma\xi\right)}$$

This requires bounds on both sides of ET^*, and an upper bound for $E\xi$. Now we have exactly

$$ET^* = \int_0^\infty \frac{1-e^{-ca}}{c} \, a e^{-(a-c\tau)^2/2\tau} \frac{da}{\sqrt{2\pi\tau^3}}$$

if $c \equiv -\gamma \equiv \sigma^{-1}(r-\delta-\sigma^2/2)$ is positive. If $c>0$, $ET^* > \frac{1}{2}$, and

$$ET^* \leq \int_0^\infty a^2 \, e^{-(a-c\tau)^2/2\tau} \frac{da}{\sqrt{2\pi\tau^3}}$$

$$\leq \frac{T}{2} + c^2T\left(2\Phi(c\sqrt{T})-1\right) + \frac{2c\sqrt{T}}{\sqrt{2\pi}} e^{-c^2T/2}$$

$$= \frac{T}{2} + c^2T\left(1-2\bar{\Phi}(c\sqrt{T})\right) + \frac{2c\sqrt{T}}{\sqrt{2\pi}} e^{-c^2T/2}$$

after some calculations.

So for $\gamma \leq 0$, we use

$$\boxed{\frac{T}{2} \leq ET^* \leq \frac{T}{2} + c^2T\left(1-2\bar{\Phi}(c\sqrt{T})\right) + \frac{2c\sqrt{T}}{\sqrt{2\pi}} e^{-c^2T/2}} \quad (c \equiv -\gamma)$$

and for $\gamma \geq 0$, we have the same bounds for $T - E^*T$, with $c = \gamma$.

As a simple upper bound,

$$\boxed{E\xi \leq \sqrt{\frac{2T}{\pi}} + \gamma^- T}$$

of opposite page is quadratic in c (recall

upper bound for E_3^0.

37 Is this quadratic non-negative definite??

* $> \frac{1}{2}$, and

Problem? The boxed expression at foot of opposite page is quadratic in c (Recall

$$\alpha = \tfrac{1}{2}\operatorname{tr} Q + \tfrac{1}{2}|Qc|^2 + \tfrac{1}{2}v^T S^{-1} v$$

$S \equiv B^T Q + QB - Q^2 \geq 0$, $v \equiv (B^T - Q)\,Qc$.) Is this quadratic non-negative definite??

Equilibrium covariance structure for potential models (28/11/95)

(i) If we make a potential model in which the underlying Markov process is the n-dimensional Gaussian diffusion

(1) $$dX = dW - BX\,dt$$

and we take the situation with

(2) $$f_i(x) = \exp\left[\tfrac{1}{2}(x-c_i)^T Q\,(x-c_i)\right]$$

in country i, our aim is to make some estimation procedure akin to GMM.

As has been shown, the bond prices satisfy

(3) $$-\log P(0,T) = \alpha T + \tfrac{1}{2}\log\det M_T + \tfrac{1}{2}\mu_0\cdot Q\mu_0 - \tfrac{1}{2}\mu_T\cdot M_T^{-1}Q\mu_T$$

where $V_T = \int_0^T e^{-sB}(e^{-sB})^T\,ds$, $M_T \equiv I - QV_T$, $\mu_T \equiv e^{-TB}X_0 - c$, and the exchange rates satisfy

(4) $$\log(Y_t^{ij}/Y_0^{ij}) = (\alpha_i-\alpha_j)t + (c_i-c_j)\cdot Q(X_t-X_0).$$

(ii) The idea is that we shall see each day the yield curve in each country, and the FX rates between the countries, and shall form an estimate of the τ-yield in country i by

(5) $$\sum_{r\geq 0}\theta^r(1-\theta)\cdot\frac{-\log P_i((n-r)\delta,(n-r)\delta+\tau)}{\tau} \equiv y_i(n\delta;\tau),$$

where δ is the time quantum. We shall assume that this is a reasonable approximation to

(6) $$\int_0^\infty \eta e^{-\eta s}\left(-\log P(n\delta-s,\ n\delta-s+\tau)/\tau\right)ds,$$

where $1/\eta = \theta\delta/(1-\theta)$; the parameter η is going to be quite small. Now assuming that X is in its stationary distn $N(0, V_\infty)$, we'll have

(7) $$\boxed{\begin{aligned} E\,y_i(n\delta;\tau) &= \alpha_i + \frac{1}{2\tau}\log\det M_\tau \\ &\quad + \frac{1}{2\tau}\left\{\operatorname{tr} M_\infty V_\tau M_\tau^{-1}Q - c_i\cdot QV_\tau M_\tau^{-1}Qc_i\right\}\end{aligned}}$$

Similarly, the exchange rates can be used to form a geometrically-weighted mean of past values. Explicitly, we form

$$(8)\quad \boxed{\rho_{ij}(n\delta) \equiv \sum_{r\geq 0} \theta^{r}(1-\theta)\left[-\log Y^{ij}_{(n-r)\delta} + \log Y^{ij}_{n\delta}\right]}$$

which we assume is a reasonable approximation to

$$(9)\quad -\int_0^\infty \eta e^{-\eta s} \log\{Y^{ij}(t-s)/Y^{ij}(t)\}\, ds.$$

If X is in equilibrium, then

$$(10)\quad E\rho_{ij}(n\delta) = E\rho_{ij}(0) = (\alpha_i - \alpha_j)\,\theta\delta/(1-\theta).$$

The updating recursion for $\rho_{ij}(n\delta)$ is

$$(11)\quad \boxed{\rho(n\delta+\delta) = +\theta\lambda_{n+1} + \theta(\rho(n\delta) - \lambda_n)} \qquad \lambda_n \equiv \log Y^{ij}_{n\delta}$$

(iii) What can we say of the covariances of the different estimators of yields and exchange rates? The random part of the integral (6) is

$$(12)\quad \tfrac{1}{2}\int_0^\infty \eta e^{-\eta s}\left[X_{n\delta-s}\cdot A_\tau X_{n\delta-s} + 2\tilde{\gamma}_\tau \cdot X_{n\delta-s}\right] ds$$

where

$$A_\tau \equiv \frac{1}{\tau}\left\{Q - e^{-\tau B^T} M_\tau^{-1} Q e^{-\tau B}\right\}$$

$$(13)$$

$$\tilde{\gamma}_\tau \equiv \frac{1}{\tau}\left[e^{-\tau B^T} M_\tau^{-1} Q c - Qc\right]$$

and the random part of the integral in (9) is

$$(14)\quad \int_0^\infty \eta e^{-\eta s}\,(c_i - c_j)\cdot Q(X_t - X_{t-s})\, ds$$

$$= (c_i - c_j)\cdot QX_t - \int_0^\infty (c_i - c_j)\cdot Q X_{t-s}\, ds\, \eta e^{-\eta s}.$$

We could alternatively express the random part of the integral (6) as

$$(15)\quad \tfrac{1}{2}\int_0^\infty \eta e^{-\eta s}(X_{n\delta-s} - \gamma_\tau)\cdot A_\tau (X_{n\delta-s} - \gamma_\tau)\, ds,$$

$$(16)\quad \gamma_\tau \equiv -A_\tau^{-1}\tilde{\gamma}_\tau$$

(17) Now $\boxed{E^x \int_0^\infty e^{-\eta t} \, v\cdot X_t \, dt = v\cdot \int_0^\infty e^{-\eta t - Bt} x \, dt = v^T(\eta + B)^{-1} x}$

and for some nnd symmetric matrix A we get

(18)
$$\boxed{\begin{aligned} &E^x \int_0^\infty e^{-\eta t}\left(\tfrac{1}{2}X_t^T A X_t\right) dt \\ &= \tfrac{1}{2} x^T \left\{ \int_0^\infty e^{-\eta t} \, e^{-tB^T} A e^{-tB} \, dt \right\} x \\ &\quad + \tfrac{1}{2}\int_0^\infty e^{-\eta t} \, \mathrm{tr}(AV_t) \, dt \end{aligned}}$$

Now to compute the covariance matrix, we need to know

$$E^\pi \int_0^\infty \eta e^{-\eta t} g(X_t) dt \, . \int_0^\infty \eta e^{-\eta s} f(X_s) \, ds$$

$$= \int \eta^2 (R_{2\eta} \, g R_\eta f + R_{2\eta} \, f R_\eta g)(x) \, \pi(dx)$$

$$= \tfrac{1}{2} (\pi, \, g R_\eta f + f R_\eta g) \, \eta$$

where f, g are either linear or quadratic. If f is of the form $v\cdot X$ and g is of the form $\frac{1}{2}x^T A x$, then $(\pi, f R_\eta g) = 0 = (\pi, g R_\eta f)$, so we just need to know

$$E^\pi (X_0^i X_0^j X_0^k X_0^l) = v_{ij} v_{kl} + v_{ik} v_{jl} + v_{il} v_{jk} \qquad [(v_{ij}) \equiv V_\infty]$$

so that if $\boxed{f(x) = \tfrac{1}{2} x^T A x \, ; \quad g(x) = \tfrac{1}{2} x^T \tilde{A} x}$

then

(19)
$$\boxed{\begin{aligned} (\pi, g R_\eta f) &= \tfrac{1}{2}\mathrm{tr}(\tilde{A} V_\infty) . \tfrac{1}{2}\int_0^\infty e^{-\eta t} \, \mathrm{tr}(AV_t) \, dt \\ &\quad + \tfrac{1}{4}\left[\mathrm{tr}(\tilde{A}V_\infty)\,\mathrm{tr}(\Gamma V_\infty) + 2\,\mathrm{tr}(\tilde{A}V_\infty \Gamma V_\infty) \right], \\ \Gamma &\equiv \int_0^\infty e^{-\eta t} e^{-tB^T} A e^{-tB} \, dt \end{aligned}}$$

Likewise if $f(x) = w\cdot x$, $g(x) = z\cdot x$,

(20) $\boxed{(\pi, g R_\eta f) = w^T(\eta + B)^{-1} V_\infty z}$

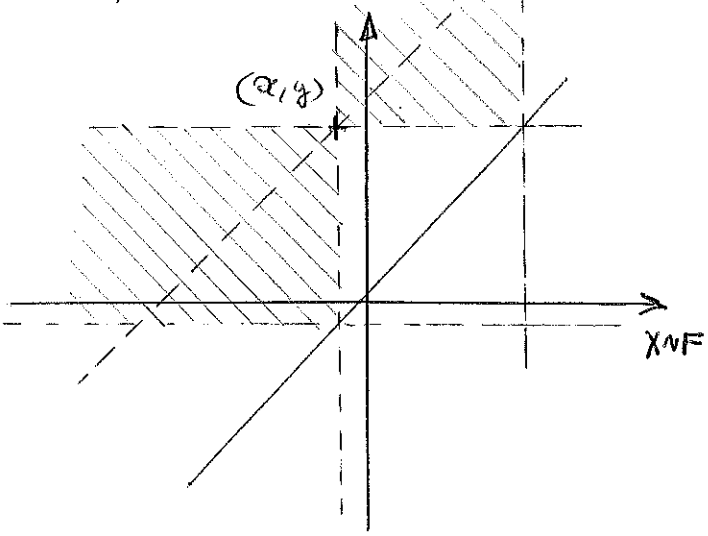
Fast coupling of random walks (30/11/95)

(i) We are going to look at the earlier coupling in the special case

F is continuous and unimodal.

There is no loss of generality taking the mode at 0, as we now shall. The earlier coupling of $X \sim F$ and $Y \sim \delta_a * F$ $(a>0)$ has the property that if the joint dist^n charges (x,y) then the shaded region has zero measure and so

$$(1) \quad \boxed{F(y) - F(x) = F_a(y) - F_a(x).}$$



The joint dist^n has

$$P(X = Y \in dx)/dx = f(x) \wedge f(x-a)$$

where f is the density of F, increasing in $(-\infty,0)$, decreasing in $\mathbb{R}^+$. Thus we have

$$P(X = Y \in dx)/dx = f(x-a), \quad x \leq m(a)$$
$$= f(x), \quad x > m(a)$$

where $m(a)$ is the crossover point of $f(x)$ and $f(x-a)$, and the remainder of the distribution is concentrated on the curve $\{(x,y) ; x < m(a) < y, (1) \text{ holds}\}$.

(ii) It seems that only the special case where f is also symmetric will work out nicely, because condition (1) holds only on the line $x+y = a$, and $m(a) = a/2$. Then for any $c > 0$

$$E(Y-X-c)^+ = \int_{a/2}^{\infty} (2y-a-c)^+ \{f(y-a) - f(y)\} dy$$
$$= \int_c^{\infty} P(Y-X > s)\, ds$$

But $P(Y-X > s) = P\big(Y > (a+s)/2\big)$, which is a convex function of a for each $s \geq 0$. Hence if φ_0 is a convex function, then

$$\varphi_1(a) \equiv \sup\{E\varphi_0(Y-X) \; ; \; Y \sim F_a, X \sim F, Y \geq X\}$$

Unimodality appears to be necessary for this result, as we see from the example

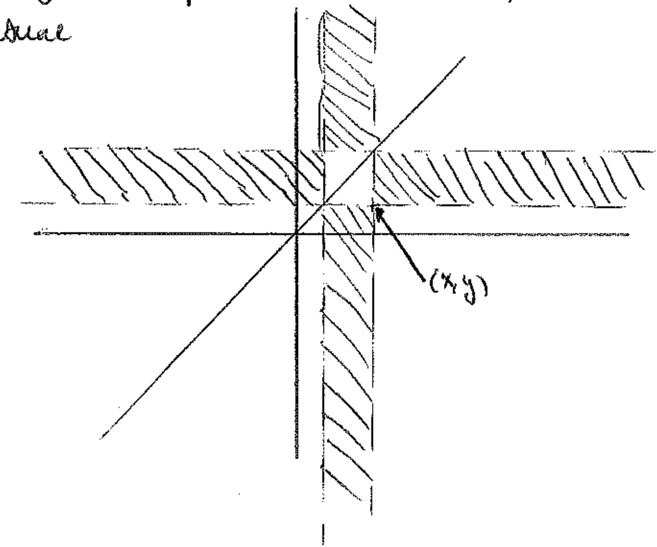
$F = p\delta_1 + (1-p)\delta_0 \quad ; \quad \psi(x) = \frac{1}{\varepsilon + |x|}$.

is attained by the earlier coupling and defines a convex function of a.

Similarly,
$$\varphi_{n+1}(a) \equiv \sup\{ E\, \varphi_n(Y-X) \ ; \ X\sim F, \ Y\sim F_a, \ Y\geq X\}$$

is a convex function

(iii) Suppose that ψ is a symmetric function, convex and decreasing in $\mathbb{R}^+$, suppose also that F is unimodal and continuous, and consider the problem
$$\max\{ E\, \psi(Y-X) \ ; \ X\sim F, \ Y\sim F_a \}.$$
Note that we are not requiring that $Y\geq X$. Nevertheless, I claim that the optimal joint distribution will have this property. Indeed, if we consider some point (x,y), $x>y$, which is charged by the optimal distribution, then the shaded region must get no mass, by the usual argument. Hence



$$F(x) - F(y) = F_a(x) - F_a(y)$$
so if we write $y = x-\delta$ then
$$F(x) - F(x-a) = F(x-\delta) - F(x-\delta-a)$$

Since for fixed a $F(x)-F(x-a)$ is itself unimodal, this relation uniquely determines x.

Similarly, if the optimal μ were to charge $(x', x'+\delta)$ we shall have to have
$$F(x'+\delta) - F(x') = F(x'+\delta-a) - F(x'-a) \quad \therefore \quad F(x'+\delta) - F(x'+\delta-a) = F(x') - F(x'-a)$$
and so $x' = x-\delta$. Thus $(x, x-\delta)$ and $(x', x'+\delta)$ are opposite corners of a square with the other two corners on the diagonal.

One candidate for a solution to the problem is the solution to the problem with the constraint $Y\geq X$. If this is denoted $\bar\mu$, and if μ^* is the solution to the problem, by considering $\int \psi(y-x)\,(\theta\bar\mu + (1-\theta)\mu^*)(dx,dy)$, this has to be decreasing in $\theta\in[0,1]$; but perturbing $\bar\mu$ towards μ^* can only be done by shifting mass off the diagonal and to points (x,y) where $F(x)-F(y) = F_a(x) - F_a(y)$, $x>y$ ... but this will surely make $E\,\psi(Y-X)$ less.

(iv) As an application of the above observations, let's notice that for any $\lambda>0$ & the function (assuming again F is symmetric)

$$V_n(a) \equiv \max\{E\exp(-\lambda|S_n'-S_n|)\ ;\ S_0'=a;\ 0=S_0\}$$

is symmetric and convex decreasing in $\mathbb{R}^+$. To see this, we proceed by induction. Clearly it's true for $n=0$. The Bellman equation is

$$V_{n+1}(a) = \max E\left[V_n(S_1'-S_1)\ ;\ S_0'=a, S_0=0\right]$$

Because F is unimodal, and V_n is symmetric, convex decreasing in $\mathbb{R}^+$, we have by (iii) that for $a>0$ the optimal joint law satisfies $S_1'\geq S_1$, and so we are solving the problem as if under the constraint $S_1'\geq S_1$. Now by (ii) the value function V_{n+1} must be convex, and it can't increase anywhere in $\mathbb{R}^+$, else by convexity it would be unbounded.

So this coupling method will maximise $E\,e^{-\lambda|S_n'-S_n|}$ for any n, and, in particular, by letting $\lambda\to\infty$, it maximises $P(S'$ and S have coupled by time $n)$ — so in the special case of a symmetric unimodal distribution, this is the fastest possible coupling!

(v) As an example, take the Cauchy distribution

$$F(dx) = \frac{dx}{\pi(1+x^2)} = f(x)\,dx$$

which is symmetric and unimodal. If initially the random walks are separated by $a>0$, then the non-negative coupling makes

$$P(X=Y\in dx) = (f(x)\wedge f(x-a))\,dx$$

and

$$P[X=a-Y\in dx] = (f(x)-f(x-a))\,dx \qquad (x<a/2)$$

The neat way to think of this is that there are two independent BMs B^1 and B^2, and $B_0^1=a$, and

$$S_n'-S_n = B^1(H_0^1\wedge H_{2n}^2)$$

where $H_a^j\equiv\inf\{u:\ B_u^j=a\}$. Pathwise, we could do the reflection coupling of two independent BMs on $\mathbb{R}$, and look at them at times which were a renewal process with lifetime dist^n the dist^n of H_1. (This embedding of the

$$E(S_n' - S_n) = \lim_{N\to\infty} \int_0^N \frac{dv}{\pi} \left\{ \frac{b(v-a)+ab}{b^2+(a-v)^2} - \frac{b(a+v)-ba}{b^2+(a+v)^2} \right\}$$

$$= \lim_{N\to\infty} \frac{1}{\pi} \left[\frac{b}{2}\log\{b^2+(v-a)^2\} - \frac{b}{2}\log\{b^2+(v+a)^2\} + a\tan^{-1}\left(\frac{v-a}{b}\right) + a\tan^{-1}\left(\frac{v+a}{b}\right) \right]_0^N$$

$$= a$$

So in fact the difference $S_n' - S_n$ is indeed a martingale here.

rws in a pair of reflection-coupling BMs works also for a Gaussian law, or a 2-sided exponential, but doesn't go a lot further.)

The interpretation in terms of two-dimensional BM gives

$$P(S_n'-S_n>0) = \int_0^\infty \frac{dv}{\pi}\left\{\frac{b}{b^2+(a-v)^2} - \frac{b}{b^2+(a+v)^2}\right\} \qquad b \equiv 2n$$

$$= \frac{2}{\pi}\tan^{-1}(a/2n)$$

$$\sim \frac{a}{n\pi} \qquad (n\to\infty)$$

This is a faster rate of coupling than we'd get for Gaussian steps, say, where the rate would be $O(n^{-1/2})$.

(vi) <u>How necessary is the symmetry assumption?</u> Suppose we only require F to be continuous unimodal, and now consider for fixed $a>0$, for each $t>0$ what are the values $(x, x+t)$ that the Rogers coupling charges. We have

$$F(x+t)-F(x) = F(x+t-a) - F(x-a)$$

$$\therefore \quad -F(x+t-a) + F(x+t) = F(x) - F(x-a)$$

This uniquely defines the x-value, $\xi(t,a)$. We have ξ is decreasing in its first argument, and $t+\xi(t,a)$ is increasing in t. We also have

$$\xi(t,a)+t = \xi(a,t)+a.$$

Now $P(Y-X>s) = P(X<\xi(s,a))$, so

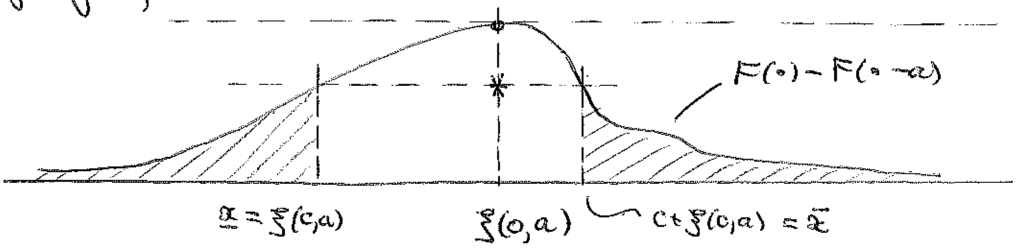
$$E\,(Y-X-c)^+ = \int_c^\infty p(X<\xi(s,a))\,ds$$

$$= \int_c^\infty P[\,Y>\xi(s,a)+s\,]\,ds$$

$$= \int_c^\infty P\left(X>\xi(s,a)+s-a\right)ds$$

$$= \int_c^\infty P(X>\xi(a,s))\,ds.$$

When is this convex in a? The answer is, "Always!!" Let's see why. For separation $a>0$, we have common mass $\int\{f(x)\wedge f(x-a)\}\,dx$, leaving $\delta(a)\equiv \int\{f(x)-f(x-a)\}^+\,dx$ unaccounted for. If we have chosen X, the Y value is a function of X, the function being picked to give Y the correct law. If you think about what is going on,



we see it's described by this picture; the missing mass is exactly the highest point $F(\xi(0,a))-F(\xi(0,a)-a) \equiv \delta(a)$ of the function, and if we pick a point U uniformly from the interval $(0, \delta(a))$, then choose $X < \xi(0,a)$ so that $F(X)-F(X-a) = U$, and $Y > \xi(0,a)$ so that $F(Y)-F(Y-a) = U$, this describes how we generate the law of (X,Y) on the event that $\{X \neq Y\}$. Thus a moment's thought will show that for $c>0$

$$\psi(c,a) \equiv E\,(Y-X-c)^+$$

$$= \int_{-\infty}^{\underline{x}} \{F(v)-F(v-a)\}\,dv + \int_{\bar{x}}^{\infty} \{F(v)-F(v-a)\}\,dv,$$

the shaded area of the picture. Here, $\underline{x}$ and $\bar{x}$ are chosen to be exactly c apart, which means that $\underline{x} = \xi(c,a)$. Thus

$$\psi(c,a) = \int_{\xi(c,a)-a}^{\xi(c,a)} F(v)\,dv - \int_{\xi(c,a)+c-a}^{\xi(c,a)+c} F(v)\,dv + a$$

$$= \int_{\xi(a,c)-c}^{\xi(a,c)+a-c} F(v)\,dv - \int_{\xi(a,c)}^{\xi(a,c)+a} F(v)\,dv + a$$

$$= \psi(a,c) + \int_{\xi(a,c)}^{\xi(a,c)+a-c} F(v)\,dv - \int_{\xi(a,c)}^{\xi(a,c)+a-c} F(v)\,dv + a - c$$

$$= \psi(a,c) + a - c.$$

But this surprising fact is enough to complete the story, since $\psi(c,a)$ is plainly convex in c!

So the conclusion is that the earlier results proved for symmetric unimodal distributions is even true without the symmetry assumption – for *any* unimodal distribution, the Rogers coupling provides the fastest possible coupling!!

<u>Optimal portfolios with bonds: Vasicek case</u> (6/12/95)

(i) Suppose an agent may invest in n stocks and a bond; for the stocks,

$$dS_t^i = S_t^i \{ \sigma_{ij}\, dW_t^j + \mu_i\, dt \} \qquad i=1,\dots,n$$

where W is n-dimensional for simplicity, and let the spot-rate process follow

$$dr_t = c \cdot dW_t + \varepsilon\, dB_t' + \beta'(m' - r_t)dt,$$

for positive constants ε, β', m', and an independent Brownian motion B'. The goal of the agent is to solve one of the optimisation problems:

(I) $\max \{ E\, U(x_T) \mid x_0 = x \}$

(II) $\max \{ E \int_0^\infty e^{-\rho t} U(c_t) dt \mid x_0 = x \}$,

where U will soon be CRRA! The story is well known; for (I), the optimal wealth process must be $x_T^* = I(\lambda \xi_T)$, $I \equiv (U')^{-1}$, and for (II) the optimal consumption stream is $c_t^* = I(\lambda e^{\rho t} \xi_t)$, with ξ the state-price density. So we shall have to identify and analyse ξ.

(ii) The change-of-measure martingale Z should solve (assuming const risk pr. for B)

$$dZ_t = Z_t \{ -\sigma^{-1}(\mu - r_t 1) \cdot dW_t - \gamma\, dB_t' \}$$

so that $\quad dW_t = dW_t^* - \sigma^{-1}(\mu - r_t 1)\, dt$; under P^*, W^* is a BM, and so

$$dr_t = c \cdot dW_t^* + \varepsilon\, dB_t^* + [\, \beta' m' - \varepsilon\gamma - c\cdot\sigma^{-1}\mu + c\cdot\sigma^{-1} 1\, r_t - \beta' r_t \,]\, dt$$

$$= c \cdot dW_t^* + \varepsilon\, dB_t^* + \beta(m - r_t)dt,$$

where $\beta = \beta' - c \cdot \sigma^{-1} 1$, $\beta m = \beta' m' - \varepsilon\gamma - c\cdot\sigma^{-1}\mu$. Thus under P^*, we have that r is a standard Vasicek process, with volatility $(|c|^2 + \varepsilon^2)^{1/2} \equiv v$.

The bond prices are

$$P(t, t+\tau) = \exp[-A(\tau) - r_t B(\tau)],$$

where $B(\tau) \equiv (1 - e^{-\beta\tau})/\beta$, $\quad A'(\tau) = m\beta\, B(\tau) - \frac{v^2}{2} B(\tau)^2$, $\quad A(0) = 0$.

(iii) State-price density?

$$\xi_t = \exp\left[-\int_0^t r_s\,ds - \int_0^t \sigma^{-1}(\mu - r_s 1)\cdot dW_s - \int_0^t \gamma \cdot dB_s' - \tfrac{1}{2}\gamma^2 t - \tfrac{1}{2}\int_0^t |\sigma^{-1}(\mu - r_s 1)|^2 ds \right].$$

It must be that for any a fixed,

$$E\left[\xi_t^a \,\middle|\, r_0 \right] = \varphi(t, r_0 ; a)$$

for a function φ yet to be discovered. We notice that

$$M_t \equiv E\left[\xi_T^a \,\middle|\, \mathcal{F}_t \right] = \xi_t^a \, \varphi(T-t, r_t)$$

so by Itô's formula

$$dM_t \doteq \xi_t^a \varphi(T-t, r_t)\Big[-a r_t + \tfrac{1}{2} a(a-1)\{\gamma^2 + |\sigma^{-1}(\mu - r_t 1)|^2\} - \frac{\dot\varphi}{\varphi} + \frac{\varphi'}{\varphi}\beta'(m' - r_t) +$$

$$+ \frac{\varphi''}{2\varphi}(|c|^2 + \varepsilon^2)\Big] dt + \xi_t^a \, \varphi' \left(-a\varepsilon\gamma - a\,\sigma^{-1}(\mu - r_t 1)\cdot c \right) dt$$

which allows us to find the equation for φ:

$$0 = \left(\frac{a(a-1)}{2}\{\gamma^2 + |\sigma^{-1}(\mu - r1)|^2\} - ar \right)\varphi - \dot\varphi + \beta'(m' - r)\varphi' + \tfrac{1}{2}\varphi'' v^2$$

$$- a\left(\varepsilon\gamma + \sigma^{-1}(\mu - r1)\cdot c \right)\varphi'.$$

This will be solved by $\varphi(t, r) = \exp(-\tfrac{1}{2} q_t r^2 - b_t r - \theta_t)$, so we have from this equation that

$$\frac{a(a-1)}{2}|\sigma^{-1}1|^2 + \tfrac{1}{2}\dot q + \tfrac{1}{2}v^2 q^2 + q(\beta' - a\, c\cdot\sigma^{-1}1) = 0$$

$$\dot b + b(v^2 q + \beta' - a\, c\cdot\sigma^{-1}1) - a(a-1)(\sigma^{-1}1, \sigma^{-1}\mu) - a$$

$$- \beta' m' q + a(\varepsilon\gamma + c\cdot\sigma^{-1}\mu) q = 0$$

$$\dot\theta + \frac{a(a-1)}{2}(\gamma^2 + |\sigma^{-1}\mu|^2) - \beta' m' b + \tfrac{1}{2}v^2(b^2 - q)$$

$$+ a(\gamma\varepsilon + (\sigma^{-1}\mu, c))\, b = 0$$

For our purposes, we shall want to compute

$$E_t\left[\xi_T^{1 - 1/R} \right],$$

$B(\tau) \equiv (1 - e^{-\beta\tau})/\beta$, of course.

so we may (and shall) <u>assume that $a<1$.</u> Also, there is the possibility of an <u>ill-posed</u> optimisation problem (for example, if $c=0=\gamma$, and we have $R\in(0,1)$, then $E\,U(x^*_T)\propto E\,\xi_T^{1-1/R}$, and this could explode.) When we solve the Riccati equation for q, setting $q=\dot\psi/v^2\psi$, we obtain

$$\ddot\psi+\lambda\dot\psi+k\psi=0 \qquad\qquad \lambda\equiv 2(\beta'-a\,c\cdot\sigma\,1),\ k\equiv a(a-1)v^2|\sigma^{-1}1|^2$$

and for real roots we must have

$$\boxed{(\beta'-a\,c\cdot\sigma^{-1}1)^2\geq a(a-1)v^2\,|\sigma^{-1}1|^2.}$$

Notice that this will always be satisfied if $0\leq a\leq 1$ (which corresponds to $R\geq 1$); in general, we require this condition to hold. The solution then is

$$\psi(x)=e^{-\lambda x/2}\left\{\tfrac{\lambda}{2}\sinh\rho x+\rho\cosh\rho x\right\} \qquad\qquad \left(2\rho\equiv\sqrt{\lambda^2-4k}\right)$$

$$\boxed{q(x)=\frac{-k\sinh\rho x}{\rho\cosh\rho x+\frac{\lambda}{2}\sinh\rho x}}$$

Solving for b gives us

$$\boxed{b(x)=\frac{2\eta\rho\sinh\rho x+(2\nu k+\lambda\eta)(\cosh\rho x-1)}{2\rho^2\cosh\rho x+\lambda\rho\sinh\rho x}}$$

where

$$\boxed{\begin{aligned}\nu&\equiv\{a\,\varepsilon\gamma+a\,c\cdot\sigma^{-1}\mu-\beta'm'\}v^{-2}=\{(a-1)\beta'm'-a\beta m\}v^{-2}\\ \eta&\equiv a+a(a-1)(\sigma^{-1}1,\sigma^{-1}\mu)\end{aligned}}$$

(iv) We could proceed to find θ, but to decide the optimal portfolio we do not need this, at least up to proportionality. If we hold π_t in the stocks, y_t in the bond (which matures at $T^*>T$) then the wealth process x_t generated satisfies

$$\boxed{dx_t=r_tx_t\,dt+\pi_t\cdot\sigma\,dW^*_t-y_t\,B(\tau)(c\cdot dW^*_t+\varepsilon\,dB^*_t)}$$

where $\tau \equiv T^* - t$. If we now compute

$$d(\xi_t x_t) = \xi_t \Big[\{\sigma^T \pi_t - x_t \, \sigma^{-1}(\mu - r_t 1) - y_t \, B(\tau) c \} \, dW_t$$

$$- (\gamma x_t + \varepsilon y_t \, B(\tau)) \, dB'_t \Big],$$

we are in a position to identify the optimal investments π_t, y_t. Indeed, taking $a \equiv 1 - 1/R$, $I(x) = x^{-1/R}$, we have

$$M_t \equiv \xi_t x_t = E_t \left[\xi_T \, I(\lambda_T \xi_T) \right]$$

$$= \lambda_T^{-1/R} \; E_t \left[\xi_T^a \right]$$

$$= \lambda_T^{-1/R} \; \xi_t^a \; \varphi(T - t, r_t),$$

so that

$$d M_t = M_t \Big[-\{a \, \sigma^{-1}(\mu - r_t 1) + (b(T-t) + r_t q(T-t)) c \} \cdot dW_t$$

$$- (a\gamma + \varepsilon (b(T-t) + r_t q(T-t))) \; dB'_t \Big].$$

Comparing coefficients, after some algebra we end up with

$$\boxed{\begin{aligned} \frac{\pi_t}{x_t} &= \frac{(\sigma\sigma^T)^{-1}(\mu - r_t 1)}{R} - \frac{\gamma}{\varepsilon R} (\sigma^T)^{-1} c \\ \frac{y_t}{x_t} B(T^* - t) &= b(T-t) + r_t \, q(T-t) - \frac{\gamma}{\varepsilon R} \end{aligned}}$$

This is consistent with the Merton result.

Pricing convertibles. (3/1/96)

(i) Consider a model in which at time 0, N shares are sold at price S_0 each, and M convertibles are sold at price C_0 each. The value of the firm at time 0 is therefore

$$V_0 = NS_0 + MC_0 .$$

The return on the firm's assets is given by $\sigma dW_t + \mu dt$. The number of unconverted convertibles at time t is denoted by m_t. A dividend on the firm's assets is paid out at rate δ to existing shareholders. The holders of the convertibles receive interest at rate ρ on their initial investment, up until the time at which they convert; at conversion, in return for the strike price K, the convertible becomes a share. The value of the firm's assets must satisfy

$$dV_t = -\delta V_t\, dt - \rho C_0\, m_t\, dt + V_t\,(\sigma dW_t + r\, dt) - K\, dm_t$$

(We are working throughout with a risk-neutral setting). The share price S_t must satisfy

$$dS_t = -\frac{\delta V_t}{N+M-m_t}\, dt + r S_t\, dt + dmg,$$

and the convertible price C_t must satisfy

$$dC_t = -\rho C_0\, dt + r C_t\, dt + dmg.$$

Clearly

$$S_t = S(t, V_t, m_t), \qquad C_t = C(t, V_t, m_t),$$

so we deduce from the two equations for dS and for dC that

$$\boxed{\begin{aligned} &\dot{S} + \tfrac{1}{2}\sigma^2 V^2 S_{VV} + [(r-\delta)V - \rho C_0 m] S_V = rS - \frac{\delta V}{N+M-m} \\ &\dot{C} + \tfrac{1}{2}\sigma^2 V^2 C_{VV} + [(r-\delta)V - \rho C_0 m] C_V = rC - \rho C_0 \end{aligned}}$$

and

$$S_m = K S_V, \qquad C_m = K C_V \qquad \text{at exercise.}$$

(ii) This is not quite right as it stands, because if we allow all of the firm's assets to be put into the risky enterprise, then it is possible that the obligations to the holders of convertibles may not be met. If at time t there are m_t unexercised convertibles, the firm must hold

$$b_t \equiv \rho C_0 m_t \left(1 - e^{-r(T-t)}\right)/r$$

in the bank in order to guarantee the coupon payments to the convertibles (if no more were exercised). The 'free' assets $Y_t \equiv V_t - b_t$ may now be invested in the risky process, and dividends paid on them at rate δ. Thus

$$dV_t = r b_t\, dt + Y_t\{\sigma dW_t + (r-\delta)\, dt\} - \rho C_0 m_t\, dt - K\, dm_t$$

and so

$$\boxed{dY_t = Y_t\{\sigma dW_t + (r-\delta)\, dt\} - \varphi(t)\, dm_t\,,} \qquad \varphi(t) \equiv K + \frac{\rho C_0}{r}\left(1 - e^{-r(T-t)}\right).$$

If we let $S_t = S(t, Y_t, m_t)$, $C_t = C(t, Y_t, m_t)$, then as before

$$\boxed{\begin{aligned} &\dot S + \tfrac{1}{2}\sigma^2 y^2 S_{yy} + (r-\delta) y S_y = rS - \frac{\delta y}{N+M-m} \\ &\dot C + \tfrac{1}{2}\sigma^2 y^2 C_{yy} + (r-\delta) y C_y = rC - \rho C_0 \end{aligned}}$$

with the boundary conditions: $\boxed{-\varphi(t) S_y + S_m = 0 = -\varphi(t) C_y + C_m}$
on the exercise boundary.

While the PDE is quite simple, it is the region which is not. If we let

$$\Omega_0 = \{(t, Y, m);\ 0 \le t \le T,\ 0 < m \le M,\ 0 < Y \le f(t,m)\},$$

then (t, Y_t, m_t) must always lie in

$$\Omega \equiv \Omega_0 \cup \{(t, Y, 0);\ 0 \le t \le T,\ 0 < Y\}.$$

The boundary function f is unknown and must be determined. On $\Omega \setminus \Omega_0$, the function S is known explicitly; in fact,

$$S(t, Y, 0) = Y/(N+M).$$

We also have that at time T, if $m > 0$ then $Y \le (N+M-m)K$, otherwise

We find
$$S(y,m) = \frac{y}{N+M-m} - \theta\, m \left(\frac{y}{N+M-m}\right)^{\nu}$$

$$C(y,m) = \frac{a}{r} + (N+M-m)\,\theta \left(\frac{y}{N+M-m}\right)^{\nu}$$

$$\theta \equiv \frac{1}{\nu}\left(\frac{\nu-1}{\nu b}\right)^{\nu-1} (N+M)^{-1}$$

There's a paper "Warrant valuation + exercise strategy" by David C. Emanuel J. Fin. Econ. 12 211–235, 1983, which solves the pricing problem assuming that $a=0$.

the convertibles would be valuable and would therefore be exercised immediately.

(iii) It seems we can get the prices for a <u>perpetual convertible</u> no trouble!

Suppose that the firm pays out at rate a for each convertible upto conversion, and suppose that $T=\infty$, so that there is no dependence on t in the expression for price of share and price of convertible;

$$S_t = S(Y_t, m_t) \quad , \quad C_t = C(Y_t, m_t),$$

and the DEs for S, C are

$$\boxed{\begin{array}{l} \mathcal{L}S = -\dfrac{\delta y}{N+M-m} \quad , \quad \mathcal{L}C = -a \\[2ex] \left.\begin{array}{c} -(K+\frac{a}{r})S_y + S_m = 0 = -(K+\frac{a}{r})C_y + C_m \\ C = S - K \end{array}\right\} \text{ at exercise} \end{array}}$$

with $\mathcal{L} \equiv -r + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2} + (r-\delta) y \frac{\partial}{\partial y}$. The exercise boundary is determined by $y = \eta(m)$, for some function η yet to be identified.

Abbreviate $\boxed{K + a/r \equiv b}$. It's clear that $S(y, m) = 0$ for $y=0$, $m>0$, that S should increase in y, and that conversion happens when $y \geq \eta(m)$.

We can actually solve the DEs for S, C;

$$\boxed{\begin{array}{l} S(y,m) = \dfrac{y}{N+M-m} - A(m)\, y^\nu \\[2ex] C(y,m) = \dfrac{a}{r} + B(m)\, y^\nu \end{array}}$$

where ν is the positive root of $\frac{1}{2}\sigma^2 x(x-1) + (r-\delta)x - r = 0$. Our boundary conditions give us

$$\boxed{\begin{array}{l} b + B(m)\,\eta(m)^\nu = \dfrac{\eta(m)}{N+M-m} - A(m)\,\eta(m)^\nu, \\[2ex] \dfrac{\eta}{(N+M-m)^2} - \eta^\nu A' = b\left\{\dfrac{1}{N+M-m} - \nu A \eta^{\nu-1}\right\}, \quad \eta B' = b\nu B \end{array}}$$

Substituting the last of these into the smooth pasting on S, we get

$$\eta^\nu \left[A' - \frac{B'A}{B} \right] = \left\{ \frac{\eta}{N+M-m} - b \right\} \frac{1}{N+M-m}$$

$$= (B+A)\,\eta^\nu . \frac{1}{N+M-m} \qquad \text{from the first}$$

whence

$$\frac{A'B - B'A}{B^2} = \left(1 + \frac{A}{B}\right) \frac{1}{N+M-m}$$

giving
$$\frac{A}{B} = \frac{\text{const}}{N+M-m} - 1 \; ;$$

since we know that $A(0) = 0$, we must have

$$\boxed{\frac{A(m)}{B(m)} = \frac{m}{N+M-m}.}$$

If we take

$$\boxed{\eta(m) = \frac{\nu b}{\nu - 1} (N+M-m),}$$

"Convert when free value per issued share gets to $\frac{\nu b}{\nu-1}$"

we find that the equations are solved by

$$\boxed{B(m) = \frac{(N+M-m)^{1-\nu}}{\nu(N+M)} \cdot \left(\frac{\nu b}{\nu-1}\right)^{1-\nu}}$$

[Check: We have $C_y = S_y$ at η, so the convex function $C+K-S$ cannot hit zero below the value $y = \eta(m)$.]

<u>Remarks</u> It's clear that we have to have $C + K \geq a/r + K \equiv b$, since we could always choose not to exercise and accept payment stream forever. Also, it's clear that $S \leq y/(N+M-m)$, which is the value of the shares if the shareholders were given the power to liquidate the firm at any time of their choosing. Thus at conversion we shall always have

$$\frac{\nu}{\nu-1} b = \frac{y}{N+M-m} \geq b .$$

(iv). We can also think of this another way. Suppose one person held all shares and all convertibles, and aimed to exercise in such a way as to maximise the current expected value of the future cash flow;

$$V(y,m) = \max E\left[\int_0^\infty a m_t e^{-rt} dt + \int_0^\infty e^{-rt} \delta Y_t \, dt + \int_0^\infty K e^{-rt} dm_t\right]$$

Thus we shall have that

$$V(Y_t, m_t) e^{-rt} + \int_0^t a m_s e^{-rs} ds + \int_0^t e^{-rs} \delta Y_s \, ds + \int_0^t K e^{-rs} dm_s$$

is a supermartingale, and a martingale under optimal play. The HJB is

$$-rV + \tfrac{1}{2}\sigma^2 y^2 V_{yy} + (r-\delta) y V_y + ma + \delta y \leq 0$$

and

$$-V_y \left(K + \tfrac{a}{r}\right) + V_m + K \leq 0$$

with equality at the conversion boundary. Now it seems reasonable to guess that

$$V(y,m) = V(y,m) \equiv m\, C(y,m) + (N+M-m)\, S(y,m) = y + ma/r$$

atleast in the no-conversion region, and if we happened to have $y > \frac{\nu}{\nu-1} b(N+M-m)$ we would instantly convert to get to the conversion boundary, so that for $y > \nu b(N+M-m)/(\nu-1)$ we have

$$V(y,m) = y + m\, a/r,$$

after a few calculations. This, of course, is a manifestation of the Modigliani-Miller theorem.

(v) An agent acting as a price-taker when the market prices of C and S are as computed above will optimally only convert when $C = S - K$, otherwise is throwing away value – so having all agents following this conversion policy is an equilibrium.

(vi) Suppose a monopolist holds all the convertibles, and $n < N$ of the shares. Then it is not too hard to see that in order to maximise the value of what he holds, he will follow exactly the same exercise policy as outlined above.

Incomplete markets: some examples (12/1/96)

(i) Take an agent with utility $U(x) = -\alpha^{-1} e^{-\alpha x}$ trading in a standard Brownian world $dS_t = S_t(\sigma dW_t + \mu dt)$ (one-dimensional for simplicity), constant interest rate r. For terminal consumption,

$$\min \; E \exp - \alpha x_T \quad \text{subj to} \quad x_0 \text{ given}$$

is achieved by $x_T = \tilde{\mu}(W_T + \tilde{\mu} T)/\alpha$, and $\qquad \tilde{\mu} \equiv \sigma^{-1}(\mu - r)$.

$$\min \; E \exp - \alpha x_T = \exp\left(-\tilde{\mu}^2 T/2 - \alpha x_0 e^{rT}\right)$$

(ii) Suppose now that the terminal payoff is shifted by $Y = cW_T + \eta$, where $\eta \sim N(0, v)$ independent of W, revealed only at time T. Then

$$\min E \exp\{-\alpha x_T - \alpha(cW_T + \eta)\}$$

$$= \min e^{\alpha^2 v/2} E \exp(-\alpha x_T - \alpha c W_T)$$

$$= \exp\left[-\frac{\tilde{\mu}^2 T}{2} + \alpha c \tilde{\mu} T - \alpha x_0 e^{rT} + \tfrac{1}{2}\alpha^2 v\right],$$

achieved by $x_T = (\frac{\tilde{\mu}}{\alpha} - c)(W_T + \tilde{\mu} T) + x_0 e^{rT}$. This gives us the agent's ask and bid prices for Y:

$$\boxed{\begin{aligned} p_A(Y) &= -c\tilde{\mu} T e^{-rT} + \tfrac{1}{2}\alpha v e^{-rT} \\ p_B(Y) &= -c\tilde{\mu} T e^{-rT} - \tfrac{1}{2}\alpha v e^{-rT} \end{aligned}}$$

(iii) Suppose now that there is a BM z independent of W; we may not trade on z, but we may observe it in forming our hedging strategies. We can proceed with the various market completions in the usual way; if $\zeta_t = \exp[-\tilde{\mu} W_t - (r + \frac{1}{2}\tilde{\mu}^2)t]$, and $Z^\theta_t = \exp[\int_0^t \theta_s dz_s - \frac{1}{2}\int_0^t \theta_s^2 ds]$ then the Lagrangian form

$$E\left[U(x_T) - x_T \lambda_\theta \zeta_T Z^\theta_T\right]$$

is maximised over x_T (with λ_θ, θ fixed) at $x^\theta_T = I(\lambda_\theta Z^\theta_T \zeta_T)$. For each θ, we'll pick λ_θ so as to make

$$x_0 = E\left[\zeta_T Z^\theta_T \, x^\theta_T\right] = e^{-rT} E^\theta\left[x^\theta_T\right]$$

and this way we get (if x^* is the actual optimising wealth process)

$$E\, U(x^*_T) \leq E\, U(x^\theta_T).$$

In our example,

$$\alpha x^\theta_T = -\log \lambda_\theta + \tilde\mu W_T + (r + \tfrac{1}{2}\tilde\mu^2)T - \int_0^T \theta_s d\zeta_s + \tfrac{1}{2}\int_0^T \theta_s^2 ds$$

so $\quad -\log\lambda_\theta = \frac{1}{2}E^\theta\int_0^T \theta_s^2 ds + \frac{1}{2}\tilde\mu^2 T - rT \qquad$ if $x_0 = 0$

Thus $\quad E\, U(x^\theta_T) = -\alpha^{-1} E\, U'(x^\theta_T) = -\alpha^{-1}\lambda_\theta\, e^{-rT}$

$$= -\alpha^{-1}e^{-rT}\exp\left[-\tfrac{1}{2}E^\theta\int_0^T\theta_s^2 ds - \tfrac{1}{2}\tilde\mu^2 T + rT\right]$$

which is minimised at $\theta = 0$. So in __this__ situation, the agent will optimally behave as if ζ was not there!

(iv) Does anything change if we are going to receive $Y = aW_T + b\zeta_T$ at time T?

We now have

$$U'(x^\theta_T + aW_T + b\zeta_T) = \lambda_\theta\, \zeta_T\, Z^\theta_T$$

$$\therefore \quad \alpha x^\theta_T = -\alpha a W_T - \alpha b \zeta_T - \log\lambda_\theta - \log \zeta_T - \log Z^\theta_T,$$

and assuming $x_0 = 0$ we shall have

$$0 = E^\theta(\alpha x^\theta_T) = \alpha a\tilde\mu T - \alpha b E^\theta\left(\int_0^T \theta_s ds\right) - \log\lambda_\theta + (r + \tfrac{1}{2}\tilde\mu^2)T - \tilde\mu^2 T$$
$$- \tfrac{1}{2}E^\theta\left[\int_0^T \theta_s^2 ds\right]$$

$$\Rightarrow \log\lambda_\theta = \alpha a\tilde\mu T - \tfrac{1}{2}E^\theta\left[\int_0^T(\theta_s + \alpha b)^2 ds\right] + (r - \tfrac{1}{2}\tilde\mu^2)T + \tfrac{1}{2}\alpha^2 b^2 T$$

So

$$E\, U(x^\theta_T + Y) = -\alpha^{-1}\exp\left[\alpha a\tilde\mu T - \tfrac{1}{2}E^\theta\left\{\int_0^T(\theta_s + \alpha b)^2 ds\right\} - \tfrac{1}{2}\tilde\mu^2 T + \tfrac{1}{2}\alpha^2 b^2 T\right\}$$

which is minimised when we take $\boxed{\theta \equiv -\alpha b}$ and the resulting x^θ is indeed feasible. Hence if x_0 is general,

$$\boxed{\sup E\, U(x_T + aW_T + b\zeta_T) = -\alpha^{-1}\exp\left[\alpha a\tilde\mu T - \tfrac{1}{2}\tilde\mu^2 T + \tfrac{1}{2}\alpha^2 b^2 T - \alpha x_0 e^{rT}\right]}$$

This coincides __exactly__ with what would happen if ζ_T were suddenly revealed at time T, as in (ii).

Questions, problems etc.

1) While reading a ms of Rene Schilling, one comes across some discussion of 1-dim'l diffusions for which $P^x(X_t > x) = p \quad \forall x, \forall t > 0$. If $p \neq \frac{1}{2}$, then one can show that such things don't exist.

2) Joe Langsam asks: If you hedge a call using an incorrect value of σ, what do you get? Can one characterise processes which can be the 3-month rate?

3) Robin Brenner asks: the trinomial tree model is incomplete, yet the Brownian limit is not – how can this be explained? Worth observing that in some sense the limit of a trinomial tree supports two independent BMs, not one!!

4) Visits to the bank? Suppose you have some consumption stream $(C_t)_{t \geq 0}$ which has to be paid for in cash, and suppose you can choose times of visits to bank and amounts of withdrawals. Each visit costs you $f(a)$, if a is the amount withdrawn. Money in bank grows at rate r. What is your best schedule for visits to bank?

5) Can one construct a local time on the set of t for which $\limsup_{h \downarrow 0} \frac{B_{t+h} - B_t}{(2h \log\log 1/h)^{1/2}} = +\infty$? Or where the Lévy modulus of continuity is violated? If so, what about an excursion story for such things? Alexander Scheid(?) in Berlin (a student of Hans) is looking at a LDP for paths analogous to the Strassen law

6) John Toland asks the following, which is apparently a tough conjecture. Suppose $D = \{ x \in \mathbb{R}^n : x_n \geq 0 \}$ and that $u : D \to \mathbb{R}$ satisfies

(i) u is positive in D° and bounded

(ii) $u = 0$ on ∂D

(iii) $\Delta u + f(u) = 0$,

where f is a function assumed to be C^1, but no more. The conjecture is that then $u(x)$ depends only on x_n.

7) Edward Fraenkel says the following is related and also tough. If $D \subset \mathbb{R}^n$ is bounded open, μ is Lebesgue measure on D, and the potential $G\mu$ of μ is constant on ∂D, then D is a ball.

If $dX_t = \sigma\sqrt{X_t}\,dW_t + \beta(\mu - X_t)\,dt$, then

$$X_t = e^{-\beta t}\,\xi\!\left(\frac{\sigma^2}{4\beta}(e^{\beta t}-1)\right) \qquad \xi \text{ is a BESQ}\left(\frac{4\beta\mu}{\sigma^2}\right) \text{ process.}$$

The BESQ(δ) transition density is

$$q_t(x,y) = \frac{1}{2t}\exp\left(-\frac{x+y}{2t}\right)\left(\frac{y}{x}\right)^{\nu/2} I_\nu\left(\frac{\sqrt{xy}}{t}\right), \qquad \nu = \frac{\delta}{2} - 1 = \frac{2\beta\mu}{\sigma^2} - 1$$

So the transition density of X is

$$\frac{\beta\, e^{\beta^2\mu t/\sigma^2}}{\sigma^2 \sinh \beta t}\,(xy)^{-\nu/2}\exp\left(-\frac{2\beta}{\sigma^2}\,\frac{x+y}{e^{\beta t}-1}\right) I_\nu\left(\frac{2\beta}{\sigma^2}\,\frac{\sqrt{xe\,y}}{e^{\beta t}-1}\right)$$

relative to the invariant measure

$$y^\nu \exp(-2\beta y/\sigma^2)\,dy$$