Approximate Kalman filtering of diffusions (31/1/96)

All of this must be well known, but it seemed quicker just to work it out. The idea is that we have some diffusion $dX_t = \sigma(X_t)\,dW_t + b(X_t)\,dt$ in $\mathbb{R}^d$, which we observe at the times $(j\delta)_{j\in\mathbb{Z}^+}$, but the observations are partial + noisy, in that we actually see $Y_n \equiv f(X_{n\delta}) + \eta_n$ where $f:\mathbb{R}^d \to \mathbb{R}^m$ is smooth. Suppose that after observing Y_0 we believe that $X \sim N(\mu_0, V_0)$.

(i) Approximate updating: we have

$$X_t - X_0 = \int_0^t \sigma(X_s)\,dW_s + \int_0^t b(X_s)\,ds$$

$$= \int_0^t \{\sigma(X_0) + (X_s - X_0)\,D\sigma(X_0)\}\,dW_s + b(X_0)\,t + o(t)$$

$$\therefore\ X_t^i - X_0^i = \sigma^i_j(X_0)\,W_t^j + \sigma^k_\ell(X_0)\,D_k\sigma^i_j(X_0)\int_0^t W_s^\ell\,dW_s^j + b^i(X_0)\,t + o(t)$$

Thus $\boxed{X_t - X_0 \sim N\big(t\,b(X_0),\ \Sigma_t(X_0)\big)}$ to first order, where

$$\Sigma_t(X_0)_{ir} = t\,\sigma\sigma^T(X_0)_{ir} + \frac{t^2}{2}\,(\sigma^k_\ell D_k \sigma^i_j)(\sigma^s_\ell D_s \sigma^r_j)(X_0).$$

Oversimplifying, we may take as a working assumption that

$$\boxed{X_{n\delta+\delta} - X_{n\delta} \sim N\big(\delta\, b(\mu_n),\ \Sigma_\delta(\mu_n)\big)}$$

(ii) If we assume the observational errors η are IID $N(0, V_\eta)$, then given $\mathcal{F}_n$, the joint density of $(X_{n\delta}, X_{(n+1)\delta}, Y_{n+1})$ is proportional to

$$\exp\Big[-\tfrac12(x_n - \mu_n)\cdot V_n^{-1}(x_n - \mu_n) - \tfrac12\big(f(x_{n+1}) - y_{n+1}\big)\cdot V_\eta^{-1}\big(f(x_{n+1}) - y_{n+1}\big)$$
$$- \tfrac12\big(x_{n+1} - x_n - \delta\, b(\mu_n)\big)\cdot \Sigma_\delta(\mu_n)^{-1}\big(x_{n+1} - x_n - \delta\, b(\mu_n)\big)\Big]$$

If we now integrate out x_n, we get the dependence on x_{n+1}, y_{n+1} is

$$\boxed{\exp\Big[-\tfrac12\big(f(x_{n+1}) - y_{n+1}\big)\cdot V_\eta^{-1}\big(f(x_{n+1}) - y_{n+1}\big) - \tfrac12\big(x_{n+1} - \delta\, b(\mu_n) - \mu_n\big)\cdot\big(V_n + \Sigma_\delta(\mu_n)\big)^{-1}\big(x_{n+1} - \delta b(\mu_n) - \mu_n\big)\Big]}$$

This suggests an approximate recursive procedure. Firstly, having observed y_{n+1}, we numerically

$$\min_{x} \tfrac{1}{2}\left(f(x)-y_{n+1}\right)\cdot V_\eta^{-1}\left(f(x)-y_{n+1}\right) + \tfrac{1}{2}(x-c_n)\left(V_n + \Sigma_\delta(\mu_n)\right)^{-1}(x-c_n)$$

where $c_n \equiv \mu_n + \delta\, b(\mu_n)$. Then we take the minimising value of x to be the (new) μ_{n+1}, and the second derivative of the function at x to be the (new) V_{n+1}^{-1}.

<u>A conjecture on quantiles (4/2/96)</u>

(i) Take some one-dimensional diffusion X, let $A(t,x) \equiv \int_0^t I_{\{X_s \le x\}}\, ds$ and let $M(t,x) \equiv \inf\{y : A(t,y) > x\}$. By familiar excursion arguments, if the diffusion starts at x_0 and $a < x_0$, we get

$$(1) \quad E^{x_0}\left[1 - e^{-\alpha A(T,a)}\right] = \frac{\psi_\lambda^-(x_0)}{\psi_\lambda^-(a)} \cdot \frac{\alpha}{\alpha+\lambda} \cdot \frac{\nu_{\lambda+\alpha}^-(a)}{\nu_{\lambda+\alpha}^-(a) + \nu_\lambda^+(a)},$$

where $T \sim \exp(\lambda)$, and $\nu_\lambda^+(a) = -D\psi_\lambda^-(a)/\psi_\lambda^-(a)$, $\nu_\lambda^-(a) = D\psi_\lambda^+(a)/\psi_\lambda^+(a)$, which are the excursion rates of marked excursions above and below a. Now we may rework the LHS as

$$\int_0^\infty \lambda e^{-\lambda t}\, dt \int_0^\infty \alpha e^{-\alpha s}\, P^{x_0}\left[s \le A(t,a)\right] ds$$

$$= \int_0^\infty \alpha e^{-\alpha s}\, ds \int_s^\infty \lambda e^{-\lambda t}\, dt\ P^{x_0}\left[M(t,s) \le a\right]$$

$$(2) \quad = \int_0^\infty \alpha e^{-(\alpha+\lambda)s} ds \int_0^\infty \lambda e^{-\lambda u}\, du\ P^{x_0}\left[M(s+u,s) \le a\right].$$

In the case where the underlying diffusion is a drifting Brownian motion, it is the known result that

$$(3) \quad P^{x_0}\left[M(s+u,s) \le a\right] = \int_{-\infty}^a P^{x_0}\left[\inf_{0\le t\le u} X_t \in dy\right] P^y\left[\sup_{0\le t\le s} X_t \le a\right];$$

(ii) <u>Could it possibly be that this is true for more general diffusions?</u> If this were true, we would have the two expressions for

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$$\psi_\lambda^-(x) = \frac{1}{\theta x} e^{-\theta x}, \quad \psi_\mu^+(x) = \frac{\sinh \gamma x}{\gamma x}$$

$$\int_0^\infty \mu e^{-\mu s} ds \int_0^\infty \lambda e^{-\lambda u} du \; P^{x_0}\left[M(s+u, s) \leq a \right]$$

$$= \frac{\psi_\lambda^-(x_0)}{\psi_\lambda^-(a)} \frac{\nu_\mu^-(a)}{\nu_\mu^-(a) + \nu_\lambda^+(a)} \qquad (a < x_0)$$

$$\stackrel{?}{=} \int_{-\infty}^a P^{x_0}\left[\underline{X}_{T(\lambda)} \in dy \right] P^y\left[\bar{X}_{T(\mu)} \leq a \right]$$

$$= \int_{-\infty}^a \frac{-\psi_\lambda^-(x_0)\, D\psi_\lambda^-(y)}{\psi_\lambda^-(y)^2} \frac{\psi_\mu^+(y)}{\psi_\mu^+(a)} dy$$

$$= \frac{\psi_\lambda^-(x_0)}{\psi_\mu^+(a)} \left\{ \frac{\psi_\mu^+(a)}{\psi_\lambda^-(a)} - \int_{-\infty}^a \frac{D\psi_\mu^+(y)}{\psi_\lambda^-(y)} dy \right\} \qquad \text{assuming } \frac{\psi_\mu^+}{\psi_\lambda^-}(-\infty) = 0$$

This holds iff

$$(4) \quad \int_{-\infty}^a \frac{D\psi_\mu^+(y)}{\psi_\lambda^-(y)} dy = \frac{\psi_\mu^+(a)}{\psi_\lambda^-(a)} \cdot \frac{\nu_\lambda^+(a)}{\nu_\lambda^+(a) + \nu_\mu^-(a)}$$

Notice that

$$\frac{d}{da} \log \frac{\psi_\mu^+(a)}{\psi_\lambda^-(a)} = \nu_\lambda^+(a) + \nu_\mu^-(a) \equiv f(a), \text{ say.}$$

Thus, to within constants, we'd have to have

$$\int_{-\infty}^a \frac{D\psi_\mu^+}{\psi_\lambda^-}(y)\, dy = \frac{1}{f(a)} \exp\left[\int^a f(x) dx \right] \nu_\lambda^+(a).$$

For a BES(3) process, with $\gamma = \sqrt{2\mu}$, $\theta = \sqrt{2\lambda}$, we'd have to have

$$\int_0^a \frac{\theta e^{-\theta y}}{\gamma y} dy \left[\gamma y \cosh \gamma y - \sinh \gamma y \right] dy = \frac{\theta}{\gamma} e^{\theta a} \frac{\sinh \gamma a}{a} \frac{1 + \theta a}{\theta + \gamma \coth \gamma a}$$

If you let $\theta \to 0$ in this, differentiate the right-hand side and let $a \downarrow 0$, then you get limit 0 on the left, limit 1 on the right:

$$\boxed{\text{so the conjecture is clearly false!}}$$

Split integral according to $s \gtrless t-u$.

(iii) If we fix some $b > a$ and investigate

$$E^x\left[e^{-\mu A(T,a)}\left(1-e^{-\nu(T-A(T,b))}\right)\right] = P^x\left[\nu\text{-killed before } \mu \text{ or } \lambda\text{-killed}\right]$$

where ν-killing happens only when the diffusion is above b, μ-killing only when below a, and λ-killing all the time, we can proceed by familiar excursion analysis. For $x \leq b$ we'll define

$$\varphi(x) = \begin{cases} \psi^+_{\lambda+\mu}(x), & x \leq a \\ k_1 \psi^+_\lambda(x) + k_2 \psi^-_\lambda(x), & a \leq x \leq b \end{cases}$$

so that φ is C^1. We get for $a \leq x \leq b$

$$\varphi(x) = c_\lambda\left[\psi^+_\lambda(x)\left\{\psi^-_\lambda(a) D\psi^+_{\lambda+\mu}(a) - \psi^+_{\lambda+\mu}(a) D\psi^-_\lambda(a)\right\} + \psi^-_\lambda(x)\left\{\psi^+_{\lambda+\mu}(a) D\psi^+_\lambda(a) - \psi^+_\lambda(a) D\psi^+_{\lambda+\mu}(a)\right\}\right]$$

where $c_\lambda^{-1} \equiv \psi^-_\lambda(a) D\psi^+_\lambda(a) - \psi^+_\lambda(a) D\psi^-_\lambda(a)$. φ is increasing. Thus for $x \leq b$,

$$E^x\left[e^{-\mu A(T,a)}\left\{1-e^{-\nu(T-A(T,b))}\right\}\right]$$

$$= \frac{\varphi(x)}{\varphi(b)} \cdot \frac{\nu}{\nu+\lambda} \cdot \frac{-D\log\psi^-_{\lambda+\nu}(b)}{-D\log\psi^-_{\lambda+\nu}(b) + D\log\varphi(b)} \qquad (*)$$

$$= \int_0^\infty \lambda e^{-\lambda t}\,dt \int_0^\infty \mu e^{-\mu s}\,ds \int_0^\infty \nu e^{-\nu u}\,du \; P\left[s > A(t,a),\; t - A(t,b) \geq u\right]$$

$$= \int_0^\infty \nu e^{-\nu u}\,du \int_0^\infty \mu e^{-\mu s}\,ds \int_u^\infty \lambda e^{-\lambda t}\,dt \; P\left[s > A(t,a),\; t-u \geq A(t,b)\right]$$

$$= \int_0^\infty \nu e^{-\nu u}\,du \int_u^\infty \lambda e^{-\lambda t}\,dt \; e^{-\mu(t-u)} \; P\left[t-u \geq A(t,b)\right]$$

$$+ \int_0^\infty \nu e^{-\nu u}\,du \int_u^\infty \lambda e^{-\lambda t}\,dt \int_0^{t-u} \mu e^{-\mu s}\,ds \; P\left[M(t,s) > a,\; M(t,t-u) > b\right]$$

$$= \int_0^\infty \nu e^{-(\nu+\lambda)u}\,du \int_0^\infty \lambda e^{-(\lambda+\mu)v}\,dv \; P\left[M(u+v,v) \geq b\right]$$

$$+ \int_0^\infty \nu e^{-(\nu+\lambda)u}\,du \int_0^\infty \mu e^{-(\mu+\lambda)s}\,ds \int_0^\infty \lambda e^{-\lambda y}\,dy \; P\left[M(u+s+y,s) > a,\; M(u+s+y,s+y) > b\right].$$

The first integral is

$$\frac{\nu}{\lambda+\nu}\cdot\frac{\lambda}{\lambda+\mu} - \int_0^\infty \nu e^{-(\nu+\lambda)u}\,du\int_0^\infty \lambda e^{-(\lambda+\mu)v}\,dv\; P\big[M(u+v,v)\le b\big]$$

$$= \frac{\nu\lambda}{(\lambda+\nu)(\lambda+\mu)}\cdot\frac{\psi^+_{\lambda+\mu}(x)}{\psi^+_{\lambda+\mu}(b)}\cdot\frac{-D\log\psi^-_{\lambda+\nu}(b)}{-D\log\psi^-_{\lambda+\nu}(b)+D\log\psi^+_{\lambda+\mu}(b)}$$

after some calculations. Thus for $a<x<b$

$$\int_0^\infty(\nu+\lambda)e^{-(\nu+\lambda)u}\,du\int_0^\infty \mu e^{-(\mu+\lambda)s}\,ds\int_0^\infty \lambda e^{-\lambda y}\,dy\; P^x\big[M(u+s+y,s)>a,\ M(u+s+y,s+y)>b\big]$$

$$= \frac{\varphi(x)}{\varphi(b)}\,\frac{-D\log\psi^-_{\lambda+\nu}(b)}{-D\log\psi^-_{\lambda+\nu}(b)+D\log\varphi(b)} - \frac{\lambda}{\lambda+\mu}\,\frac{\psi^+_{\lambda+\mu}(x)}{\psi^+_{\lambda+\mu}(b)}\cdot\frac{-D\log\psi^-_{\lambda+\nu}(b)}{-D\log\psi^-_{\lambda+\nu}(b)+D\log\psi^+_{\lambda+\mu}(b)}$$

In the special case of BM with drift c, we have

$$\psi^+_\lambda(x)=e^{\alpha x},\quad \psi^+_{\lambda+\mu}(x)=e^{\tilde\alpha x} \qquad \alpha=\sqrt{c^2+2\lambda}-c, \text{ etc.}$$

$$\psi^-_\lambda(x)=e^{-\beta x},\quad \psi^-_{\lambda+\nu}(x)=e^{-\tilde\beta x}, \qquad \beta=\sqrt{c^2+2\lambda}+c$$

so $C_\lambda^{-1}=(\alpha+\beta)\,e^{-2ca}$ and so after some calculations, for $a\le x\le b$

$$\varphi(x)=\frac{e^{\tilde\alpha a}}{\alpha+\beta}\Big\{e^{\alpha(x-a)}(\tilde\alpha+\beta)+e^{-\beta(x-a)}(\alpha-\tilde\alpha)\Big\},$$

so the RHS of the box above is

$$\frac{\varphi(x)}{\varphi(b)}\;\frac{\tilde\beta\{e^{\alpha\delta}(\tilde\alpha+\beta)-e^{-\beta\delta}(\tilde\alpha-\alpha)\}}{\tilde\beta\{e^{\alpha\delta}(\tilde\alpha+\beta)-e^{-\beta\delta}(\tilde\alpha-\alpha)\}+\alpha(\tilde\alpha+\beta)e^{\alpha\delta}+\beta(\tilde\alpha-\alpha)e^{-\beta\delta}} - \frac{\lambda}{\lambda+\mu}\,e^{-\tilde\alpha(b-x)}\,\frac{\tilde\beta}{\tilde\beta+\tilde\alpha}$$

$(\delta\equiv b-a)$.

Returning to (*) for the special case of drifting Brownian motion, we obtain on substituting that

$$\boxed{\begin{aligned} &E^{0}\ e^{-\mu A(T,a)}\left[1-e^{-\nu(T-A(T,b))}\right] \\ &= \frac{\nu}{\nu+\lambda}\ \frac{\tilde{\beta}\left\{e^{-\alpha a}(\tilde{\alpha}+\beta)+e^{\beta a}(\alpha-\tilde{\alpha})\right\}}{(\alpha+\tilde{\beta})(\tilde{\alpha}+\beta)\,e^{\alpha\delta}+(\tilde{\beta}-\beta)(\alpha-\tilde{\alpha})\,e^{-\beta\delta}} \end{aligned}}$$

$\delta \equiv b-a$, $\alpha \equiv \sqrt{c^2+2\lambda}-c$, $\tilde{\alpha} \equiv \sqrt{c^2+2\lambda+2\mu}-c$, $\beta \equiv \alpha+2c$, $\tilde{\beta} \equiv \sqrt{c^2+2\lambda+2\nu}+c$.

Another look at utility from possession (7/2/96)

Let's look again at an agent whose holding of some good at time t obeys

$$dx_t = -\delta x_t\, dt + l_t\{\sigma\, dW_t + \mu dt\}$$

where l_t is the rate of working at time t; the agent aims to

$$\max\ E \int_0^\infty e^{-\rho t}\ U(x_t, l_t)\, dt$$

where we'll require that x, l are non-negative. Assume also that $\sigma = 1$, as we may choose the scale of l to absorb this. We have

$$x_t = e^{-\delta t}(x_0 + \int_0^t l_s e^{\delta s}(dW_s + \mu ds))$$

So if we consider a feasible perturbation η to the optimal l, the FOC gives

$$0 \geq E \int_0^\infty e^{-\rho t}\Big[U_x(t)\, e^{-\delta t}\int_0^t \eta_s e^{\delta s}(dW_s + \mu ds) + U_l(t)\ \eta_t\Big]\, dt$$

where $U_x(t) = U_x(x_t^*, l_t^*)$, for example. Rearranging,

$$0 \geq E \int_0^\infty \eta_t \Big\{e^{-\rho t}U_l(t) + \mu \int_t^\infty U_x(s)\, e^{-\delta(s-t)-\rho s}\, ds\Big\}\, dt$$

$$+ E\Big[\int_0^\infty e^{-\rho t}U_x(t)\, e^{-\delta t}\, dt \cdot \int_0^\infty e^{\delta s}\eta_s\, dW_s\Big].$$

Now if we suppose that we have the integral representation

$$\int_0^\infty e^{-\rho t}\, U_x(t)\, e^{-\delta t}\, dt = a + \int_0^\infty \theta_u\, dW_u,$$

we have the useful form of the FOC:

$$0 \geq E \int_0^\infty e^{\delta t}\eta_t\Big\{\theta_t + \mu \int_t^\infty U_x(s)\, e^{-(\rho+\delta)s}\, ds + e^{-(\rho+\delta)t}\, U_l(t)\Big\}\, dt$$

Thus

$$\boxed{\theta_t + e^{-(\rho+\delta)}U_l(t) + \mu\ E\Big[\int_t^\infty U_x(s)\, e^{-(\rho+\delta)s}\, ds\Big] \leq 0}$$

with equality when $l_t^* > 0$. If we use the notation $M_t = a + \int_0^t \theta_u\, dW_u$, we therefore have

$$0 \geq \theta_t + e^{-(\rho+\delta)t} U_\ell(t) + \mu \left(M_t - \int_0^t U_x(s) e^{-(\rho+\delta)s} ds \right).$$

If we make the simplifying assumption that equality holds throughout, we find

$$d\theta_t = -\mu \theta_t \, dW_t + \mu U_x(t) e^{-(\rho+\delta)t} dt - d\left(e^{-(\rho+\delta)t} U_\ell(t)\right)$$

$$= -\mu \theta_t \, dW_t + dz_t,$$

say. Now if $Y_t \equiv \exp(-\mu W_t - \frac{1}{2}\mu^2 t)$, this is soluble in the usual way:

$$\theta_t = Y_t \left(\theta_0 + \int_0^t Y_s^{-1} dz_s \right)$$

$$= Y_t \Big(\theta_0 + \int_0^t Y_s^{-1} \mu U_x(s) e^{-(\rho+\delta)s} ds - Y_t^{-1} e^{-(\rho+\delta)t} U_\ell(t) + U_\ell(0)$$

$$+ \int_0^t e^{-(\rho+\delta)s} U_\ell(s) Y_s^{-1} (\mu \, dW_s + \mu^2 ds) \Big)$$

$$= Y_t \Big(-\mu a + \mu \int_0^t Y_s^{-1} e^{-(\rho+\delta)s} \left\{ U_x(s) ds + U_\ell(s)(dW_s + \mu ds) \right\}$$

$$- Y_t^{-1} e^{-(\rho+\delta)t} U_\ell(t) \Big)$$

Perpetual callable convertibles (13/2/96)

(a) We'll consider the following situation: N shares, M convertibles initially, and

(i) up to conversion or surrender each convertible receives a cash-flow $a \geq 0$. At conversion, the holder pays K and receives a share. At surrender, the holder receives K^*.

(ii) The firm is obliged to hold riskless bonds sufficient to meet the coupon payments of the live convertibles for ever. Thus if there are m_t live convertibles at time t, the value of the riskless bonds held will be $a m_t r^{-1}$, where r is the riskless rate. The value of the firm in excess of this can be invested in risky assets

(iii) If the value of the firm V_t is

$$V_t = Y_t + m_t \frac{a}{r},$$

then dividends are paid out to the shares at rate δY_t.

The process Y denotes the 'free value' of the firm, and we shall have

$$dY_t = Y_t\left(\sigma dW_t + (r-\delta)\, dt\right) - (K + \tfrac{a}{r})\, dm_t$$

If the share price is $S_t \equiv S(Y_t, m_t)$, convertible is $C(Y_t, m_t)$, then we have to have

$$\boxed{\begin{aligned} &\mathcal{L}S + \frac{\delta y}{N+M-m} = 0, \quad \mathcal{L}C + a = 0 \\ &\left.\begin{aligned} -(K+\tfrac{a}{r})S_y + S_m = 0 &= -(K+\tfrac{a}{r})C_y + C_m \\ C &= S - K \end{aligned}\right\} \text{ at conversion} \end{aligned}}$$

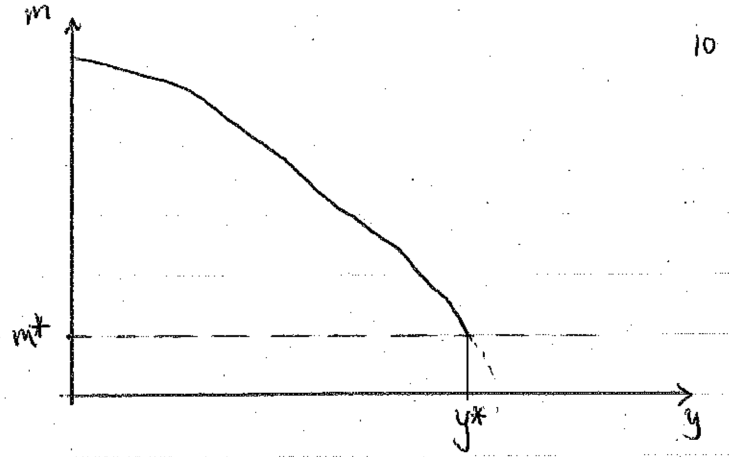
where $\mathcal{L} \equiv -r + \frac{1}{2}\sigma^2 y^2\, \partial^2/\partial y^2 + (r-\delta)\, y\, \partial/\partial y$.

Clearly

$$V_t = Y_t + m_t \frac{a}{r} = (N+M-m_t) S_t + m_t C_t.$$

(b) Let's suppose that the conversion boundary is at $y = \eta(m)$, so that we shall always have $y \leq \eta(m)$, at least while $m > 0$. As m decreases, η should increase.

Let's also suppose that we run until first $Y = y^*$, and then at that time the company calls all remaining convertibles.



Suppose that $y^* = y(m^*)$
Now if we've just reached $Y = y^*$, and the convertibles are called, suppose that $m' \leq m^*$ of the convertibles get surrendered, and the remaining $m^* - m'$ get converted. Then the share price after this will be

$$\frac{\left\{ y^* + m^* \frac{a}{r} - m' K^* + (m^* - m') K \right\}}{N + M - m'}$$

so the holder(s) of the convertibles will rather convert than surrender so long as

$$\frac{y^* + m^* \frac{a}{r} + m^* K - m'(K + K^*)}{N + M - m'} - K \geq K^*$$

ie

$$\boxed{y^* + m^* \frac{a}{r} + m^* K \geq (N+M)(K+K^*)}$$

So, in fact, all holders of convertibles will be indifferent between surrender and conversion if this inequality holds with equality. The firm would never call if K^* were greater than the market value of the convertible (they'd be handing over money if so) and because S and C are traded assets, it has to be that

$$\boxed{y^* + m^* \left(\frac{a}{r} + K \right) = (N+M)(K+K^*).}$$

The solution to the PDEs for S, C must be

$$S = \frac{y}{N+M-m} + \alpha(m)\, y^\nu \qquad \text{where } \tfrac{1}{2}\sigma^2 \nu(\nu-1) + (r-\delta)\nu - r = 0, \quad \nu > 1$$

$$C = \frac{a}{r} + \beta(m)\, y^\nu$$

for some functions α, β. As we did without the call feature, we can solve the boundary matching conditions together with the conditions at (y^*, m^*) to reach the conclusion that the solutions are <u>exactly the same as before</u>, that is,

$$\boxed{\begin{aligned} \eta(m) &= \frac{\nu}{\nu-1}\left(\frac{a}{r}+K\right)(N+M-m) \\ C(y,m) &= \frac{a}{r} + y^{\nu}\,\frac{(N+M-m)^{1-\nu}}{\nu(N+M)}\left(\frac{\nu b}{\nu-1}\right)^{1-\nu} \qquad \left(b \equiv \frac{a}{r}+K\right) \\ S(y,m) &= \frac{y}{N+M-m} - \frac{y^{\nu}\,m}{\nu(N+M)}\,(N+M-m)^{-\nu}\left(\frac{\nu b}{\nu-1}\right)^{1-\nu} \end{aligned}}$$

It follows easily that the critical values are

$$\boxed{\begin{aligned} m^* &= \frac{N+M}{b}\Big[\nu b - (\nu-1)(K+K^*)\Big]^+ \\ y^* &= \nu(N+M)\left(K^* - a/r\right) \end{aligned}}$$

So for it to be worth calling at all, we need $K^* \leq \frac{\nu}{\nu-1} b - K$. At call, $C = K^*$, and feasible values of K^* are

$$0 \leq K^* - \frac{a}{r} \leq \frac{b}{\nu-1}.$$

(c) <u>If we don't assume that the bonds are protected, how does it look?</u> This time, if V_t is the value of the firm's assets at time t, we shall have

$$dV_t = V_t\left(\sigma\, dW_t + (r-\delta)\,dt\right) - a\, m_t\, dt - K\, dm_t,$$

at least up to the moment when the company goes bust. Clearly we have

$$m_t\, C(V_t, m_t) + (N+M-m_t)\, S(V_t, m_t) = V_t.$$

Now suppose that ψ^+ (ψ^-) is the increasing (decreasing) solution to

$$\mathcal{G}\psi = r\psi \qquad\qquad \mathcal{G} \equiv \tfrac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + ((r-\delta)x - 1)\frac{d}{dx}$$

with $\psi^+(1) = 1$, $\psi^-(0) = 1$. Then for any $\lambda > 0$, $\psi^\pm(x/\lambda)$ solves

$$\tfrac{1}{2}\sigma^2 x^2 f''(x) + ((r-\delta)x - \lambda) f'(x) - r f(x) = 0,$$

If we take the "protected bonds" version, on the previous page, we also have
$C_y = (N+M)^{-1}$ at conversion; this is smooth pasting, of course

so we can express the share and convertible prices in these terms:

$$\boxed{\begin{aligned} C(v,m) &= \theta(m)\,\psi^+\!\left(\frac{v}{am}\right) + \frac{a}{r}\left(1-\psi^-\!\left(\frac{v}{am}\right)\right) \\ S(v,m) &= \frac{v}{N+M-m} - \frac{m}{N+M-m}\,C(v,m) \end{aligned}}$$

$$0 \le v \le \eta(m)$$

for some function θ. We have used the fact that $C(0+,m)=0$. The conditions to be satisfied at the conversion boundary are summarised as

$$\boxed{\begin{aligned} (N+M)C &= \left[\theta(m)\,\psi^+(\eta/am) + \frac{a}{r}\left(1-\psi^-(\eta/am)\right)\right](N+M) \\ &= \eta - K(N+M-m), \\ 0 &= KC_v - C_m \\ &= \left(K+\frac{\eta}{m}\right)\left\{\theta(m)\,D\psi^+\!\left(\frac{\eta}{am}\right) - \frac{a}{r}\,D\psi^-(\eta/am)\right\} - am\,\theta'(m)\,\psi^+(\eta/am) \end{aligned}}$$

Now if we differentiate the first one with respect to m and reorganise, we discover that

$$\theta'(m)\,\psi^+\!\left(\frac{\eta}{am}\right) = \frac{mK+\eta}{m(N+M)}$$

or equivalently

$$\boxed{C_v(\eta,m) \equiv \frac{1}{am}\left\{\theta(m)\,D\psi^+\!\left(\frac{\eta}{am}\right) - \frac{a}{r}\,D\psi^-\!\left(\frac{\eta}{am}\right)\right\} = \frac{1}{N+M}}$$

Can we describe the conversion boundary more explicitly? Well, if we fix a value p for $\eta(m)/m$, and write

$$\begin{pmatrix} \psi^+(p/a) & \psi^-(p/a) \\ D\psi^+(p/a) & D\psi^-(p/a) \end{pmatrix} \equiv \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

the conditions which have to be matched at conversion can be more neatly expressed as

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\begin{pmatrix} \theta \\ -a/r \end{pmatrix} = \frac{m}{N+M}\begin{pmatrix} K+p \\ a \end{pmatrix} - \begin{pmatrix} K+a/r \\ 0 \end{pmatrix}$$

We can similarly handle a situation where for some Mkv process X the spot rate is $r(X_t)$ and the coupons on the convertibles are at rate $\rho(X_t)$. We have

$$dC_t = dm_t + \{r(X_t)C_t - \rho(X_t)\}dt$$

and $dV_t = V_t\{\sigma dW_t + (r(X_t)-\delta)dt\} - m_t\rho(X_t)dt - K dm_t$

and $$C_t(N+M) \geq V_t - (N+M-m_t)K$$

This gives us that $e^{-R_t}C_t + \int_0^t \rho(X_s)e^{-R_s}ds$ is a martingale (at least up to conversion or the collapse of the firm), so if we have horizon T, and T^* denotes the time the firm collapses, we have

$$C_t = C(X_t, V_t, m_t; t)$$

where

$$C(x,v,m;t) = \sup_{0\leq\tau\leq(T-t)\wedge T^*} E^{x,v}\left((N+M)^{-1}\left(V_\tau - (N+M-m)K\right)^+ e^{-R_\tau} + \int_0^\tau e^{-R_s}\rho(X_s)ds \right)$$

So if $\varphi(x) = E^x \int_0^\infty \exp(-R_s)\rho(X_s)ds$, we shall have

$$C(x,v,m;t) - \varphi(x) = \sup_{0\leq\tau\leq(T-t)\wedge T^*} E^{x,v}\left[e^{-R_\tau}\left(\frac{(V_\tau - (N+M-m)K)^+}{N+M} - \varphi(X_\tau)\right)\right]$$

This doesn't seem to help a lot in general; we probably have to go numerical pretty soon. Overall, there seem to be a few clear principles for the pricing of convertibles in this context.

(i) Since

$$V_t = m_t C_t + (N+M-m_t) S_t$$

and we know the SDE for V, we need only understand the pricing of the convertibles and the share prices will then follow.

(ii) Since we always have

$$C_t \geq S_t - K$$

with equality at conversion, rearranging the value equation gives

$$(N+M)C_t \geq V_t - K(N+M-m_t).$$

This allows us to price the convertible by solving for m fixed the American-style pricing equation

$$\max_{t \leq \tau \leq T} E\left[e^{-r\tau} \{V_\tau - K(N+M-m)\}^+ \;\middle|\; V_t = v, m_t = m \right]$$

This makes the boundary conditions on the function C much easier to handle; indeed, if we just solve by backward recursion, we don't even <u>see</u> the boundary conditions!

(iii) For optimal calling policy, the firm calls once C reaches K^*. The evolution of C up til that time remains unchanged. Clearly the firm wouldn't call when $C < K^*$, as this is just handing over money; the firm wants the call to force conversions (which generate money for the firm) rather than surrenders (which require the firm to pay out money). On the other hand, once the price has reached K^* why wait for it to go any higher? Since everyone will convert when we call at $C = K^*$, we generate no more capital for the firm by waiting, and we will actually lose by having to pay out coupons meantime!

(iv) If the dividends on the shares are paid discretely, then conversions would only ever occur just immediately prior to a dividend payment. The paper of Constantinides deals with this

How would we build one of these models? Take a BM $(W_t)_{t\geq 0}$ on some $(\Omega, (\mathcal{F}_t), \mathbb{P})$, and the given initial term structure, and suppose that for $0 \le t \le \delta$, $W_t = W_t^\delta$.
Then solve

$$dx_{t\delta} = \gamma_{t\delta}\, x_{t\delta} \left[dW_t^\delta + \frac{\gamma_{t\delta}\, x_{t\delta}}{1+x_{t\delta}}\, dt \right], \quad x_{0\delta} = \frac{P(0,\delta)}{P(0,2\delta)} - 1$$

This gives us

$$P(t, 2\delta) = \frac{P(t,\delta)}{1+x_{t\delta}} \qquad (0 \le t \le \delta)$$

and

$$\left.\frac{d\mathbb{P}^\delta}{d\mathbb{P}^{2\delta}}\right|_{\mathcal{F}_\delta} = (1+x_{\delta\delta}) \cdot \frac{P(0,2\delta)}{P(0,\delta)},$$

from which we can get $W^{2\delta}$. Now solve

$$d\, x_{t,2\delta} = \gamma_{t,2\delta}\, x_{t,2\delta} \left[dW_t^{2\delta} + \frac{\gamma_{t,2\delta}\, x_{t,2\delta}\, dt}{1+x_{t,2\delta}} \right] \qquad (0 \le t \le \delta)$$

$$1 + x_{0,2\delta} = P(0,2\delta)/P(0,3\delta)$$

from which we can obtain

$$P(t, 3\delta) = \frac{P(t,2\delta)}{1+x_{t,2\delta}} = \frac{P(t,\delta)}{(1+x_{t\delta})(1+x_{t,2\delta})} \qquad (0 \le t \le \delta)$$

and also

$$\left.\frac{d\mathbb{P}^{2\delta}}{d\mathbb{P}^{3\delta}}\right|_{\mathcal{F}_\delta} = (1+x_{\delta,2\delta}) \cdot \frac{P(0,3\delta)}{P(0,2\delta)}$$

Proceeding thus, we can obtain $\mathbb{P}^{k\delta}\big|_{\mathcal{F}_\delta}$ for all $k = 1, 2, \ldots$

The next step would be to suppose that $dW_t^{2\delta} = dW_t$ for $\delta \le t \le 2\delta$, and now build the solutions over $[\delta, 2\delta]$.
We shall be able to build $P(j\delta, k\delta)$ for $0 \le j \le k$ uniquely from this data, as well as the $\mathbb{P}^{k\delta}$, forward measures.
How to fit in intermediate values?

The market model: what's going on? (18/2/96)

(i) In the work of Miltersen, Sandmann & Sondermann (see also Brace, Gatarek & Musiela) the idea is to try to set things up so that for some fixed $\delta > 0$, for all $T > 0$

$$\left(\frac{P(t,T)}{P(t,T+\delta)} - 1 \right)_{0 \le t \le T} \equiv \left(x(t,T) \right)_{0 \le t \le T}$$

is a log-Gaussian process, so that $x(T,T)$ (which would be the interest paid at time $T+\delta$, fixed at time T, on borrowed sum 1) will be log-Gaussian (in the $(T+\delta)$-forward measure $\mathbb{P}^{T+\delta}$). This then justifies the Black formula for a caplet. Notice that

$$\frac{P(t,T)}{P(t,T+\delta)} \equiv \left. \frac{d\mathbb{P}^T}{d\mathbb{P}^{T+\delta}} \right|_{\mathcal{F}_t} \quad \text{is a } \mathbb{P}^{T+\delta}\text{-martingale},$$

so this leads us to represent x by

$$d\,x(t,T) = \gamma(t,T)\, x(t,T)\, dW_t^{T+\delta}$$

assuming a fundamentally one-dimensional Brownian world behind everything. Likewise, since $P(t,T+\delta)/P(t,T) \equiv (1+x(t,T))^{-1}$ is a $\mathbb{P}^T$-martingale, we discover that

$$d\left(\frac{1}{1+x(t,T)} \right) = -\frac{1}{(1+x_{tT})^2} \left\{ \gamma_{tT}\, x_{tT}\, dW_t^{T+\delta} - \frac{\gamma_{tT}^2\, x_{tT}^2\, dt}{1+x_{tT}} \right\}$$

from which

$$dW_t^{T+\delta} - \frac{\gamma_{tT}\, x_{tT}\, dt}{1+x_{tT}} = dW_t^T.$$

If you set $y(t,T) = \frac{1}{2}\log(x_{tT})$, you obtain

$$dy(t,T) = \tfrac{1}{2}\gamma_{tT}\, dW_t^{T+\delta} - \tfrac{1}{4}\gamma_{tT}^2\, dt$$

$$= \tfrac{1}{2}\gamma_{tT}\, dW_t^T + \tfrac{1}{4}\gamma_{tT}^2 \tanh y_{tT}\, dt$$

so under $\mathbb{P}^{T+\delta}$, y is a deterministic time change of a BM with drift -1, and under $\mathbb{P}^T$ it is the same time change of a BM whose drift is either 1 or -1, with equal probability. This explains why Miltersen et al are getting to the answer via expectations in $\mathbb{P}^T$, rather than $\mathbb{P}^{T+\delta}$, as preferred by BGM.

(ii) What one would <u>really</u> like, and what is all one really needs, is that for all T

the $\mathbb{P}^T$-law of $\dfrac{1}{P(t,T)} - 1 \equiv \xi_{tT}$ is log-Gaussian

Assuming we think to achieve this by taking

$$d\xi_{tT} = \xi_{tT} \{ \gamma_{tT}\, dW_t^T + \mu_{tT}\, dt \}, \qquad \gamma, \mu \text{ deterministic}$$

we have that

$$P(0,T) \cdot e^{\int_0^t r_s ds} \; P(t,T)^{-1} \equiv \left. \frac{d\mathbb{P}}{d\mathbb{P}^T} \right|_{\mathcal{F}_t} \quad \text{is a } \mathbb{P}^T\text{-martingale}$$

from which

$$r_t (1+\xi_{tT})\, dt + \xi_{tT} (\gamma_{tT}\, dW_t^T + \mu_{tT}\, dt) \quad \text{is a } \mathbb{P}^T\text{-martingale},$$

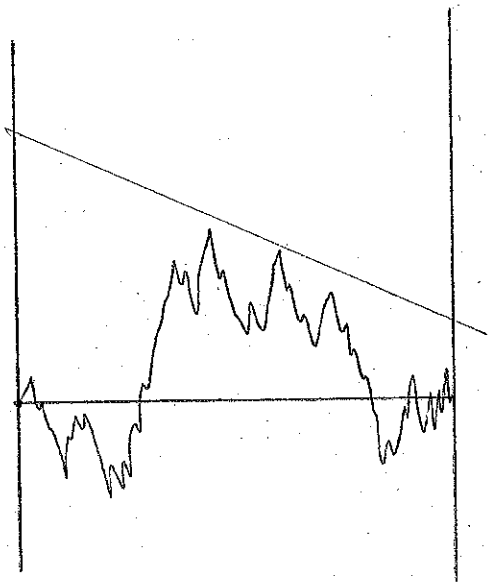
whence

$$r_t + (1-P_{tT}) \mu_{tT} = 0 \qquad (t \leq T)$$

so that

$$P_{tT} = 1 + r_t / \mu_{tT}$$

– but if we let $T \to \infty$, it's impossible to get limit 0 if r is random in any way..!



Some observations on $S(S-X)$ (6/3/96)

(i) If we take a Brownian bridge $(b_t)_{0\le t\le 1}$ from 0 to 0 and now define

$$S(x) := \sup_{0\le t\le 1} \{ b_t + xt \}$$

for each $x\in\mathbb{R}$, then S is an increasing convex function, $S(x) \geq x^+$. The concave dual $S^*(t) \equiv \inf\{S(x) - xt\}$ is $-\infty$ if $t\notin[0,1]$, else it is the least concave majorant of b. Both of the functions S and S^* are piecewise linear. The slope of S at x gives the time in the unit interval at which $b_t + tx$ is maximal.

If we consider $S(x) - x/2$, then $(S(-x)+\frac{x}{2}) \overset{\mathcal{D}}{=} (S(x) - x/2)$ as processes.

If we were told the Brownian bridge b, and the value of $S(x)(S(x)-x)$, could we deduce x? If we're told $S(x)(S(x)-x) \equiv (S(x)-x/2)^2 - x^2/4 = a$, then $S(x) - x/2 = \sqrt{a+x^2/4}$, so we could identify some discrete set of x's where these two convex functions agree, but in general there is not going to be a unique point.

(ii) Let $\tau(x)$ be the first time that $b_t + xt = S(x)$. Joint law of $(S(0),\tau(0))$?

$$P[\, b \text{ first reaches } a \text{ before time } t, \text{ never reaches } a+\varepsilon\,]$$

$$= P\left[\, (1-s)B\left(\tfrac{s}{1-s}\right) > a \text{ for some } s\le t, \text{ and always} \le a+\varepsilon \,\right] \qquad u = \frac{s}{1-s},\quad s = \frac{u}{1+u}$$

$$= P\left[\, B_u > a(1+u) \text{ for some } u \le \tfrac{t}{1-t}, \text{ and } B_v \le (a+\varepsilon)(1+v)\ \forall v \,\right]$$

$$= \int_0^{t/(1-t)} a e^{-(a+au)^2/2u} \frac{du}{(2\pi u^3)^{1/2}} \left\{1 - \exp\left[-2(a+\varepsilon)\varepsilon(1+u)\right]\right\}$$

Dividing by ε and letting $\varepsilon\downarrow 0$ gives

$$P[\, S(0)\in da,\ \tau(0)\le t\,]/da = \int_0^{t/(1-t)} a\exp\left\{-\frac{a^2(1+u)^2}{2u}\right\}\frac{du}{\sqrt{2\pi u^3}}\cdot 2a(1+u)$$

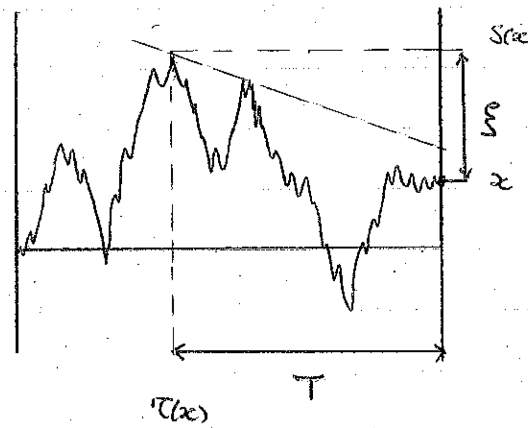
so that if we differentiate wrt t we shall obtain

$$\boxed{P\left[S_0 \in da, \tau_0 \in dt\right]/da\,dt = \frac{2a^2}{(t(1-t))^{3/2}} \exp\left[-\frac{a^2}{2t(1-t)}\right]\frac{1}{\sqrt{2\pi}}}$$

(iii) Similar calculations give more generally for $a > x$

$$\boxed{P\left[S(x) \in da, \tau(x) \in dt\right]/da\,dt = \frac{2a(a-x)}{\sqrt{2\pi}\,(t(1-t))^{3/2}} \exp\left[-\frac{(a-xt)^2}{2t(1-t)}\right].}$$

(iv) Suppose we fix some particular x, and let $T \equiv 1-\tau(x)$, $\xi \equiv S(x) - x$. The graph of $b_t + xt$ looks like:



We have the identity in law

$$\left(S(x) - \{b_{1-s} + x(1-s)\}\right)_{0 \le s \le T}$$

$$\stackrel{\mathcal{D}}{=} \left(\frac{T-s}{T} R\left(\frac{sT}{T-s}\right)\right)_{0 \le s \le T}$$

where R is a BES(3) process started at ξ. For $0 < a < \xi/T$,

$$P\left[S(x) - \{b_{1-s} + x(1-s)\} \ge a(T-s) \text{ for all } 0 \le s \le T\right]$$

$$= P\left[b_u + xu \le S(x) - a(u - \tau(x)) \text{ for all } \tau(x) \le u \le 1\right]$$

$$= P\left[\frac{T-s}{T} R\left(\frac{sT}{T-s}\right) \ge a(T-s) \text{ for all } 0 \le s \le T\right]$$

$$= P\left[R_u \ge aT \text{ for all } u\right]$$

$$= \frac{aT}{\xi}.$$

Given that the inf of R is aT, what can we say of the time it hits the inf? It's going to be the law of the hitting time of aT for a BM, of course!

(V) So we can assemble this into a description of the least concave majorant of the Brownian bridge as follows.

Pick $t_0 \equiv \tau(0)$ uniformly on $[0,1]$

Take an independent $N(0,I)$ in $\mathbb{R}^3$, V, say, and let

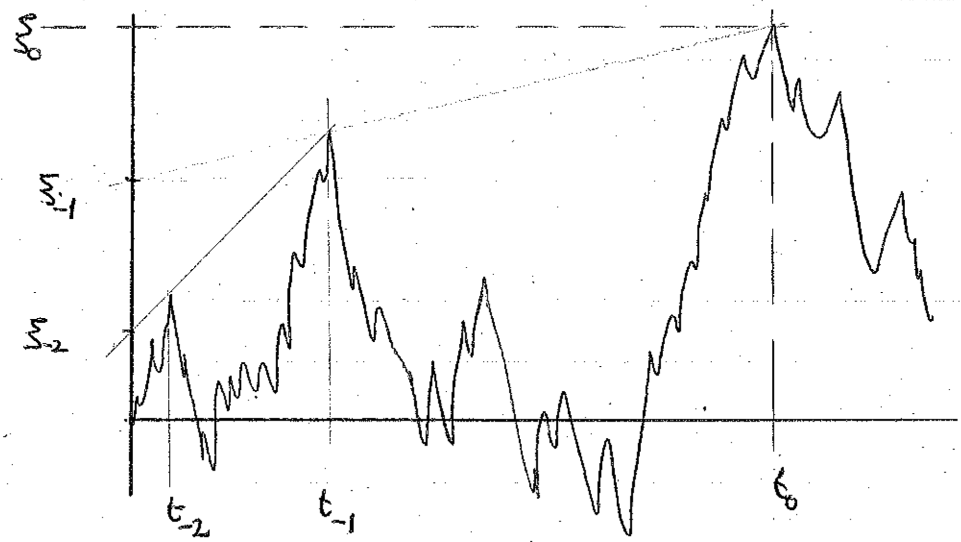
$$\xi_0 \equiv S_0 = \sqrt{t_0(1-t_0)}\ |V|.$$

The least concave majorant is the inf of a sequence of straight lines and we may describe the distribution of these in the following way. Sequentially for $n = 1, 2, 3, \ldots$,

Pick ξ_{-n} uniformly in $[0, \xi_{-n+1}]$

Take $Z_{-n} \sim N(0,1)$ and now let

$$0 < t_{-n} = \frac{(\xi_{-n+1} - \xi_{-n})^2 \quad t_{-n+1}}{Z_{-n}^2 \ t_{-n+1} \ + (\xi_{-n+1} - \xi_{-n})^2} < t_{-n+1}$$



Take
$$\eta_{-n} \equiv \xi_{-n} + \frac{t_{-n}}{t_{-n+1}} (\xi_{-n+1} - \xi_{-n})$$

with the convention that $\eta_0 = \xi_0 = S(0)$. The LCM is made up of the line segments joining the points (t_{-n}, η_{-n}) to (t_{-n+1}, η_{-n+1}) $(n = 1, 2, \ldots)$ together with the similarly defined line segments to the right of t_0. The recursion for the t_n is more simply stated:

$$\boxed{\frac{1}{t_{-n}} = \frac{1}{t_{-n+1}} + \frac{Z_{-n}^2}{(\xi_{-n+1} - \xi_{-n})^2}}$$

Yet another way we can describe it is as

$$\frac{1}{t_{-n}} = \frac{1}{t_{-n+1}} + \frac{1}{H_n}$$

where H_n is the time taken for a Brownian motion to get up to $\xi_{-n+1} - \xi_{-n}$

Law of inf and final value of a killed diffusion (7/3/96)

(i) Suppose we have a Markov pr X', generator $\mathcal{G}'$, and the spot rate process is $r'_t \equiv \varphi(X'_t)$. Now suppose you are going to pay out at rate $\underline{r}'_t \equiv \inf_{s\le t} r'_s$; what is the current value of this cash flow if $X'_0 = x_0$? Clearly, it's

$$E^{x_0} \int_0^\infty \exp(-R_s)\, \underline{r}'_s\, ds \qquad R_t = \int_0^t r'_u\, du$$

$$= E^{x_0}\left[\int_0^{R_\infty} e^{-t}\, \underline{r}'(\tau_t)/r'(\tau_t)\, dt\right] \qquad \tau_t \equiv \inf\{s;\, R_s > t\}$$

If we time change by τ, we have $X_t \equiv X'(\tau_t)$ is a Mkv pr with generator $\mathcal{G} \equiv \varphi(x)^{-1}\mathcal{G}'$. For simplicity,

<u>assume</u> $P^x(R_\infty = \infty) = 1$ <u>for all</u> x

so that the value of the cash-flow is simply

$$E^{x_0}\left[\ \underline{r}(T)/r(T)\ \right]$$

where $r_t = \varphi(X_t)$, $\underline{r}_t \equiv \inf_{s\le t} r_s$, and $T \sim \exp(\lambda)$ independent of X — of course, $\lambda = 1$ in this case, but let's keep it general because it makes it easier to check the calculations.

(ii) To get an example where more can be done, let's assume that X is a one-dimensional diffusion in natural scale, and φ is strictly positive, strictly increasing. What's the joint law of $(\underline{X}(T), X(T))$? For $x < x_0$, $x < y$,

$$P^{x_0}\left[\ \underline{X}(T) < x,\ X(T) \in dy\right] = \frac{\psi_\lambda^-(x_0)}{\psi_\lambda^-(x)} \cdot \lambda r_\lambda(x,y)\, m(dy)$$

$$= \frac{\lambda\, \psi_\lambda^-(x_0)\ \psi_\lambda^+(x)\, \psi_\lambda^-(y)}{\psi_\lambda^-(x)} \cdot c_\lambda\ m(dy)$$

$$\therefore\ P^{x_0}\left[\underline{X}(T) \in dx,\ X(T) \in dy\right] = \frac{\psi_\lambda^-(x_0)}{\psi_\lambda^-(x)} \cdot \frac{\psi_\lambda^-(y)}{\psi_\lambda^-(x)} \cdot \lambda c_\lambda\left[\psi_\lambda^-(x) D\psi_\lambda^+(x) - \psi_\lambda^+(x) D\psi_\lambda^-(x)\right] dx\, m(dy)$$

$$\boxed{\therefore\ P^{x_0}\left[\underline{X}(T) \in dx,\ X(T) \in dy\right] = 2\lambda\ \frac{\psi_\lambda^-(x_0)}{\psi_\lambda^-(x)}\ \frac{\psi_\lambda^-(y)}{\psi_\lambda^-(x)}\ dx\ m(dy)}$$

If we re-express in terms of the diffusion before it got put into natural scale, we shall find

$$\int_a^{x_0} s'(x)\,dx \int_x^b \frac{2\lambda\, \varphi_\lambda^-(x_0)\, \varphi_\lambda^-(z)}{\varphi_\lambda^-(x)^2} \quad \frac{\varphi(x)}{\varphi(z)} \quad \frac{dz}{(\sigma^2 s')(z)}$$

if the original diffusion is in (a,b), with generator $\mathcal{G} = \frac{1}{2}\sigma(x)^2 D^2 + b(x) D$, and where $\varphi_\lambda^- = \psi_\lambda^- \circ s$.

Slightly less approximate Kalman filtering of diffusions (27/3/96)

Referring back to the situation on p1, we have

$$\left(X_{n\delta+\delta} - X_{n\delta} \mid X_{n\delta} \right) \sim N\left(\delta\, b(X_{n\delta}),\ \Sigma_\delta(X_{n\delta}) \right)$$

where $\Sigma_\delta(x)_{ij} \equiv \delta\, \sigma^i_r \sigma^j_r(x) + \frac{1}{2}\delta^2 \ (\sigma^k_\ell D_k \sigma^i_r)(\sigma^m_\ell D_m \sigma^j_r)(x)$. It may be too oversimplified to suppose that if $(X_{n\delta} \mid \mathcal{Y}_{n\delta}) \sim N(\mu_n, V_n)$ then we simply replace $X_{n\delta}$ by μ_n.

Taking Taylor expansion up to order 2, we'd have for general $f \in C^2$

$$E\left[f(X_{n\delta}) \mid \mathcal{Y}_{n\delta} \right] \doteq f(\mu_n) + \tfrac{1}{2}\,\mathrm{tr}\left(V_n\, D^2 f(\mu_n) \right)$$

so that

$$E\left[X_{(n+1)\delta} \mid \mathcal{Y}_{n\delta} \right] \doteq \mu_n + \delta\, b(\mu_n) + \tfrac{1}{2}\,\mathrm{tr}\left(V_n\, D^2 b(\mu_n) \right) . \delta \equiv c_n .$$

Next

$$\mathrm{var}\left(X_{(n+1)\delta} \mid \mathcal{Y}_{n\delta} \right) = E\left[\Sigma_\delta(X_{n\delta}) \mid \mathcal{Y}_{n\delta} \right] + \mathrm{var}\left(X_{n\delta} + \delta b(X_{n\delta}) \mid \mathcal{Y}_{n\delta} \right)$$

$$= \Sigma_\delta(\mu_n) + \tfrac{1}{2}\,\mathrm{tr}\left(V_n\, D^2 \Sigma_\delta(\mu_n) \right)$$

$$+ \left(I + \delta D b(\mu_n) \right) V_n \left(I + \delta D b(\mu_n) \right)^T \equiv \tilde{V}_n ,$$

by expanding the second term as a Taylor series, up to order 1.

So the slightly less approximate method for updating would be to observe the y_{n+1} vector and then

$$\min_x \left[\tfrac{1}{2}\left(y - f(x) \right) \cdot V_\eta^{-1} \left(y - f(x) \right) + \tfrac{1}{2} (x - c_n) \cdot \tilde{V}_n^{-1} (x - c_n) \right]$$

and set μ_{n+1} to be the minimising value of x, and V_{n+1}^{-1} to be the second derivative of the objective function at the minimum.

How to handle numerical solution of knock-out options (24/4/96)

In risk-neutral measure, asset price S_t obeys

$$S_t = S_0 \exp\{\sigma W_t + ct\} \equiv S_0 \exp(X_t)$$

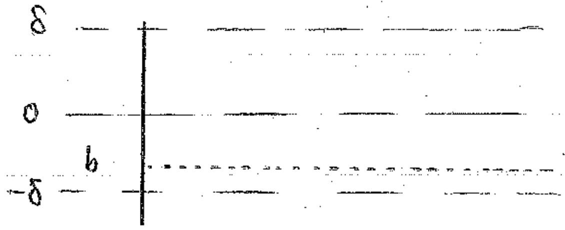
An option pays off $\varphi(S_T)$ at time T so long as $a(t) \le X_t \le b(t)$ for all $t \le T$. Let's suppose the barriers are actually constant. Then we approximate the process X with a Markov chain on $\delta\mathbb{Z}$ by time-changing using the additive functional

$$A_t = \sigma^{-2} \sum_{j\in\mathbb{Z}} L(t, j\delta)\,\delta$$

$$\left.\begin{array}{ll} \text{Rate of excursions from } 0 \text{ to } \delta \text{ is} & c\sigma^{-2}\, e^{2c\delta/\sigma^2}\{e^{2c\delta/\sigma^2}-1\}^{-1} \\ \text{" \quad " \quad " } 0 \text{ to } -\delta \text{ is} & c\sigma^{-2}\,\{e^{2c\delta/\sigma^2}-1\}^{-1} \end{array}\right] \text{ if } c>0$$

The jump chain has an 'up' prob^y $p = e^{c\delta/\sigma^2}/2\cosh(c\delta/\sigma^2) \equiv 1-q$ and in time interval $[0,T]$ the chain will make a $\mathcal{P}(cT\sigma^{-2}\coth(c\delta/\sigma^2))$ number of jumps.

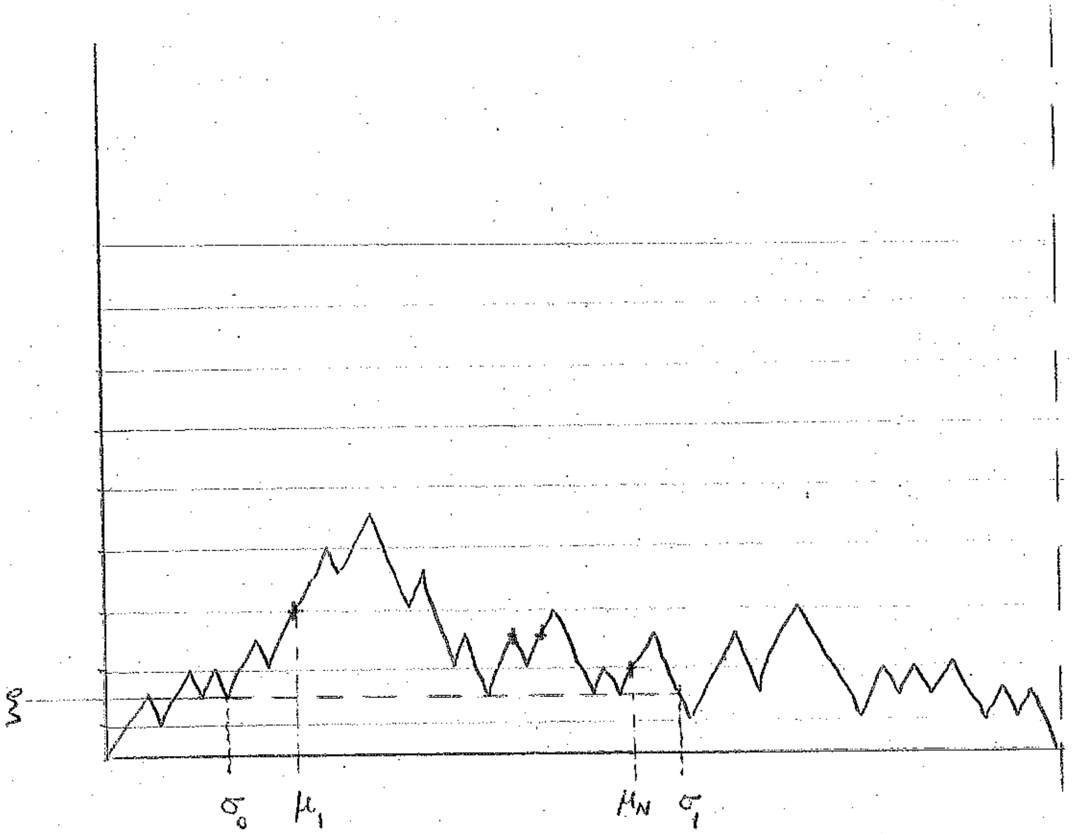
How to deal with barrier in binomial tree or simulation?



If we knock out at barrier b, $-\delta < b < 0$, then the up prob^y should simply be changed to

$$\tilde{p} \equiv \frac{s(0)-s(b)}{s(\delta)-s(b)} = \frac{e^{-2cb/\sigma^2}-1}{e^{-2cb/\sigma^2}-e^{-2c\delta/\sigma^2}}$$

with knockout happening with the complementary probability.



ξ
σ_0
μ_1
μ_N
σ_1

Decomposing marked Brownian excursions? (6/5/96)

(i) Let's consider the random-walk version of a conjecture of David Hobson. Take a random-walk excursion $(S_n)_{0\le n\le \tau}$, with $S_0=0$, $S_1=1$, and $\tau\equiv\inf\{k: S_k=0,\ k>1\}$. Insert marks at each time-point $n\ge 1$ with prob'y $1-\theta$, independently. Then

$$\varphi_k \equiv P^k(\text{reach } 0 \text{ without a mark}) \equiv E^k[\theta^\tau] = \gamma^k$$

where γ solves $\theta\gamma^2 - 2\gamma + \theta = 0$; $\gamma = (1-\sqrt{1-\theta^2})/\theta$.

(ii) We condition the excursion to contain at least one mark, and now define
$\xi \equiv \inf\{S_n : \mu_1 \le n \le \mu_N\}$, where μ_1 is time of first mark, μ_N time of last.

The principal claim is that if we excise the time-interval $(\sigma_0, \sigma_1]$ and close up the excursion, marking the point where excision took place, then we are looking at a rw excursion conditioned to have exactly one mark in it.

What does such an excursion look like?

$$\tilde\varphi_k \equiv P^k[\text{exactly one mark by time } \tau] = E^k[\tau(1-\theta)\theta^{\tau-1}]$$

$$= (1-\theta)\frac{\partial}{\partial\theta}\gamma^k$$

$$= k\gamma^k \frac{1-\theta}{\theta\sqrt{1-\theta^2}}$$

After the mark, we have to reach 0 with no mark at all, so we see a rw with up-probability $\gamma/2$ and down-probability $1/(2\gamma)$. But before? If we let A denote the event that there is exactly one mark in $(S_n)_{1\le n\le\tau}$, we have before

$$P(X_{n+1}=k+1,\ n+1 \text{ is marked} \mid A, X_n=k) = \tfrac{1}{2}(1-\theta)\varphi_{k+1}/\tilde\varphi_k = \frac{\theta\sqrt{1-\theta^2}}{2k}\gamma$$

$$P(X_{n+1}=k+1,\ n+1 \text{ not marked} \mid A, X_n=k) = \frac{\gamma\theta}{2}\,\frac{k+1}{k}$$

$$P(X_{n+1}=k-1,\ n+1 \text{ is marked} \mid A, X_n=k) = \frac{\theta\sqrt{1-\theta^2}}{2k}\gamma^{-1}$$

$$P(X_{n+1}=k-1,\ n+1 \text{ not marked} \mid A, X_n=k) = \frac{\theta}{2\gamma}\,\frac{k-1}{k}$$

From this, if μ_1 is the time of the first mark, we get

$$P(\mu_1 = n \mid \mu_1 \geq n, A, X_n = k) = \sqrt{1-\theta^2} \big/ \{ k + \sqrt{1-\theta^2} \}$$

(iii) If now we consider what happens for the excursion conditioned to have at least one mark in it, if $\tilde{A}$ denotes that event, then

$$P[\text{excursion happens at } n \mid X_n = k,\ \tilde{A},\ \text{no excursion before } n]$$

$$= \{ (1-\theta)\gamma^k + \theta . P^k(\text{at least two upward excursions from } k, \text{ after last of which get to}$$

$$0 \text{ without mark; first-last excns marked} \mid \text{no excursion before } n) \} / (1-\gamma^k)$$

$$= \{ (1-\theta)\gamma^k + \theta \cdot \tfrac{1}{2}(1-\gamma) . \left(\sum_{r \geq 0} (\tfrac{1}{2})^r \right) \cdot \tfrac{1}{2}(1-\gamma) . \gamma^k \} / (1-\gamma^k)$$

$$= \left(1 - \theta + \tfrac{1}{2}\theta(1-\gamma)^2 \right) / (\gamma^{-k} - 1) .$$

This is not the same as the above expression for the probability that the excursion happens at n given that it hasn't happened before, in the case of RW with exactly one mark.

Nevertheless, in the Brownian situation, one can show that the result is true. Both David and I have proofs.

Why do we need cash? (20/5/96)

(i) Take a very simple model where an agent tries to

$$\max \ E \int_0^\infty e^{-\delta t} U(c_t)\,dt$$

where consumption is paid for out of cash, and the cash holding at time t, x_t, and y_t, the value of savings account satisfy

$$dy_t = r y_t dt - dA_t$$

$$dx_t = -c_t\,dt + dA_t - \int \mu(dt,dz)\, \varphi(x_{t-}, z)$$

where A_t is the net amount shifted from savings to cash by time t, and μ is the random measure describing shocks to the wallet. Assume it's compensated by $\nu(dx)\,dt$. As soon as cash goes negative, you're compelled to dip into savings to restore it immediately to being positive (you may choose how much to draw). If you run out of both, you stop, so will need $U(0) > -\infty$ for well-posed problem.

Since there are no costs for moving funds from x to y or y to x, the value function of this problem is a function of $w \equiv x+y$ alone, and we have

$$\int_0^t e^{-\delta s} U(c_s)\,ds + e^{-\delta t} V(w_t) \quad \text{is a supermg, mg under optimal control,}$$

so that

$$\boxed{\sup_{\substack{c\geq 0\\ x+y=w}} \Big[U(c) - \delta V(w) + (ry - c)V'(w) + \int \nu(dz)\big\{ V(w - \varphi(x,z)) - V(w)\big\} \Big] = 0}$$

One might have (for example)

$$\varphi(x,z) = z + \varepsilon (z-x)^+$$

(ii) We could try taking $U(c) = c^\theta$ for some $\theta \in (0,1)$, and seek solution of the form $V(w) = a w^\theta$. It's completely hopeless, though.

(iii) Maybe simpler is the idea of a fixed penalty ε each time you run out of cash? Then we get

$$\sup_{\substack{c\geq 0\\ x+y=w}} \Big[U(c) - \delta V(w) + (ry-c)V'(w) + \int \nu(dz)\big\{ V(w - z - \varepsilon I_{\{z > x\}}) - V(w)\big\}\Big] = 0$$

We then have (with $g(x,w) \equiv \int \nu(dz)\left[V(w) - V(w-z-\varphi(x,z))\right]$)

$$\max_{c\geq 0,\, x\in[0,w]} \quad U(c) - \delta V(w) + (r(w-x)-c)V'(w) - g(x,w) = 0$$

or again

$$\tilde{u}(V'(w)) - \delta V(w) + r(w-\xi(w))V'(w) - g(\xi(w), w) = 0,$$

where $\xi(w)$ is the maximising value of x for given w; when $0 < x < w$, it solves

$$-r V'(w) - g_x(\xi(w), w) = 0.$$

If we define f by $\boxed{V'(f(\lambda)) = \lambda}$ then we have

$$\tilde{u}(\lambda) - \delta V(f(\lambda)) + r(f(\lambda) - \xi(f(\lambda)))\lambda - g(\xi(f(\lambda)), f(\lambda)) = 0$$

so differentiating wrt λ gives us

$$-I(\lambda) - \delta\lambda f'(\lambda) + r\lambda f'(\lambda) + r f(\lambda) - r\lambda\xi'(f(\lambda))f'(\lambda) - r\xi(f(\lambda))$$
$$- g_x(\xi(f(\lambda)), f(\lambda))\,\xi'(f(\lambda))f'(\lambda) - g_w(\xi\circ f, f)\cdot f'(\lambda) = 0$$

or again (at least when $x\in(0,w)$)

$$-I(\lambda) + (r-\delta)\lambda f'(\lambda) - r\xi(f(\lambda)) + r f(\lambda) - g_w(\xi(f(\lambda)), f(\lambda))\, f'(\lambda) = 0$$

However, this is no easier.

(iv) If we had $dx_t = -c_t\,dt + dA_t + \sigma dW_t + b\,dt$ as the dynamics for x, with a cost ε each time you have to go to the bank, we'd have that A has only discrete jumps.

Mixed Markov chains (27/5/96)

Suppose that at time t we have $n_j(t)$ particles of type j, $j = 1, \dots, K$, and that the total number N of particles is fixed. Suppose that at rate ρ_N a particle meets another named particle, and then if the first is of type i and the second of type k then the i-particle changes to a j-particle at rate q^k_{ij}. Thus

$$n \equiv (n_1, \dots, n_K) \mapsto n - e_i + e_j \quad \text{at rate} \quad n_i(n_k - \delta_{ik}) \rho_N \, q^k_{ij}$$

This is a finite statespace Markov chain.

(i) When is this chain irreducible? This might be different depending on the value of N. For example: Take at least five types arranged in a circle, and use these rules. When you meet one of own type j, move to $j-1$ or $j+1$. If you meet a $(j+1)$ you go to $j+2$, when you meet a $j-1$ you go to $j-2$, and if you meet some i which is more than 1 away, you either go to $(j-1)$, $(j+1)$ or to i. This way, two particles started in state 1 will always remain adjacent, but from 3 particles in state 1 you can always reach all other states.

However, it is clear that if we have irreducibility when there are N particles, there will be irreducibility with $N+1$.

(ii) More generality will be needed if we are to explain an environment in which individuals meet and trade, because the two parties will both change simultaneously. We must also allow for spontaneous change of type, and the creation and death of particles, if we want to explain the most general situation. So we envisage a world where the state at time t is $\underline{n}(t) \equiv (n_1(t), \dots, n_K(t))$ and changes

$$n \mapsto n - e_i - e_j + e_k + e_\ell \quad \text{at rate} \quad n_i(n_j - \delta_{ij}) \, q(ij; k, \ell)$$

$$n \mapsto n - e_i + e_j \quad \text{at rate} \quad n_i \, \alpha_{ij}$$

$$n \mapsto n + e_i \quad \text{at rate} \quad \lambda_i$$

$$n \mapsto n - e_i \quad \text{at rate} \quad n_i \mu_i .$$

The appropriate technology appears to be generating functions. If we take $\theta_i \in (0,1]$, $i = 1, \dots, K$, and define

$$\varphi(t,\theta;\underline{n}) \equiv E\left[\prod_{j=1}^{k} \theta_j^{\eta_j(t)} \;\middle|\; \underline{\eta}(0)=\underline{n}\right]$$

then we see that, with $f(n) \equiv \prod \theta_j^{n_j}$, we shall have

$$Qf(n) = \sum_{(i,j)}\sum_{(k,\ell)} n_i(n_j-\delta_{ij})\, q(i,j;k,\ell)\left\{\frac{\theta_k\theta_\ell}{\theta_i\theta_j}-1\right\} f(n)$$

$$+ \sum_{i,j}{}' n_i\,\alpha_{ij}\left(\frac{\theta_j}{\theta_i}-1\right) f(n) + \sum_i \lambda_i(\theta_i-1) f(n)$$

$$+ \sum_i{}' n_i\,\mu_i(\theta_i^{-1}-1) f(n)$$

whence

$$\frac{\partial\varphi}{\partial t} = P_t Qf$$

$$= \sum_{(i,j)}\sum_{(k,\ell)} q(i,j;k,\ell)(\theta_k\theta_\ell-\theta_i\theta_j)\frac{\partial^2\varphi}{\partial\theta_i\partial\theta_j}$$

$$+ \sum_{i,j}\alpha_{ij}(\theta_j-\theta_i)\frac{\partial\varphi}{\partial\theta_i} + \sum_i \lambda_i(\theta_i-1)\varphi$$

$$+ \sum_i \mu_i(1-\theta_i)\frac{\partial\varphi}{\partial\theta_i}$$

which gives us

$$\boxed{\frac{\partial\varphi}{\partial t} = \frac{1}{2}\sum_i\sum_j{}' a_{ij}(\theta)\frac{\partial^2\varphi}{\partial\theta_i\partial\theta_j} + \sum_i{}' \beta_i(\theta)\frac{\partial\varphi}{\partial\theta_i} + \sum_i \lambda_i(\theta_i-1)\varphi}$$

where

$$\boxed{\begin{aligned} a_{ij}(\theta) &\equiv 2\sum_{k,\ell} q(i,j;k,\ell)(\theta_k\theta_\ell-\theta_i\theta_j) \\ \beta_i(\theta) &\equiv \sum_j \alpha_{ij}(\theta_j-\theta_i) + \mu_i(1-\theta_i) \end{aligned}}$$

This looks very like a diffusion equation... but no particular reason to have $a \geq 0$

$$V_x = (x+y)^{-R}\left[f(\theta) + \frac{1-\theta}{1-R} f'(\theta) \right], \quad V_y = (x+y)^{-R}\left[f(\theta) - \frac{\theta}{1-R} f'(\theta) \right]$$

$$V_{xx} = \frac{(x+y)^{-R-1}}{1-R}\left[-R(1-R) f(\theta) - 2R(1-\theta) f'(\theta) + (1-\theta)^2 f''(\theta) \right]$$

$$\tilde{U}(\lambda) = \frac{R}{1-R} \lambda^{(R-1)/R}$$

Why cash again (13/6/96)

Let's follow up on the Brownian disturbances, so if y_t is amount in the savings account at time t, and x_t is the cash held, we'll suppose we have the dynamics

$$\begin{cases} dx_t = -c_t\,dt + (x_t + \gamma y_t)(\sigma dW_t + b\,dt) \\ dy_t = r y_t\,dt \end{cases}$$

at least while there is no moving of money between accounts. The agent aims to find

$$\max E\left[\int_0^\infty e^{-\delta t} U(c_t)\,dt \,\Big|\, x_0 = x, y_0 = y\right] \equiv V(x,y)$$

where we shall always assume $U(c) = c^{1-R}/(1-R)$

(i) Proportional costs. This is probably the only case we'll have a chance with. The dynamics are

$$\begin{cases} dx_t = -c_t\,dt + (x_t + \gamma y_t)(\sigma dW_t + b\,dt) + (1-\varepsilon)dA_t - (1+\tilde{\varepsilon})\,d\tilde{A}_t \\ dy_t = r y_t\,dt - dA_t + d\tilde{A}_t \end{cases}$$

where $\varepsilon, \tilde{\varepsilon} > 0$, and A_t (resp. $\tilde{A}_t$) represents total amount shifted out of savings (resp into savings) by time t.

We seek a solution of the form

$$V(x,y) = \frac{(x+y)^{1-R}}{1-R} \cdot f\left(\frac{x}{x+y}\right) \equiv \frac{(x+y)^{1-R}}{1-R} f(\theta) \qquad \left(\theta = \frac{x}{x+y}\right)$$

to the HJB equation

$$\tilde{U}(V_x) - \delta V + b(x+\gamma y)V_x + \frac{\sigma^2}{2}(x+\gamma y)^2 V_{xx} + r y V_y = 0$$

with the conditions

$$(1-\varepsilon)V_x - V_y \le 0, \qquad V_y - (1+\tilde{\varepsilon})V_x \le 0.$$

Hence

$$\boxed{\begin{aligned} &\frac{R}{1-R}\left[f + \frac{1-\theta}{1-R}f'\right]^{(R-1)/R} - \frac{\delta}{1-R}f + b(\theta+\gamma-\gamma\theta)\left[f + \frac{1-\theta}{1-R}f'\right] \\ &+ \frac{\sigma^2}{2(1-R)}(\theta+\gamma-\gamma\theta)^2\left[(1-\theta)^2 f'' - 2R(1-\theta)f' - R(1-R)f\right] + r(1-\theta)\left[f - \frac{\theta}{1-R}f'\right] = 0 \end{aligned}}$$

and

$$\varepsilon \ge \frac{f'}{(1-R)f + (1-\theta)f'} \ge -\tilde{\varepsilon}$$

Let's note that this is <u>very</u> similar to the Davis-Norman situation; if we withdrew consumption from y instead, and with $\gamma = 0$, we'd have the same situation.

An alternative formulation suggested by the preceding form of HJB is to take

$$V(x,y) = y^{1-R} \, g\left(\frac{x}{x+y}\right) \equiv y^{1-R} \, g(\theta)$$

and then we have

$$\frac{R}{1-R}\left[(1-\theta)^2 g'(\theta)\right]^{(R-1)/R} - \delta g + b(\theta+\gamma-\gamma\theta)(1-\theta) g'(\theta)$$

$$+ \frac{\sigma^2}{2}(\theta+\gamma-\gamma\theta)^2 \left\{(1-\theta)^2 g'' - 2(1-\theta) g'\right\} + r(1-R) g - r\theta(1-\theta) g' = 0.$$

In fact, it's even better to use x/y as the variable; we then have (with $z = x/y$, $V(x,y) = y^{1-R} h(z)$)

$$\boxed{\frac{R}{1-R} h'(z)^{(R-1)/R} - \delta h(z) + b(z+\gamma) h'(z) + \frac{\sigma^2}{2}(z+\gamma)^2 h''(z) + r(1-R) h(z) - r z h'(z) = 0}$$

Still hard to work with, though.

A change of variables to $\xi_t \equiv x_t e^{-rt}$, $\eta_t \equiv y_t e^{-rt}$ gives us

$$\begin{cases} d\xi_t = -\tilde{c}_t dt + (\xi_t + \gamma\eta_t)(\sigma dW_t + b dt) - r\xi_t dt + (1-\varepsilon) da_t - (1+\tilde{\varepsilon}) d\tilde{a}_t \\ d\eta_t = d\tilde{a}_t - da_t \end{cases}$$

where $\tilde{c}_t \equiv e^{-rt} c_t$, and we want to

$$\max \; E\left[\int_0^\infty e^{-\delta t + r(1-R)t} \, \tilde{c}_t^{\,1-R} \frac{dt}{1-R}\right]$$

The solution will be that $z_t \equiv \xi_t / \eta_t$ is a diffusion in some interval,

$$\boxed{dz_t = -\varphi(z_t)\, dt + (z_t+\gamma)(\sigma\, dW_t + b\, dt) - r z_t\, dt + dL_t - d\tilde{L}_t}$$

where $V(\xi,\eta) \equiv \eta^{1-R} g(\xi/\eta)$ and $\varphi(z) \equiv g'(z)^{-1/R}$, and g solves

$$\boxed{\frac{R}{1-R} g'(z)^{(R-1)/R} - (\delta - r + rR) g(z) + \frac{1}{2}\sigma^2 (z+\gamma)^2 g''(z) + (bz + b\gamma - rz) g'(z) = 0}$$

(ii) <u>An approximation.</u> This is very crude, but suppose we are going to consume from cash according to the level of our cash holding, so that the cash held at time t, x_t, solves

$$dx_t = \sigma(x_t)\,dW_t + \mu(x_t)\,dt + dA_t$$

where A is some increasing process which denotes cash transferred from other investments.*
Our aim is to choose A so that we get

$$V(x) = \min E\left[\int_0^\infty e^{-rt}\,dA_t \,\middle|\, X_0 = x\right].$$

Cash must always be non-negative. If we define

$$\psi_-(x) = E^x\left(e^{-rH_0}\right), \quad \psi_+(x) = 1/E^0\left(\exp - rH_x\right)$$

for the diffusion with generator $\mathcal{G} \equiv \frac{1}{2}\sigma^2 D^2 + \mu D$, reflected at 0, we can say that the valuef^n solves

$$\mathcal{G}V - rV = 0 \qquad \text{where } A \text{ is not active}$$

and $V' = -1$ in intervals across which the optimally-controlled diffusion jumps (and in such intervals, $\mathcal{G}V - rV \geq 0$). Always $V' \geq -1$, and A grows continuously at points where $V' = -1$, $\mathcal{G}V' > 0$.

<u>If ψ_- is convex, then the optimal behaviour is to reflect off 0, but otherwise do nothing.</u>

More generally, this looks quite a tough little problem. For $0 < x < y$, by comparing starting at y then running till hit x with optimal rule from y, we see

$$V(y) \leq \frac{\psi_-(y)}{\psi_-(x)} V(x) \qquad \therefore \quad V(y)/\psi_-(y) \text{ is decreasing.}$$

Similarly, by comparing the optimal policy from $x < y$ to the policy which reflects off 0 until you hit y, then switch to optimal policy, we conclude that

$$\frac{1}{\psi_+(x)}\left\{V(x) + \frac{\psi_-(x)}{D\psi_-(0)}\right\} \quad \text{is increasing,}$$

and, indeed, it's easily seen to be negative everywhere.

---

* It may be that $\mu(x) = b(x) - c$, where c is a constant desired rate of consumption, say

Pooling risks in the presence of transactions costs (23/7/96)

(i) Let's consider a world with riskless interest rate r, and a single stock, whose mid-price S_t is $S_t = S_0 \exp[\sigma W_t + (b - \frac{1}{2}\sigma^2)t]$. In the open market, you get $(1-\delta_-)S_t$ if you sell it, and you must pay $(1+\delta_+)S_t$ to buy it at time t.

A group of N agents get together and agree that they will trade amongst themselves costlessly at price $\tilde{S}_t \equiv p_t S_t$, where $1-\delta_- \leq p_t \leq 1+\delta_+$ always.

Agent i holds ν_t^i shares at time t and x_t^i in the bank account, so his wealth at time t will be

$$w_t^i = \nu_t^i \tilde{S}_t + x_t^i$$

and the self-financing condition is that

$$dw_t^i = \nu_t^i \, d\tilde{S}_t + r x_t^i \, dt - c_t^i \, dt + \varepsilon_i x_t^i \, dB_t^i ,$$

which incorporates random shocks to the cash holdings of each agent, and the consumption withdrawals c_t^i. The aggregate cash $x_t \equiv \sum_i x_t^i$ of the pool thus satisfies

$$dx_t = (r x_t - c_t)dt + \sum_1^N \varepsilon_i x_t^i \, dB_t^i \qquad (c_t \equiv \textstyle\sum_i c_t^i).$$

Minimising the volatility of the noise term subject to $\sum x^i = x$ gives

$$x_t^i = \varepsilon_i^{-2} x_t / (\textstyle\sum \varepsilon_j^{-2}) \equiv \theta_i x_t , \text{ say,}$$

and

$$dx_t = (r x_t - c_t)dt + x_t \gamma \, dB_t ,$$

where $dB \equiv \sum \varepsilon_i^{-1} dB^i / (\sum \varepsilon_i^{-2})^{\frac{1}{2}}$, $\gamma \equiv (\sum \varepsilon_j^{-2})^{-\frac{1}{2}}$.

Now let's assume that the aggregation looks like this, so that the dynamics of the pool's total wealth x_t in the bank, and the market value $y_t \equiv \nu_t S_t$ of their holding of shares satisfy

$$\boxed{\begin{aligned} dx_t &= (r x_t - c_t)\,dt + \gamma x_t \, dB_t - (1+\delta_+)\,dA_t^+ + (1-\delta_-)\,dA_t^- \\ dy_t &= y_t\,(\sigma dW_t + b\,dt) + dA_t^+ - dA_t^- \end{aligned}}$$

When $\gamma = 0$, this is just the classical (Davis-Norman) dynamics.

The goal is to discover whether there can be an equilibrium in which this is the nature of the aggregation. [It's not possible – see p 37].

(ii) Assume that all agents have utility $U(x) = x^{1-R}/(1-R)$, and that the utility of the pool is represented in this form too. Let's understand more about the optimal behavior of the pool. If

$$V(x,y) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \;\middle|\; x_0 = x,\ y_0 = y\right],$$

we have the HJB equation as usual;

$$\tilde{U}(V_x) - \rho V + \tfrac{1}{2}\gamma^2 x^2 V_{xx} + r x V_x + \tfrac{1}{2}\sigma^2 y^2 V_{yy} + b y V_y = 0, \qquad \left[\tilde{U}(\lambda) \equiv \tfrac{R}{1-R}\lambda^{(R-1)/R}\right]$$

$$(1-\delta_-) V_x \le V_y \le (1+\delta_+) V_x .$$

By scaling, $V(x,y) = y^{1-R}\, v(z)$, $z \equiv x/y$. From this, we obtain the DE for v:

$$\left.\begin{aligned} &\frac{R}{1-R}\, v'(z)^{R/(R-1)} - \rho v + \tfrac{1}{2}(\gamma^2+\sigma^2) z^2 v'' + (r + R\sigma^2 - b) z v' + \left(b(1-R) - \tfrac{1}{2}\sigma^2 R(1-R)\right) v = 0 \\ &1 + z - \delta_- \le \frac{v}{v'}(1-R) \le 1 + z + \delta_+ \end{aligned}\right\}$$

or, more simply,

$$\boxed{\begin{aligned} &\frac{R}{1-R}(v')^{R/R-1} + \tfrac{1}{2}\alpha z^2 v'' + \beta z v' + \lambda v = 0 \\ &1 + z - \delta_- \le \frac{v}{v'}(1-R) \le 1 + z + \delta_+ \end{aligned}}$$

with

$$\alpha \equiv \gamma^2 + \sigma^2, \quad \beta \equiv r + R\sigma^2 - b, \quad \lambda \equiv (b - \tfrac{1}{2}R\sigma^2)(1-R) - \rho .$$

So the pool performs exactly as for the Davis-Norman model, but with different parameters. We have that for critical $a < b$ to be determined

$$v(z) = v(a)\left(\frac{1-\delta_- + z}{1-\delta_- + a}\right)^{1-R} \qquad (z \le a)$$

$$= v(b)\left(\frac{1+\delta_+ + z}{1+\delta_+ + b}\right)^{1-R} \qquad (z \ge b)$$

We can deduce certain things about the nature of the solution; for example,

$$c_t^{-R} = V_x(x_t, y_t) = y_t^{-R} v'(z_t),$$

so that

$$\boxed{c_t = y_t \, v'(z_t)^{-1/R} = (x_t + y_t)\,(1+z_t)^{-1} v'(z_t)^{-1/R}}$$

so you optimally consume proportionally to wealth, with the ratio depending on z only.

Also,

$$\boxed{dz_t = z_t\,(\gamma dB_t - \sigma dW_t) + (r - b + \sigma^2) z_t\,dt - v'(z_t)^{-1/R}\,dt + d\tilde{A}_t^- - d\tilde{A}_t^+},$$

so that under optimal control, z is an autonomous diffusion, regulated to remain in the interval $[a,b]$.

(iii) Is it possible to find a price $\tilde{S}_t \equiv S_t p_t$ of the form $S_t\, p(z_t)$ which would make this equilibrium work, i.e., so that agent i would hold wealth $x_t^i = \theta^i x_t$ in his (risky) bank account?

For absence of arbitrage, will need $p' = 0$ at a and at b. We get

$$d\tilde{S}_t = \tilde{S}_t \Big[\sigma dW_t + b\,dt + \frac{p'}{p}\big\{ z_t(\gamma dB - \sigma dW + (r-b)\,dt) - v'(z_t)^{-1/R} dt \big\}$$

$$+ \frac{p''}{2p}\, z_t^2\, (\gamma^2 + \sigma^2)\,dt \Big],$$

and so

$$dz_t\, d\tilde{S}_t = \tilde{S}_t\, z_t \left[(\sigma^2 + \gamma^2)\, z_t\, p'(z_t)/p(z_t) - \sigma^2 \right] dt$$

$$dz_t\, dB_t^i = \gamma^2 z_t \; dt/\varepsilon_i$$

$$d\tilde{S}_t\, d\tilde{S}_t = \tilde{S}_t^2 \cdot \left\{ \gamma^2 z^2 (p'/p)^2 + \sigma^2 (1 - z p'/p)^2 \right\} dt$$

For brevity, we'll write $\mu(z) \equiv z p'(z)/p(z)$. Now let's define the value function

$$F(w,z) \equiv \max E\left[\int_0^\infty e^{-\rho t}\, U(c_t^i)\,dt \;\Big|\; w_0^i = w,\ z_0 = z \right]$$

for agent i. In the interests of keeping the acreage of formulae under control, let's write more briefly

$\xi = x/w \quad , \quad \eta = \nu S^2/w$

$$d\tilde{S} = \tilde{S}\left[\sigma(1-\mu)\,dW + \gamma\mu\, dB + \mu_S\, dt\right]$$

$$dz = z(\gamma dB - \sigma dW) + \mu_z\, dt + d\tilde{A}_t^- - d\tilde{A}_t^+$$

where $\mu_S \equiv b + (r-b)\mu - p' v(z)^{-1/R}/p + \frac{1}{2}\frac{p'' z^2}{p}(\gamma^2+\sigma^2)$,

$$\mu_z \equiv z(r-b+\sigma^2) - v'(z)^{-1/R}.$$

We therefore have

$$dw^i = \nu\tilde{S}\left[\sigma(1-\mu)dW + \gamma\mu\, dB + \mu_S\, dt\right] + x^i(\varepsilon_i dB^i + r\,dt) - c^i\, dt.$$

Now if we introduce the value function

$$F(w,z) \equiv \max E\left[\int_0^\infty e^{-\rho t} U(c_t^i)\,dt \,\middle|\, w_0^i = w,\ z_0 = z\right],$$

we find that F will solve

$$\sup_{c,x,\nu} U(c) - \rho F + F_w\left(\nu\tilde{S}\mu_S + rx - c^i\right) + F_z\mu_z + \frac{1}{2}F_{ww}\Big(\nu^2\tilde{S}^2(\sigma^2(1-\mu)^2 + \gamma^2\mu^2)$$

$$+ 2\nu\tilde{S}\varepsilon_i x\,\gamma\mu\,\frac{\gamma}{\varepsilon_i} + \varepsilon_i^2 x^2\Big] + F_{wz}\left[\nu z\tilde{S}\{-\sigma^2(1-\mu) + \gamma^2\mu\} + xz\gamma^2\right]$$

$$+ \frac{1}{2}F_{zz}\, z^2(\gamma^2+\sigma^2) \qquad = 0$$

with F_z vanishing at $z=a$, $z=b$. From scaling, we know that F must have the form

$$F(w,z) = w^{1-R}\, h(z)/(1-R)$$

so we obtain

$$\sup_{\substack{\xi,\eta\\ \xi+\eta=1}}\Bigg[\frac{R}{1-R}h^{(R-1)/R} - \frac{\rho h}{1-R} + (r\xi + \mu_S\eta)h + \frac{h'}{1-R}\mu_z$$

$$- \frac{Rh}{2}\left\{\eta^2(\gamma^2\mu^2 + \sigma^2(1-\mu)^2) + 2\eta\xi\gamma^2\mu + \varepsilon_i^2\xi^2\right\}$$

$$+ h'\left[\eta z(\mu\gamma^2 - \sigma^2(1-\mu)) + \gamma^2 z\xi\right] + \frac{1}{2}z^2 h''\frac{\gamma^2+\sigma^2}{1-R}\Bigg] = 0,$$

where we have $\xi \equiv x/w$, $\eta \equiv \nu\tilde{S}/w$, which explains why $\xi+\eta=1$ arises as a condition.

Optimising this over ξ gives

$$\boxed{\xi = \frac{(r-\mu_s)h + Rh(1-\mu)(\sigma^2(1-\mu)-\gamma^2\mu) + zh'(1-\mu)(\gamma^2+\sigma^2)}{Rh\left(\varepsilon_i^2-\gamma^2+(\sigma^2+\gamma^2)(1-\mu)^2\right)}}$$

and

$$0 = \frac{R}{1-R}h^{(R-1)/R} - \frac{\rho h}{1-R} + \mu_s h + \frac{h'}{1-R}\mu_z - \tfrac{1}{2}Rh(\gamma^2\mu^2+\sigma^2(1-\mu)^2)$$
$$+h'z(\mu\gamma^2-\sigma^2(1-\mu)) + \tfrac{1}{2}z^2h''\frac{\gamma^2+\sigma^2}{1-R}$$
$$+\frac{\left\{(r-\mu_s)h + Rh(1-\mu)(\sigma^2(1-\mu)-\gamma^2\mu) + zh'(1-\mu)(\gamma^2+\sigma^2)\right\}^2}{2Rh\left\{\varepsilon_i^2-\gamma^2+(\sigma^2+\gamma^2)(1-\mu)^2\right\}}$$

Of course, it's unlikely we could solve this, but we do see (as really must have been obvious from a long way back!) that

$$\boxed{\text{the optimal } \xi_t \equiv x_t^i/w_t^i \text{ is a function only of } z_t.}$$

If the conjective form of the solution, that $x_t^i = \theta^i x_t$ for all t, actually is true, we have two expressions

$$x_t^i = \theta^i x_t = \xi(z_t)\, w_t^i .$$

This clearly can't happen; if we do Itô expansion of $\xi(z_t)\, w_t^i$, there is a spare term in dB^i which cannot be matched against the martingale terms in the expansion of $\theta^i x_t$.

Conclusion: The conjectured pooling of risk does not arise if we assume that individual agents behave as price takers and then invest and consume optimally.

Pooling resources: general situation (31/7/96)

(i) We are going to consider N agents, whose bank accounts x^i_t at time t satisfy

$$dx^i_t = x^i_t\,(r\,dt + \varepsilon_i\, dB^i_t) - c^i_t\, dt$$

for independent Brownian motions B^i, and consumption streams c^i. They also hold shares which are traded on open market at mid-price $S_t = \exp[\sigma W_t + (b-\frac{1}{2}\sigma^2)t]$, but which it costs the agents $(1+\delta_+)S_t$ to buy, and they get only $(1-\delta_-)S_t$ when they sell. The agents are able to trade the shares costlessly among themselves at some price $\tilde{S}_t$, and the return is $d\eta_t \equiv d\tilde{S}_t/\tilde{S}_t$. The wealth process of agent i thus satisfies

$$dw^i_t = w^i_t\, d\eta_t + \varphi^i_t\,(r\,dt + \varepsilon_i\, dB^i_t - d\eta_t) - c^i_t\, dt.$$

Let's suppose that the return $d\eta_t$ takes the form

$$d\eta_t = \alpha(w_t)\, dW_t + \theta(w_t)\cdot dB_t + \mu(w_t)\, dt$$

with coefficients depending on the vector $w_t \equiv (w^i_t)_{i=1}^N$, and that each agent is behaving optimally as a price-taker faced with this price system*. The aim is to find a pricing system in which market clearing occurs, in this sense: the total number of shares held by the pool remains constant most of the time, but when it changes, the total holdings of the pool's bank accounts gets reduced by the transactions costs incurred in the open-market trading.

(ii) The individual agent solves a conventional optimal investment/consumption problem, and ends up with HJB equations for $V \equiv V^i$

$$\sup_{c^i,\varphi^i}\Big[U_i(c^i) - \rho_i V^i + \{(w^j-\varphi^j)\,\mu + r\varphi^j - c^j\}\, V^i_j$$

$$+ \tfrac{1}{2} V^i_{jk}\{ (w^j-\varphi^j)(w^k-\varphi^k)(\alpha^2+|\theta|^2) + (w^j-\varphi^j)\,\theta_k\varepsilon_k\varphi^k$$

$$+ (w^k-\varphi^k)\,\theta_j\varepsilon_j\varphi^j + \delta_{jk}\,\varepsilon_j\varepsilon_k\,\varphi^j\varphi^k\}\Big] = 0,$$

where the φ^j and c^j $(j\neq i)$ must solve their respective HJB equations. By performing the maximisation, we obtain

---

* Agent i tries to obtain $V^i(w) \equiv \sup E\left[\int_0^\infty e^{-\rho_i t}\, U_i(c^i_t)\, dt \,\middle|\, w_0 = w\right]$

$$\boxed{\begin{aligned} & U_i'(c_i) = V_i^i \\ & -\varphi^i\left[\varepsilon_i^2 - 2\theta_i\varepsilon_i + \alpha^2 + |\theta|^2\right] V_{ii}^i = (r-\mu)V_i^i + \sum_{j\neq i} V_{ji}^i \left\{(\varepsilon_i\theta_i - \alpha^2 - |\theta|^2)(w^j - \varphi^j) - \varepsilon_j\theta_j\varphi^j\right\} \end{aligned}}$$

and the form of the HJB equation will be

$$\tilde{U}_i(V_i^i) - \rho_i V^i + \sum_{j\neq i}\left((w^j-\varphi^j)\mu + r\varphi^j - c^j\right)V_j^i$$

$$+ \frac{1}{2}\sum_{j\neq i}\sum_{k\neq i} V_{jk}^i\left((\alpha^2+|\theta|^2)(w^j-\varphi^j)(w^k-\varphi^k) + 2(w^j-\varphi^j)\varepsilon_k\theta_k\varphi^k + \varepsilon_j\varepsilon_k\varphi^j\varphi^k\delta_{jk}\right)$$

$$- \frac{1}{2V_{ii}^i}\left[(r-\mu)V_i^i + \sum_{j\neq i} V_{ji}^i\left\{(\varepsilon_i\theta_i - \alpha - |\theta|^2)(w^j-\varphi^j) - \varepsilon_j\theta_j\varphi^j\right\}\right]^2\left(\varepsilon_i^2 - 2\theta_i\varepsilon_i + \alpha^2 + |\theta|^2\right)^{-1} = 0,$$

together with some boundary conditions.

(iii) If we consider the market clearing conditions, we must have

$$\psi_t \equiv \psi(w_t) \equiv \sum_i (w_t^i - \varphi_t^i)/\tilde{S}_t$$

should be the number of shares held by the pool at time t. Most of the time we expect this to be constant. When ψ grows, we shall have to have

$$dx_t \equiv d\left(\sum \varphi_t^i\right) = rx_t\,dt + \sum \varphi_t^i\,\varepsilon_i\,dB_t^i - \tilde{S}_t\,d\psi_t,$$

and $\tilde{S}_t = (1+\delta_+)S_t$ at such times, $\tilde{S}_t = (1-\delta_-)S_t$ at times when ψ decreases. This tallies with what's happening in terms of market mid value $z_t \equiv \psi_t S_t + x_t$, because

$$dz_t = \psi_t\,dS_t + S_t\,d\psi_t + dx_t$$

$$= \psi_t\,dS_t + S_t\,d\psi_t + rx_t\,dt + \sum \varphi_t^i\,\varepsilon_i\,dB_t^i - (1+\delta_+)S_t\,d\psi_t$$

$$= \psi_t\,dS_t + rx_t\,dt + \sum \varphi_t^i\,\varepsilon_i\,dB_t^i - \delta_+ S_t\,d\psi_t,$$

which is exactly what the dynamics should be.

__NB__ Neither S nor $\tilde{S}$ has a local time component, yet $\tilde{S}/S$ remains in $[1-\delta_-, 1+\delta_+]$, and reaches the endpoints — so it has to be kept in by some BESQ kind of effect... !!

Pooling with log utility (10/8/96)

(i) Suppose an agent with log utility gets the chance to invest in an n-vector of risky assets with return

$$\sigma_t \, dW_t + \mu_t \, dt,$$

where σ is $n \times n$, but let's suppose there is no riskless asset available. By adjoining a riskless asset with return $\gamma_t \, dt$, we complete the market, and the wealth process x_t of the agent satisfies

$$dx_t = \theta_t \cdot (\sigma_t \, dW_t + (\mu_t - \gamma_t 1)\, dt) + \gamma_t x_t \, dt - c_t \, dt,$$

where θ is the portfolio of risky assets, c is consumption, and if $\beta_t \equiv \exp(-\int_0^t \gamma_s ds)$ and $\tilde{x}_t \equiv \beta_t x_t$, we get

$$d\tilde{x}_t = \beta_t \theta_t \cdot \sigma_t \, dW_t^* - \tilde{c}_t \, dt \qquad (\tilde{c}_t \equiv \beta_t c_t)$$

where $dW_t^* \equiv dW_t + \sigma_t^{-1}(\mu_t - \gamma_t 1)\, dt$ is BM in risk-neutral probability. As is completely standard, if the agent tries to max $E(\int_0^\infty e^{-\rho t} U(c_t)\, dt)$ the optimal c will satisfy

$$e^{-\rho t} U'(c_t) = \lambda \zeta_t$$

for some constant λ. This being the case, the P^*-martingale

$$M_t \equiv E_t^* \left[\int_0^\infty \tilde{c}_t \, dt \right] = \int_0^t \tilde{c}_s \, ds + E_t \left[\int_t^\infty \zeta_s c_s \, ds \right] / \zeta_t \beta_t^{-1}$$

$$= \int_0^t \tilde{c}_s \, ds + \frac{\beta_t}{\zeta_t} \cdot \int_t^\infty e^{-\rho s} \lambda^{-1} ds$$

$$= \int_0^t \tilde{c}_s \, ds + \frac{\beta_t e^{-\rho t}}{\lambda \rho \zeta_t} = \int_0^t \tilde{c}_s \, ds + \tilde{x}_t$$

allows us to deduce that under optimal control $(a_t \equiv \sigma_t \sigma_t^T)$

$$\boxed{x_t = \frac{e^{-\rho t}}{\lambda \rho \zeta_t}, \quad c_t = \rho x_t, \quad \theta_t = x_t \, a_t^{-1}(\mu_t - \gamma_t 1)}$$

For zero investment in the (fictitious) riskless asset, the condition is

$$\boxed{\gamma_t = (1^T a_t^{-1} \mu_t - 1) / 1^T a_t^{-1} 1}$$

$$\sigma_t^i = \begin{pmatrix} \varepsilon_i \; e_i \\ \psi \end{pmatrix}$$

In the special case $n=2$, $a^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$, $\Delta \equiv \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$,

$$\boxed{x_t^{-1}\theta_t = \begin{pmatrix} \Delta \\ -\Delta \end{pmatrix} \frac{\mu_1 - \mu_2}{\alpha_{11} + \alpha_{12} + \alpha_{21} + \alpha_{22}} + \begin{pmatrix} \alpha_{11} + \alpha_{12} \\ \alpha_{21} + \alpha_{22} \end{pmatrix} \frac{1}{\alpha_{11} + \alpha_{12} + \alpha_{21} + \alpha_{22}}}$$

(ii) Suppose now we have agents $1, 2, \dots, N$, and agent i can invest in his own (risky) bank account with return $\varepsilon_i\, dV_t^i + r\,dt$, or in the 'pool share' with return

$$dy = \psi \cdot dW + \kappa\, dt$$

where the market mid price S_t of the share is $dS_t = (\sigma\, dW_t^0 + b\,dt)\, S_t$, and the characteristics of the pool price remain to be determined.

We have

$$(a_t^i)^{-1} = \Delta_i^{-1} \begin{pmatrix} |\psi|^2 & -\varepsilon_i\psi_i \\ -\varepsilon_i\psi_i & \varepsilon_i^2 \end{pmatrix} \qquad \Delta_i \equiv \varepsilon_i^2 (|\psi|^2 - \psi_i^2)$$

so that

$$\boxed{\theta_t^i / x_t^i = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{(r - \kappa)}{|\psi - \varepsilon_i e_i|^2} + \begin{pmatrix} |\psi|^2 - \varepsilon_i\psi_i \\ \varepsilon_i(\varepsilon_i - \psi_i) \end{pmatrix} \cdot \frac{1}{|\psi - \varepsilon_i e_i|^2}},$$

and

$$\boxed{\gamma_t^i = \frac{(|\psi|^2 - \varepsilon_i\psi_i) r + (\varepsilon_i - \psi_i)\varepsilon_i \kappa - \varepsilon_i^2(|\psi|^2 - \psi_i^2)}{|\psi - \varepsilon_i e_i|^2}}$$

Also,

$$\begin{aligned} d\zeta_t^i / \zeta_t^i &= -\gamma_t^i\, dt - (\sigma_t^i)^{-1}(\mu - \gamma_t^i 1) \cdot dW_t \\ &= -\gamma_t^i\, dt - (a_t^i)^{-1}(\mu - \gamma_t^i 1) \cdot \sigma_t^i\, dW_t \\ &= -\gamma_t^i\, dt - (\theta_t^i / x_t^i) \cdot \sigma_t^i\, dW_t \end{aligned}$$

If we define two $(N+1)$-vectors

$$u_i \equiv \frac{\varepsilon_i e_i - \psi}{|\varepsilon_i e_i - \psi|}, \quad v_i \equiv \left\{ \psi - \frac{e_i \langle \psi, u_i \rangle}{\langle u_i, e_i \rangle} \right\}$$

we have that u_i and v_i are orthogonal; and

$$(\theta^i/x^i)^T \sigma^i = \frac{r-\kappa}{|\psi - \varepsilon_i e_i|} u_i + \frac{\varepsilon_i \langle u_i, e_i\rangle}{|\psi - \varepsilon_i e_i|^2} v_i .$$

Total cash holding in the share will be

$$y_t \equiv \sum_{i=1}^N (0,1)\theta^i_t = \sum_{i=1}^N \frac{x^i_t}{|\psi_t - \varepsilon_i e_i|^2} \{ \kappa_t - r + \varepsilon_i(\varepsilon_i - \psi^2_t) \}$$

and while the pool is not dealing with the market it has to be that

$$dy_t = y_t \, d\eta_t = y_t(\psi_t \, dW_t + \kappa_t dt)$$

For simplicity, let's assume until further notice that $\boxed{\varepsilon_i = \varepsilon \text{ for all } i}$

$$dx^i_t / x^i_t = \frac{\theta^i_t}{x^i_t} \sigma^i_t dW_t + \left\{ \gamma^i_t - \rho + \left| (\sigma^i_t)^T \theta^i_t / x^i_t \right|^2 \right\} dt$$

$$= \left(\frac{r-\kappa}{|\psi - \varepsilon e_i|} u_i^T + \frac{\varepsilon \langle u_i, e_i\rangle}{|\psi - \varepsilon e_i|^2} v_i^T \right) dW_t$$

$$+ \left\{ \gamma^i_t - \rho + \frac{(r-\kappa)^2}{|\psi - \varepsilon_i e_i|^2} + \frac{\varepsilon_i^2 (\varepsilon_i - \psi_i)^2}{|\psi - \varepsilon_i e_i|^6} (|\psi|^2 - |\psi_i|^2) \left\{ 1 + \frac{|\psi|^2 - \psi_i^2}{(\varepsilon - \psi_i)^2} \right\} \right\}$$

(iv) At this level of generality, it seems to be wholly intractable, so let's suppose we're going to search for a solution with $\varepsilon_i = \varepsilon$ for all i

$$\boxed{\psi^0 = \alpha, \quad \psi^i = \varphi \text{ for } i = 1, \dots, N}$$

In this case,

$$\gamma^i_t = \gamma_t \text{ for all } i, \qquad \theta^i_t / x^i_t \equiv \pi_t \equiv (\pi^1_t, \pi^2_t) \text{ for all } i$$

so if $x_t \equiv \sum_{i=1}^N x^i_t$ denotes the total wealth of the pool, then

$$dx_t = -\rho x_t dt + x_t \pi_t \cdot \mu_t dt + x_t \pi^2_t \psi_t dW_t + \varepsilon \sum x^i_t \pi^1_t dW^i_t$$

$$= x_t \left[-\rho dt + \pi_t \cdot \mu_t dt + \pi^2_t \psi_t \cdot dW_t + \varepsilon \pi^1_t dM_t \right]$$

where $dM_t = \left(\sum x^i_t dW^i_t \right) / x_t$.

Likewise, if we set

$$\pi_t^2 \equiv \frac{K_t - r + \varepsilon^2 - \varepsilon\varphi_t}{|\psi_t - \varepsilon q|^2} \equiv Y_t / Z_t$$

then

$$\frac{d\pi_t^2}{\pi_t^2} = \frac{dY_t}{Y_t} - \frac{dZ_t}{Z_t} + \frac{d\langle Z \rangle_t}{Z_t^2} - \frac{dY}{Y} \cdot \frac{dZ}{Z}$$

$$= \frac{dK_t - \varepsilon d\varphi_t}{K_t - r + \varepsilon^2 - \varepsilon\varphi_t} - \frac{2\alpha\, d\alpha + 2(N\varphi - \varepsilon)d\varphi + d\langle\alpha\rangle + N\, d\langle\varphi\rangle}{Z_t}$$

$$+ 4\, \frac{\alpha^2 d\langle\alpha\rangle + 2\alpha(N\varphi - \varepsilon)\, d\langle\alpha,\varphi\rangle + (N\varphi - \varepsilon)^2\, d\langle\varphi\rangle}{Z_t^2}$$

$$- (dK - \varepsilon d\varphi)(2\alpha d\alpha + 2(N\varphi - \varepsilon)d\varphi) \;/\, YZ$$

gives π^2 as an exponential semimg. We should have

$$d(\pi^2 x) = \pi^2 x \quad (\psi dW + K dt)$$

max $U(c) - \lambda c$ when $e^{-\gamma c} = \lambda$: $\tilde{u}(\lambda) = -\lambda/\gamma + \frac{\lambda}{\gamma} \log(\lambda)$

Modelling returns as a Lévy process (16/9/96)

Ernst Eberlein finds that by expressing $S_t = S_0 \exp[Z_t]$, where Z is a suitable Lévy process, one gets impressive matching of market data. For pricing, there are too many EMMs, so he chooses one (in a somewhat arbitrary way). Better would be to postulate an agent with utility U, representative of the market, trying to

$$\max E \int_0^\infty \exp(-\delta t)\, U(c_t)\, dt,$$

where the wealth process X solves

$$dX_t = rX_t\, dt + \theta_t \{ dS_t/S_{t-} - r dt \} - c_t\, dt$$

From now on, assume

$$U(x) = -\gamma^{-1} e^{-\gamma x}, \quad d\tilde{Z} \equiv dS/S_- \text{ has Brownian part } \sigma W_t, \text{ drift } b, \text{ Lévy measure } \mu,$$

concentrated on $(-1, \infty)$

Usual tale: let

$$V(x) \equiv \sup E\left[\int_0^\infty e^{-\delta t} U(c_t)\, dt \;\middle|\; X_0 = x \right]$$

so that the HJB gives (assuming $\tilde{Z}$ has finite variation for simplicity)

$$\sup_{c,\theta} \left[U(c) - \delta V + (rX - r\theta - c)V' + b\theta V' + \tfrac12 \sigma^2\theta^2 V'' + \int \{ V(x+\theta z) - V(x) \} \mu(dz) \right] = 0$$

$$\therefore \quad \tilde{u}(V') - \delta V + (rX - r\theta + b\theta)V' + \tfrac12 \sigma^2 \theta^2 V'' + \int (V(x+\theta z) - V(x))\, \mu(dz) \le 0,$$

with equality at optimal θ.

We have here

$$V(x) = -\kappa\, e^{-\gamma x r}, \quad \tilde{u}(\lambda) = \frac{\lambda}{\gamma}(\log\lambda - 1), \quad c(\lambda) = -\frac{1}{\gamma}\log\lambda$$

for some constant $\kappa > 0$, so this will allow us to simplify:

$$\sup_\theta \; \kappa r \{ \log(\gamma\kappa r \cancel{e^{-\gamma r x}}) - 1 \} + \delta\kappa + \gamma\kappa r(\cancel{rx} - r\theta + b\theta) - \frac{\kappa}{2}(\sigma\theta\gamma r)^2$$
$$- \kappa \int (e^{-\theta\gamma r z} - 1)\, \mu(dz) \qquad = 0.$$

So if we introduce

$$\psi(\alpha) = \tfrac12 \sigma^2 \alpha^2 + b\alpha + \int (e^{\alpha z} - 1)\, \mu(dz)$$

we get more simply

$$\sup_\theta \; r\{\log(\gamma\kappa r) - 1\} + \delta - \psi(-\theta\gamma r) - r^2\theta\gamma = 0. \qquad \text{Set } \tilde{\psi}(x) \equiv \psi(x) - rx.$$

So if $\tilde{\psi}$ is minimised at α^*, we have $\boxed{\theta^* = -\alpha^*/\gamma r}$ and

$$\boxed{c_t^* = rX_t - \frac{1}{\gamma}\log(\gamma\kappa r)}$$

This gives us a simple story for the optimal wealth process:

$\tilde{Z}_t - rt$ is a mg

$$dX_t = \theta^* \left[d\tilde{Z}_t - r\,dt \right] + \frac{1}{\gamma} \log(\gamma\kappa r)\, dt,$$

so that

$$X_t = x_0 + \theta^* \tilde{Z}_t + \frac{1}{\gamma}\{\log(\kappa\gamma r) + \alpha^*\}\, t$$

Now we have that the EMM is given by

$$e^{-rt} \left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t} = e^{-rt} \rho_t \propto e^{-\delta t} u'(c_t) \propto e^{-\delta t} \exp(-\gamma r X_t)$$

$$\propto \exp\left[\alpha^* \tilde{Z}_t - \delta t - r\{\log(\kappa\gamma r) + \alpha^*\} t \right]$$

Hence

$$\boxed{\rho_t = \exp[\, \alpha^* \tilde{Z}_t - \psi(\alpha^*)\, t \,]}$$

Notice several interesting features of the solution:

(i) The risk-neutral measure doesn't depend on agent's preferences

(ii) Under P^*, $\tilde{Z}_t - rt$ is a martingale

(iii) The Esscher transform choice of measure used by Ernst is justified in this simple equilibrium framework.

(iv) Optimal consumption is

$$c_t^* = r X_t + \frac{1}{\gamma r}\{ \delta - r - \tilde{\psi}(\alpha^*) \}$$

Mixed Markov chains: simple examples (9/10/96)

(i) Let's consider the evolution of the distribution of the types in a large population governed by

$$\dot{x}_i(t) = \sum_j \sum_k x_j(t)\, x_k(t)\, q^k_{ji}$$

where each Q^k is a Q-matrix. Firstly, let's just suppose there are 2 types,

$$Q^k = \begin{pmatrix} -\alpha_k & \alpha_k \\ \beta_k & -\beta_k \end{pmatrix}$$

and $x_1(t) + x_2(t) = 1$, so we get

$$\dot{x}_1 = -x_1^2 \alpha_1 + x_1 x_2 (\beta_1 - \alpha_2) + x_2^2 \beta_2$$

$$= x_1^2 (\beta_2 - \alpha_1 - \beta_1 + \alpha_2) + x_1 (\beta_1 - \alpha_2 - 2\beta_2) + \beta_2$$

This quadratic takes values $\beta_2 > 0$ at $x_1 = 0$, and $-\alpha_1 < 0$ at $x_1 = 1$, so there is a unique root ρ_1 in $(0,1)$. If ρ_2 is the other root, and $a \equiv \beta_2 - \alpha_1 - \beta_1 + \alpha_2$, we get

$$\dot{x}_1 = a\,(x_1 - \rho_1)(x_1 - \rho_2)$$

which has explicit solution

$$\boxed{x_1(t) = \frac{\rho_1 (x_1(0) - \rho_2) - \rho_2 (x_1(0) - \rho_1)\, e^{-a(\rho_2 - \rho_1)t}}{x_1(0) - \rho_2 - (x_1(0) - \rho_1)\, e^{-a(\rho_2 - \rho_1)t}}}$$

(Note that $a(\rho_2 - \rho_1) > 0$ if $a \neq 0$)

(ii) To gain insight into the 3-type case, let's suppose we have the structure

$$Q^1 = \begin{pmatrix} -\alpha_1 & \alpha_1 & 0 \\ \beta_1 & -\beta_1 & 0 \\ \gamma_1 & 0 & -\gamma_1 \end{pmatrix}, \quad Q^2 = \begin{pmatrix} -\alpha_2 & \alpha_2 & 0 \\ \beta_2 & -\beta_2 & 0 \\ 0 & \gamma_2 & -\gamma_2 \end{pmatrix}, \quad Q^3 = \begin{pmatrix} -\theta & 0 & \theta \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{pmatrix}.$$

If the population were all in $\{1, 2\}$ they'd stay there, or if they were all type 3, but in between there is a conflict; will the 3's die out, dominate, or co-exist?

$$\begin{pmatrix} -\alpha_1 & \alpha_1 & 0 \\ \beta_1 & -\beta_1 & 0 \\ \gamma_1 & 0 & -\gamma_1 \end{pmatrix} \quad \begin{pmatrix} -\alpha_2 & \alpha_2 & 0 \\ \beta_2 & -\beta_2 & 0 \\ 0 & \gamma_2 & -\gamma_2 \end{pmatrix} \quad \begin{pmatrix} -\theta & 0 & \theta \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left.\begin{aligned}\dot{x}_1 &= -\alpha_1 x_1^2 + \beta_2 x_2^2 + x_1 x_2(\beta_1 - \alpha_2) + x_1 x_3(-\theta + \gamma_1)\\ \dot{x}_2 &= \alpha_1 x_1^2 - \beta_2 x_2^2 + x_1 x_2(-\beta_1 + \alpha_2) + x_2 x_3(-\lambda + \gamma_2)\end{aligned}\right\}$$

What are the fixed points of this equation? We'd have to have $\dot{x}_1 + \dot{x}_2 = 0$, which gives us that

$$\boxed{x_1(\gamma_1 - \theta) = x_2(\lambda - \gamma_2)} \qquad (1)$$

(so we only can have a fixed point with $x_1 > 0$, $x_2 > 0$ if $(\gamma_1 - \theta)(\lambda - \gamma_2) > 0$). Writing $m^{-1} \equiv (\lambda - \gamma_2)/(\gamma_1 - \theta)$, assuming this is positive, we get that the unique fixed point with $x_1\, x_2\, x_3 > 0$ is given by

$$\boxed{x_1 = \frac{\theta - \gamma_1}{\beta_2 m^2 + m(\beta_1 - \alpha_2) - \alpha_1 + (1+m)(\theta - \gamma_1)} = m^{-1} x_2}$$

BUT this is only going to be OK if the values of x_1, x_2 satisfy $x_1 > 0, x_2 > 0$, $x_1 + x_2 < 1$. Assuming (wmlog) that $\boxed{\theta - \gamma_1 > 0,}$ we shall need (for $x_1 < 1$) that

$$\beta_2 m^2 + m(\beta_1 - \alpha_2 + \theta - \gamma_2) - \alpha_1 > 0$$

and for $x_1 + x_2 < 1$ we need the stronger condition

$$\boxed{\beta_2 m^2 + m(\beta_1 - \alpha_2) - \alpha_1 > 0} \qquad (2)$$

We can intepret these somehow. If we're in equilibrium at (x_1, x_2, x_3), the flux from $\{1,2\}$ to $\{3\}$ is $x_1 x_3\, \theta + x_2 x_3\, \lambda$, which has to balance the flux in the reverse direction, $x_1 x_3\, \gamma_1 + x_2 x_3\, \gamma_2$. That's where (1) comes from.

Next, we would have the flux $1 \to 2$ will be $\alpha_1 + m\alpha_2$ $\quad (m \equiv x_2/x_1)$ and flux $2 \to 1$ is $(\beta_1 + m\beta_2)$ (or at least proportional to these). Condition (2) can be rephrased as

$$m(\beta_1 + m\beta_2) > \alpha_1 + m\,\alpha_2$$

which just says that if we maintained the ratio $1:m$, the chain just in $\{1,2\}$ would want to spend more time in 1, and that is the state from which you pass more easily to 3 than back.

If $X_t = \sigma W_t + \mu t \equiv \tilde{W}(\sigma^2 t) + c.\sigma^2 t$ $\qquad (c \equiv \mu\sigma^{-2})$
we shall have correspondingly

$$q_t(x,y) = \exp\{c(y-x) - \tfrac{1}{2}c^2\sigma^2 t\} \sum_{k\in\mathbb{Z}} \{ p_{\sigma^2 t}(x+2ka, y) - p_{\sigma^2 t}(x+2ka, y) \}$$

$$= \exp\{c(y-x) - \tfrac{1}{2}c^2\sigma^2 t\} \; \frac{2}{a} \sum_{k\geq 1} \sin\frac{k\pi x}{a} \sin\frac{k\pi y}{a} \exp\left(-k^2 \frac{\pi^2\sigma^2 t}{2a^2}\right)$$

and

$$h_t(x) \equiv \int_0^a q_t(x,y)\,dy$$

$$= \sum_{k\in\mathbb{Z}} \left[e^{2kac} \left\{ \Phi\left(\frac{a-x-2ka-c\sigma^2 t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{-x-2ka-c\sigma^2 t}{\sigma\sqrt{t}}\right) \right\} \right.$$
$$\left. - e^{2kac-2cx} \left\{ \Phi\left(\frac{a+x-2ka-c\sigma^2 t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{x-2ka-c\sigma^2 t}{\sigma\sqrt{t}}\right) \right\} \right]$$

$$= \exp\left[-cx - \tfrac{1}{2}c^2\sigma^2 t\right] \sum_{k\geq 1} \frac{2}{a} \sin\left(\frac{k\pi x}{a}\right) \exp\left(-k^2 \frac{\sigma^2\pi^2 t}{2a^2}\right) \frac{\frac{k\pi}{a}\left(1 - e^{ca}(-1)^k\right)}{c^2 + k^2\pi^2/a^2}$$

First exit of a drifting BM from an interval (10/10/96)

(i) Let's consider a BM with drift $X_t = W_t + ct$ in $[0, a]$, killed when it exits. From Cowk (p 47) we obtain the result

$$q_t(x,y) = \exp\left(c(y-x) - \tfrac{1}{2}c^2t\right) \sum_{n\geq 1} \frac{2}{a} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \exp\left\{-\frac{t}{2}\left(\frac{n\pi}{a}\right)^2\right\}$$

for the killed transition density, so

$$h_t(x) \equiv \int_0^a q_t(x,y)\,dy = \sum_{n\geq 1} \frac{2}{a} \sin\left(\frac{n\pi x}{a}\right) \exp\left[-cx - \tfrac{1}{2}c^2t - \tfrac{1}{2}t\left(\frac{n\pi}{a}\right)^2\right] \left(\frac{n\pi}{a}\right) \frac{1 - e^c(-1)^n}{c^2 + n^2\pi^2/a^2}$$

So if we want to know

$$E^x\left[L(T,\xi)\,;\; T < \tau\right],$$

where $\tau = \inf\{u : X_u \notin [0,a]\}$, $x, \xi \in (0,a)$, we need to have

$$\int_0^T q_t(x,\xi)\, h_{T-t}(\xi)\, dt$$

$$= \sum_{n\geq 1}\sum_{m\geq 1} \int_0^T dt\, \frac{4}{a^2} \sin\frac{n\pi x}{a} \sin\frac{n\pi\xi}{a} \sin\frac{m\pi\xi}{a}\, \frac{m\pi}{a}\, \frac{1-e^c(-1)^m}{c^2+(m\pi/a)^2} \exp\left[-\frac{t}{2}\left(\frac{n\pi}{a}\right)^2 - \frac{T-t}{2}\left(\frac{m\pi}{a}\right)^2\right].$$

$$\cdot \exp\left(-cx - \tfrac{1}{2}c^2T\right)$$

$$= \frac{4}{a^2} e^{-cx-\frac{1}{2}c^2T} \sum_{m,n\geq 1} \left(\sin\frac{n\pi x}{a} \sin\frac{n\pi\xi}{a} \sin\frac{m\pi\xi}{a}\right) \frac{m\pi}{a}\, \frac{1-e^c(-1)^m}{c^2+(m\pi/a)^2}.$$

$$\cdot \frac{a^2}{\pi^2(m^2-n^2)}\left[e^{-(n\pi/a)^2 T/2} - e^{-(m\pi/a)^2 T/2}\right]$$

(ii) This isn't very helpful. But perhaps we could approximate by assuming that we're going to condition the process never to leave $(0,a)$. This amounts to h-transforming the semigroup (q_t) by the harmonic function

$$h(t,x) = \exp\left[\tfrac{1}{2}\theta^2 t - cx\right] \sin(\pi x/a) \qquad \theta^2 \equiv c^2 + \pi^2/a^2.$$

This gives the new ~~transition~~ f^n as

$$\tilde{q}_t(x,y) \equiv \frac{q_t(x,y)\,h(t,y)}{h(0,x)} = \frac{\sin(\pi y/a)}{\sin(\pi x/a)} \sum_{n\geq 1} \frac{2}{a} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \exp\left(-\frac{t}{2}(n^2-1)\pi^2/a^2\right)$$

which is independent of the drift, rather amazingly.

$$\beta\sqrt{t} - \frac{x}{\sqrt{t}} \equiv z \qquad \therefore \sqrt{t} = \left(z + \sqrt{z^2 + 4\beta x}\right)/2\beta$$

$$\frac{dt}{2\sqrt{t}} = \frac{1}{2\beta}\left[1 + \frac{z}{\sqrt{z^2+4\beta x}}\right]dz = \frac{\sqrt{t}\,dz}{\sqrt{z^2+4\beta x}}$$

Easily, we get

$$\boxed{\begin{aligned} e^{cx}\int_0^T x\, e^{-x^2/2t - \frac{1}{2}c^2 t}\,\frac{dt}{\sqrt{2\pi t^3}} &= P^0\left[H_x < T\right] \qquad \text{for BM with drift } c > 0 \\ &= \bar{\Phi}\left(c\sqrt{T} - x/\sqrt{T}\right) + e^{2cx}\,\bar{\Phi}\left(c\sqrt{T} + x/\sqrt{T}\right) \end{aligned}}$$

$$z^2 + 4\beta x \equiv v^2 \quad , \quad z\,dz = v\,dv$$

It may well be better to use the 'reflection principle' form of q:

$$\tilde{q}_t(x,y) = e^{-\pi^2 t/2a^2} \frac{\sin(\pi y/a)}{\sin(\pi x/a)} \sum_{n\in\mathbb{Z}} \left\{ p_t(x+2na, y) - p_t(-x+2na, y) \right\}$$

We therefore need to compute (with $\pi/a \equiv \beta$ for short) assuming $x>0$

$$\psi(T,x) \equiv \int_0^T e^{-\frac{1}{2}\beta^2 t} e^{-x^2/2t} \frac{dt}{\sqrt{2\pi t}} = e^{-\beta x} \int_0^T \exp\left[-\tfrac{1}{2}\left(\beta\sqrt{t} - x/\sqrt{t}\right)^2 \right] \frac{dt}{\sqrt{2\pi t}}$$

$Z \equiv \beta\sqrt{T} - x/\sqrt{T}$

$$= e^{-\beta x} \int_{-\infty}^{Z} \exp(-\tfrac{1}{2}z^2) \sqrt{\frac{2}{\pi}} \frac{z + \sqrt{z^2+4\beta x}}{2\beta} \cdot \frac{dz}{\sqrt{z^2+4\beta x}}$$

$$= \frac{e^{-\beta x}}{2\beta} \int_{-\infty}^{Z} e^{-z^2/2} \left(1 + \frac{z}{\sqrt{z^2+4\beta x}}\right) \sqrt{\frac{2}{\pi}}\, dz$$

$$= \frac{e^{-\beta x}}{\beta} \left\{ \Phi(Z) + \int_{-\infty}^{Z} \frac{z\, dz\, e^{-z^2/2}}{\sqrt{z^2+4\beta x}\cdot\sqrt{2\pi}} \right\}$$

$$= \frac{e^{-\beta x}}{\beta} \left\{ \Phi(Z) - \int_{Z}^{\infty} \frac{z\, dz\, e^{-z^2/2}}{\sqrt{z^2+4\beta x}\cdot\sqrt{2\pi}} \right\}$$ by antisymmetry of the integrand

$$= \frac{e^{-\beta x}}{\beta} \left\{ \Phi(Z) - \int_{(Z^2+4\beta x)^{\frac{1}{2}}}^{\infty} \exp(-\tfrac{1}{2}v^2 + 2\beta x) \frac{v\, dv}{\sqrt{2\pi}\, v} \right\}$$

$$= \frac{e^{-\beta x}}{\beta} \Phi(Z) - \frac{e^{\beta x}}{\beta} \bar{\Phi}\left(\sqrt{Z^2+4\beta x}\right)$$

$$= \frac{e^{-\beta|x|}}{\beta} \Phi\left(\beta\sqrt{T} - \frac{|x|}{\sqrt{T}}\right) - \frac{e^{\beta|x|}}{\beta} \bar{\Phi}\left(\beta\sqrt{T} + \frac{|x|}{\sqrt{T}}\right)$$

using finally the assumption $x>0$, and the symmetry of the integral in x. We can re-express ψ as

$$\psi(T,x) = \exp\left(-\frac{x^2}{2T} - \beta^2 T/2\right) \int_0^\infty \frac{dz}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2} - \frac{z|x|}{\sqrt{T}}\right\} \frac{2\sinh(\beta z\sqrt{T})}{\beta}$$

$$= \int_0^\infty dz'\, e^{-\beta^2 T/2}\, p_T(0, z'+|x|) \frac{2\sinh\beta z'}{\beta}$$

We can write

$$\tilde{q}_t(x,y) = e^{-\beta^2 t/2} \frac{\sin \beta y}{\sin \beta x} \Big[\sum_{k\geq 0} \{ p_t(x-2ka, y) - p_t(2ka+2a-x, y) \}$$
$$- \sum_{k \geq 0} \{ p_t(-2ka - x, y) - p_t(2ka + 2a + x, y) \} \Big]$$

and each of the terms in each of the sums is non-negative; so

$$\int_0^T \tilde{q}_t(x,y)\, dt = \frac{\sin \beta y}{\sin \beta x} \Big[\sum_{k\geq 0} \{ \psi(T, x-y-2ka) - \psi(T, x+y-2ka-2a) \}$$
$$- \sum_{k\geq 0} \{ \psi(T, -x-y-2ka) - \psi(T, -x+y-2ka-2a) \} \Big]$$

and this should be reasonably easy to handle.

(iii) For the application in mind, we are interested in $\int_0^T \tilde{q}_t(x,y)\,dt$ for y very close to 0, so let's rewrite

$$\tilde{q}_t(x,y) = e^{-\beta^2 t/2} \frac{\sin(\beta y)}{\sin(\beta x)} \sum_{n\in\mathbb{Z}} \{ p_t(x+2na, y) - p_t(x+2na, -y) \}$$

and then

$$\int_0^T \tilde{q}_t(x,y) = \frac{\sin\beta y}{\sin\beta x} \sum_{n\in\mathbb{Z}} \{\psi(T, x+2na-y) - \psi(T, x+2na+y)\}$$

$$= \frac{\sin\beta y}{\sin\beta x}\Big[\sum_{n\geq 0} \int_0^\infty dz\, e^{-\beta^2 T/2} \frac{2\sinh\beta z}{\beta} \{p_T(0, x+2na-y+z) - p_T(0, x+2na+y+z)\}$$

$$- \sum_{n\geq 1} \int_0^\infty dz\, e^{-\beta^2 T/2} \frac{2\sinh\beta z}{\beta} \{ p_T(0, z+2na-x-y) - p_T(0, z+2na-x+y)\}\Big]$$

Take the square bracket, divide by y, and let $y \downarrow 0$; we get

$$e^{-\beta^2 T/2}\Big[\sum_{n\geq 0} \int_0^\infty dz \frac{x+2na+z}{T} p_T(0, x+2na+z) \frac{4\sinh\beta z}{\beta} - \sum_{n\geq 1} \int_0^\infty dz \frac{z+2na-x}{T} p_T(0, z+2na-x) \frac{4\sinh\beta z}{\beta}\Big]$$

$$= e^{-\beta^2 T/2}\Big[\sum_{n\geq 0} \int_0^\infty 4\cosh\beta z\, p_T(0, z+x+2na)\,dz - \sum_{n\geq 1} \int_0^\infty 4\cosh\beta z\, p_T(0, z+2na-x)\,dz\Big]$$

$$= e^{-\beta^2 T/2} \sum_{n\geq 0} \int_0^\infty 4\cosh\beta z \{p_T(0, z+2na+x) - p_T(0, z+2na+2a-x)\}\,dz$$

which is a sum of <u>non-negative</u> terms;

$$= 2\sum_{n\geq 0} \exp(-(2n+1)a\beta T)\Big[e^{(a-x)\beta T}\, \bar{\Phi}\Big(\frac{(2n+1)a-\beta T}{\sqrt{T}} - \frac{a-x}{\sqrt{T}}\Big)$$

$$- e^{-(a-x)\beta T}\, \bar{\Phi}\Big(\frac{(2n+1)a-\beta T}{\sqrt{T}} + \frac{a-x}{\sqrt{T}}\Big)\Big]$$

Provided $2na > \beta T \equiv \pi T/a$, we have an upper bound for the n^{th} term in the sum, (using $\cosh x \leq e^x$):

$$\leq \frac{8(a-x)}{2na-\beta T} \exp\Big[-\frac{2n^2a^2}{T} - \beta^2 T/2\Big]$$

which drops very rapidly; we'll only need a few terms in the sum.

(iv) How about one-sided barriers? Taking a BM with drift c and conditioning it to stay in $\mathbb{R}^+$ produces the diffusion with generator

$$\mathcal{G}_+ \equiv \tfrac{1}{2}D^2 + |c| \coth |c|x \, D = \frac{1}{h}\mathcal{G}(h\,\cdot) \qquad , \text{where } h \equiv 1-e^{-2|c|x}.$$

Thus the ~~transition~~ semigp is

$$q_t(x,y) = \{ p_t(x,y) - p_t(-x,y) \} \frac{h(y)}{h(x)} e^{c(y-x)-\frac{1}{2}c^2t}$$

$$\sim y^2 \frac{4x|c|}{\sqrt{2\pi t^3}} \exp\left[-(x+|c|t)^2/2t \right] \cdot \left(\frac{1}{1-e^{-2|c|x}} \right)$$

So, <u>assuming $c>0$</u> (if not, write $|c|$ in place of c) we shall have

$$\int_0^T q_t(x,y)\,dt \sim y^2 \frac{4c}{1-e^{-2cx}} \int_0^T x \exp\left(-\frac{x^2}{2t} - cx - \tfrac{1}{2}c^2t\right) \frac{dt}{\sqrt{2\pi t^3}}$$

$$= y^2 \, 2c \operatorname{csech}(xc) \frac{\partial}{\partial x}\left[-\psi(T,x)\right]$$

using the notation of p 49, replacing β there by c here;

$$= \frac{2cy^2}{\sinh cx} \left\{ e^{-cx}\bar{\Phi}\left(\frac{x}{\sqrt{T}} - c\sqrt{T}\right) + e^{cx}\bar{\Phi}\left(\frac{x}{\sqrt{T}} + c\sqrt{T}\right) \right\}$$

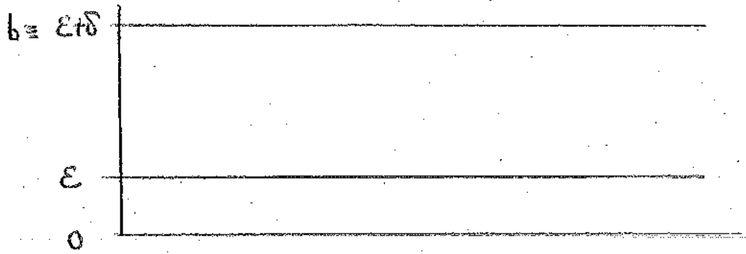
Note that this is certainly no good for x small, but we're interested in the situation where x is $O(1)$.

In fact, we don't need to do any asymptotics in this case:

$$\int_0^T q_t(x,y)\,dt = \frac{\sinh cy}{\sinh cx} \left\{ \psi(T,x-y) - \psi(T,x+y) \right\},$$

using the expression for ψ on p 49, replacing β by c.

(v) Now that we've found an expression for the expected amount of local time at a level before T, given that the barrier(s) not crossed, we need to compute the mean time "lost" by this edge effect.



So we have a BM with drift $c>0$ and we want to know the mean local time at ε before exit $[0,b]$, given that you exit at b. Clearly, it's just

$$2\left\{ \frac{s'(\varepsilon)}{s(b)-s(\varepsilon)} + \frac{s'(\varepsilon)}{s(\varepsilon)-s(0)} \right\}^{-1} = \frac{(e^{2c\varepsilon}-1)(1-e^{-2c\delta})}{c\,(e^{2c\varepsilon}-e^{-2c\delta})}$$

So we shall have roughly

$$\boxed{\begin{aligned} &E^{x}\left[\text{no of visits to } \varepsilon \text{ before } T \,\middle|\, \text{barrier(s) not crossed}\right] \\ &\quad \doteqdot E^{x}\left[L(T,\varepsilon) \,\middle|\, \text{barrier(s) not crossed}\right] \cdot \frac{c\,(e^{2c\varepsilon}-e^{-2c\delta})}{(e^{2c\varepsilon}-1)(1-e^{-2c\delta})} \end{aligned}}$$

Now we have to know the mean time to exit $[0,b]$ given that we go out at b. Solve

$$\tfrac{1}{2}f'' + (c\coth cx)f'(x) + 1 = 0, \qquad f(1)=0$$

$$= \frac{1}{2\sinh^2(cx)} D\left(\sinh^2 cx\, f'(x)\right) + 1$$

$$\therefore\ (\sinh^2 cx)\, f'(x) = -2\int^x \sinh^2 cy\, dy = x - \frac{\sinh 2cx}{2c} + \text{const}$$

$$\therefore\ f'(x) = \frac{x}{\sinh^2 cx} - \frac{\cosh cx}{c\sinh cx} + \text{const}.\ \operatorname{cosech}^2 cx$$

$$\Rightarrow\ f(x) = A - \frac{x\coth cx}{c} + \frac{\text{const}}{c}\coth cx.$$

For boundedness at 0, and $f(b)=0$, we get

$$\boxed{E^{\varepsilon}\left(\tau \mid X(\tau)=b\right) = \left(b\coth cb - \varepsilon\coth c\varepsilon\right)/c}$$

Mean time to exit $(\varepsilon-\delta, \varepsilon+\delta)$ given start at ε is computed similarly: it's

$$\frac{\delta}{c}\tanh c\delta.$$

Hence the expected shortfall at each visit to ε will be

$$\boxed{\frac{\delta}{c}\tanh c\delta - \frac{b\coth cb - \varepsilon\coth c\varepsilon}{c}} \equiv \Delta(\varepsilon,\delta)$$

(vi) Finally, how do we scale in σ? If $X_t = \sigma W_t + \alpha t \overset{\mathcal{D}}{=} W(\sigma^2 t) + \frac{\alpha}{\sigma^2} \cdot \sigma^2 t \equiv \tilde{X}(\sigma^2 t)$ then we are effectively carrying out the above analysis replacing T by $\sigma^2 T$, and keeping the barrier(s) the same, but altering drift to α/σ^2.

The expected no. of visits to the grid point nearest the barrier will change only by change of parameters, and the discrepancy for each visit will be $\sigma^{-2}\Delta(\varepsilon,\delta)$.

(vii) Summary Suppose we take drifting BM $X_t = \sigma W_t + \alpha t$ with either a single barrier at 0 or a pair of barriers, at 0 and $a>0$, and we then condition it not to cross the barrier(s) before time T. If we construct the embedded random walk with mesh δ, the number of steps of this which are fitted into $[0,T]$ (if we condition on not crossing the barrier(s)) will get inflated. Roughly, we should increase T to T', where

$$\sigma^2(T'-T) = E^x[\text{no of visits to } \varepsilon \mid \text{no crossings}] \cdot \Delta(\varepsilon,\delta) \quad + \ ?$$

where $x\in(0,a)$ is the starting point, $\varepsilon\in(0,\delta)$ is the lowest grid point still above 0, and the "+?" is the corresponding correction for the upper barrier (if there is one), and

$$\Delta(\varepsilon,\delta) = \frac{\delta}{c}\tanh c\delta - \left[\frac{(\varepsilon+\delta)\coth c(\varepsilon+\delta) - \varepsilon\coth c\varepsilon}{c}\right], \qquad c = |\alpha|/\sigma^2 .$$

Next,

$$E^x[\text{no. of visits to } \varepsilon \mid \text{no crossings}]$$
$$\doteqdot E^x[L(T,\varepsilon) \mid \text{no crossings}] \frac{c(e^{2c\varepsilon} - e^{-2c\delta})}{(e^{2c\varepsilon}-1)(1-e^{-2c\delta})},$$

and in the case of a single barrier,

$$E^x[L(T,\varepsilon) \mid \text{no crossings}] \doteqdot \frac{\sinh c\varepsilon}{\sinh cx}\left[\psi(\sigma^2T, c, x-\varepsilon) - \psi(\sigma^2T, c, x+\varepsilon)\right],$$

for a double barrier

$$E^x[L(T,\varepsilon) \mid \text{no crossings}] \doteqdot \frac{\sin(\pi\varepsilon/a)}{\sin(\pi x/a)}\Big[\sum_{k\ge 0}\{\psi(\sigma^2T, \tfrac{\pi}{a}, x-\varepsilon-2ka) - \psi(\sigma^2T, \tfrac{\pi}{a}, x+\varepsilon-2ka-2a)\}$$

$$- \sum_{k\ge 0}\{\psi(\sigma^2T, \tfrac{\pi}{a}, -x-\varepsilon-2ka) - \psi(\sigma^2T, \tfrac{\pi}{a}, -x+\varepsilon-2ka-2a)\}\Big],$$

and the last piece of the picture:

$$\psi(T,\beta,x) \equiv \frac{e^{-\beta|x|}}{\beta}\,\bar{\Phi}\left(\frac{|x|}{\sqrt{T}} - \beta\sqrt{T}\right) - \frac{e^{+\beta|x|}}{\beta}\,\bar{\Phi}\left(\frac{|x|}{\sqrt{T}} + \beta\sqrt{T}\right),$$

with $\bar{\Phi}(t) = \int_t^\infty e^{-y^2/2}\,dy/\sqrt{2\pi}$, of course.

Questions, problems

1) Andreas Pechtl (DG Bank, Frankfurt) challenges me to find the $r>0$ for which $\int_0^\infty (1+x^r)^{-r}\, dx = 1$.

2) Michel Crouhy (crouhy@cibc.ca) wants to understand pricing convertible bonds issued by Japanese Companies with payments in CHF.

3) Rudiger Frey has worked on the problem of equilibrium pricing, where one (group of) trader(s) is maximising expected utility of consumption, whereas there are Black-Scholes traders who are writing options and hedging via BS. Another question concerns an economy where a big trader can shift the prices of assets; how would this trader hedge?

$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k\geq 0} \left(\frac{x^2}{4}\right)^k / k!\, \Gamma(k+\nu+1)$ solves $x^2 f'' + x f' = (x^2+\nu^2) f$.

BES(ν) trans. density $p_t^\nu(x,y) = \frac{y}{t} e^{-(x^2+y^2)/2t} \left(\frac{y}{x}\right)^\nu I_\nu\left(\frac{xy}{t}\right)$