A question on BES(3) (14/10/96)

(i) Jonathan Warren states that if R is a BES(3) process, $R_0 = 0$, then

$$\frac{1}{R_t} = \hat{R}\left(\int_t^\infty \frac{ds}{R_s^4}\right)$$

defines another BES(3) process $\hat{R}$. How can we see this?

Suppose that we were instead to start the BES(3) process R at $\varepsilon > 0$, and take

$$A_t \equiv \int_0^t R_s^{-4}\, ds, \qquad \alpha_t \equiv \inf\{u : A_u > t\}.$$

Then

$$\left(1/R(\alpha_t)\right)_{0 \le t \le A_\infty}$$ is a BM started at $\frac{1}{\varepsilon}$ and run until it hits 0

by standard results on time change. By Williams' representation, we therefore define a BES(3) process $\hat{R}$ run until its last exit from $1/\varepsilon$ by writing

$$\hat{R}_t = 1/R \circ \alpha(A_\infty - t)$$

Now we have easily

$$\hat{R}\left(\int_t^\infty R_s^{-4}\, ds\right) = \frac{1}{R_t} \qquad \text{for } 0 \le t \le A_\infty$$

Letting $\varepsilon \downarrow 0$ is now no problem, if we simply represent $(R_t)_{t \ge 0} = \left(R^*(H_\varepsilon^* + t)\right)_{t \ge 0}$, where R^* is a BES(3) started from 0, first hitting ε at H_ε^*.

(ii) Let's define $\hat{R}$ by

$$\frac{1}{R_t} = \hat{R}(A_t), \qquad A_t \equiv \int_t^\infty R_s^{-4}\, ds,$$

where R is a BES(3), $R_0 = 0$. Let τ be inverse to A, and suppose that we define another BES(3) $\tilde{R}$ by the usual time inversion:

$$\tilde{R}_t = t\, R(1/t),$$

so that in terms of this $A_t = \tilde{A}(1/t) \equiv \int_0^{1/t} u^2 \tilde{R}_u^{-4}\, du$, $\tau_t = 1/\tilde{\tau}_t$

and

$$\boxed{\frac{t}{\tilde{R}_t} = \hat{R}(\tilde{A}_t) \equiv \hat{R}\left(\int_0^t u^2 \tilde{R}_u^{-4}\, du\right)}$$

This shows that $\hat{R}$ can be obtained in a conventional way as a time change of

$\tilde{R}$, and we do the usual change of time to learn that

$$f(\hat{R}_t) - \int_0^t \left\{ \tfrac{1}{2} f''(\hat{R}_s) + \frac{\tilde{\tau}_s}{\hat{R}_s^{\,3}} f'(\hat{R}_s) \right\} ds \qquad \text{is a martingale}$$

in the filtration of $\tilde{R} \circ \tilde{\tau}$. However, it is not immediately evident that $\hat{R}$ is a BES(3) if we do this; the only way it could be is if the filtration of $\hat{R}$ is strictly smaller than that of $\tilde{R} \circ \tilde{\tau}$, and, when we drop down to that smaller filtration we do indeed get a BES(3).

(iii) In order to understand this, consider

$$E \int_0^\infty g(t, \tau_t, \hat{R}_t)\, dt = E \int_0^\infty g(A_s, s, \tfrac{1}{R_s}) \frac{ds}{R_s^4}$$

$$= \int_0^\infty ds \int_0^\infty \frac{2x^2 e^{-x^2/2s}}{\sqrt{2\pi s^3}} \frac{dx}{x^4} \int_0^\infty \frac{x^{-1} e^{-1/2x^2u}}{\sqrt{2\pi u^3}}\, du \; g(u, s, 1/x)$$

because, conditional on $R_s = r$, the law of A_s is the law of first time to hit $1/r$ for BM; hence

$$E\, g(\tau_t, \hat{R}_t) = \int_0^\infty ds \int_0^\infty \frac{dx}{x^3} \frac{2 \exp\left(-x^2/2s - \tfrac{1}{2x^2 t}\right)}{2\pi (st)^{3/2}} g(s, 1/x)$$

$$= \int_0^\infty dv \int_0^\infty dy \frac{4y^2 \exp(-v^2/2 - y^2/2t)}{\sqrt{2\pi}\,.\,\sqrt{2\pi t^3}} g\left(\frac{1}{y^2 v^2}, y\right)$$

Hence

$$\boxed{(\hat{R}_t, \tau_t) \overset{\mathcal{D}}{=} \left(R_t, \frac{1}{R_t^2 B_1^2}\right)}$$

for each $t > 0$, where B is a Brownian motion indept of R.

(iv) Note $\tau_t = \int_t^\infty \hat{R}_s^{-4}\, ds$, so that $\mathcal{L}(\tau_t \mid \hat{R}_u : u \le t)$ is like the hitting time of 0 for BM started at $\hat{R}_t^{-1}$, so $E(\tilde{\tau}_t \mid \hat{R}_u : u \le t) = \hat{R}_t^2$ and so the optional projection of the drift $\tilde{\tau}_t / \hat{R}_t^3$ is $1/\hat{R}_t$, as required.

GMM estimation of a Markov-modulated Poisson process (3/11/96)

(i) Suppose we have an ergodic Markov process X, and independent standard Poisson processes $\tilde N_1, \dots, \tilde N_K$. Define

$$N^i_t \equiv \tilde N_i(\Lambda^i_t), \quad \Lambda^i_t \equiv \int_0^t f_i(X_s)\,ds, \quad Y^i(\alpha) \equiv \int_{-\infty}^0 \alpha e^{\alpha t}\, dN^i_t .$$

If we see $\left((Y^i(\theta_{ij}))_{j=1}^{n_i} \right)_{i=1}^K$, how would we estimate the Mkv process X, and the functions f_i?

Suppose for simplicity that X is a finite Markov chain with a known structure (perhaps a birth-and-death chain, for simplicity). We have

$$E[\,Y^i(\alpha)\,] = (\pi, f_i) \qquad (\pi \text{ is invariant law of } X)$$

and that

$$M^i_t(\alpha) \equiv \int_0^t \alpha e^{\alpha s}\left(dN^i_s - f_i(X_s)\,ds\right) = e^{\alpha t}\,Y^i_t(\alpha) - \int_0^t \alpha e^{\alpha s}\, d\Lambda^i_s .$$

is a martingale, so

$$E\left[M^i_t(\alpha)\, M^j_t(\beta) \right] = E\Big[e^{(\alpha+\beta)t}\, Y^i_t(\alpha)\, Y^j_t(\beta) - e^{\alpha t} Y^i_t(\alpha) \int_0^t \beta e^{\beta s}\, d\Lambda^j_s$$
$$- e^{\beta t} Y^j_t(\beta) \int_0^t \alpha e^{\alpha s}\, d\Lambda^i_s + \int_0^t \alpha e^{\alpha s} d\Lambda^i_s \int_0^t \beta e^{\beta s} d\Lambda^j_s \Big]$$

$$= e^{(\alpha+\beta)t}\, E\left[Y^i_t(\alpha) Y^j_t(\beta)\right] - E\left[\int_0^t \beta e^{\beta s} M^i_s(\alpha)\,ds + \int_0^t \alpha e^{\alpha s} M^j_s(\beta)\,ds\right]$$
$$- E\left[\int_0^t \alpha e^{\alpha s} d\Lambda^i_s \cdot \int_0^t \beta e^{\beta s} d\Lambda^j_s \right]$$

$$= E\left([M^i(\alpha), M^j(\beta)]_t \right) = \delta_{ij}\ \alpha\beta\ E\left(\int_0^t e^{(\alpha+\beta)s} f_i(X_s)\,ds\right)$$

Hence on multiplying by $e^{-(\alpha+\beta)t}$, letting $t \to \infty$, we obtain

$$E\left[Y^i(\alpha)\, Y^j(\beta) \right] = \delta_{ij} \frac{\alpha\beta}{\alpha+\beta}(\pi, f_i) + E\left[\int_{-\infty}^0 \alpha e^{\alpha s} d\Lambda^i_s \cdot \int_{-\infty}^0 \beta e^{\beta s} d\Lambda^j_s \right]$$

$$\boxed{\therefore\quad \operatorname{cov}\left(Y^i(\alpha), Y^j(\beta) \right) = \delta_{ij} \frac{\alpha\beta}{\alpha+\beta}(\pi, f_i) + \operatorname{cov}\left(\int_{-\infty}^0 \alpha e^{\alpha s} d\Lambda^i_s ,\ \int_{-\infty}^0 \beta e^{\beta s} d\Lambda^j_s \right)}$$

$$= \delta_{ij} \frac{\alpha\beta}{\alpha+\beta}(\pi, f_i) + \frac{\alpha\beta}{\alpha+\beta}\left(\pi,\ f_j \hat R_\alpha f_i + f_i \hat R_\beta f_j\right) - (\pi, f_i)(\pi, f_j)$$

where $(\hat{R}_\alpha)$ is the resolvent of the reversed Mkv process. But since $(\pi, f\hat{R}_\alpha g) = (\pi, gR_\alpha f)$, we have also

$$\boxed{\operatorname{cov}\left(Y^i(\alpha), Y^j(\beta)\right) = \delta_{ij}\frac{\alpha\beta}{\alpha+\beta}(\pi, f_i) + \frac{\alpha\beta}{\alpha+\beta}\left(\pi, f_i R_\alpha f_j + f_j R_\beta f_i\right) - (\pi, f_i)(\pi, f_j)}$$

Thus the covariance matrix of the observed variables is known, if the underlying chain and the intensities f_i are known.

(ii) How would we estimate? One simple-minded way would be to consider the observed variables to be approximately normally distributed, so that the log-likelihood would be

$$-\tfrac{1}{2}\log\det V - \tfrac{1}{2}(\eta-\bar{\eta})\cdot V^{-1}(\eta-\bar{\eta})$$

where η is the observed vector, $\bar{\eta}$ is its mean value (as a function of the f_i and the chain) and V is its covariance matrix. This expression should be maximised over the unknown variables.

(iii) Observe also that in practice it would be fairly easy to keep a running record of

$$\int_{-\infty}^{t}\alpha e^{\alpha(s-t)} Y^j_{s-}(\beta)\,dN^i_s = \int_{-\infty}^{t}\alpha e^{\alpha(s-t)}\left(\int_{(-\infty,s)}\beta e^{\beta(u-s)}\,dN^j_u\right)dN^i_s \equiv \eta^{ij}_t(\alpha,\beta),$$

if we were keeping a running record of the $Y^i_t(\alpha)$. In equilibrium, we can compute

$$E\,\eta^{ij}_t(\alpha,\beta) = \beta(\pi, f_j R_\beta f_i),$$

by the same sort of techniques as above.

[Special case: $dX_t = dW_t - bX_t\,dt$, a one-dim^l OU, $f(x) = \kappa(x-c)^2$. Then

$$E\,Y(\alpha) = \kappa(c^2 + 1/2b), \quad E\,\eta(\alpha,\beta) = \frac{3\beta\kappa^2}{(\beta+2b)(2b)^2} + \frac{\kappa^2c^2}{b}\,\frac{3\beta+b}{\beta+b} + \kappa^2c^4 + \frac{\kappa^2}{2b(\beta+2b)}$$

$$= \{E\,Y(\beta)\}^2 + \frac{\beta\kappa^2}{4b^2}\left\{\frac{2}{\beta+2b} + \frac{8c^2b}{\beta+b}\right\}$$

Shares in the context of the potential approach (11/11/96)

(i) Here is a question which Theo Dijkstra asked! Suppose we have the Markov process $dX_t = d\tilde{W}_t - BX_t\,dt$ as the base Markov process, and we use the exponential-quadratic model for term structure:

$$f(x) \equiv \exp\left[\tfrac{1}{2}(x-c)\cdot Q(x-c)\right] \equiv R_\alpha g(x)$$

so that

$$r_t = r(X_t) = \tfrac{1}{2}(x-v)\cdot S(x-v),$$

where $S \equiv B^TQ + QB - Q^2$, $v \equiv S^{-1}(B^T - Q)Qc$, $\alpha \equiv \tfrac{1}{2}\left(|Qc|^2 + \mathrm{tr}\,Q + v\cdot Sv\right)$.

Suppose now we introduce a share, determined by

$$dS_t = S_t\left\{a\cdot dW^*_t + r_t\,dt\right\},$$

with $a \in \mathbb{R}^d$ fixed, $dW^* = d\tilde{W} - \nabla\log f(X)\,dt = d\tilde{W} - Q(X-c)\,dt$, where W^* is a BM in the risk-neutral measure $\mathbb{P}^*$. How would we price a call option on this share?

We have

$$S_t = S_0 \exp\left[a\cdot W^*_t + \int_0^t r_s\,ds - \tfrac{1}{2}|a|^2 t\right]$$

$$= S_0 \exp\left[a\cdot X_t + \int_0^t\left\{a\cdot\left(BX_u - Q(X_u - c)\right) + \tfrac{1}{2}(X_u - v)\cdot S(X_u - v)\right\}du - \frac{|a|^2 t}{2}\right]$$

(ii) There are various equivalent ways to compute what we need, but perhaps neatest is to calculate the law of S_T in the T-forward measure. We need for $\theta \in \mathbb{R}$

$$\tilde{\mathbb{E}}_t \exp\left\{\tfrac{1}{2}(X_T - c)\cdot Q(X_T - c) + \theta\left(a\cdot X_T + \int_0^T\left\{a\cdot(BX_u - QX_u + Qc) + \tfrac{1}{2}(X_u - v)\cdot S(X_u - v)\right\}du\right)\right\}$$

$$= \exp\left(\tfrac{1}{2}X_t\cdot V_t X_t + \gamma_t\cdot X_t + \psi_t + A_t\right) \equiv \exp(Y_t), \quad \text{say}$$

The game (as usual!) is to find V, γ and ψ. Here,

$$A_t \equiv \theta\int_0^t\left\{a\cdot(BX_u - QX_u + Qc) + \tfrac{1}{2}(X_u - v)\cdot S(X_u - v)\right\}du$$

We need $dY_t + \tfrac{1}{2}d\langle Y\rangle_t$ to be a $\hat{\mathbb{P}}$-martingale.

But

$$dY_t + \tfrac{1}{2} d\langle Y \rangle_t = \tfrac{1}{2} X \cdot \dot{V} X \, dt + VX \, dX + \tfrac{1}{2} \mathrm{tr} V \, dt + \dot{\gamma} X dt + \gamma dX + \dot{\psi} \, dt$$
$$+ \theta \left[a \cdot (BX - QX + Qc) + \tfrac{1}{2} (X - v) \cdot S (X - v) \right] dt$$
$$+ \tfrac{1}{2} |VX|^2 \, dt$$

whence

$$\left.\begin{aligned} &\dot{V} - (VB + B^T V) + \theta S + V^2 = 0 \\ &\dot{\gamma} - B^T \gamma + \theta (B^T - Q) a - S v = 0 \\ &\tfrac{1}{2} \mathrm{tr} V + \dot{\psi} + \theta \, a \cdot Qc + \tfrac{1}{2} \theta \, v \cdot S v = 0 \end{aligned}\right|$$

with $V_T = Q$, $\gamma_T = -Qc + \theta a$, $\psi_T = 0$. This matrix Riccati equation doesn't have a solution in general.

(iii) Special case: $B = \beta I$; then Q is diagonal, wlog, and $S = 2\beta Q - Q^2$, and we need to solve

$$\dot{V} - 2\beta V + \theta (2\beta - Q) Q + V^2 = 0, \qquad V_T = Q.$$

In fact, we can cope with a little more generality, viz, B and Q both diagonal. Solving the one-dimensional equation

$$-\dot{x} - 2bx + k + x^2 = 0, \qquad x_0 = q$$

has solution

$$x_t = + \frac{\lambda q + (k + qb) \tanh \lambda t}{\lambda + (b - q) \tanh \lambda t} \qquad \lambda \equiv \sqrt{b^2 - k}$$

Unfortunately, even in this easy case the formulae aren't too clean.

(iv) Possible approach to pricing of options.

How about using Fourier methods? We have

$$\int_{-\infty}^{\infty} e^{\varepsilon x} (K - e^x)^+ e^{itx} \, dx = \int_{-\infty}^{\log K} \left(K e^{(\varepsilon + it)x} - e^{(1 + \varepsilon + it)x} \right) dx$$

$$= K e^{(\varepsilon + it) \log K} \left(\frac{1}{\varepsilon + it} - \frac{1}{1 + \varepsilon + it} \right),$$

which is an L^2 function. Using this,

$$\boxed{E\left[Z (K - e^X)^+ e^{\varepsilon X} \right] = \int_{-\infty}^{\infty} \frac{dt \; K e^{(\varepsilon + it) \log K}}{(2\pi)(\varepsilon + it)(1 + \varepsilon + it)} E\left[Z e^{-itX} \right].}$$

In the specific example, we have to take $Z = \exp[-\alpha T + \frac{1}{2}(X_T - c)\cdot Q(X_T - c) - \frac{1}{2}(X_0 - c)\, Q\,(X_0 - c)]$, and $X = a\cdot W^*_T - \frac{1}{2}|a|^2 T + \int_0^T r_u\, du$. We compute

$$\tilde{E}\Big[\exp\big\{\tfrac{1}{2}(X_T-c)\cdot Q(X_T-c) + i\theta\big(a\cdot W^*_T + \int_0^T r_u\,du\big)\Big]$$

$$= \tilde{E}\exp\Big\{\tfrac{1}{2}(X_T-c)\cdot Q(X_T-c) + i\theta\Big(a\cdot(X_T-X_0) + \int_0^T\{a\cdot(BX_u - Q(X_u-c)) + r_u\}\,du\Big)\Big\}$$

$$= E_0 \exp\Big[-\tfrac{1}{2}X_T\cdot BX_T + \tfrac{1}{2}X_0\cdot BX_0 + \tfrac{T}{2}\cdot \mathrm{tr}\,B - \tfrac{1}{2}\int_0^T |BX_u|^2\,du$$
$$+ \tfrac{1}{2}(X_T-c)\cdot Q(X_T-c) + i\theta\, a(X_T-X_0)$$
$$+ i\theta a\int_0^T\{BX_u - Q(X_u-c))\,du + i\theta\int_0^T r_u\,du\Big]$$

where we assume B is <u>symmetric</u>, and P_0 is the law of X as BM. This gives us $\exp(\frac{1}{2}X_0\cdot BX_0 - i\theta a\cdot X_0)$ times $\exp(\frac{1}{2}T\,\mathrm{tr}B + i\theta a\,QcT)$ times

$$E_0\exp\Big[-\tfrac{1}{2}X_T\cdot BX_T + \tfrac{1}{2}(X_T-c)\cdot Q(X_T-c) + i\theta a X_T$$
$$+\int_0^T\big\{i\theta a\,(BX_u - QX_u) + i\theta r_u - \tfrac{1}{2}|BX_u|^2\big\}\,du\Big]$$

This is a bit closer, but probably still needs numerical work...

Markovian intensities (21/11/96)

(i) We'll model the intensity of the i^{th} asset by $f_i(x) \equiv (x_i)^2 . K_i$, where x will be an N-vector, which diffuses according to

$$dX = dW - BX\,dt$$

where B is symmetric. The invariant density is $\propto \exp(-x \cdot Bx)$. By applying a rotation if need be, let's suppose that $B \equiv \mathrm{diag}(\beta_i)$ is diagonal. We're going to need to compute things like

$$\mu_{ij}(m,n,\alpha) \equiv \int_0^\infty e^{-\alpha t} E\left(X_i(0)^m X_j(t)^n \right) dt$$

where X_i are solutions to $dX_i = dW^i - \beta_i X_i\,dt$. We shall therefore get

For $i \neq j$: $\quad \mu_{ij}(m,n,\alpha) = 0$ unless m and n are both even,

$$\mu_{ij}(2k, 2\ell, \alpha) = \frac{1}{\alpha} \frac{(2k)!}{k!} (4\beta_i)^{-k} \frac{(2\ell)!}{\ell!} (4\beta_j)^{-\ell}$$

For $i = j$: this time,

$$E\, X_0^m X_t^n = 0 \quad \text{if } n+m \text{ is odd}$$

$$= E\left[X(0)^m \left(e^{-\beta t} X(0) + Z \right)^n \right], \qquad Z \sim N\left(0, \frac{1-e^{-2\beta t}}{2\beta}\right)$$

$$= \sum_{\ell=0}^{[n/2]} \binom{n}{2\ell} E\, Z^{2\ell}\, E\left(X(0)^{n+m-2\ell} \right) e^{-(n-2\ell)\beta t}$$

$$= \sum_{\ell=0}^{[n/2]} \binom{n}{2\ell} \frac{(2\ell)!}{\ell!} \left(\frac{1-e^{-2\beta t}}{4\beta} \right)^\ell e^{-(n-2\ell)\beta t} \frac{(n+m-2\ell)!}{\left(\frac{n+m}{2}-\ell\right)!} \left(\frac{1}{4\beta}\right)^{(n+m-2\ell)/2}$$

otherwise.

So if $n+m = 2q$, we get

$$\boxed{\mu_{ii}(m,n,\alpha) = \sum_{\ell=0}^{[n/2]} \sum_{r=0}^{\ell} \binom{n}{2\ell}\binom{\ell}{r} \frac{(2\ell)!}{\ell!} \frac{(2q-2\ell)!}{(q-\ell)!} (4\beta)^{-q} (-1)^r \left(\alpha + (n+2r-2\ell)\beta\right)^{-1}}$$

and if $n+m$ is odd, then $\mu_{ii}(m,n,\alpha) = 0$

Our current application is going to be exclusively in terms of $m, n = 0, 1, 2$, and for this we get

For $i \neq j$

$$\mu_{ij}(0,2,\alpha) = 3/4\alpha\beta_j^2$$

$$\mu_{ij}(2,2,\alpha) = \frac{9}{16\alpha}(\beta_i\beta_j)^{-2}$$

and for $i = j$:

$$\mu_{ii}(1,1,\alpha) = \frac{1}{(\alpha+2\beta_i)\ 2\beta_i}$$

$$\mu_{ii}(0,2,\alpha) = \frac{1}{2\alpha\beta_i}$$

$$\mu_{ii}(2,2,\alpha) = (4\beta_i)^{-1}\left[\frac{8}{\alpha+2\beta_i} + \frac{4}{\alpha}\right]$$

(ii) If we return to the processes

$$\boxed{\eta_t^{ij}(\alpha,\beta) \equiv \int_{(-\infty,t]} \alpha e^{\alpha(s-t)}\, Y_{s-}^j(\beta)\, dN_s^i}$$

introduced earlier, then $E\,\eta_t^{ij}(\alpha,\beta) = \beta(\pi, f_j R_\beta f_i)$, but what about higher moments? Covariation with the $Y^i(\beta)$??

First calculation:

$$E\,\eta_0^{ij}(\alpha,\beta)\,Y_0^k(\gamma) = E\int_{(-\infty,0]}\alpha e^{\alpha s}\,Y_{s-}^j(\beta)\,dN_s^i \int_{(-\infty,0]}\gamma e^{\gamma u}\,dN_u^k$$

$$= E\int_{(-\infty,0]}dN_s^i\int_{(-\infty,s)}dN_t^j\int_{(-\infty,0]}dN_u^k\ \alpha\beta\gamma\, e^{\alpha s+\gamma u+\beta(t-s)}$$

$$= E\int_{(-\infty,0]}\int_{(-\infty,s)}\int_{(-\infty,0]}\alpha\beta\gamma\, e^{\alpha s+\gamma u+\beta(t-s)}\Big\{f_i(X_s)f_j(X_t)f_k(X_u)\,ds\,dt\,du$$

$$+\ \delta_{s-u}\,f_k(X_u)\,f_j(X_t)\,dt\,ds\,du\,\delta_{ki} + \delta_{u-t}\,f_k(X_u)\,f_i(X_s)\,ds\,dt\,du\,\delta_{jk}\Big\}$$

$$= I_1 + I_2 + I_3,$$

say.

Split these terms further to give

$$I_1 = E\int_0^\infty du \int_u^\infty ds \int_s^\infty dt \; f_i(\hat{X}_s) f_j(\hat{X}_t) f_k(\hat{X}_u) \; \alpha\beta\gamma e^{-\alpha s - \gamma u - \beta(t-s)}$$

$$+ E\left[\int_0^\infty ds \int_s^\infty du \int_s^\infty dt \quad \dots\dots \right]$$

$$= I_1' + I_1'' .$$

It's now not hard to compute

$$I_1' = \frac{\alpha\beta\gamma}{\alpha+\gamma}(\pi, f_k \hat{R}_\alpha f_i \hat{R}_\beta f_j) = \frac{\alpha\beta\gamma}{\alpha+\gamma}(\pi, f_j R_\beta f_i R_\alpha f_k),$$

$$I_1'' = \frac{\alpha\beta\gamma}{\alpha+\gamma}(\pi, f_j R_\beta f_k R_{\alpha+\beta} f_i + f_k R_\gamma f_j R_{\gamma+\beta} f_i),$$

$$I_2 = \delta_{ik} \frac{\alpha\beta\gamma}{\alpha+\gamma}(\pi, f_j R_\beta f_k),$$

$$I_3 = \delta_{jk} \frac{\alpha\beta\gamma}{\alpha+\gamma}(\pi, f_k R_{\gamma+\beta} f_i),$$

So assembling all this gives

$$\boxed{E\,\eta_0^{ij}(\alpha,\beta)\, y_0^k(\gamma) = \frac{\alpha\beta\gamma}{\alpha+\gamma}\Big(\pi, f_j R_\beta f_i R_\alpha f_k + f_j R_\beta f_k R_{\alpha+\beta} f_i + f_k R_\gamma f_j R_{\gamma+\beta} f_i + \delta_{ik} f_j R_\beta f_k + \delta_{jk} f_k R_{\gamma+\beta} f_i \Big)}$$

<u>Second calculation</u> : Find $E\,\eta_0^{ij}(\alpha,\beta)\,\eta_0^{kl}(\lambda,\mu)$. We have

$$\left.\begin{aligned} \eta_0^{ij}(\alpha,\beta) &= \int_{(-\infty,0]} \left(\int_{(-\infty,s)} \beta e^{\beta(u-s)} dN_u^j \right) \alpha e^{\alpha s} dN_s^i \\ \eta_0^{kl}(\lambda,\mu) &= \int_{(-\infty,0]} \left(\int_{(-\infty,t)} \mu e^{\mu(v-t)} dN_v^l \right) \lambda e^{\lambda t} dN_t^k \end{aligned}\right\}$$

As we always have $u < s \leq 0$, $v < t \leq 0$. We need to consider the possible cross terms in the expectation, particularly, we need to compute

$$E\left[dN_u^j \, dN_s^i \, dN_v^l \, dN_t^k \,\middle|\, \mathcal{F} \right]$$

By considering the possible ways that two or more of the time variables could be the

Same, we get

$$E\left[dN_u^j \, dN_s^i \, dN_v^l \, dN_t^k \,\middle|\, \mathcal{X} \right] = f_j(X_u) f_i(X_s) f_l(X_v) f_k(X_t)\, du\,ds\,dv\,dt$$

$$+ \delta_{ik}\delta_{jl}\,\delta_{t-s}\,\delta_{u-v}\, ds\,dv\, f_i(X_s) f_l(X_v) + \delta_{s-v}\,\delta_{il}\, f_j(X_u) f_i(X_s) f_k(X_t)\, du\,ds\,dt$$

$$+ \delta_{t-u}\,\delta_{jk}\, f_j(X_u) f_i(X_s) f_l(X_v)\, du\,ds\,dv + \delta_{s-t}\,\delta_{ik}\, f_j(X_u) f_i(X_s) f_l(X_v)\, du\,ds\,dv$$

$$+ \delta_{u-v}\,\delta_{jl}\, f_j(X_u) f_i(X_s) f_k(X_t)\, du\,ds\,dt.$$

Thus we may write

$$\boxed{E\left[\eta_0^{ij}(\alpha,\beta)\,\eta_0^{k,l}(\lambda,\mu) \right] = J_1 + J_2 + J_3 + J_4 + J_5 + J_6}$$

according to this decomposition, and compute each term separately.

$$J_1 = \frac{\alpha\beta\lambda\mu}{\alpha+\lambda}\left(\pi,\ f_l R_\mu f_k R_\lambda f_j R_{\lambda+\beta} f_i + f_l R_\mu f_j R_{\beta+\mu} f_k R_{\beta+\lambda} f_i + f_j R_\beta f_l R_{\mu+\beta} f_k R_{\lambda+\beta} f_i\right)$$

$$+ \frac{\alpha\beta\lambda\mu}{\alpha+\lambda}\left(\pi,\ f_j R_\beta f_i R_\alpha f_l R_{\alpha+\mu} f_k + f_j R_\beta f_l R_{\beta+\mu} f_i R_{\mu+\alpha} f_k + f_l R_\mu f_j R_{\beta+\mu} f_i R_{\alpha+\mu} f_k\right)$$

$$J_2 = \frac{\alpha\beta\lambda\mu}{\alpha+\lambda}\left(\pi,\ f_l R_{\beta+\mu} f_i\right)\delta_{ik}\,\delta_{jl}$$

$$J_3 = \frac{\alpha\beta\lambda\mu}{\alpha+\lambda}\left(\pi,\ f_j R_\beta f_i R_{\alpha+\mu} f_k\right)\delta_{il}$$

$$J_4 = \frac{\alpha\beta\lambda\mu}{\alpha+\lambda}\left(\pi,\ f_l R_\mu f_j R_{\beta+\lambda} f_i\right)\delta_{jk}$$

$$J_5 = \frac{\alpha\beta\lambda\mu}{\alpha+\lambda}\left(\pi,\ f_l R_\mu f_j R_{\beta+\mu} f_i + f_j R_\beta f_l R_{\beta+\mu} f_i\right)\delta_{ik}$$

$$J_6 = \frac{\alpha\beta\lambda\mu}{\alpha+\lambda}\left(\pi,\ f_l R_{\beta+\mu} f_k R_{\lambda+\beta} f_i + f_l R_{\beta+\mu} f_i R_{\alpha+\mu} f_k\right)\delta_{jl}$$

End correction in binomial pricing (29/4/96)

Just for the record, here is the calculation in the appendix to the paper with Emily.
We start at 0 with a BM with drift, $X_t = W_t + ct$, and kill when it reaches $\{-\delta, \delta\}$.
What's the Green function?

If $\tau \equiv \inf\{t : |X_t| = \delta\}$, we get

$$E^x L(\tau, x) = \left\{ \frac{s'(x)}{2}\left(\frac{1}{s(\delta)-s(x)} + \frac{1}{s(x)-s(-\delta)} \right)\right\}^{-1} \qquad s(x) \equiv -e^{-2cx}$$

$$= \frac{2\big(s(x)-s(-\delta)\big)\big(s(\delta)-s(x)\big)}{s'(x)\ \big(s(\delta)-s(-\delta)\big)}$$

so that

$$E^0 L(\tau, x) = \begin{cases} \big(s(\delta)-s(x)\big)\big(s(0)-s(-\delta)\big)\ \kappa / s'(x) & (x \geq 0) \\ \big(s(\delta)-s(0)\big)\ \big(s(x)-s(-\delta)\big)\ \kappa / s'(x) & (x < 0) \end{cases}$$

for suitable choice of κ $\big(2/(s(\delta)-s(-\delta))$, in fact$\big)$. Now

$$\int_0^\delta E^0 L(\tau,x)\,dx = \int_0^\delta \frac{\kappa}{2c}\big(s(0)-s(-\delta)\big)\ \left(e^{-2cx} - e^{-2c\delta}\right) e^{2cx}\,dx$$

$$= \frac{\kappa}{2c}\left(e^{2c\delta}-1\right)\left\{\delta - \frac{1-e^{-2c\delta}}{2c}\right\}$$

and

$$\int_{-\delta}^0 E^0 L(\tau,x)\,dx = \frac{\kappa\big(s(\delta)-s(0)\big)}{2c}\int_{-\delta}^0 \left(e^{2c\delta} - e^{-2cx}\right) e^{2cx}\,dx$$

$$= \frac{\kappa}{2c}\left(1-e^{-2c\delta}\right)\left\{\frac{e^{2c\delta}-1}{2c} - \delta\right\}$$

so adding

$$E^0 \tau = \frac{\kappa}{2c}\cdot \delta\left(e^{2c\delta} - 2 + e^{-2c\delta}\right)$$

and thus for $x \geq 0$

$$\frac{E^0 L(\tau,x)}{E^0 \tau} = \frac{e^{2c\delta} - e^{2cx}}{\delta\left(e^{2c\delta}-1\right)}$$

and for $x < 0$

$$\frac{E^0 L(\tau,x)}{E^0 \tau} = \frac{e^{2c(x+\delta)} - 1}{\delta\left(e^{2c\delta}-1\right)}$$

How about the values to be used at the end of the tree? At position x, we use not $(e^x - K)^+$, but instead

$$\int_{-\delta}^{\delta} h(y)\,(e^{x+y} - K)^+\,dy$$

where

$$h(y) \equiv E^0 L(\tau, y)/E^0\tau = \begin{cases} \beta\,(e^{2c\delta} - e^{2cy}), & y \geq 0 \\ \beta\,(e^{2c(\delta+y)} - 1), & y \leq 0 \end{cases}$$

with $\beta^{-1} \equiv \delta(e^{2c\delta} - 1)$. We have then for $e^{x+\delta} \leq K$ the value 0, and if $e^{x-\delta} \geq K$ we get

$$-K + e^x \;\beta\,(e^{(2c+1)\delta} - 1)(1 - e^{-\delta})\; 2c/(2c+1).$$

If $e^x \leq K \leq e^{x+\delta}$, and $b \equiv -x + \log K$, $0 \leq b \leq \delta$, we shall have

$$\int_b^{\delta} h(y)\,(e^{x+y} - e^{x+b})\,dy$$

$$= \beta\, e^{x+2c\delta+\delta}\left\{1 - e^{-(\delta-b)} - (\delta - b)\,e^{-(\delta-b)} - \frac{1 - e^{-(2c+1)(\delta-b)}}{2c+1} + e^{-(\delta-b)}\,\frac{1 - e^{-2c(\delta-b)}}{2c}\right\}$$

If $e^{x-\delta} \leq K \leq e^x$, and $b \equiv -x + \log K$, we shall get

$$\int_b^{\delta} h(y)\left(e^{x+y} - e^{x+b}\right) dy$$

$$= \beta\, e^{x+2c\delta+\delta}\left\{1 - e^{-\delta} - \delta e^{-\delta} - \frac{1 - e^{-(2c+1)\delta}}{2c+1} + \frac{e^{-\delta}(1 - e^{-2c\delta})}{2c}\right\}$$

$$+ \; e^x(1 - e^b)\,\frac{\beta}{2c}\left\{2c\delta\, e^{2c\delta} - e^{2c\delta} + 1\right\}$$

$$+ \; \beta e^x\left\{\frac{e^{2c\delta}}{2c+1}\left(1 - e^{(2c+1)b}\right) - \frac{e^{b+2c\delta}}{2c}\left(1 - e^{2cb}\right) - 1 + e^b - b e^b\right\}$$

$$\equiv \beta e^{x+2c\delta+\delta}\left[1 - e^{-\delta} - \delta e^{b-\delta} - \frac{1 - e^{-(2c+1)\delta}}{2c+1} + \frac{e^{b-\delta}(1 - e^{-2c\delta})}{2c}\right]$$

$$+ \beta e^x\left[\frac{e^{2c\delta}\{1 - e^{(2c+1)b}\}}{2c+1} - \frac{e^{b+2c\delta}}{2c}(1 - e^{2cb}) - 1 + e^b - b e^b\right].$$

An example. Take $S_t = \exp\left(rt + \sigma W(\tau_t) - \frac{1}{2}\sigma^2 \tau_t\right)$ where τ is a subordinator, so
$E \exp(z X_t) = \exp t\left(rz + \psi_s\left(\frac{1}{2}\sigma^2 z(z-1)\right)\right)$, where ψ_s is the Lévy transform of τ.

If we use $\psi_s(z) = \log\left(\alpha/\alpha - z\right)$, so it's a gamma process, and with $T=1$, we get

$$E(K - S_T)^+ = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \frac{K e^{z \log K}}{z(1+z)} \frac{\alpha e^{-rz}}{\alpha - \frac{1}{2}\sigma^2 z(z+1)}$$

$$= \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \frac{K e^{az}}{z(1+z)} \frac{2\alpha}{\sigma^2} \frac{1}{(\theta_2 - z)(\theta_1 + z)}$$

$a \equiv -r + \log K$,
$\theta_1, \theta_2 > 0$ satisfy
$\theta_1 \theta_2 = 2\alpha/\sigma^2$
$\theta_1 - \theta_2 = 1$

$$= K \theta_1 \theta_2 \left\{ \frac{1}{\theta_1 \theta_2} - \frac{e^{-a}}{(\theta_2 + 1)(\theta_1 - 1)} - \frac{e^{-a\theta_1}}{\theta_1 (1 - \theta_1)(\theta_2 + \theta_1)} \right\}$$

$$= K \left\{ 1 - \frac{\theta_1 e^{-a}}{1 + \theta_2} + \frac{e^{-a\theta_1}}{\theta_1 + \theta_2} \right\}$$

Lévy returns + Saddlepoint approximations (16/12/96)

Suppose that we take a model for share price process

$$S_t = S_0 \exp(X_t)$$

where $E\, e^{zX_t} \equiv \exp(t\,\psi(z))$. Assume we're working with a risk-neutral probability, so that $\psi(1) = r$, the constant riskless rate. To price a put option, we need to evaluate

$$E\,(K - e^{X_T})^+ = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \frac{K e^{z \log K}}{z(1+z)} E\left(e^{-zX_T}\right) \qquad \text{for } c>0$$

$$= \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \frac{K \exp(T\psi(-z) + z\log K)}{z(1+z)}$$

Let's pick c to minimise $T\psi(-x) + x\log K$. This value of c will be >0 iff $\boxed{-T\psi'(0) + \log K < 0}$ which we'll assume for now.

Abbreviating $\varphi(z) \equiv T\psi(-z) + z\log K$, we now try to pick m so that

$$\tfrac{1}{2}w^2 - \tfrac{1}{2}m^2 = m(w-m) + \tfrac{1}{2}(w-m)^2 = \varphi(z+c)$$

(The quadratic on the left is minimised at 0 to the minimum value of φ.) This relationship between w and z can (hopefully) be extended to an analytic function and then

$$E\,(K - e^{X_T})^+ = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{K\exp(\varphi(z+c))}{(z+c)(1+z+c)}$$

$$= \int_{\Gamma} \frac{dz}{dw} \cdot \frac{dw}{2\pi i} \frac{K\exp(\frac{1}{2}w^2 - \frac{1}{2}m^2)}{(z(w)+c)(1+z(w)+c)}$$

where Γ is a curve passing through the origin;

$$\doteqdot \int_{-ib}^{ib} \frac{dz}{dw} \frac{dw}{2\pi i} \frac{K\exp(\frac{1}{2}w^2 - \frac{1}{2}m^2)}{(z(w)+c)(1+z(w)+c)}$$

Now $w = \varphi'(z+c)\dfrac{dz}{dw} = \dfrac{dz}{dw}\left\{\varphi''(c).z + o(z)\right\}$ $\therefore$ $\dfrac{dz}{dw} = \dfrac{w}{z\,\varphi''(c)} + o(z)$

whence $\dfrac{dz}{dw}(0) = (\varphi''(c))^{-\frac{1}{2}}$, and this gives the approximation

$$E\,(K - e^{X_T})^+ \doteqdot \int_{-ib}^{ib} \varphi''(c)^{-\frac{1}{2}} \frac{dw}{2\pi i} \frac{K\exp(+\frac{1}{2}w^2 - \frac{1}{2}m^2)}{(w\varphi''(c)^{-\frac{1}{2}} + c)(1+c+w\varphi''(c)^{-\frac{1}{2}})}$$

$$= K e^{(c^2 v - m^2)/2} \left\{ \Phi(-c\sqrt{v}) - \Phi\left(-(1+c)\sqrt{v}\right) \right\}.$$

$$\doteq \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} \frac{K\exp(\frac{1}{2}w^2\varphi''(c) - \frac{1}{2}m^2)}{(w+c)(w+c+1)}$$

$$= \int_0^\infty e^{-y^2/2v} \frac{dy}{\sqrt{2\pi v}} K e^{-cy}(1-e^{-y})\, e^{-\frac{1}{2}m^2} \qquad v \equiv \varphi''(c)$$

$$= \int_0^\infty \exp\left[-(y+cv)^2/2v + \tfrac{1}{2}c^2 v - \tfrac{1}{2}m^2\right] \frac{dy}{\sqrt{2\pi v}} K(1-e^{-y})^+$$

$$= E\,(K-e^{\xi})^+ \cdot \exp(\tfrac{1}{2}c^2 v - \tfrac{1}{2}m^2)$$

where $\xi \sim N(\log K + cv, v)$. Notice that

(i) if the Lévy process is drifting BM, the approximation is exact

(ii) the assumption $\log K < \psi'(0)T$ says we're dealing with a small-K regime.

The next case to deal with is where $-1 < c < 0$, or again $\boxed{T\psi'(0) < \log K < T\psi'(1)}$

We have

$$E(K-e^{X_T})^+ = \int_{1-i\infty}^{1+i\infty} \frac{dz}{2\pi i} \frac{K\exp\varphi(z)}{z\,(1+z)}$$

$$= \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \frac{K\exp\varphi(z)}{z(1+z)} + K$$

$$\doteq \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} \frac{K\exp\left[\frac{1}{2}w^2\varphi''(c) - \frac{1}{2}m^2\right]}{(c+w)(1+c+w)} + K$$

by the same change-of-variables argument as before;

$$= K - Ke^{-m^2/2}\left\{ e^{c^2v/2}\,\Phi(-c\sqrt{v}) + e^{(1+c)^2 v/2}\,\Phi(-(1+c)\sqrt{v}) \right\}$$

and again in the case of drifting BM, this expression is exact.

The final case is where $c < -1$, that is $\boxed{\log K > T\psi'(1)}$ and we have similarly

$$E(K-e^{X_T})^+ = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \frac{K\exp\varphi(z)}{z(1+z)} + K - e^{rT}$$

$$\doteq K - e^{rT} + \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} \frac{K \exp\left(\frac{1}{2}w^2\varphi''(c) - \frac{1}{2}m^2\right)}{(c+w)(1+c+w)} \qquad c \equiv -1-b < -1$$

$$= K - e^{rT} + \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} \frac{K \exp\left(\frac{1}{2}w^2\varphi''(c) - \frac{1}{2}m^2\right)}{(1+b-w)(b-w)}$$

$$= K - e^{rT} + K \int_0^\infty e^{-bx}(1-e^{-x}) \frac{e^{-x^2/2v}}{\sqrt{2\pi v}}\, dx \; e^{-m^2/2}$$

$$= K - e^{rT} + K\, e^{b^2v/2 - m^2/2}\, E\left(1 - e^{\xi}\right)^+, \qquad \xi \sim N((1+c)v, v)$$

$$= K - e^{rT} + K \exp\left(\frac{b^2v}{2} - \frac{m^2}{2}\right) \bar{\Phi}\left(-(1+c)\sqrt{v}\right) - K \exp\left(\frac{c^2v}{2} - \frac{m^2}{2}\right) \bar{\Phi}\left(-c\sqrt{v}\right).$$

Again, this is exact for the BS formula.

<u>Note:</u> $\int_b^\infty e^{\lambda y}\, p_T(\xi, y)\, dy = e^{\lambda\xi + \lambda^2 T/2}\, \bar{\Phi}\left(\frac{b-\xi-\lambda T}{\sqrt{T}}\right) \equiv h(\lambda, \xi)$, say.

Down-and-out call with linear barrier function (18/12/96)

The log price is $X_t = \sigma W_t + \mu t$, $\mu \equiv r - \sigma^2/2$, and we'll suppose that KO happens if ever $X_t < \alpha + \beta t$. Thus

$$E\left[(S_0 e^{X_T} - K)^+ : X_t \geq \alpha + \beta t \text{ for all } 0 \leq t \leq T \right]$$

$$= E\left((S_0 e^{\sigma W_T + \mu T} - K)^+ : W_t + \left(\frac{\mu - \beta}{\sigma}\right) t \geq \frac{\alpha}{\sigma} \quad \forall\, 0 \leq t \leq T \right)$$

$(a \equiv \alpha/\sigma)$

$(\gamma \equiv (\mu - \beta)/\sigma)$

$$= E\left[e^{\gamma W_T - \frac{1}{2}\gamma^2 T} (S_0 e^{\sigma W_T + \tilde{\mu} T} - K)^+ : W_t \geq a \quad \forall\, 0 \leq t \leq T \right] \qquad (\tilde{\mu} \equiv \mu - \gamma\sigma)$$

$$\left(\theta \equiv \frac{1}{\sigma}\left\{ \log \frac{K}{S_0} - \tilde{\mu} T \right\}\right)$$

$$= \int_{a \vee \theta}^{\infty} e^{\gamma y - \frac{1}{2}\gamma^2 T} (S_0 e^{\sigma y + \tilde{\mu} T} - K)^+ \{ p_T(0,y) - p_T(2a,y) \} dy$$

assuming that $a < 0$ (otherwise the problem is trivial)

$$= S_0 e^{(\tilde{\mu} - \frac{1}{2}\gamma^2)T} \{ h(\sigma+\gamma, 0) - h(\sigma+\gamma, 2a) \} - K e^{-\gamma^2 T/2} \{ h(\gamma, 0) - h(\gamma, 2a) \}$$

$$= S_0 e^{rT} \left[\bar{\Phi}\left(\frac{a \vee \theta - (\sigma+\gamma)T}{\sqrt{T}} \right) - \bar{\Phi}\left(\frac{a \vee \theta - 2a - (\sigma+\gamma)T}{\sqrt{T}} \right) e^{2a(\sigma+\gamma)} \right]$$

$$- K \left[\bar{\Phi}\left(\frac{a \vee \theta - \gamma T}{\sqrt{T}} \right) - e^{2a\gamma} \bar{\Phi}\left(\frac{a \vee \theta - 2a - \gamma T}{\sqrt{T}} \right) \right]$$

Further functionals which may help in GMM estimation of MMPP (2/1/97)

(i) Why not try

$$\frac{\alpha\beta}{\beta-\alpha}\int_{-\infty}^{t}\left\{ e^{\alpha(s-t)} - e^{\beta(s-t)} \right\} dN_s \ ?$$

This has mean equal to $E\lambda_0$, and doesn't jump when a new observation arrives. In fact, it's easy to see that it can also be written as

$$(\beta-\alpha)^{-1}\left\{ \beta\, Y_t(\alpha) - \alpha\, Y_t(\beta) \right\}$$

which has mean (π, f), and second moment

$$\frac{\alpha\beta}{2(\alpha+\beta)}(\pi,f) + \frac{\beta^2\alpha^2}{\beta^2-\alpha^2}\left\{ \frac{1}{\alpha}(\pi, fR_\alpha f) - \frac{1}{\beta}(\pi, fR_\beta f) \right\}.$$

For the special case considered at the foot of p.4, this gives us a second moment of

$$\frac{\alpha\beta}{2(\alpha+\beta)}(\pi,f) + (\pi,f)^2 + \frac{K^2\alpha\beta}{2b^2(\alpha+\beta)}\left[\frac{\alpha+\beta+2b}{(\alpha+2b)(\beta+2b)} + \frac{4c^2b}{(\alpha+b)(\beta+b)}(\alpha+\beta+b) \right]$$

(ii) How about measuring how far apart Y has moved over a time interval, using

$$\int_0^\infty \gamma e^{-\gamma u}\, du \left(Y_t(\alpha) - Y_{t-u}(\beta) \right)^2 \ ?$$

$$= Y_t(\alpha)^2 - 2Y_t(\alpha)\frac{\beta Y_t(\gamma) - \gamma Y_t(\beta)}{\beta-\gamma} + \frac{2\gamma}{2\beta-\gamma}\left\{ \eta_t(\gamma,\beta) - \eta_t(2\beta,\beta) \right\}$$

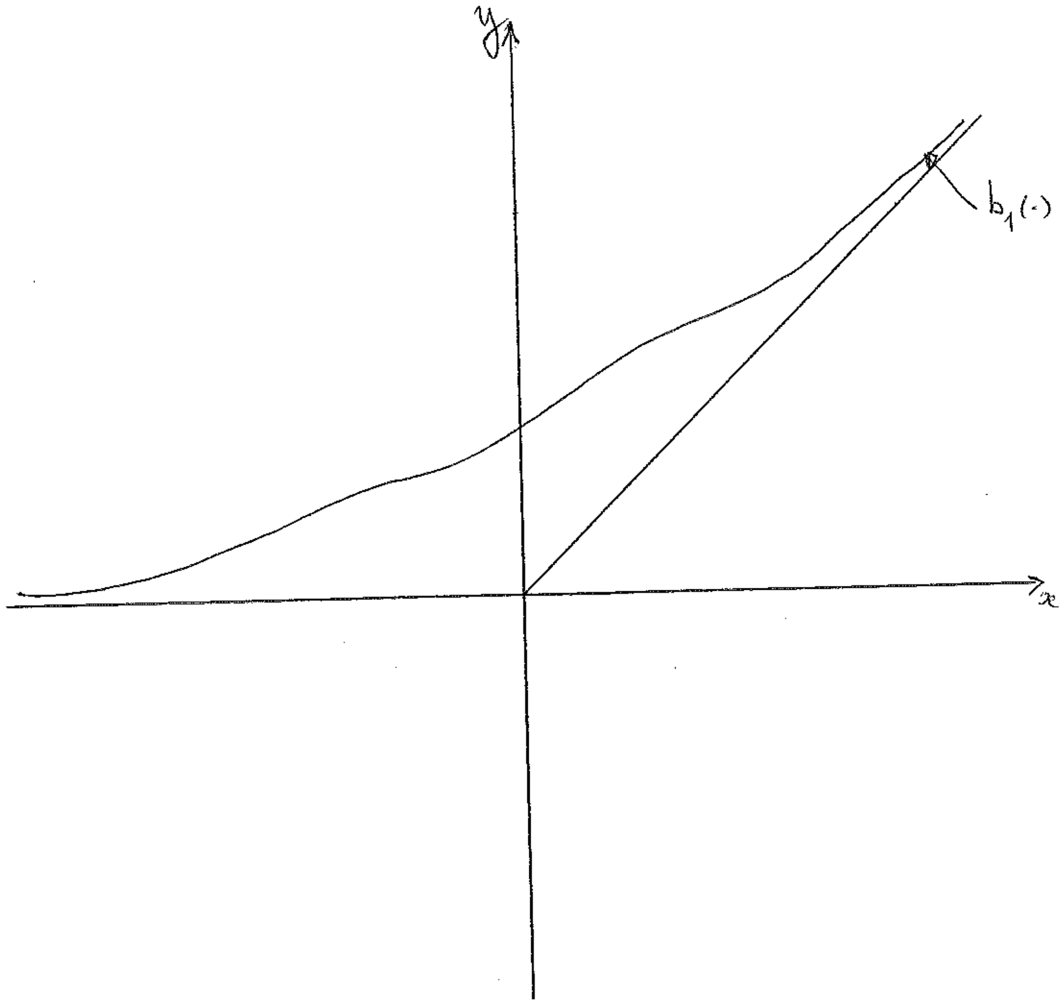
$$+ \frac{1}{2\beta-\gamma} Y_t(\gamma) - \frac{\gamma}{2\beta(2\beta-\gamma)} Y_t(2\beta)$$

So even this is expressed in terms of the Y's and η's we were working with before. This means it is unlikely that we will get a better determination of the parameters than with just the Y's and η's.

If we write $\theta_1 = Kc^2$, $\theta_2 = K/2b$, we have $E\,Y(\alpha) = \theta_1 + \theta_2$ and

$$E\,\eta(\alpha,\beta) = (\theta_1+\theta_2)^2 + \beta\theta_2^2\left\{ \frac{2}{\beta+2b} + \frac{4\theta_1/\theta_2}{\beta+b} \right\}$$

Thus if $b/\beta \gg 1$, we have no chance of accurate estimation of b... so we'll have to be using some β which is of order no bigger than b...



y
x
b₁(·)

Maximum Maximum: perturbation analysis. (13/2/97)

(i) The problem is thus. If M is a martingale and the laws ν_i of $M(i)$ are specified $(i=1,2)$, $M_0=0$, can one find the stochastically largest law of $\bar{M}_2 \equiv \sup\{M_s : s \le 2\}$? Is this even a well-posed question?

Clearly, in the time interval $[0,1]$ we do the Azema-Yor embedding of law ν_1, because this makes $\bar{M}_1$ as big as it can be. Then conditional on M_1, we choose the laws of M_2, and then do AY embedding of that.

Assume laws have densities; let $p(x,y)$ be the joint density of $X \equiv M_1$ and $Y \equiv M_2$, let $b_1(\cdot)$ be the barycentre function of ν_1, let $p(y|x)$ be the conditional density of Y given X, and let $b(\cdot|x)$ be the barycentre function of that conditional law. We may therefore express

$$\bar{M}_2 = b(Y|X) \vee b_1(X).$$

So for any increasing differentiable $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi(0)=0$, we get

$$E\left[\varphi(\bar{M}_2)\right] = \iint p(x,y)\,dx\,dy\,\varphi(b_1(x) \vee b(y|x))$$

$$= \int p_1(x)\,dx \int p(y|x)\,\varphi\left(b_1(x) \vee b(y|x)\right)dy$$

(ii) Suppose we had found the optimal joint density p, and we considered changing it for some fixed $x_1 < x_2$ to

$$\hat{p}(x_1,y) = p(x_1,y) + \eta(y), \qquad \hat{p}(x_2,y) = p(x_2,y) - \eta(y),$$

where

$$\int \eta\,dy = \int y\,\eta\,dy = 0,$$

in order to keep the marginals unaffected, and also to make the mean of $p(\cdot|x)$ unaltered.

Notationally simpler is to take a univariate density f and perturb to $f+\eta$, and consider the effect on $E\,\varphi(c \vee b(Y))$. We have

$$\int f(y)\,\varphi(c \vee b(y))\,dy = \varphi(c) + \int f(y)\left\{\int_c^{c \vee b(y)} \varphi'(z)\,dz\right\}dy$$

$$= \varphi(c) + \int_c^\infty \varphi'(z)\left(\int_{b^{-1}(z)}^\infty f(y)\,dy\right)dz,$$

so we need to understand the effect of perturbation on b. The small change in b is given by

$$\tilde{b}(x) = \int_x^\infty (y - b(x))\,\eta(y)\,dy \,/\, \bar{F}(x) \quad + b(x),$$

so that the small change in $E\,\varphi(c \vee b(Y))$ is given by

$$\int \eta(y)\, \varphi(c \vee b(y))\, dy + \int f(y)\, I_{\{b(y)>c\}}\, \varphi'(b(y)) \left(\int_y^\infty (v - b(y))\, \eta(v)\, \frac{dv}{\bar{F}(y)} \right) dy$$

$$= \int \eta(y) \left\{ \varphi(c \vee b(y)) + \int_{-\infty}^{y} dz\; f(z)\, \varphi'(b(z))\, I_{\{b(z)>c\}}\, \frac{(y - b(z))}{\bar{F}(z)} \right\} dy$$

$$= \int_{b^{-1}(c)}^{\infty} \eta(y) \left\{ \int_{b^{-1}(c)}^{y} \varphi'(b(z)) \left(b'(z) + f(z)\, \frac{y - b(z)}{\bar{F}(z)} \right) dz \right\} dy$$

$$= \int_{b^{-1}(c)}^{\infty} \eta(y) \int_{b^{-1}(c)}^{y} \varphi'(b(z))\, \frac{f(z)}{\bar{F}(z)}\, (y - z)\, dz\, dy \qquad (*)$$

So if we return to the situation where we perturb p to $\tilde{p}$, the change to first order in the payoff will be

$$\int_{b^{-1}(b_1(x_1)|x_1)}^{\infty} \eta(y) \int_{b^{-1}(b_1(x_1)|x_1)}^{y} \varphi'(b(z|x_1))\, \frac{p(z|x_1)}{\bar{P}(z|x_1)}\, (y-z)\, dz\, dy$$

$$- \int_{b^{-1}(b_1(x_2)|x_2)}^{\infty} \eta(y) \int_{b^{-1}(b_1(x_2)|x_2)}^{y} \varphi'(b(z|x_2))\, \frac{p(z|x_2)}{\bar{P}(z|x_2)}\, (y-z)\, dz\, dy.$$

$$\equiv \int \eta(y)\, g(y)\, dy,$$

say. Now the feasible perturbations η are those such that $\int \eta(t)\,dt = 0 = \int t\,\eta(t)\,dt$, and $p(x_1, y) + \eta(y) \geq 0, \quad p(x_2, y) - \eta(y) \geq 0$

Thus if $A_i = \{ y : p(x_i, y) > 0 \}$, if we restrict the support of η to $A_1 \cap A_2$ we conclude that for some constants a, m

$$g(y) = a + my \qquad \text{in } A_1 \cap A_2$$

Likewise,

$$g(y) \leq a + my \qquad \text{in } A_2$$
$$g(y) \geq a + my \qquad \text{in } A_1$$

Writing $\beta_i \equiv b^{-1}(b_1(x_i)|x_i)$ we have

$$g(y) = \int_{\beta_1}^{y \vee \beta_1} dt \int_{\beta_1}^{t} \varphi'(b(z|x_1))\, \frac{p(z|x_1)}{\bar{p}(z|x_1)}\, dz - \int_{\beta_2}^{y \vee \beta_2} \int_{\beta_2}^{t} \varphi'(b(z|x_2))\, \frac{p(z|x_2)}{\bar{P}(z|x_2)}\, dz$$

The perturbation (*) can also be expressed as

$$\int_{b^{-1}(c)}^{\infty} \eta(y) \left\{ \int_c^{b(y)} \varphi'(v) \frac{y - b^{-1}(v)}{v - b^{-1}(v)} \, dv \right\} dy$$

$$= \int_c^{\infty} \frac{\varphi'(v)\, dv}{v - b^{-1}(v)} \int_{b^{-1}(v)}^{\infty} \eta(y)(y - b^{-1}(v))\, dy$$

$$= \int_c^{\infty} \frac{\varphi'(v)\, dv}{v - b^{-1}(v)} \int_{b^{-1}(v)}^{\infty} dt \; \bar{\eta}(t).$$

So for feasible perturbations η we have to have

$$0 \geq \int_{\beta_1}^{\infty} \eta(y) \int_{b_1(x_1)}^{b(y|x_1)} \varphi'(v) \frac{y - b^{-1}(v|x_1)}{v - b^{-1}(v|x_1)} \, dv\, dy - \int_{\beta_0}^{\infty} \eta(y) \int_{b_1(x_2)}^{b(y|x_2)} \varphi'(v) \frac{y - b^{-1}(v|x_2)}{v - b^{-1}(v|x_2)} \, dv\, dy$$

$$= \int_{\beta_1}^{\infty} \eta(y)\, dy \int_{\beta_1}^{y} \varphi'(b(z|x_1)) \frac{y - z}{b(z|x_1) - z} Db(z|x_1)\, dz - \cdots$$

and if we restrict η to be supported in $(\beta_1 \vee \beta_2, \infty)$, <u>assuming</u> $p(y|x_1)p(y|x_2) > 0$ <u>there</u>, the conclusion has to be

$$\varphi'(b(z|x_1)) \frac{Db(z|x_1)}{b(z|x_1) - z} = \varphi'(b(z|x_2)) \frac{Db(z|x_2)}{b(z|x_2) - z}$$

so that $b(z|x_1) = b(z|x_2)$ for $z > \beta_1 \vee \beta_2$, since $\varphi \uparrow$ was assumed arbitrary.

This coupling appears to be known: Neil O'Connell quotes W. Feller "The fundamental limit theorems in probability" BAMS $\underset{\sim}{51}$ 800–832, 1945

A question of Richard Arratia on random permutations (12/3/97).

(i) In a random permutation of $\{1, 2, \dots, n\}$, let $C_r(n)$ denote the number of cycles of length r. It is known that

$$P(C_1(n) = a_1, C_2(n) = a_2, \dots, C_n(n) = a_n) = \prod_{r=1}^{n} \left(\frac{1}{r}\right)^{a_r} \frac{1}{a_r!} \qquad \left(a_i \geq 0, \ \Sigma i a_i = n\right)$$

– see the paper of Arratia & Tavaré Ann. Prob 1992, 1567–1591 – and it can be shown that if (Z_i) are independent Poisson, $E Z_i = 1/i$, then for fixed b

$$(C_1(n), \dots, C_b(n)) \Rightarrow (Z_1, Z_2, \dots, Z_b)$$

The paper of A&T is concerned with a very rapid total-variation convergence of the laws to the Poisson limit; if $d_b(n) = \| \mathcal{L}(C_1(n), \dots, C_b(n)) - \mathcal{L}(Z_1, \dots, Z_b) \|$, then A&T prove if $b/n \to 0$, then

$$d_b(n) \leq F(b/n), \qquad F(x) = \sqrt{2\pi m}\, \frac{2^{m-1}}{(m-1)!} + \frac{1}{m!} + 3\left(\frac{x}{e}\right)^{-e}, \qquad m \equiv [x].$$

(ii) Here's an interesting way to re-express the problem. Take a permutation – for example

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 5 & 8 & 6 & 1 & 4 & 2 & 9 \end{pmatrix}$$

and write out the cycle containing 1, starting with 1:

$$(1\ 3\ 5\ 6)$$

Find the smallest number not yet written (2 in this case) and write out the cycle starting there, but write it to the <u>left</u> of what's already written;

$$(2\ 7\ 4\ 8)\ (1\ 3\ 5\ 6)$$

Then continue with all remaining elements until we end up with a representation in terms of cycles:

$$(9)(2\ 7\ 4\ 8)(1\ 3\ 5\ 6).$$

Now drop the brackets, and we see a different permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 7 & 4 & 8 & 1 & 3 & 5 & 6 \end{pmatrix}$$

What we've defined is a 1-1 mapping of Σ_n to Σ_n, which carries the <u>number of disjoint cycles</u> into <u>the number of downward ladder points</u>, and a cycle of length l gets mapped to a gap of length l between downward ladder points!

We could therefore take a sequence $U_1, U_2, \dots$ of IID $U(0,1)$s, and use the permutation induced by $(U_1, \dots, U_n)$ to give us the cycle decomposition of a random permutation.

(iii) Let's write ν_b for the number of times in the sequence (U_i) that two succeeding downward ladder points are separated by b, and write $Z = \prod_{r=1}^{m} \theta_r^{\nu_r}$. Then if we define

$$\varphi(\theta,x) \equiv E[Z \mid U_1 = x],$$

we shall have

$$\varphi(\theta,x) = \left\{ \sum_{r=1}^{m} (1-x)^{r-1} x\, \theta_r + (1-x)^m \right\} \int_0^x \frac{dy}{x}\, \varphi(\theta,y)$$

$$\equiv h(x) \int_0^x \varphi(\theta,y)\, dy$$

where $h(x) \equiv x^{-1} \left\{ \sum_{r=1}^{m} (1-x)^{r-1} x\, \theta_r + (1-x)^m \right\}$. Then if $g(x) \equiv \varphi(\theta,x)/h(x)$, we see $g'(x) = h(x)\, g(x)$ so that

$$g(x) = \exp\left[- \int_x^1 h(v)\, dv + a(\theta) \right].$$

Notice that $x h(x) \to 1$ as $x \to 0$, and $h(x) - \frac{1}{x} = \sum_{r=1}^{m} (1-x)^{r-1} \theta_r - \sum_{j=0}^{m-1} (1-x)^j$ so that

$$h(x) = \frac{1}{x} + \sum_{r=1}^{m} (1-x)^{r-1} (\theta_r - 1).$$

Hence

$$g(x) = x \exp\left[- \int_x^1 \sum_{r=1}^{m} (\theta_r - 1)(1-v)^{r-1}\, dv + a(\theta) \right]$$

and so

$$\varphi(\theta,x) = h(x) \cdot x \cdot \exp\left[- \int_x^1 \sum_{r=1}^{m} (\theta_r - 1)(1-v)^{r-1}\, dv + a(\theta) \right]$$

Since $\varphi(\theta, 0) = 1$, we deduce that $a(\theta) = \sum_{r=1}^{m} (\theta_r - 1)/r$, therefore

$$\boxed{\varphi(\theta,x) = \left\{ 1 + \sum_{r=1}^{m} (1-x)^{r-1} x (\theta_r - 1) \right\} \exp\left[\int_0^x \sum_{r=1}^{m} (\theta_r - 1)(1-v)^{r-1}\, dv \right]}$$

and

$$\boxed{E[Z] = \int_0^1 \varphi(\theta,x)\, dx = \int_0^1 h(x) \exp\left[- \int_x^1 h(v)\, dv + a(\theta) \right] = e^{a(\theta)}}$$

which establishes the limit law of $(\nu_1, \dots, \nu_b)$ to be independent Poissons, means $(1, \dots, 1/b)$

It appears that this approach won't give the really rapid convergence of A + T.

A question on transaction costs (13/3/97)

Here's a question of Ray Rishel on transaction costs, which is completely deterministic! Suppose you can have money in one of two accounts with rates of interest $r < R$, but it costs to shift from one to the other. If x_t is the amount in the account with rate r at time t, and y_t the amount in the other, the dynamics are given by

$$dx_t = r x_t \, dt - (1+\varepsilon') dL_t + (1-\varepsilon) dK_t - c_t \, dt$$

$$dy_t = R y_t \, dt + dL_t - dK_t$$

If we aim to max $\int_0^\infty e^{-\delta t} U(c_t)\, dt$, with $U(x) = x^{1-\rho}/(1-\rho)$, what should we do?

(i) It's clear we would never move money from y to x when $x > 0$, but once $x = 0$, we move from y to x and consume. So once $x = 0$, we get dynamics

$$dy_t = R y_t \, dt - \frac{c_t}{1-\varepsilon} dt$$

The budget constraint is $y_0 = \int_0^\infty e^{-Rt} \frac{c_t}{1-\varepsilon} dt$, and taking the Lagrangian form of the problem leads to

$$e^{-\delta t} U'(c_t^*) = \frac{\lambda e^{-Rt}}{1-\varepsilon} \quad \therefore \quad c_t^* = \left(\frac{\lambda}{1-\varepsilon}\right)^{-1/\rho} e^{-(\delta - R)t/\rho}$$

To match the budget constraint, we take

$$\lambda = y_0^{-\rho} (1-\varepsilon)^{1-\rho} \left(R - \frac{R-\delta}{\rho}\right)^{-\rho}$$

and have to assume $R - (R-\delta)/\rho > 0$ for well-posed problem. This gives us a maximised payoff of

$$y_0^{1-\rho} \cdot K \equiv y_0^{1-\rho} \frac{(1-\varepsilon)^{1-\rho}}{1-\rho} \left(\frac{\rho}{\delta - R(1-\rho)}\right)^{\rho} \equiv y_0^{1-\rho} \frac{(1-\varepsilon)^{1-\rho}}{1-\rho} B^{-\rho}$$

(ii) If initially we start off with $x_0 > 0$, it may be advantageous to move some immediately to y. If x_0 were too small, we expect that we'll be drawing consumption too soon to make it worth moving anything to y straightaway. So if we let τ denote the time when x reaches 0, assuming we start at (x_0, y_0) with no immediate transfer, we have

$$x_0 = \int_0^\tau e^{-rt} c_t \, dt$$

So if we fix τ, we'll choose c to maximise $\int_0^\tau e^{-\delta t} U(c_t)\, dt$ in the same style as above;

$$c_t^* = \frac{x_0 b}{1 - e^{-b\tau}} e^{-(\delta - r)t/\rho}, \qquad b \equiv \frac{\delta - r(1-\rho)}{\rho},$$

and the contribution to the payoff is

$$\int_0^\tau e^{-\delta t}\, U(c_t^*)\, dt = \frac{x_0^{1-\rho}}{1-\rho} \left(\frac{1-e^{-b\tau}}{b}\right)^\rho .$$

Thus if we fix τ and consume from x until we run out, we end up with payoff

$$\frac{x_0^{1-\rho}}{1-\rho}\left(\frac{1-e^{-b\tau}}{b}\right)^\rho + e^{-\delta\tau}\left(y_0 e^{R\tau}\right)^{1-\rho} \kappa$$

<u>Case 1: $0<\rho<1$.</u> In this case, $b>B$, and maximisation of the payoff amounts to maximising

$$\left(\frac{x_0}{y_0(1-\varepsilon)}\right)^{1-\rho}\left(\frac{B}{b}\right)^\rho (1-e^{-b\tau})^\rho + e^{-\rho B\tau}$$

which comes down to solving

$$\frac{x_0\, b}{y_0(1-\varepsilon)B}\;\frac{1}{1-e^{-b\tau}} = e^{(b-\rho B)\tau/(1-\rho)} \qquad (*)$$

and this always has a unique solution.

<u>Case 2: $\rho>1$.</u> This time, we are trying to minimise

$$\left(\frac{x_0}{y_0(1-\varepsilon)}\right)^{1-\rho}\left(\frac{B}{b}\right)^\rho (1-e^{-b\tau})^\rho + e^{-\rho B\tau},$$

and, once again, there's a unique minimising τ.

Either way, we end up with a unique value $\tau^* = \tau^*(x_0, y_0)$ to maximise the payoff. We are in a region where there is no initial jump if by perturbing along $z \mapsto (x_0-(1+\varepsilon')z,\ y_0+z)$ we don't improve. Thus the first-order condition is

$$-(1+\varepsilon')\, x_0^{-\rho}\left(\frac{1-e^{-b\tau^*}}{b}\right)^\rho + (1-\rho)\kappa\, y_0^{-\rho}\, e^{-\rho B\tau^*} \le 0,$$

which can be reexpressed as

$$(1+\varepsilon')^{1/\rho}\;\frac{1-e^{-b\tau^*}}{x_0 b} \le \frac{e^{-B\tau^*}}{y_0 B}\,(1-\varepsilon)^{\frac{1}{\rho}-1}$$

i.e.

$$\left(\frac{1+\varepsilon'}{1-\varepsilon}\right)^{1/\rho} \le \exp\left[\frac{R-r}{1-\rho}\,\tau^*\right]$$

This determines the critical value of τ^* at which it just becomes worthwhile to make an initial jump, and hence (from $(*)$) the slope of the critical line at which we change from no initial jump to an initial jump.

Working in terms of time-to-go τ, we get

$$\dot{v}_\tau = v_\tau^2 - 2\beta v_\tau + \alpha \equiv (v_\tau - \theta_1)(v_\tau - \theta_2)$$

$$\theta_1 = \beta + \theta$$
$$\theta_2 = \beta - \theta$$

$$\Rightarrow v_\tau = \frac{\theta_2 e^{2\theta\tau}(q-\theta_1) - \theta_1(q-\theta_2)}{(q-\theta_1)e^{2\theta\tau} - (q-\theta_2)}$$

$$= \theta_1 - \frac{g'(\tau)}{g(\tau)} \qquad \text{where } g(\tau) \equiv (q-\theta_1)e^{2\theta\tau} + \theta_2 - q$$

Thus

$$\dot{\gamma}_\tau = \rho + \gamma_\tau\left(\theta_1 - \beta - \frac{g'(\tau)}{g(\tau)}\right) = \rho + \gamma_\tau\left(\theta - \frac{g'(\tau)}{g(\tau)}\right)$$

$$\Rightarrow \frac{d}{d\tau}\left[g_\tau \gamma_\tau e^{-\theta\tau}\right] = \rho e^{-\theta\tau} g(\tau).$$

$$\Rightarrow \gamma_\tau g_\tau e^{-\theta\tau} = -2\theta\lambda + \frac{\rho}{\theta}(q-\theta_1)(e^{\theta\tau}-1) + \frac{\rho}{\theta}(\theta_2 - q)(1 - e^{-\theta\tau})$$

A quadratic-functional calculation (27/3/97)

(i) Suppose we have a one-dimensional OU process $dx_t = dW_t - \beta x_t\, dt$, $\beta > 0$. To compute

$$E_t \exp\Big[\tfrac{1}{2} q\, x_T^2 + \lambda x_T + \int_0^T (\tfrac{1}{2}\alpha x_s^2 + \rho x_s)\, ds \Big]$$

$$= \exp\Big[\tfrac{1}{2} v_t x_t^2 + \gamma_t x_t + \kappa_t + \int_0^t (\tfrac{1}{2}\alpha x_s^2 + \rho x_s)\, ds \Big] \equiv e^{Y_t},$$

we shall have to have

$$dY_t + \tfrac{1}{2} d\langle Y\rangle_t = (v_t x_t + \gamma_t)\, dx_t + (\tfrac{1}{2} v_t + \tfrac{1}{2}\dot{v}_t x_t^2 + \dot{\gamma}_t x_t + \dot{\kappa}_t + \tfrac{1}{2}\alpha x_t^2 + \rho x_t)\, dt$$
$$+ \tfrac{1}{2} (v_t x_t + \gamma_t)^2\, dt$$

is a martingale. This gives the equations

$$\left.\begin{aligned} &\dot{v}_t - 2\beta v_t + \alpha + v_t^2 = 0 && v_T = q \\ &-\beta\gamma_t + \dot{\gamma}_t + \rho + \gamma_t v_t = 0 && \gamma_T = \lambda \\ &\tfrac{1}{2} v_t + \dot{\kappa}_t + \tfrac{1}{2}\gamma_t^2 = 0 && \kappa_T = 0 \end{aligned}\right\}$$

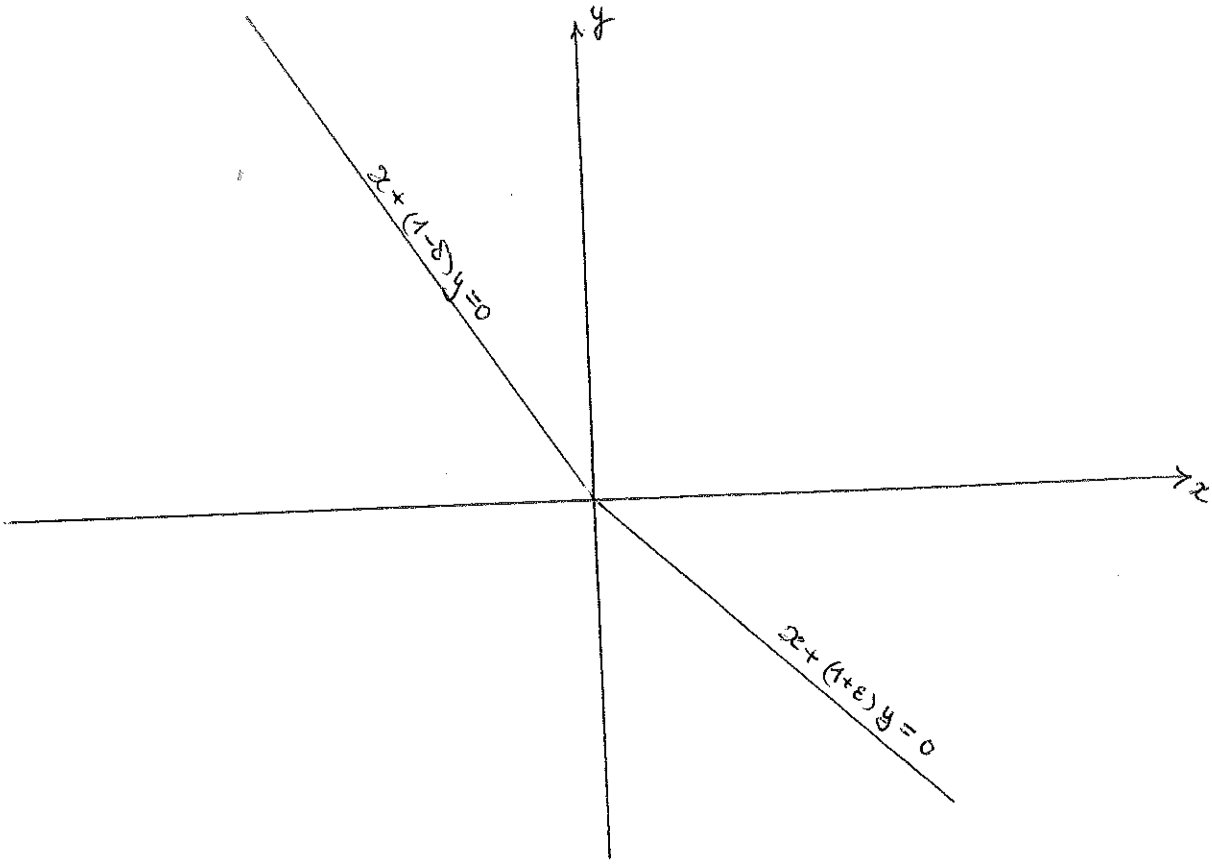
which is solved by

$$v_t = \frac{(\beta-\theta)\, e^{2\theta\tau}(q-\beta-\theta) - (\beta+\theta)(q-\beta+\theta)}{e^{2\theta\tau}(q-\beta-\theta) - (q-\beta+\theta)} \qquad \begin{pmatrix} \tau \equiv T-t \\ \theta \equiv \sqrt{\beta^2-\alpha} \end{pmatrix}$$

$$\gamma_t = \frac{\theta\lambda + \rho\theta^{-1}\big[\theta \sinh\theta\tau + (\beta-q)\{\cosh(\theta\tau) - 1\}\big]}{\theta\cosh\theta\tau + (\beta-q)\sinh\theta\tau}$$

$$\kappa_t = \frac{\beta\tau}{2} - \frac{1}{2}\log\Big[\frac{\theta\cosh\theta\tau + (\beta-q)\sinh\theta\tau}{\theta}\Big]$$
$$+ \Big[\rho^2 y^2 \log y\, (\theta-q+\beta)^2 + 4\rho y(\theta-q+\beta)\{\rho(\beta-q) - \lambda\theta^2\}$$
$$+ 2\rho^2(\theta-q+\beta)(\theta+q-\beta) - 2(\lambda\theta^2 - \rho(\beta-q))^2$$
$$+ \rho^2 \log y\, (\theta^2 - (\beta-q)^2)\Big] \Big/ \Big(((\beta+\theta-q)y^2 + \theta-\beta+q)(\beta+\theta-q)\theta^3\Big)$$
$$- \frac{2\rho^2(\beta-q) - 2\rho^2\theta - \theta(\lambda\theta+\rho)^2}{\theta^3(\beta+\theta-\rho)}$$

where $y \equiv e^{\theta\tau}$.



$$\tilde{U}(a) \equiv \sup(U(c) - ac) = a^{1-1/R} \frac{R}{1-R}$$

We know that for each ξ, $V(x,y,\xi)$ is increasing in x, y and is concave, and if $V(x,y,\xi) \equiv y^{1-R} v(\frac{x}{y},\xi) \equiv y^{1-R} V(\frac{x}{y}, 1, \xi)$, clearly v is increasing and concave. Thus $c_t = y_t \, v'(\frac{x_t}{y_t}, \xi_t)^{-1/R} = w_t (1-p_t) \, v'(\frac{p_t}{1-p_t}, \xi_t)^{-1/R}$,

and $t \mapsto v'(t,\xi)^{-1/R}$ is increasing

A question of Ray Rishel (2/4/97).

This is a nice question posed by Ray Rishel. Let's summarise what we know about it.

(i) Let x_t denote amount of cash in bank at time t, let y_t denote the value of holding in shares at time t, and suppose that we have to pay ~~transactions~~ costs, and that the return on the share is a Markov chain taking values in a finite set $I \subseteq \mathbb{R}$, with $I \cap \{x<r\} \neq \emptyset \neq I \cap \{x>r\}$ to avoid triviality. Thus the dynamics will be

$$(1)\qquad \left.\begin{aligned} dx_t &= (r x_t - c_t)\,dt - (1+\varepsilon)\,dL_t + (1-\delta)\,dK_t \\ dy_t &= y_t \xi_t\,dt + dL_t - dK_t \end{aligned}\right\}$$

and we'll insist that the process stays in the solvency region

$$(2)\qquad \{(x,y) : x + (1-\delta)y \geq 0,\ x + (1+\varepsilon)y \geq 0\}.$$

The objective is to

$$\max E \int_0^\infty e^{-\rho t}\, U(c_t)\,dt$$

where $U'(x) = x^{-R}$ for some $R>0$. If $V(x,y,\xi)$ denotes the value function (depending on x, y and the state ξ of the Markov chain), then the HJB equation will be

$$\sup_{\substack{c\geq 0\\ \lambda \geq 0\\ \kappa\geq 0}} \Big[U(c) - \rho V + (rx-c)V_x + y\xi V_y + \lambda\big(V_y - (1+\varepsilon)V_x\big) + \kappa\big(-V_y + (1-\delta)V_x\big) + QV \Big] = 0$$

from which we obtain

$$(3)\qquad \boxed{\begin{aligned} &\tilde{U}(V_x) - \rho V + rxV_x + y\xi V_y + QV = 0 \\ &(1+\varepsilon)V_x \geq V_y \geq (1-\delta)V_x \end{aligned}} \qquad (\leq 0 \text{ in jump region})$$

(ii) Exploit scaling by introducing variables

$$\boxed{p_t \equiv \frac{x_t}{x_t+y_t}, \qquad w_t \equiv x_t + y_t}$$

so that $\boxed{V(x,y,\xi) = w^{1-R}\,\psi(p,\xi)}$.

and so (at least away from times of jumps)

$$dp_t = (1-p_t)\left[\left(rp_t - \frac{c_t}{w_t}\right)dt - (1+\varepsilon)\frac{dL_t}{w_t} + (1-\delta)\frac{dK_t}{w_t}\right]$$
$$- p_t\left[(1-p_t)\xi_t\,dt + \frac{dL_t}{w_t} - \frac{dK_t}{w_t}\right]$$

(4) $$= \left\{(r-\xi)p_t(1-p_t) - (1-p_t)\frac{c_t}{w_t}\right\}dt - (1+\varepsilon(1-p_t))\frac{dL_t}{w_t} + (1-\delta(1-p_t))\frac{dK_t}{w_t}$$

which is actually an <u>autonomous</u> dynamic for p, once we notice that $c_t = w_t\{(1-R)\psi + (1-p)\psi'\}^{-1/R}$.

The HJB equation now becomes

(5) $$\frac{R}{1-R}\{(1-R)\psi + (1-p)\psi'\}^{1-\frac{1}{R}} + (r-\xi)\{(1-R)p\psi + (1-p)p\psi'\}$$
$$+(\xi(1-R) - \rho)\psi + Q\psi = 0$$ (≤ 0 in jump region)

$$\psi' \le \frac{\delta(1-R)}{1-\delta+\delta p}\psi$$ i.e. $(1-\delta+\delta p)^{R-1}\psi(p)$ is decreasing (if $R\in(0,1)$)

$$\psi' \ge -\frac{\varepsilon(1-R)}{1+\varepsilon-\varepsilon p}\psi$$ i.e. $(1+\varepsilon-\varepsilon p)^{R-1}\psi(p)$ is increasing (if $R\in(0,1)$)

and $p_* \equiv -\frac{1-\delta}{\delta} \le p \le p^* \equiv \frac{1+\varepsilon}{\varepsilon}$ is the region for p.

Away from times of jumps, the evolution of wealth is

$$dw_t = w_t\left\{rp_t - \frac{c_t}{w_t} + (1-p_t)\xi_t\right\}dt - w_t\left\{\varepsilon\frac{dL_t}{w_t} + \delta\frac{dK_t}{w_t}\right\},$$

so we have a <u>skew-product decomposition</u>.

(iii) What can we say of jumping behaviour? If $I_> \equiv \{\xi\in I : \xi > r\}$, $I_< = I\setminus I_>$, and we assume $r\notin I$ (no real loss, because if we were in state $\xi = r$, it's obvious how to proceed), let's notice that <u>if we're in $I_>$</u>, we would never move wealth from y to x except if we'd reached the boundary $x+(1-\delta)y$ of the solvency region. We might move wealth from x to y for an initial jump, but thereafter we would let the process evolve. So there's some $\beta(\xi)$ for each $\xi\in I_>$ such that we keep p always in $[p_*, \beta(\xi)]$ until the state ξ changes. From the dynamics of p, it's clear that p decreases while in $\mathbb{R}^+$; if it should decrease to p_*, then consume at rate

$$c_t = w_t\,p_*(r-\xi).$$

This rate of consumption keeps us just on the boundary of the solvency region without need for dK intervention, so the best thing is to keep $dK = 0 = dL$ and consume at this rate.

What happens if $\xi \in I_<$? There will be some $\alpha(\xi)$ such that we keep $p \in [\alpha(\xi), p^*]$ while in state ξ. This may require an initial jump on entering the state. It may also require continuous intervention (dK) when $p = \alpha(\xi)$, but if this were to happen, it must be (from the dynamics of p) that p remains at $\alpha(\xi)$ once there. Otherwise, p will increase when below some asymptotic value $p(\xi)$, and decrease while above; should it be the case that $p(\xi) = p^*$, then when p reaches p^*, consume at rate $c_t = w_t\, p^*(r - \xi)$ and don't intervene.

(iv) As a method of solving the problem, we could consider the value of the game in two modified situations (assuming $0 < R < 1$):

(a) After n^{th} change of state, everything stops, and no more consumption happens;

(b) After n^{th} change of state, transactions costs are eliminated.

If we actually had no transactions costs, the wealth would satisfy

$$(6) \qquad dw_t = w_t (r \vee \xi_t)\, dt - c_t\, dt,$$

and the value function is of the form $w^{1-R} f(\xi)$. The HJB eq^n becomes

$$(7) \qquad \frac{R}{1-R}\left[(1-R) f(\xi)\right]^{1-1/R} - \rho f(\xi) + (1-R)(r \vee \xi) f(\xi) + Q f(\xi) = 0,$$

which should be easy enough to solve (how many solutions?) If we abbreviate $\tilde{r} \equiv r \vee \xi$, and imagine we try to max $E \int_0^\tau e^{-\rho s} U(c_s)\, ds$, where τ is first jump time, we

$$(8) \qquad \max E \int_0^\tau e^{-\rho s} U(c_s)\, ds = \max \int_0^\infty e^{-\rho s - q s} U(c_s)\, ds \quad \text{s.t.} \quad \int_0^\infty e^{-\tilde{r} s} c_s\, ds = w_0$$

so using Lagrangian, we find that for some λ, $c_s = I(\lambda e^{(\rho + q - \tilde{r})s})$, and in fact $\lambda_0^{-1/R} = w_0 \{\tilde{r} + (\rho + q - \tilde{r})/R\}$. Thus the value for this simple problem is just

$$(9) \qquad \frac{w_0^{1-R}}{1-R} \left\{ \tilde{r} + \frac{\rho + q - \tilde{r}}{R} \right\}^{-1/R}$$

where $q \equiv q_\xi$. This gives us a first guess for the function f. Or perhaps an even simpler lower bound is obtained by assuming $\xi \le r$ always:

$$(10) \qquad \frac{w_0^{1-R}}{1-R} \left(r + \frac{\rho - r}{R} \right)^{-1/R}$$

We could now attempt to build up bounds on the value function:

(11) $$\underline{V}^n(x,y,\xi) \le V(x,y,\xi) \le \bar{V}^n(x,y,\xi)$$

where

(12) $$\tilde{u}(\underline{V}^n_x) - \rho\underline{V}^n + rx\underline{V}^n_x + y\xi\underline{V}^n_y - q_\xi\underline{V}^n = \sum_{\zeta\ne\xi} q_{\xi\zeta}\underline{V}^{n-1}(x,y;\zeta)$$

and $(1+\epsilon)\underline{V}^n_x \ge \underline{V}^n_y \ge (1-\delta)\underline{V}^n_x$,

and the similar thing for $\bar{V}^n$, ie solving HJB when we go to regime $(n-1)$ after first jump. This simplifies life somewhat, because (12) can be solved separately for each ξ. If we can actually do this, we'd have a solution, but I guess the problem is with the boundary conditions on (12), or, perhaps easier, the corresponding version of (5)

(V) There's another approach to the problem, which is to use the invariant-measure methodology of Dick Stockbridge. The presence of jumps complicates matters somewhat, but let's replace a jump by a rapid slide and then the paths of (x,y) will be continuous:

$$\left.\begin{aligned} \dot{x}_t &= rx_t - c_t - (1+\epsilon)l_t + (1-\delta)k_t \\ \dot{y}_t &= \xi_t y_t + l_t - k_t \end{aligned}\right\} \qquad 0 \le l, k \le M.$$

We have (with $p_t \equiv x_t/(x_t+y_t)$, $w_t \equiv x_t+y_t$) that

$$\boxed{\begin{aligned} \dot{p}_t &= (r-\xi_t)p_t(1-p_t) - (1-p_t)\tilde{c}_t - (1+\epsilon(1-p_t))\frac{l_t}{w_t} + (1-\delta(1-p_t))\frac{k_t}{w_t} \\ \dot{w}_t &= w_t\left\{ rp_t + \xi_t(1-p_t) - \tilde{c}_t - \epsilon\frac{l_t}{w_t} - \delta\frac{k_t}{w_t}\right\} \end{aligned}}$$

where $\tilde{c}_t \equiv c_t/w_t$ is the consumption rate per unit of wealth. The payoff is

$$E\left[\int_0^\infty e^{-\rho t} w_t^{1-R}\,\tilde{c}_t^{\,1-R}\frac{dt}{1-R}\,\Big|\,\xi_0, p_0, w_0=1\right]$$

$$= E\left[\int_0^\infty \exp\left\{-\rho t + (1-R)\int_0^t (rp_s + \xi_s(1-p_s) - \tilde{c}_s - \epsilon\lambda_s - \delta\kappa_s)\,ds\right\}\tilde{c}_t^{\,1-R}\frac{dt}{1-R}\,\Big|\,\xi_0, p_0, w_0=1\right]$$

where we write $\lambda_t \equiv w_t^{-1}l_t$, $\kappa_t \equiv k_t/w_t$. As another abbreviation, let's put

If we have a formulation which embraces D-N and Rishel,

$$\left.\begin{aligned} dx_t &= \{ rx_t - c_t - (1+\varepsilon)l_t + (1-\delta)k_t \} \, dt \\ dy_t &= y_t(\sigma(\xi_t)dW_t + \alpha(\xi_t)\,dt) + (l_t - k_t)\,dt \end{aligned}\right\}$$

we find that

$$dp_t = p_t(1-p_t)\left[r - \alpha(\xi_t) + (1-p_t)\,\sigma(\xi_t)^2\right] dt - p_t(1-p_t)\,\sigma(\xi_t)\,dW_t$$

$$- \tilde{c}_t\,(1-p_t)\,dt - (1+\varepsilon(1-p_t))\frac{l_t}{w_t}\,dt + (1-\delta(1-p_t))\frac{k_t}{w_t}\,dt$$

$$dw_t = w_t\left\{(1-p_t)\,\sigma(\xi_t)\,dW_t + (1-p_t)\,\alpha(\xi_t)\,dt + (rp_t - \tilde{c}_t)\,dt - \varepsilon\frac{l_t}{w_t}\,dt - \delta\frac{k_t}{w_t}\,dt\right\}$$

$$m_1^{\gamma}(dp) = \int_{\tilde{c}}\int_{\lambda}\int_{\kappa} \mu_{\gamma}(dp, d\tilde{c}, d\lambda, d\kappa)$$

$$m_2^{\gamma}(dp) = \int_{\tilde{c}}\int_{\lambda}\int_{\kappa} \mu_{\gamma}(dp, d\tilde{c}, d\lambda, d\kappa)\,\tilde{c}$$

etc.

$$A_t \equiv -\rho t + (1-R)\int_0^t (r\,p_s + \xi_s(1-p_s) - \tilde{c}_s - \varepsilon\lambda_s - \delta\kappa_s)\,ds \equiv \int_0^t \alpha_s\,ds.$$

Thus the payoff is

$$E\left[\int_0^\infty e^{A_t}\,\tilde{c}_t^{\,1-R}\,\frac{dt}{1-R}\,\Big|\,\xi_0, p_0, w_0 = 1\right].$$

If we now take some nice test function f, and consider the martingale

$$f(p_t,\xi_t)\,e^{A_t} - \int_0^t e^{A_s}\left[f_1'(p_s,\xi_s)\,\dot{p}_s + Qf(p_s,\xi_s) + \alpha_s\,f(p_s,\xi_s)\right]ds$$

we shall have

$$-f(p_0,\xi_0) = E\Big[\int_0^\infty e^{A_t}\Big[f_1'(p_t,\xi_t)\big\{(r-\xi_t)p_t(1-p_t) - (1-p_t)\tilde{c}_t - (1+\varepsilon(1-p_t))\lambda_t$$
$$+ (1-\delta(1-p_t))\kappa_t\big\} + Qf(p_t,\xi_t)$$
$$+ (-\rho + (1-R)\{r p_t + \xi_t(1-p_t) - \tilde{c}_t - \varepsilon\lambda_t - \delta\kappa_t\}\,f(p_t,\xi_t)\Big]\,dt\Big]$$

$$\equiv \sum_z \iiiint \mu_z(dp, d\tilde{c}, d\lambda, d\kappa)\Big[f_1'(p,z)\big\{(r-z)p(1-p) - (1-p)\tilde{c} - (1+\varepsilon(1-p))\lambda$$
$$+ (1-\delta(1-p))\kappa\big\} + Qf(p,z)$$
$$+ (-\rho + (1-R)(rp + z(1-p) - \tilde{c} - \varepsilon\lambda - \delta\kappa))\,f(p,z)\Big],$$

which defines the occupation measure $\mu_z(dp, d\tilde{c}, d\lambda, d\kappa)$. The constraint can also be expressed as

$$-f(p_0,\xi_0) = \sum_z\Big[\int m_1^z(dp)\,\big\{(r-z)p(1-p)f_1'(p,z) + Qf(p,z) + \{-\rho + (1-R)(rp + z(1-p))\}f(p,z)\big\}$$
$$+ \int m_2^z(dp)\,\big\{-(1-p)f_1'(p,z) - (1-R)\,f(p,z)\big\}$$
$$+ \int m_3^z(dp)\,\big\{-(1+\varepsilon(1-p))f_1'(p,z) - \varepsilon(1-R)\,f(p,z)\big\}$$
$$+ \int m_4^z(dp)\,\big\{(1-\delta(1-p))\,f_1'(p,z) - \delta(1-R)f(p,z)\big\}\Big]$$

If we represent the value f^n $V(x,y,\xi) = (x+y)^{1-R} f(p,\xi)$, $p \equiv x/(x+y)$, then the HJB equations for this problem are

$$\frac{R}{1-R}\left[(1-R)f + (1-p)f'\right]^{1-1/R} - \rho f + rp\left[(1-R)f + (1-p)f'\right] + \alpha(1-p)\left[(1-R)f - pf'\right]$$
$$+\tfrac{1}{2}\sigma^2(1-p)^2\left[p^2 f'' + 2Rpf' - R(1-R)f\right] + \alpha f = 0.$$

$$\sigma(p,\xi) \equiv -p(1-p)\,\sigma(\xi), \quad \beta(p,\xi) \equiv \left\{rp - \alpha(\xi)p + Rp(1-p)\sigma(\xi)^2\right\}(1-p)$$

$$\tan\theta = \frac{x-y}{x+y} = 2p-1 \quad \therefore \; p = \tfrac{1}{2}(1+\tan\theta), \; 1-p = \tfrac{1}{2}(1-\tan\theta)$$

(vi) Let's broaden the formulation so that

$$dx_t = [\, r x_t - c_t - (1+\varepsilon)\, l_t + (1-\delta)\, k_t \,]\, dt$$

$$dy_t = y_t\, [\, \sigma(\xi_t)\, dW_t + \alpha(\xi_t)\, dt \,] + (l_t - k_t)\, dt$$

which embraces the Davis-Norman and Rishel problems as special cases. We then have for $p = x/(x+y)$

$$\boxed{\begin{aligned} dp_t &= -p_t(1-p_t)\sigma(\xi_t)\, dW_t + (1-p_t)\left[p_t r - \tilde{c}_t - p_t\alpha(\xi_t) + p_t(1-p_t)\sigma(\xi_t)^2 \right] dt \\ &\qquad - (1+\varepsilon(1-p_t))\, \lambda_t\, dt + (1-\delta(1-p_t))\, \kappa_t\, dt \\ dw_t &= w_t \left[(1-p_t)\sigma(\xi_t)\, dW_t + \big((1-p_t)\alpha(\xi_t) + r p_t - \tilde{c}_t - \varepsilon\lambda_t - \delta\kappa_t \big)\, dt \right] \end{aligned}}$$

Our payoff is

$$E \int_0^\infty \exp\Big[-\rho t + (1-R)\Big(\int_0^t (1-p_s)\sigma(\xi_s)\, dW_s + \int_0^t \big\{ (1-p_s)\alpha(\xi_s) + r p_s - \tilde{c}_s - \varepsilon\lambda_s - \delta\kappa_s - \tfrac{1}{2}(1-p_s)^2\sigma(\xi_s)^2 \big\}\, ds \Big) \Big]\, \tilde{c}_t^{1-R} \frac{dt}{1-R}$$

$$= \hat{E} \int_0^\infty \exp\Big[A_t - \int_0^t (1-R)\tilde{c}_s\, ds - \varepsilon \int_0^t (1-R)\lambda_s\, ds - \int_0^t \delta(1-R)\kappa_s\, ds \Big]\; \tilde{c}_t^{1-R}\, \frac{dt}{1-R}$$

where $\hat{P}$ is the probability under which $dW_s = d\hat{W}_s + (1-R)(1-p_s)\sigma(\xi_s)\, ds$, and

$$A_t = -\rho t + \int_0^t \Big[(1-p_s)\alpha(\xi_s)(1-R) + (1-R) r p_s - \tfrac{R}{2}(1-R)(1-p_s)^2\sigma(\xi_s)^2 \Big]\, ds$$

$$\equiv \int_0^t b(p_s, \xi_s)\, ds,$$

for brevity. So now the ~~story~~ is

$$\boxed{\begin{aligned} dp_t &= \sigma(p_t, \xi_t)\, d\hat{W}_t + \beta(p_t, \xi_t)\, dt - (1-p_t)\tilde{c}_t\, dt - (1+\varepsilon(1-p_t))\lambda_t\, dt \\ &\qquad + (1-\delta(1-p_t))\, \kappa_t\, dt \\ &\max\ \hat{E} \int_0^\infty \exp\Big[+ \int_0^t b(p_s, \xi_s)\, ds - (1-R)\int_0^t (\tilde{c}_s + \varepsilon\lambda_s + \delta\kappa_s)\, ds \Big]\; \tilde{c}_t^{1-R}\, \frac{dt}{1-R} \end{aligned}}$$

In fact, a better variable for this problem would appear to be $\theta_t \equiv \tan^{-1}(2p-1)$; in this case we have for the dynamics and the optimisation problem

$$\text{drift} = \frac{\cos 2\theta}{4\cos\theta}\left[\cancel{-4\sigma^2 \sin\theta \cos^2\theta} + 2r\cos\theta + R\sigma^2\cos\theta - 2\alpha\cos\theta - R\sigma^2\sin\theta \right]$$

$$= \frac{1}{4}\cos 2\theta \left[-\sigma^2 \cos 2\theta \tan\theta + 2r + R\sigma^2 - 2\alpha - R\sigma^2 \tan\theta \right]$$

cfe of $\tilde{c}$ is $-(1-\tan\theta)\cos^2\theta$

$$d\theta = 2\cos^2\theta\Big[\sigma(p,\xi)\,d\hat{W} + \beta(p,\xi)\,dt - (1-p)\,\tilde{c}\,dt - (1+\varepsilon(1-p))\lambda\,dt + (1-\delta(1-p))\kappa\,dt\Big]$$
$$- \cancel{4}\sin\theta\cos^3\theta\;\sigma(p,\xi)^2\,dt$$

and

$$\max E\int_0^\infty \exp\Big[\int_0^t b(p_s,\xi_s)\,ds - (1-R)\int_0^t(\tilde{c}_s+\varepsilon\lambda_s+\delta\kappa_s)\,ds\Big]\;\tilde{c}_t^{\,1-R}\,\frac{dt}{1-R}.$$

In general terms, we'll have some underlying diffusion η satisfying

$$d\eta = \sigma(\eta,\xi)\,dW + \gamma(\eta,\xi)\,dt - \psi_1(\eta,\xi)\,\tilde{c}\,dt - \psi_2(\eta,\xi)\,\lambda\,dt - \psi_3(\eta,\xi)\,\kappa\,dt$$

and we aim to maximise

$$E\int_0^\infty \exp\Big[\int_0^t b(\eta_s,\xi_s)\,ds - \int_0^t\{\varphi_1(\eta_s,\xi_s)\tilde{c}_s + \varphi_2(\eta_s,\xi_s)\lambda_s + \varphi_3(\eta_s,\xi_s)\kappa_s\}\,ds\Big]\;\tilde{c}_t^{\,1-R}\,\frac{dt}{1-R}$$
$$\equiv E\int_0^\infty e^{A_t}\,\tilde{c}_t^{\,1-R}\,\frac{dt}{1-R}$$

The constraints will be that for a large enough family of test functions f

$$-f(\eta_0,\xi_0) = \iint \mu(d\eta,d\xi)\Big\{\tfrac12\sigma(\eta,\xi)^2 f''(\eta,\xi) + \gamma(\eta,\xi)f'(\eta,\xi) + b(\eta,\xi)f(\eta,\xi) + Qf(\eta,\xi)\Big\}$$
$$- \iint \mu_c(d\eta,d\xi)\Big\{\varphi_1(\eta,\xi)f(\eta,\xi) + \psi_1(\eta,\xi)f'(\eta,\xi)\Big\}$$
$$- \iint \mu_\lambda(d\eta,d\xi)\Big\{\varphi_2(\eta,\xi)f(\eta,\xi) + \psi_2(\eta,\xi)f'(\eta,\xi)\Big\}$$
$$- \iint \mu_\kappa(d\eta,d\xi)\Big\{\varphi_3(\eta,\xi)f(\eta,\xi) + \psi_3(\eta,\xi)f'(\eta,\xi)\Big\}$$

(vii) <u>Markov-chain approximation?</u> If we were to approximate the diffusion θ by a Markov chain on an equally-spaced grid of spacing δ, we'd have at gridpoint θ, $p = \tan\theta$, that the jump rates would be approximately

<u>up</u>: $2\cos^2\theta\big\{(r + R(1-p)\sigma(\xi)^2)p + (1-\delta(1-p))\kappa\big\}\,\delta^{-1} + \frac{1}{2\delta^2}4\cos^4\theta\,.\,p^2(1-p)^2\sigma(\xi)^2$

<u>down</u>: $2\cos^2\theta\Big[\alpha(\xi)p(1-p) + \tilde{c}(1-p) + (1+\varepsilon(1-p))\lambda + \sin\theta\cos\theta\; p^2(1-p)^2\sigma(\xi)^2\Big] + \frac{1}{2\delta^2}4\cos^4\theta\; p^2(1-p)^2\sigma(\xi)^2$

If $\delta \ll 1$, this will give the correct infinitesimal mean and variance, approximately.

For $t < T$,

$$(I+TV)^{-1}X_T - (I+tV)^{-1}X_t = \int_t^T (I+sV)^{-1}\,d\hat{W}_s + \{(I+tV)^{-1} - (I+TV)^{-1}\}\, V^{-1}a$$

$$\therefore \quad X_T + V^{-1}a = (I+TV)\left[(I+tV)^{-1}(X_t + V^{-1}a) + \int_t^T (I+sV)^{-1}\,d\hat{W}_s \right]$$

$$\sim N\left((I+TV)(I+tV)^{-1}(X_t + V^{-1}a),\ (I+TV)(I+tV)^{-1} \cdot (T-t) \right)$$

Optimal investment in an uncertain market (5/5/97)

(i) Suppose you can invest in shares,

$$dS^i_t / S^i_t = \sigma_{ij}(dW^j_t + \alpha_j dt) \qquad (i = 1, \dots, d)$$

where $\alpha \sim N(a, V)$ are not known. Let $X^j_t \equiv (W^j_t + \alpha_j t)$. The agent is attempting to maximise $E\, U(w_T)$, where $U(x) \equiv x^{1-R}/(1-R)$. The posterior law of μ given $\mathcal{F}_t$ is $N((I+tV)^{-1}(a+VX_t),\ (I+tV)^{-1}V)$, and we have that

$$dX_t = d\hat{W}_t + (I+tV)^{-1}(a+VX_t)\,dt$$

so that

$$d((I+tV)^{-1}X_t) = (I+tV)^{-1}d\hat{W}_t + (I+tV)^{-2}a\,dt.$$

Here, $\hat{W}$ is a $\mathcal{F}_t$-Brownian motion. If Z_t is change-of-measure martingale which takes us to risk-neutral probability we shall have

$$\frac{dZ_t}{Z_t} = \{b - (V^{-1}+tI)^{-1}(V^{-1}a + X_t)\}\, d\hat{W}_t \qquad (b \equiv r\sigma^{-1}1)$$

so that

$$d(\log Z_t) = \frac{dZ_t}{Z_t} - \tfrac{1}{2}|b - (V^{-1}+tI)^{-1}(V^{-1}a + X_t)|^2 dt \qquad V_t \equiv (V^{-1}+tI)^{-1}$$

$$= \{b - V_t(V^{-1}a+X_t)\}\{dX_t - V_t(V^{-1}a+X_t)dt\}$$
$$- \tfrac{1}{2}|b - V_t(V^{-1}a+X_t)|^2 dt$$

$$= \{b - V_t(V^{-1}a+X_t)\}dX_t - (b - V_t(V^{-1}a+X_t))\cdot(b + V_t(V^{-1}a+X_t))$$

$$= b\,dX_t - d\{\tfrac{1}{2}(V^{-1}a+X_t)V_t(V^{-1}a+X_t)\} + \tfrac{1}{2}\mathrm{tr}\,V_t\,dt - \tfrac{1}{2}|b|^2 dt$$

from which

$$Z_t = \exp\Big[\, b\cdot X_t - \tfrac{1}{2}|b|^2 t + \tfrac{1}{2}\mathrm{tr}\int_0^t (V^{-1}+sI)^{-1}ds - \tfrac{1}{2}(V^{-1}a+X_t)V_t(V^{-1}a+X_t)$$
$$+ \tfrac{1}{2}(V^{-1}a)\cdot V(V^{-1}a)\ \Big]$$

$$= \exp\Big[\, b\cdot X_t - \tfrac{1}{2}|b|^2 t + \tfrac{1}{2}\int_0^t \mathrm{tr}((V^{-1}+sI)^{-1})ds + \tfrac{1}{2}t\, a\cdot(I+tV)^{-1}a - a\cdot(I+tV)^{-1}X_t$$
$$- \tfrac{1}{2}X_t\cdot V_t X_t\ \Big]$$

State price deflator $\zeta_t = e^{-rt}Z_t$, so optimal wealth at time T will be $\lambda\,\zeta_T^{-1/R}$.

$V_{tT} = (T-t)\, V_t\, V_T^{-1}$

Introduce $Y_t \equiv X_t + V^{-1}a$, so that

$$Z_t = \exp\Big[-\tfrac{1}{2}(Y_t - V_t^{-1}b)V_t(Y_t - V_t^{-1}b) + \tfrac{1}{2}(b-a)V^{-1}(b-a) + \tfrac{1}{2}\sigma \int_0^t V_s\,ds \Big]$$

and

$$\mathcal{L}(Y_T|\mathcal{F}_t) = N(V_tV_T^{-1}Y_t, (T-t)V_tV_T^{-1}).$$

To compute the optimal wealth $W_t^* = E[\lambda \zeta_T^{1-1/R}|\mathcal{F}_t]\,\zeta_t^{-1}$, we calculate

$$E\left[\zeta_T^{1-1/R}|\mathcal{F}_t\right] = (\text{fn of } t).\exp\Big[-\tfrac{1}{2}(1-\tfrac{1}{R}).(\eta - V_T^{-1}b)V_T\big(I + \tfrac{R-1}{R}(T-t)V_t\big)^{-1}(\eta - V_T^{-1}b)\Big]$$

where $\eta \equiv V_tV_T^{-1}Y_t$.

Hence we have

$$W_t^* = (\text{fn of } t).\exp\Big[\tfrac{1}{2}(Y_t - V_t^{-1}b)V_t(Y_t - V_t^{-1}b) - \tfrac{1}{2}(1-\tfrac{1}{R})(Y_t - V_t^{-1}b)V_t.$$

$$\{I + \tfrac{R-1}{R}(T-t)V_t\}^{-1}V_tV_T^{-1}(Y_t - V_t^{-1}b)\Big]$$

Thus gives

$$\frac{dW^*}{W^*} = \Big(\big[V_t - (1-\tfrac{1}{R})V_t\{1 + \tfrac{R-1}{R}(T-t)V_t\}^{-1}V_tV_T^{-1}\big](Y_t - V_t^{-1}b), dY_t\Big) + \text{others}$$

$$= V_t\{R + (R-1)(T-t)V_t\}^{-1}(Y_t - V_t^{-1}b).dW_t + \text{rest}$$

$$= \Big((\sigma^T)^{-1}V_t\{R + (R-1)(T-t)V_t\}^{-1}(Y_t - V_t^{-1}b), \frac{dS}{S}\Big) + \text{rest}.$$

$$= \Big((\sigma^T)^{-1}\{R + (R-1)(T-t)V_t\}^{-1}(V_tY_t - b), \frac{dS}{S}\Big) + \text{rest}.$$

Now $V_tY_t = E(\alpha|\mathcal{F}_t)$, so for the case $R=1$ we recover the result of Browne & Whitt, and Lakner, that we do the same as for the Merton problem, using the certainty-equivalent drift.

(ii) To find out what may be the impact on the payoff, we must calculate

$$E\left[\zeta_T^{1-1/R}\right] = e^{-r(1-1/R)T}E\left[Z_T^{1-1/R}\right].$$

$$= \frac{W_0^{1-R}}{1-R} \exp\Big[(R-1)\{-rT + \tfrac{1}{2}\operatorname{tr}\int_0^T V_s\,ds\} - \frac{(R-1)}{2}\,\frac{I}{R}\,(b-a)\cdot(I + (1-\tfrac{1}{R})TV)^{-1}(b-a) \Big].$$

$$\det(I + \beta TV)^{-R/2} \qquad (\beta = 1-\tfrac{1}{R})$$

Eg in one dimension, $\mu = \sigma a = 0.2$, $r = 0.05$, $\sigma = 0.25$, $R = 4$, $V = 16\times 10^{-2}$,

$\therefore (\mu - r)/\sigma = 0.6$

When $t = 1$, $\dfrac{\text{optimal payoff}}{\text{best Merton payoff}} = 0.99599$

$= 0.9863$, $t = 2$

$= 0.9735$, $t = 3$

$= 0.9588$, $t = 4$

$= 0.9433$, $t = 5$

$= 0.8662$, $t = 10$

$= 0.7446$, $t = 20$

Now $Y_T \sim N(V_T^{-1} a, T(I+TV))$ so

$$E\, \zeta_T^{1-1/R} = \exp\Big[(1-\tfrac{1}{R})\big\{-rT + \tfrac{1}{2}(b-a)V^{-1}(b-a) + \tfrac{1}{2}\mathrm{tr}\textstyle\int_0^T V_s\,ds\big\}\Big] E \exp\big(-\tfrac{1}{2}(1-\tfrac{1}{R})\,|\theta+k|^2\big)$$

where $\theta \sim N(0, TV)$, $k \equiv V_T^{-1/2}(a-b)$.

The expectation comes out to be

$$\det\big(I + (1-\tfrac{1}{R})TV\big)^{-1/2} \exp\Big[\tfrac{1}{2}\gamma^2 - \tfrac{1}{2}k^2(1-\tfrac{1}{R})\Big],$$

and $\gamma \equiv (1-\tfrac{1}{R})\,\Xi k$, $\Xi^{-1} \equiv 1-\tfrac{1}{R} + (TV)^{-1}$; thus the optimal expected utility is

$$\frac{W_0^{1-R}}{1-R}\left\{E\big(\zeta_T^{1-1/R}\big)\right\}^R$$

$$= \frac{W_0^{1-R}}{1-R} \exp\Big[(R-1)\big\{-rT + \tfrac{1}{2}(b-a)V^{-1}(b-a) + \tfrac{1}{2}\mathrm{tr}\textstyle\int_0^T V_s\,ds\big\}\Big] \exp\big(\tfrac{1}{2}R\gamma^2 - \tfrac{1}{2}(R-1)k^2\big).$$
$$\det\big(I + (1-\tfrac{1}{R})TV\big)^{-R/2}$$

(iii) How would this compare with a (Merton) constant proportions policy? Let's assume $R \geq 1$, otherwise the problem is ill-posed for big enough T.

The wealth would satisfy

$$dW_t = W_t\{r\,dt + \theta(\sigma\, dW_t + \sigma\alpha\, dt - r1\, dt)\}$$

so that $W_t = w_0 \exp\big[rt + \theta\cdot\sigma W_t + \theta\cdot(\sigma\alpha - r1)t - \tfrac{1}{2}|\sigma^T\theta|^2 t\big]$, and thus

$$E\left[\frac{W_T^{1-R}}{1-R}\right] = \frac{W_0^{1-R}}{1-R} \exp\Big[(1-R)T\big\{r + \theta^T(\sigma a - r1) - \tfrac{1}{2}\theta^T(R\sigma\sigma^T + (R-1)T\sigma V_0\sigma^T)\theta\big\}\Big]$$

So the best choice of θ would be

$$\boxed{\theta^* = \big(R\sigma\sigma^T + T(R-1)\sigma V_0\sigma^T\big)^{-1}(\sigma a - r1)}$$

with expected utility

$$\frac{W_0^{1-R}}{1-R} \exp\Big[(1-R)T\big\{r + \tfrac{1}{2}(\sigma a - r1)^T(R\sigma\sigma^T + T(R-1)\sigma V_0\sigma^T)^{-1}(\sigma a - r1)\big\}\Big]$$

Discretising Davis-Norman-Rishel (23/5/97)

The controlled process satisfies an SDE of the form

$$dz = v(z,\xi)\, d\hat{W} + g(z,\xi)\, dt - \psi_1(z,\xi)\, \tilde{c}\, dt - \psi_2(z,\xi)\, \lambda\, dt - \psi_3(z,\xi)\, \kappa\, dt,$$

where $\tilde{c}, \lambda, \kappa \geq 0$, and $\psi_2 \geq 0 \geq \psi_3$. The chain ξ jumps independently according to some Q-matrix Q.

The objective is to

$$\min E \int_0^\infty \exp\Big\{ \underbrace{\int_0^t \{ b(z_s,\xi_s) - \varphi_1(z_s,\xi_s)\tilde{c}_s - \varphi_2(z_s,\xi_s)\lambda_s - \varphi_3(z_s,\xi_s)\kappa_s \} ds}_{A_t} \Big\}\, \tilde{c}_t^{\,-(R-1)}\, dt$$

where we shall be assuming

$$\boxed{R > 1, \quad \text{and} \quad v > 0 \quad \text{everywhere}}$$

In this case, $\varphi_1, \varphi_2, \varphi_3 < 0$. We've going to take the interval $[a, b]$ in which the diffusion lives, and discretise it into $N+2$ points,

$$z_j = a + j\Delta, \quad j = 0, 1, \dots, N+1, \qquad (\Delta \equiv (b-a)/(N+1))$$

We shall now make the jump rates

$$z_j \mapsto z_{j+1} \quad \text{rate} \quad \frac{g(z_j,\xi)^+ - \psi_3(z_j,\xi)\kappa + \psi_1(z_j,\xi)^+\tilde{c}}{\Delta z} + \frac{v(z_j,\xi)^2}{2(\Delta z)^2} \qquad (j = 2,\dots,N)$$

$$z_j \mapsto z_{j-1} \quad \text{rate} \quad \frac{g(z_j,\xi)^- + \psi_1(z_j,\xi)^-\tilde{c} + \psi_2(z_j,\xi)\lambda}{\Delta z} + \frac{v(z_j,\xi)^2}{2(\Delta z)^2} \qquad (j = 2,\dots,N)$$

which matches drift & volatility in the middle of the interval.

The ends of the interval need somewhat different treatment, since it is forbidden to reach a or b in this case; if we did, the volatility would take us immediately out of the interval unless we immediately closed out our position to zero cash and zero shares - but then we'd have no more consumption, with infinite penalty! So we have to have jump rates $z_1 \mapsto z_0$ and $z_N \mapsto z_{N+1}$ both 0, and, I propose,

$$z_1 \mapsto z_2 \quad \text{at rate} \quad v(z_1,\xi)^2/(\Delta z)^2$$

$$z_N \mapsto z_{N-1} \quad \text{at rate} \quad v(z_N,\xi)^2/(\Delta z)^2$$

$$-\frac{v(z_N,\xi)^2}{\Delta_\xi} \quad + g(z_N,\xi) - \psi_2(z_N,\xi)\lambda - \psi_1(z_N,\xi)\tilde{c}$$

which at least matches the volatility at these points.

The variables of the problem are $m(i,\xi)$, $m_c(i,\xi)$, $m_\lambda(i,\xi)$, $m_\kappa(i,\xi)$, which are discounted mean times in the different states;

$$m_c(i,\xi) = E^{\nu_0}\left[\int_0^\infty e^{A_t} I_{\{Z_t = i,\ \xi_t = \xi\}} c_t \, dt\right],$$

for example.

We shall have that $m_\lambda(1,\xi) = m_\kappa(N,\xi) = 0$, and

$$\frac{v(z_1,\xi)^2}{\Delta z} \cdot m(1,\xi) = g(z_1,\xi)\, m(1,\xi) - \psi_3(z_1,\xi)\, m_\kappa(1,\xi) - \psi_1(z_1,\xi)\, m_c(1,\xi)$$

$$\frac{v(z_N,\xi)^2}{\Delta z} \cdot m(N,\xi) = g(z_N,\xi)\, m(N,\xi) - \psi_2(z_N,\xi)\, m_\lambda(N,\xi) - \psi_1(z_N,\xi)\, m_c(N,\xi)$$

which determines $m_\kappa(1,\xi)$, $m_\lambda(N,\xi)$ from $m(1,\xi)$, $m_c(1,\xi)$, $m(N,\xi)$, $m_c(N,\xi)$.

Thus if there are N_ξ states of the chain ξ, the problem has

$$4\, N \cdot N_\xi - 4 N_\xi = 4(N-1) N_\xi \quad \text{free variables.}$$

<u>What goes on at the ends of the interval?</u>

We may continue to consume, but the choice of κ, λ must be such as to keep the net jump rates correct. At the left-hand end, z_1, we have chosen up-rate $\beta = \left(\frac{v(z_1,\xi)}{\Delta z}\right)^2$ to give correct vol with a zero down rate, so

$$\text{net drift} = g(z_1,\xi) - \psi_3(z_1,\xi)\kappa - \psi_1(z_1,\xi)\tilde{c} = \beta \cdot \Delta z$$

$$\therefore \quad \kappa = -\frac{1}{\psi_3}\left\{\beta \cdot \Delta z + \psi_1 \tilde{c} - g\right\}$$

This is notional, but its effect is felt through the discounting. The similar analysis at z_N gives us

$$\lambda = \frac{1}{\psi_2}\left\{\frac{v(z_N,\xi)^2}{\Delta z} - \psi_1 \tilde{c} + g\right\}$$

$$\max_{\theta} \left(1+\frac{\theta}{x}\right)^{1-R} v\left(\frac{y-\theta}{x+\theta}\right) = \max_{t} (1+t)^{1-R} v\left(\frac{A-t}{1+t}\right) = \max_{t} (1+t)^{1-R} v\left(\frac{1+S}{1+t} - 1\right)$$

$$= (1+S)^{1-R} \max_{w} w^{1-R} v\left(\frac{1}{w} - 1\right)$$

$$= (1+S)^{1-R} (\pi^*)^{1-R} v\left(\frac{1-\pi^*}{\pi^*}\right).$$

A primitive model of liquidity effects (11/6/97)

(i) If x_t denotes the amount in the bank at time t, y_t denotes the value of shares held at time t, we assume the dynamics

$$\left.\begin{aligned} dx_t &= (r x_t - c_t)\,dt + \theta_t\, dN_t \\ dy_t &= y_t(\sigma dW_t + \alpha dt) - \theta_t\, dN_t \end{aligned}\right\}$$

where N is a Poisson process rate λ. The idea is that only at the times of this Poisson process are we allowed to buy or sell shares. If $U(x) = x^{1-R}/(1-R)$, we assume that the objective is to find

$$\max\ E\left[\int_0^\infty \exp(-\rho t)\, U(c_t)\, dt \;\middle|\; x_0 = x,\ y_0 = y\right] \equiv V(x,y)$$

and let's assume that we insist $x_t \geq 0$, $y_t \geq 0$ for all t (else we could get big trouble!)

The HJB equation for V will be

$$\sup_{c,\theta}\left\{ U(c) - \rho V + (rx - c)V_x + \tfrac{1}{2}\sigma^2 y^2 V_{yy} + \alpha y V_y + \lambda\{V(x+\theta, y-\theta) - V(x,y)\}\right\} = 0.$$

Exploiting scaling, we know that

$$V(x,y) = x^{1-R}\, v(y/x) \equiv y^{1-R}\, \psi(x/y) \qquad \begin{pmatrix} s \equiv y/x \\ \equiv 1/z \end{pmatrix}$$

So we have the two equivalent forms of HJB:

$$\boxed{\sup_{c,\theta}\left[\frac{U(c)}{x^{1-R}} - \rho v + \left(r - \frac{c}{x}\right)\{(1-R)v(s) - s v'(s)\} + \tfrac{1}{2}\sigma^2 s^2 v''(s) + \alpha s v'(s) + \lambda\left(1+\frac{\theta}{x}\right)^{1-R} v\!\left(\frac{y-\theta}{x+\theta}\right) - \lambda v(s)\right] = 0}$$

$$\sup_{c,\theta}\left[\frac{U(c)}{y^{1-R}} - \rho\psi + \left(rz - \frac{c}{y}\right)\psi'(z) + \tfrac{1}{2}\sigma^2\{z^2\psi'' + 2Rz\psi' - R(1-R)\psi\}\right.$$
$$\left. + \alpha\{(1-R)\psi - z\psi'\} + \lambda\left(1 - \frac{\theta}{y}\right)^{1-R}\psi\!\left(\frac{x+\theta}{y-\theta}\right) - \lambda\psi(z)\right] = 0$$

Hence

$$\left(\frac{c}{x}\right)^{-R} = (1-R)v(s) - s v'(s), \qquad \left(\frac{c}{y}\right)^{-R} = \psi'(z)$$

at optimality, and optimising over θ, we find that the best thing to do is always to jump to make the proportion of wealth in the bank equal to some π^*.

$$\tilde{u}(\lambda) = \frac{R}{1-R}\,\lambda^{1-1/R}$$

(ii) The dynamics of $z_t \equiv x_t/y_t$ is given by

$$dz_t = -\sigma z_t\, dW_t + \left\{ \sigma^2 z_t + (r-\alpha) z_t - z_t \frac{c_t}{w_t} \right\} dt + \theta_t (1+z_t)\, dN_t$$

The dynamics of $p_t \equiv x_t/(x_t+y_t)$ is given by

$$dp_t = -p(1-p)\sigma\, dW + \left\{ rp - \frac{c}{w} - rp^2 + p\frac{c}{w} - \alpha p(1-p) + \sigma^2 p(1-p)^2 \right\} dt + \theta \frac{dN_t}{W_t}.$$

(iii) Let's consider what happens if we start with $y=0$, working with the first form of HJB:

$$\tilde{U}\left((1-R)v - sv'\right) - \rho v + r\left[(1-R)v - sv'\right] + \tfrac{1}{2}\sigma^2 v'' + s\alpha v' + \lambda (1+s)^{1-R} \underbrace{(\pi^*)^{1-R} v\left(\tfrac{1-\pi^*}{\pi^*}\right)}_{\equiv \kappa^*} - \lambda v = 0$$

If we take the rule that we jump to proportion π in the bank whenever we get the chance, and assuming we start at 0, we get

$$v(0) = \sup E\left[\int_0^{\tau_1} e^{-\rho t} c_t^{1-R} \frac{dt}{1-R} + e^{-\rho\tau_1} (x_{\tau_1})^{1-R} \underbrace{(\pi)^{1-R}\, v\left(\tfrac{1-\pi}{\pi}\right)}_{\equiv \kappa} \right]$$

Now the best choice of c is to have $\left(\frac{c_t}{x_t}\right)^{-R} = (1-R)v(0)$ $\therefore$ $\boxed{c = \gamma x, \quad \gamma \equiv (1-R)^{-1/R} v(0)^{-1/R}}$

and we conclude

$$\boxed{\begin{aligned} v(0) &= \int_0^\infty e^{-(\rho+\lambda)t + (1-R)(r-\gamma)t} \left\{ \frac{\gamma^{1-R}}{1-R} + \lambda\kappa \right\} dt \\ &= \left\{ \frac{\gamma^{1-R}}{1-R} + \lambda\kappa \right\} \Big/ \left\{ \rho + \lambda - (1-R)(r-\gamma) \right\} \end{aligned}}$$

Hence

$$\boxed{\begin{aligned} &\gamma^{-R}\left[\lambda + \rho + (R-1)r - R\gamma\right] \\ &= \lambda\kappa(1-R) \end{aligned}}$$

This will hold for $R>1$, and $R<1$

(iv) Now what can we get from the behaviour as $x \downarrow 0$? Two cases need to be considered.

<u>$0<R<1$</u> In this case, if we start with $x=0$, we consume nothing until the first jump. Thus if our policy is always to rebalance to proportion π in the bank,

$$V(0,y) = E\left[e^{-\rho\tau_1} y_{\tau_1}^{1-R} \psi\left(\frac{\pi}{1-\pi}\right) \right]$$

$$= y^{1-R} \psi\left(\frac{\pi}{1-\pi}\right) \lambda \left\{ \lambda + \rho - (1-R)(\alpha - \tfrac{1}{2}R\sigma^2) \right\}^{-1}$$

Thus

$$\boxed{\psi(0) \equiv \lim_{s\to\infty} s^{-(1-R)} v(s) = \psi\left(\frac{\pi}{1-\pi}\right) \lambda \left\{ \lambda + \rho - (1-R)(\alpha - \tfrac{1}{2}R\sigma^2) \right\}^{-1}}$$

$\underline{R>1}$. In this case, it seems that if $x_0=0$, we have $V(x_0,y)=-\infty$, because if we consume anything at all before τ_1, there's positive probability that the shares will crash before you can sell them, and you'll end up with negative wealth. Thus

$$s^{R-1}v(s) \to -\infty \qquad \text{as } s\uparrow\infty$$

$$v(s) \uparrow 0 \qquad \text{as } s\uparrow\infty.$$

Can we obtain more precise information? Perhaps, but first let's try something else.

(v) Without restricting R, let's consider the situation where we begin with $x_0=1$, $y_0=\varepsilon$, and consider the perturbation of V which results.

We have

$$y_t = \varepsilon \exp\{\sigma W_t + (\alpha-\tfrac{1}{2}\sigma^2)t\}$$

$$\frac{c_t}{x_t} = \{(1-R)v(s_t) - s_t v'(s_t)\}^{-1/R} \qquad (s_t \equiv y_t/x_t)$$

$$= \{(1-R)v(0) - s_t R v'(0)\}^{-1/R} \qquad \text{to first order}$$

$$= ((1-R)v(0))^{-1/R}\left\{1 + s_t \frac{v'(0)}{(1-R)v(0)}\right\} \qquad \text{to first order}$$

$$\equiv \gamma + \beta s_t \qquad [\beta \equiv \gamma^{-1}\ v'(0)/(1-R)v(0).]$$

Hence

$$\frac{dx}{dt} = rx_t - \gamma x_t - \beta y_t$$

$$\Rightarrow \boxed{\begin{aligned} x_t &= e^{(r-\gamma)t}\left\{1-\beta\int_0^t e^{(\gamma-r)s}y_s\,ds\right\} \\ c_t &= \gamma x_t + \beta y_t \\ &= \gamma e^{(r-\gamma)t} + \beta y_t - \gamma\beta e^{(r-\gamma)t}\int_0^t e^{(\gamma-r)s}y_s\,ds \end{aligned}}$$

Hence

$$v(\varepsilon) = E\left[\int_0^{\tau_1} e^{-\rho t}\frac{\gamma^{1-R}}{1-R}e^{(1-R)(r-\gamma)t}\left\{1-\beta\int_0^t e^{(\gamma-r)s}y_s\,ds + \beta\gamma^{-1}e^{(\gamma-r)t}y_t\right\}^{1-R}dt\right.$$

$$\left. + e^{-\rho\tau_1}\kappa\,(x_{\tau_1}+y_{\tau_1})^{1-R}\right]$$

$$\doteq E\int_0^\infty e^{-(\lambda+\rho+(1-R)(\gamma-r))t}\frac{\gamma^{1-R}}{1-R}\left\{1+(1-R)\beta\left(\gamma^{-1}e^{(\gamma-r)t}y_t - \int_0^t e^{(\gamma-r)s}y_s\,ds\right)\right\}dt$$

$$+ E\int_0^\infty \lambda\kappa\, e^{-(\lambda+\rho+(1-R)(\gamma-r))t}\left\{1+(1-R)\left\{e^{(\gamma-r)t}y_t - \beta\int_0^t e^{(\gamma-r)s}y_s\,ds\right\}\right\}dt$$

Maple confirms →

$[\varphi \equiv \lambda + \rho + (1-R)(\gamma - r)]$

Hence to first order

$$v(\varepsilon) - v(0) \doteq E \int_0^\infty e^{-\varphi t} \Big\{ \gamma^{1-R} \beta \Big(\gamma^{-1} e^{(\gamma - r)t} y_t - \int_0^t e^{(\gamma - r)s} y_s \, ds \Big)$$

$$+ \lambda \kappa (1-R) \Big(e^{(\gamma - r)t} y_t - \beta \int_0^t e^{(\gamma - r)s} y_s \, ds \Big) \Big\} dt$$

So dividing by ε and letting $\varepsilon \downarrow 0$ we shall obtain

$$v'(0) = \int_0^\infty e^{-\varphi t} \Big[\gamma^{1-R} \beta \Big\{ \gamma^{-1} e^{(\gamma + \alpha - r)t} - \int_0^t e^{(\gamma + \alpha - r)s} ds \Big\}$$

$$+ \lambda \kappa (1-R) \Big\{ e^{(\gamma + \alpha - r)t} - \beta \int_0^t e^{(\gamma + \alpha - r)s} ds \Big\} \Big]$$

$[a \equiv \alpha + \gamma - r]$

$$= \gamma^{1-R} \beta \frac{\varphi - \gamma}{\gamma \varphi (\varphi - a)} + \lambda \kappa (1-R) \frac{\varphi - \beta}{\varphi (\varphi - a)}.$$

But let's recall that

$$\boxed{\gamma = \big((1-R) v(0) \big)^{-1/R}, \qquad \beta = v'(0) \, \gamma^{R-1}}$$

So

$$\varphi(\varphi - a) v'(0) = \gamma^{-R} \beta (\varphi - \gamma) + \lambda \kappa (1-R)(\varphi - \beta);$$

If we gather together all terms in $v'(0)$ we obtain

$$\boxed{v'(0) \big[\varphi(\varphi - a) - \gamma^{-1}(\varphi - \gamma) + \lambda \kappa (1-R) \gamma^{R-1} \big] = \lambda \kappa (1-R) \varphi}$$

which we combine with the other boundary condition

$$\boxed{\varphi v(0) = \frac{\gamma^{1-R}}{1-R} + \lambda \kappa}$$

and this ought to determine $v(0)$, $v'(0)$ for a given κ.
From the equation determining γ on p 40, $\gamma^{-R}(\varphi - \gamma) = \lambda \kappa (1-R)$, so our equation for $v'(0)$ collapses to

$$\boxed{v'(0)(\varphi - a) = \lambda \kappa (1-R) \equiv \gamma^{-R}(\varphi - \gamma) = (1-R) v(0) (\varphi - \gamma)}$$

From Maple,

$$\boxed{v''(0) = \frac{(1-R) \lambda \kappa R \big[(\varphi - a)^2 - (1-R) \gamma^{1+R} \lambda \kappa \big]}{(\varphi - a)^2 (2\sigma^2 + 2a - \varphi)}}$$

If we use $V(x,y) = (x+y)^{1-R}\,\varphi\left(y/x+y \right)$, $p \equiv y/(x+y)$, we get

$$\frac{R}{1-R}\left\{(1-R)\,\varphi - p\varphi\right\}^{1-1/R} - (\rho+\lambda)\varphi + (1-R)\left[r(1-p) + \alpha p - \tfrac{1}{2}\sigma^2 R\, p^2\right]\varphi$$

$$+\, p(1-p)\left[\alpha - r - pR\sigma^2\right]\varphi' + \tfrac{1}{2}\sigma^2 p^2(1-p)^2\varphi'' + \lambda\bar{\varphi} = 0$$

(vi) What is probably better is to write

$$V(x,y) = (x+y)^{1-R}\, g\left(\frac{x}{x+y}\right) \equiv w^{1-R}\, g(t)$$

We end up with the HJB equation

$$\boxed{\begin{aligned}&\frac{R}{1-R}\Big[(1-R)g + (1-t)g'\Big]^{1-1/R} - (\rho+\lambda) g + (1-R)\{rt + \alpha(1-t) - \tfrac12\sigma^2 R(1-t)^2\} g \\ &+ t(1-t)\big(r-\alpha+\sigma^2 R(1-t)\big) g' + \tfrac12\sigma^2 t^2(1-t)^2 g'' + \lambda\bar g = 0\end{aligned}}$$

where $\bar g = \sup\{g(t) : 0\le t\le 1\}$

(vii) For $R>1$, we see that for fixed y $V(x,y)\to-\infty$ as $x\downarrow 0$. But always $V(x,y)\ge V(x,0)\ge -\text{const}.\, x^{1-R}$, so we may alternatively consider expressing V as $V(x,y) = x^{1-R}\, h(y/(x+y))$; if we do this, we shall have

$$\frac{R}{1-R}\Big\{(1-R)h(s) - s(1-s)h'(s)\Big\}^{1-1/R} - (\rho+\lambda)h(s) + r\Big\{(1-R)h(s) - s(1-s)h'(s)\Big\}$$
$$+\alpha s(1-s)h'(s) + \tfrac12\sigma^2 s^2\Big((1-s)^2 h''(s) - 2(1-s)h'(s)\Big) + \lambda h(1-s)^{R-1} = 0.$$

There doesn't seem to be a lot to choose between these.

Either way, the strategy appears to be

(i) Use Maple to find the Taylor expansion near 0 of the solution

(ii) Use an ODE solver to continue the solution away from 0.

This seems to be a powerful combined numerical method.

(viii) What happens $(R>1)$ as $x\downarrow 0$, with $y>0$ held fixed? Essentially, all the utility is coming from $[0,\tau_1]$, so we are faced with the problem

$$\max E\left[\int_0^{\tau_1} e^{-\rho t}\, c_t^{1-R}\,\frac{dt}{1-R}\,\Big|\, X_0 = x\right] \equiv V(x) = \frac{a\, x^{1-R}}{1-R}.$$

Solving this gives $\quad c_t^* = \dfrac{\lambda+\rho+r(R-1)}{R}\, x_t \equiv a^{-1/R}\, x_t,$

as we can see from the equation for h above.

Building block Markov processes for potential applications (25/6/97)

(i) Let's see what happens (in one dimension) if we solve the SDE

$$dX_t = dZ_t - b X_t dt$$

where Z is some Lévy process. We get $X_t = e^{-bt} X_0 + \int_0^t e^{-b(t-s)} dZ_s$, and the question arises as to what the law of the stochastic part is. If

$$E e^{\theta Z_t} = e^{t\psi(\theta)}, \qquad (\mathrm{Re}\,\theta = 0)$$

then easily

$$E \exp\left(\theta \int_0^t e^{-bu} dZ_u\right) = \exp\left(\int_0^t \psi(\theta e^{-bs}) ds\right)$$

which requires us to compute this latter. Assuming no drift and no Brownian part (which we can easily add in later), let's suppose ψ is given in the form

$$\psi(\theta) = \int (e^{\theta x} - 1)\, \mu(dx)$$

So we need to compute

$$\int_0^t \left(\int (\exp(\theta e^{-bs} x) - 1)\, \mu(dx)\right) ds$$

$$= \int_0^t \left\{ \int_0^\infty \mu(dx) \int_0^x \theta e^{-bs} e^{y\theta e^{-bs}} dy - \int_{-\infty}^0 \mu(dx) \int_x^0 \theta e^{-bs} e^{y\theta e^{-bs}} dy \right\} ds$$

$$= \int_0^t \left\{ \int_0^\infty \theta e^{-bs} e^{y\theta e^{-bs}} \mu(y,\infty) dy - \int_{-\infty}^0 \theta e^{-bs} e^{y\theta e^{-bs}} \mu(-\infty,y) dy \right\} ds$$

$$e^{-bs} \equiv v$$

$$= \int_{e^{-bt}}^1 \frac{dv}{b} \left\{ \int_0^\infty \theta e^{y\theta v} \mu(y,\infty) dy - \int_{-\infty}^0 \theta e^{y\theta v} \mu(-\infty,y) dy \right\}$$

$$= \int_0^\infty \mu(y,\infty) \left\{ e^{y\theta} - e^{y\theta e^{-bt}} \right\} \frac{dy}{by} - \int_{-\infty}^0 \mu(-\infty,y) \left(e^{y\theta} - e^{y\theta e^{-bt}} \right) \frac{dy}{by}$$

Thus a congenial choice of the tails of μ seems to be essential for 'nice' expressions for the law of X_t.

(ii) Let's see what happens if for some $\alpha > 0$, $\beta \in (0,1)$, we take μ conc. on $(0,\infty)$

$$\boxed{\mu(y,\infty) = e^{-\alpha y} y^{-\beta}}$$

We would then have

$$\boxed{\psi(\theta) = \theta(\alpha - \theta)^{\beta - 1} \Gamma(1-\beta)}$$

After some calculations, we reach the conclusion that

$$\boxed{\int_0^t \psi(\theta e^{-bs})\, ds = \frac{\Gamma(1-\beta)}{\beta b}\left\{-(\alpha-\theta)^\beta + (\alpha-\theta e^{-bt})^\beta\right\}}$$

(iii) As a special limiting case, we could use $\tilde{\mu}(y) = e^{-\alpha y}$, so we need to compute

$$\int_0^\infty e^{-\alpha y}\left(e^{\theta y} - e^{\lambda\theta y}\right)\frac{dy}{by} \qquad \lambda \equiv e^{-bt}$$

$$= \int_0^\infty e^{-\alpha y}\left(e^{-vy} - e^{-v\lambda y}\right)\frac{dy}{by} \qquad v \equiv -\theta$$

$$= -\int_0^\infty e^{-\alpha y}\int_{\lambda v}^{v} e^{-ty}\,dt\,\frac{dy}{b}$$

$$= -\int_{\lambda v}^{v}\frac{dt}{b}\,\frac{1}{\alpha+t} = -\frac{1}{b}\left\{\log(\alpha+v) - \log(\alpha+\lambda v)\right\} = \frac{1}{b}\log\frac{\alpha+\lambda v}{\alpha+v}$$

for $v \equiv -\theta > 0$. Thus if we temporarily throw away the downward jumps, we shall obtain

$$\boxed{E\, e^{-sX_t} = \left(\frac{\alpha+se^{-bt}}{\alpha+s}\right)^{1/b}}$$

(iv) If we assume wlog that $EZ_1 = 0$, $\int (|x|\wedge 1)\mu(dx) < \infty$, and try the quadratic model

$$f(x) = k + \tfrac{1}{2}q\,(x-c)^2$$

we get $\mathcal{G} f(x) = -\beta q\, x^2 + \beta q\, cx + \frac{1}{2}q\, EZ_1^2$, so that

$$(\lambda - \mathcal{G}) f(x) = q\,\frac{\lambda+2\beta}{2}\left(x - \frac{\lambda c+\beta c}{\lambda+2\beta}\right)^2 - \frac{q}{2}\,\frac{(\lambda c+\beta c)^2}{\lambda+2\beta} + \lambda k + \frac{\lambda q}{2}c^2 - \frac{1}{2}q\,EZ_1^2$$

So we require

$$\lambda k = \frac{q}{2}\left\{\frac{(\lambda c+\beta c)^2}{\lambda+2\beta} - \lambda c^2 + EZ_1^2\right\}$$

$$= \frac{q}{2}\left\{\frac{\beta^2 c^2}{\lambda+2\beta} + EZ_1^2\right\}$$

(v) Think of it as follows: If U is an open nbd of 0, then in equilibrium X has Lévy measure $\nu(U^c) = \int_0^\infty ds\; \mu(e^{\beta s}U^c)$.

Higher dimensions? Either independent components or spherically symmetric would be reasonably tractable, but probably not much else would be. We could get dependence via Brownian bits.

More on the liquidity problem (3/7/97)

(i) Let's consider what would happen if the agent chose to consume at rate $\tilde{\gamma} x_t$ and at the reset times always moved back to have a proportion π of wealth in the bank. In that case,

$$V(\pi w, (1-\pi)w) = a w^{1-R}/(1-R)$$

for some constant a which needs to be determined, and

$$V(x_0, y_0) = E\left[\int_0^{\tau_1} e^{-\rho t} \left(\tilde{\gamma} x_0 e^{(r-\tilde{\gamma})t}\right)^{1-R} \frac{dt}{1-R} + e^{-\rho\tau_1} \frac{a}{1-R} \left(x_0 e^{(r-\tilde{\gamma})\tau_1} + y_0 S_{\tau_1} \right)^{1-R} \right]$$

where $S_t \equiv \exp\{\sigma W_t + (\alpha - \frac{1}{2}\sigma^2)t\}$. If we abbreviate $\rho + (R-1)(r-\tilde{\gamma}) \equiv \beta$, we shall thus have

$$V(x_0, y_0) = \frac{(\tilde{\gamma} x_0)^{1-R}}{1-R} \frac{1}{\beta+\lambda} + \frac{a}{1-R} E\left[e^{-\beta\tau_1} \left(x_0 + y_0 \exp\left(\sigma W_{\tau_1} + (\alpha - \tfrac{1}{2}\sigma^2 - r + \tilde{\gamma})\tau_1\right)\right)^{1-R} \right]$$

$$= \frac{(\tilde{\gamma} x_0)^{1-R}}{1-R} \{\lambda + \rho + (R-1)(r-\tilde{\gamma})\}^{-1} + \frac{a x_0^{1-R}}{1-R} E\left[e^{-\beta\tau_1} \varphi\left((\alpha - \tfrac{1}{2}\sigma^2 - r + \tilde{\gamma})\tau_1 + \log \tfrac{y_0}{x_0}, \sigma^2\tau_1 \right)\right]$$

where $\varphi(\mu, v) = E\,(1+e^Z)^{1-R}$, when $Z \sim N(\mu, v)$.

When $x_0 = \pi$, $y_0 = 1-\pi$, we learn that

$$\boxed{a\left\{ 1 - \pi^{1-R} E\left[e^{-\beta\tau_1} \varphi\left((\alpha - \tfrac{1}{2}\sigma^2 - r + \tilde{\gamma})\tau_1 + \log \tfrac{1-\pi}{\pi}, \sigma^2\tau_1 \right)\right] \right\} = (\tilde{\gamma}\pi)^{1-R} \{\lambda + \rho + (R-1)(r-\tilde{\gamma})\}^{-1}}$$

(ii) Obviously, we'll need to know rather more about φ before we can get further. We have

$$\varphi(\mu, v) = \int_{-\infty}^{\infty} e^{-(y-\mu)^2/2v} (1+e^y)^{1-R} \, dy/\sqrt{2\pi v}$$

$$= \int_0^\infty \frac{dy}{\sqrt{2\pi v}} e^{-(y-\mu)^2/2v} e^{(1-R)y} \sum_{n\geq 0} \frac{e^{-ny}\,\Gamma(2-R)}{n!\,\Gamma(2-R-n)} + \int_{-\infty}^0 e^{-(y-\mu)^2/2v} \sum_{n\geq 0} \frac{e^{ny}\,\Gamma(2-R)}{n!\,\Gamma(2-R-n)} \frac{dy}{\sqrt{2\pi v}}$$

$$= \sum_{n\geq 0} \frac{\Gamma(2-R)}{n!\,\Gamma(2-R-n)} e^{-\mu^2/2v} \left[e^{\beta_n^2/2} \bar{\Phi}(\beta_n) + e^{\alpha_n^2/2} \bar{\Phi}(\alpha_n) \right]$$

where $\beta_n \equiv (v(n+R-1) - \mu)/\sqrt{v}$, $\alpha_n \equiv (\mu + nv)/\sqrt{v}$. However, we can do better than this.

If we consider the random variable $\sigma W_T + \kappa T$, where $T \sim \exp(\beta+\lambda)$, it has the density

$$\frac{\eta}{\sqrt{\kappa^2 + 2\eta\sigma^2}} \exp\left[\frac{\kappa y}{\sigma^2} - |y| \frac{\sqrt{\kappa^2 + 2\eta\sigma^2}}{\sigma^2} \right] \qquad (\eta \equiv \beta + \lambda)$$

Abbreviating $b \equiv -\log(y_0/x_0)$, $\theta \equiv (\kappa^2 + 2\eta\sigma^2)^{1/2}$, we need to evaluate

$$\int_{-\infty}^{\infty} \frac{\eta}{\theta} \exp\left[\frac{\kappa y}{\sigma^2} - \frac{|y|\theta}{\sigma^2} \right] (1 + e^{y-b})^{1-R} \, dy$$

$$= \frac{\eta}{\theta} \sum_{n \geq 0} \frac{\Gamma(2-R)}{n!\,\Gamma(2-R-n)} \left\{ \int_b^{\infty} \exp\left[\frac{\kappa y - \theta|y|}{\sigma^2} + (1-R-n)(y-b) \right] dy + \int_{-\infty}^{b} \exp\left[\frac{\kappa y - \theta|y|}{\sigma^2} + n(y-b) \right] dy \right\}$$

Case 1: $b \geq 0$ The term in $\{\ \}$ is

$$\frac{e^{(\kappa-\theta)b/\sigma^2}}{\frac{\theta-\kappa}{\sigma^2} + n + R - 1} + \int_0^b \exp\left(\frac{\kappa-\theta}{\sigma^2} y + n(y-b) \right) dy + \frac{e^{-nb}}{(\theta+\kappa)\sigma^{-2} + n}$$

$$= \frac{e^{-(\theta-\kappa)b/\sigma^2}}{(\theta-\kappa)\sigma^{-2} + n + R - 1} + \frac{e^{(\kappa-\theta)b/\sigma^2} - e^{-nb}}{(\kappa-\theta)\sigma^{-2} + n} + \frac{e^{-nb}}{(\theta+\kappa)\sigma^{-2} + n}$$

Case 2: $b \leq 0$ Now the term in $\{\ \}$ becomes

$$\frac{e^{(n+R-1)b}}{(\theta-\kappa)\sigma^{-2} + n + R - 1} + \frac{e^{(n+R-1)b} - e^{(\kappa+\theta)\sigma^{-2} b}}{(\kappa+\theta)\sigma^{-2} + 1 - R - n} + \frac{e^{(\kappa+\theta)b/\sigma^2}}{(\theta+\kappa)\sigma^{-2} + n}$$

(iii) We need to understand the behaviour as $\lambda \to \infty$ of

$$E\, (\pi + (1-\pi) e^{X(T)})^{1-R} \qquad [X_t = \sigma W_t + \kappa t,\ \kappa \equiv \alpha - \tfrac{1}{2}\sigma^2 - r + \tilde{\delta},\ T \sim \exp(\eta),\ \eta \equiv \beta + \lambda]$$

$$= E\,(1 + (1-\pi)(e^{X(T)} - 1))^{1-R}$$

Now if we set $H \equiv \inf\{u : X_u = \log 2\}$ we can estimate for $R \in (0,1)$

$$E\left[\{1 + (1-\pi)(e^{X(T)} - 1)\}^{1-R} : H \leq T \right]$$

$$\leq E\left[(2-\pi)^{1-R} + (1-R)(1-\pi)(2-\pi)^{-R} (e^{X(T)} - 2) \,;\, H \leq T \right] \qquad \text{(concavity)}$$

$$= \underbrace{P[H \leq T]}_{= \exp\left[-\log 2 \cdot \frac{\sqrt{\kappa^2 + 2\sigma^2\eta} - \kappa}{\sigma^2} \right]} \left\{ (2-\pi)^{1-R} - 2(1-R)(1-\pi)(2-\pi)^{-R} + (1-R)(1-\pi)(2-\pi)^{-R} \cdot 2 \underbrace{E\, e^{X(T)}}_{\eta(\eta - \kappa - \frac{1}{2}\sigma^2)^{-1}} \right\}$$

and so this tends to zero faster than any power of η. Similarly, for $R > 1$ we have easily $E\left[(\pi + (1-\pi)e^{X(T)})^{1-R} ; H \leq T \right] \leq \pi^{1-R} P(H \leq T)$ which goes rapidly to zero. The conclusion is that for large λ we may neglect the contribution to the expectation on the event $\{H \leq T\}$. The choice of $\log 2$ is not crucial - any positive value would do.

... Continued p 51 ...

Volatility of returns in tick-data model (8/7/97)

(i) In a simple model for share quotes, the quotes come at rate $\lambda_t = f(X_t)$ where X is some underlying Markov process. There's an underlying 'notional' log price process ξ_t which we'll assume is $\xi_t = \sigma W_t + \mu t$, and the actual quote at time t will be $\xi_t + \varepsilon$, where ε is independent of everything, with some distribution.

Suppose we decide to look at share prices every $\delta > 0$ of time. What does this mean? If $\{\tau_i : i \in \mathbb{Z}\}$ are the times of quotes, we chop the time axis into pieces $I_j = ((j-1)\delta, j\delta]$, $j \in \mathbb{Z}$ and we define $T_j = \sup\{\tau_i : \tau_i \le j\delta\}$. The change of log-price ascribed to time interval I_j will be

$$Y_j \equiv \xi(T_j) + \varepsilon - (\xi(T_{j-1}) + \varepsilon')$$

where ε is the error on the quote at time T_j, ε' the error on the quote at time T_{j-1}. Note that $T_j = T_{j-1}$ is possible, that is, there is <u>no</u> change during I_j. Note that it is also possible that $T_{j-1} \notin I_{j-1}$.

(ii) The mean of Y_j is

$$E\left[\sigma\{W(T_j) - W(T_{j-1})\} + \mu(T_j - T_{j-1}) + \varepsilon - \varepsilon' : T_j > T_{j-1}\right]$$

$$= \mu\, E\left[T_j - T_{j-1} ; T_j > T_{j-1}\right],$$

so we need to compute this. The value is in fact just $\mu\delta$, assuming that X is stationary ergodic. Why? If we run time backward from time $j\delta$, then T_j is the first quote we come to, T_{j-1} is the first we come to after δ. So $E(j\delta - T_j) = E(j\delta - \delta - T_{j-1})$

$$\Rightarrow \quad E\,T_j = \delta + E\,T_{j-1}.$$

(iii) How about the second moment? We have

$$E\left[\sigma^2(T_j - T_{j-1}) + \mu^2(T_j - T_{j-1})^2 + (\varepsilon - \varepsilon')^2 : T_j > T_{j-1}\right]$$

$$= \sigma^2\delta + \mu^2 E(T_j - T_{j-1})^2 + 2\,\mathrm{var}(\varepsilon)\,.\, P[T_j > T_{j-1}]$$

$$= \sigma^2\delta + \mu^2\delta^2 + \mu^2 E(T_j - \delta - T_{j-1})^2 + 2\,\mathrm{var}(\varepsilon)\; P[T_j > T_{j-1}]$$

This is essentially impossible to take further at a general level. However, the specific assumption on ξ might be relaxed a bit; we could assume that ξ is a process with stationary increments, so $E(\xi_t - \xi_s) = \mu(t-s)$, $\mathrm{var}(\xi_t - \xi_s) = v(t-s)$ for some function v. Then we should have

$$\theta = \sqrt{\beta^2 + \sigma^2 \gamma}$$

$$\begin{cases} E\, Y_j = \mu\delta & \text{as before} \\ E\, Y_j^2 = E\left[v(T_j - T_{j-1}) + \mu^2 (T_j - T_{j-1})^2 \right] + 2\,\mathrm{var}(\varepsilon)\, P[T_j > T_{j-1}] \end{cases}$$

(iv) <u>Example</u> Let's try taking

$$dX = \sigma\, dW - \beta X\, dt$$

$\left[\begin{array}{l} \text{Invariant law} \\ N(0, \sigma^2/2\beta) \end{array} \right]$

in n-dimensions, with scalar σ, β. We next compute

$$E^x \exp - \int_0^t \tfrac{1}{2} \gamma\, |X_s - a|^2\, ds = \exp\left[-\tfrac{1}{2} v(t)\, |x|^2 - x\cdot b(t) - k(t) \right]$$

in the usual way:

$$\left.\begin{array}{rr} -\tfrac{1}{2}\gamma + \tfrac{1}{2}\dot{v} + \beta v + \tfrac{1}{2}\sigma^2 v^2 & = 0 \\ \gamma a + \dot{b} + \beta b + \sigma^2 v\, b & = 0 \\ -\tfrac{n}{2}\sigma^2 v - \tfrac{1}{2}\gamma |a|^2 + \dot{k} + \tfrac{1}{2}\sigma^2 b^2 & = 0 \end{array}\right\} \Rightarrow$$

$$v(t) = \frac{\gamma \sinh\theta t}{\theta\cosh\theta t + \beta\sinh\theta t}$$

$$b(t) = \frac{\gamma a}{\theta}\, \frac{\beta(1-\cosh\theta t) - \theta\sinh\theta t}{\theta\cosh\theta t + \beta\sinh\theta t}$$

$$k(t) = \frac{n}{2}\log\frac{\theta\cosh\theta t + \beta\sinh\theta t}{\theta} + \tfrac{1}{2}\gamma^2 |a|^2 t$$
$$- \frac{1}{2}\left(\frac{\sigma\gamma a}{\theta}\right)^2 \frac{2\beta(1-\cosh\theta t) + \theta^2 t\cosh\theta t + (\beta t - 1)\theta\sinh\theta t}{\theta(\theta\cosh\theta t + \beta\sinh\theta t)}$$

$\left(\theta \equiv \sqrt{\beta^2 + \sigma^2\gamma}\right)$

Heuristics for the Merton problem (11/7/97)

(i) Suppose the agent follows the policy of consuming at some constant γ times wealth, and keeps constant θ proportion of wealth in risky asset. Then

$$dw_t = r w_t\,dt + \theta w_t \{\sigma\, dW_t + (\alpha - r)\,dt\} - \gamma w_t\,dt$$

$$\therefore\ w_t = w_0 \exp\left[\sigma\theta W_t + \{(\alpha-r)\theta - \tfrac{1}{2}\sigma^2\theta^2 + r - \gamma\}t\right]$$

and the value of the payoff is

$$E\left[\int_0^\infty e^{-\rho t}\, c_t^{1-R}\,\frac{dt}{1-R}\right]$$

$$= \frac{(\gamma w_0)^{1-R}}{1-R}\, E\int_0^\infty \exp\{-\rho t + (1-R)(\sigma\theta W_t + \{(\alpha-r)\theta - \tfrac{1}{2}\sigma^2\theta^2 + r - \gamma\}t)\}\,dt$$

$$= \frac{(\gamma w_0)^{1-R}}{1-R}\left\{\rho + (R-1)\left((\alpha-r)\theta + r - \gamma - \tfrac{1}{2}R\sigma^2\theta^2\right)\right\}^{-1} \qquad (1)$$

From this, we can easily get $\theta^* = (\alpha - r)/\sigma^2 R$ [optimised payoff is $\dfrac{\gamma^{-R}\, w_0^{1-R}}{1-R}$]

$$\gamma^* = R^{-1}\left[\rho + (R-1)\left(r + (\alpha-r)^2/2\sigma^2 R\right)\right]$$

as per usual. If we take the function

$$\log\gamma + \frac{1}{R-1}\log\left\{\rho + (R-1)\left((\alpha-r)\theta + r - \gamma - \tfrac{1}{2}R\sigma^2\theta^2\right)\right\}$$

and expand about (θ^*, γ^*), we shall find that the first order terms in $(\gamma-\gamma^*)$ and $(\theta-\theta^*)$ vanish (of course) and the second order terms are

$$-(\gamma-\gamma^*)^2\,\frac{2\sigma^4 R^5}{\left[2\rho\sigma^2 R + (\alpha-r)^2(R-1) + 2R\sigma^2(R-1)\right]^2}$$

$$-(\theta-\theta^*)^2\,\frac{\sigma^4 R^3}{2\rho\sigma^2 R + (\alpha-r)^2(R-1) + 2R\sigma^2(R-1)} \qquad \text{(Maple).}$$

Thus if we erroneously used the policy with θ, γ instead of θ^*, γ^*, this is approximately the decrease in initial log wealth that we incur.

(ii) We may represent $X_T = \xi_+ - \xi_-$ where $\xi_+ \equiv \bar{X}_T$, $\xi_- = -\underline{X}_T$ are independent exponential variables with parameters $\omega_\pm \equiv \sigma^{-2}\left[\sqrt{\kappa^2 + 2\eta\sigma^2} \mp \kappa\right]$. Hence writing $c \equiv \log 2$ we have

$$P[\bar{X}_T \leq c] = 1 - e^{-\omega_+ c} = 1 - \exp\left\{-\frac{c}{\sigma^2}\left(\sqrt{\kappa^2+2\eta\sigma^2} - \kappa\right)\right\} \to 1 \quad (\eta \to \infty)$$

Thus we shall have

$$E\left[\left(1 + (1-\pi)(e^{X_T} - 1)\right)^{1-R} : H > T\right]$$

$$= \sum_{n \geq 0} \frac{\Gamma(2-R)}{n!\,\Gamma(2-R-n)} \cdot (1-\pi)^n \; E\left[(e^{X_T}-1)^n : H > T\right].$$

But

$$E\left[e^{mX_T} : H > T\right] = E\, e^{-m\xi_-} \cdot \int_0^c \omega_+ e^{-\omega_+ x + mx}\, dx = \frac{\omega_+ \omega_-}{(\omega_- + m)(\omega_+ - m)}\left(1 - e^{-c(\omega_+ - m)}\right)$$

$$= \frac{\eta}{\eta - \kappa m - \frac{1}{2}\sigma^2 m^2}\left\{1 - e^{-c(\omega_+ - m)}\right\}$$

Now we're thinking of η as being very large, so the $e^{-c(\omega_+ - m)}$ is negligible, and we've got

$$E\left[(e^{X_T}-1)^n : H > T\right] = \sum_{r=0}^{n} \binom{n}{r}(-1)^{n-r} \frac{\eta}{\eta - \kappa r - \frac{1}{2}\sigma^2 r^2} + o(\eta^{-N})$$

Now dumping the $o(\eta^{-N})$ term, we find from Maple for the first few terms that

n	lead term in η
1	$\frac{\kappa + \frac{1}{2}\sigma^2}{\eta} = \frac{\kappa + b}{\eta}$ for short, $b \equiv \frac{1}{2}\sigma^2$
2	$\sigma^2/\eta = 2!\; b/\eta$
3	$3!\; 2b(3b+\kappa)\eta^{-2}$
4	$4!\; b^2/\eta^2$
5	$5!\, .\, 3b^2(5b+\kappa)/\eta^3$
6	$6!\, .\, b^3/\eta^3$
7	$7!\, .\, 4b^3(7b+\kappa)/\eta^4$

So only the first couple of terms have $O(\eta^{-1})$ contributions, so that

$$\boxed{E\left[\left(1+(1-\pi)(e^{X_T}-1)\right)^{1-R} : H > T\right] = 1 + (1-R)(1-\pi)\frac{\kappa + \frac{1}{2}\sigma^2}{\eta} - (1-R)R\,(1-\pi)^2\,\frac{\frac{1}{2}\sigma^2}{\eta} + O(\eta^{-2})}$$

Now let's return that to the earlier relation for a from pp 46–47; from there, we had

$$a = \frac{(\tilde{\gamma}\pi)^{1-R}}{\beta+\lambda} + \frac{\lambda}{\beta+\lambda}\cdot a \cdot E\left[\left(1+(1-\pi)(e^{X_\tau}-1)\right)^{1-R}\right] \qquad (\eta \equiv \beta+\lambda)$$

So

$$(\beta+\lambda)a = (\tilde{\gamma}\pi)^{1-R} + \lambda a\left[1 + (1-R)(1-\pi)\frac{\kappa+\frac{1}{2}\sigma^2}{\eta} - (1-R)R(1-\pi)^2\frac{\sigma^2}{2\eta} + O(\eta^{-3})\right]$$

Hence in the limit as $\lambda\to\infty$ we'd be seeing

$$\left[\beta + (1-R)R(1-\pi)^2\frac{\sigma^2}{2} - (1-R)(1-\pi)(\kappa+\tfrac{1}{2}\sigma^2)\right]a = (\tilde{\gamma}\pi)^{1-R}$$

Substituting $\beta = \rho+(R-1)(r-\tilde{\gamma})$, $(1-\pi)=\theta$, $\kappa+\frac{1}{2}\sigma^2 = \alpha-r+\tilde{\gamma}$ (which is how they're defined) and $\tilde{\gamma}\pi = \gamma$ reduces this to expression (1) on p 50, so that's encouraging.

(iii) We can of course do higher order approximations, and it's perhaps of interest to record the next term in the expansion. We get $(\varepsilon \equiv 1/\lambda)$

$$a = \frac{\gamma^{1-R}}{\rho+(R-1)\left((\alpha-r)\theta + r-\gamma - Rb\theta^2\right)} + \varepsilon\,\gamma^{1-R}\,\theta(R-1)\Big\{R(R+1)(R+2)b^2\theta^3 +$$

$$+ 2R(R+1)b(r-\tilde{\gamma}-\alpha-2b)\theta^2 + R\Big((\alpha-r)^2 - \rho b + \tilde{\gamma}^2 + \tilde{\gamma}\big(2(\alpha-r)+(R+3)b\big)$$

$$-(R+3)br + 2b(b+2\alpha)\Big)\theta + (\tilde{\gamma}-r)(\tilde{\gamma}-r+\alpha)R + (\alpha-\rho)(r-\tilde{\gamma}-\alpha)\Big\}.$$

$$\left\{\rho+(R-1)\left((\alpha-r)\theta + r-\gamma-Rb\theta^2\right)\right\}^{-2}$$

<u>Where is this optimised?</u> Obviously, quite close to $\theta=\theta^*$, $\gamma=\gamma^*$. In fact,

$$\theta-\theta^* = -\frac{\varepsilon}{4R^3b^2}(R\sigma^2-\alpha+r)^{-3}\Big[16R^3b^3(R\sigma^2+\alpha-r)\rho^2 + 4Rb\rho\Big\{rR^3\sigma^6(R-2)$$

$$+(\alpha-r)\Big((r-\alpha)^3 + 2Rb\big(2(r-\alpha)^2 + R(r-\alpha)^2 + 2Rb(2R(R+1)b+2r+3rR-4\alpha)\big)\Big)\Big\}$$

$$-2R^4r^2\sigma^8(R-1) + (\alpha-r)\Big\{2R^3\sigma^6\big(2R^2(r^2+b^2)+r^2(1-3R)-2bRr\big) +$$

$$+(\alpha-r)\Big(8R^3b^3(-12Rb+8r-7Rr+5rR^2) + (\alpha-r)\Big(4R^2b^2(26Rb+2R^3b+Rr-3rR^2-4r)$$

$$+(\alpha-r)\Big\{-2Rb(25Rb+R^3b+6R^2b+2rR-2r-2rR^2) + (\alpha-r)\big(Rb(13+R^2+4R)-(1+R)(\alpha-r)\big)$$

$$\Big\}\Big)\Big)\Big\}.\Big]$$

and

$$\gamma - \gamma^* = -\frac{\varepsilon}{16}\frac{(\alpha - r)}{R^4 b^2 (R\sigma^2 - \alpha + r)^2}\Big[(R^2-1)t^5 - R\sigma^2(R-1)(2R^2+R+9)t^4$$

$$+ 4R\sigma^2(R-1)\left(2R^3 b + 6Rb - r(R^2-R+1) + \rho\right)t^3$$

$$+ 4R^2\sigma^4\left(4R(1-R)b + 3rR^3 - 4rR^2 - \rho - R\rho + r\right)t^2$$

$$+ 4R^2\sigma^4\left(R^2\left(2R(b+r)^2 + r^2 - \sigma^2\right) + (\rho - r)\left(-r + \rho + 2R(1+R)r + 4R^3 b\right)\right)t$$

$$+ 4R^3\sigma^6\left(\rho - r + R(2\rho - r)\right)\left(\rho - (1-R)r\right)\Big]$$

with the aid of Maple.

Cash-in-advance in continuous time (25/8/97)

Let's suppose that the holding of cash at time t is x_t, and the amount in the bank is y_t, governed by equations

$$\begin{cases} dx_t = c_t^{\alpha}\,\sigma dW_t - c_t\,dt + dA_t \\ dy_t = r y_t\,dt - (1+\varepsilon)\,dA_t \end{cases}$$

where A is increasing and $\alpha = \frac{1}{2}$ or $\alpha = 1$. The agent aims to find

$$\max E\left[\int_0^\infty e^{-\rho t}\,U(c_t)\,dt \;\middle|\; x_0 = x,\, y_0 = y\right] \equiv V(x,y),$$

where U is CRRA.

(i) If $\alpha = 1$, we have the scaling $V(\lambda x, \lambda y) = \lambda^{1-R}\,V(x,y)$, so we could have the representation $\quad V(x,y) = y^{1-R}\,v(x/y)$.

The HJB equations here are

$$\sup_{c \geq 0}\left[\, U(c) - \rho V(x,y) + \tfrac{1}{2}\sigma^2 c^2\, V_{xx}(x,y) - c\,V_x(x,y) + ry\,V_y(x,y)\,\right] = 0,$$

$$V_x - (1+\varepsilon)V_y \leq 0 \quad , = \text{ if } x = 0.$$

Thus in terms of v we shall have

$$\boxed{\begin{aligned} &\sup_{t\geq 0}\left[\, \frac{t^{1-R}}{1-R} - \rho\, v\!\left(\tfrac{x}{y}\right) + \tfrac{1}{2}\sigma^2 t^2\, v''\!\left(\tfrac{x}{y}\right) - t\, v'\!\left(\tfrac{x}{y}\right) + r\,(1-R)\,v\!\left(\tfrac{x}{y}\right) - r\,\tfrac{x}{y}\,v'\!\left(\tfrac{x}{y}\right)\right] = 0 \\ &(1+\theta+\varepsilon\theta)\,v'(\theta) \leq (1-R)(1+\varepsilon)\,v(\theta), \quad \text{equal if } \theta \equiv \tfrac{x}{y} \text{ is zero.} \end{aligned}} \qquad (1)$$

(ii) It could also be argued that we should take $\alpha = \frac{1}{2}$, in which case for some BM $\tilde{W}$ we get

$$x_t = x_0 + \sigma\,\tilde{W}(C_t) - C_t + A_t\,.$$

Since $x_t \geq 0\ \forall t$ is needed, if we define

$$k_t \equiv \sup_{s\leq t}\,(s - \sigma\tilde{W}_s - x_0)^+$$

then we shall have to have that $A_t = k(C_t)$. If Γ is the inverse to C, $\Gamma' \equiv \gamma$, we convert the budget constraint $(1+\varepsilon)\int_0^\infty \exp(-rs)\,dA_s = y_0$ into the condition

$$\frac{y_0}{1+\varepsilon} = \int_0^\infty \exp(-r\Gamma_s)\,dk_s$$

and the objective $\qquad\qquad (\Gamma' \equiv \gamma = 1/c(\Gamma))$

$$\int_0^\infty e^{-\rho t}\,c_t^{1-R}\,\frac{dt}{1-R} = \int_0^\infty e^{-\rho\Gamma_s}\,c(\Gamma_s)^{-R}\,\frac{ds}{1-R} = \int_0^\infty e^{-\rho\Gamma_s}\,\gamma_s^R\,ds\,.$$

A possible approach is to look for the value f^n

$$V(x,y) \equiv \sup E\left[\int_0^\infty e^{-\rho\Gamma_t} \gamma_t^R \frac{dt}{1-R} \,\middle|\, x_0 = x,\ y_0 = y \right]$$

The dynamics of $\tilde{x}_t \equiv x(\Gamma_t)$, $\tilde{y}_t \equiv y(\Gamma_t)$ are simply

$$\left.\begin{aligned} d\tilde{x}_t &= \sigma\, d\tilde{W}_t - dt + dk_t \\ d\tilde{y}_t &= r\gamma_t \tilde{y}_t\, dt - (1+\varepsilon)\, dk_t \end{aligned}\right\}$$

So the HJB equation resulting is

$$\sup_\gamma \left[\frac{\gamma^R}{1-R} - \rho\gamma V + \tfrac{1}{2}\sigma^2 V_{xx} - V_x + r\gamma y V_y \right] = 0, \qquad V_x \le (1+\varepsilon) V_y .$$

This gives

$$\gamma^* = \left\{ \frac{1-R}{R} \left(\rho V - r y V_y \right) \right\}^{1/R-1},$$

and

$$\boxed{\left\{ \frac{1-R}{R} \left(\rho V - r y V_y \right) \right\}^{R/(R-1)} + \tfrac{1}{2}\sigma^2 V_{xx} - V_x = 0.} \qquad (2)$$

Any hope of solving this?

(iii) Another approach would be via a perturbation analysis: if we think that γ is the optimal process (and it seems only natural that γ should be a f^n of $\tilde{x}, \tilde{y}$ alone) then we perturb to $\gamma + \eta$ and look for FOCs. Lagrangian form is for some $\mathcal{F}_\infty$-meas r.v. Z

$$E\left[\int_0^\infty e^{-\rho\Gamma_t} \gamma_t^R \frac{dt}{1-R} - Z \int_0^\infty e^{-r\Gamma_s} dk_s \right]$$

So FOC is

$$0 = E\left[\int_0^\infty \frac{e^{-\rho\Gamma_t}}{1-R} \gamma_t^R \left\{ \frac{R}{\gamma_t}\eta_t - \rho \int_0^t \eta_s\, ds \right\} dt + rZ \int_0^\infty e^{-r\Gamma_s} \left(\int_0^s \eta_u\, du \right) dk_s \right]$$

$$= E\left[\int_0^\infty \eta_s\, ds \left\{ \frac{R}{1-R} \gamma_s^{R-1} e^{-\rho\Gamma_s} - \rho \int_s^\infty \gamma_t^R e^{-\rho\Gamma_t} \frac{dt}{1-R} + \int_s^\infty rZ e^{-r\Gamma_t} dk_t \right] \right.$$

$$\therefore \quad \boxed{\gamma_t^{R-1} e^{-\rho\Gamma_t} = E_t\left[\frac{\rho}{R} \int_t^\infty \gamma_s^R e^{-\rho\Gamma_s} ds - \frac{r(1-R)}{R} \int_t^\infty Z e^{-r\Gamma_s} dk_s \right]} \qquad (3)$$

So if $Z_t \equiv E_t(Z)$, we shall have that

$$M_t \equiv \gamma_t^{R-1} e^{-\rho\Gamma_t} + \int_0^t \frac{\rho}{R} \gamma_s^R e^{-\rho\Gamma_s} ds - \frac{r(1-R)}{R} \int_0^t Z_s e^{-r\Gamma_s} dk_s \quad \underline{\text{is a martingale.}}$$

If we now write $\gamma_t \equiv f(\tilde{x}_t, \tilde{y}_t)$ we can do the Itô expansion of M:

$$dM_t = e^{-\rho\Gamma_t}\left[\frac{1-R}{R}\rho f^R dt + (R-1) f^{R-2} df + \frac{(R-1)(R-2)}{2} f^{R-3} d\langle f\rangle \right] - \frac{r(1-R)}{R} Z_t e^{-r\Gamma_t} dk_t$$

$$\doteq e^{-\rho\Gamma_t}\left\{ \frac{1-R}{R}\rho f^R + (R-1) f^{R-2}\left(-f_x + r\tilde{y} f_y + \tfrac{1}{2}\sigma^2 f_{xx}\right) + \frac{(R-1)(R-2)}{2} f^{R-3}\sigma^2 f_x^2 \right\} dt$$

$$+ \left\{ e^{-\rho\Gamma_t}(R-1) f^{R-2}\left(f_x - (1+\epsilon) f_y\right) - \frac{r(1-R)}{R} Z_t e^{-r\Gamma_t} \right\} dk_t$$

Thus we learn that f solves the PDE

$$\boxed{\frac{1-R}{R}\rho f^3 + (R-1) f\left(\tfrac{1}{2}\sigma^2 f_{xx} - f_x + r\tilde{y} f_y\right) + \frac{\sigma^2}{2}(R-1)(R-2) f_x^2 = 0,}$$

There is also a further condition, based on the fact that the c/e of $dk = 0$. But this is no nicer.

(iv) We could deduce an expression for V from (2), which we rewrite as

$$\psi(x,y) + \tfrac{1}{2}\sigma^2 V_{xx}(x,y) - V_x(x,y) = 0$$

The Green's f^n for $\sigma\tilde{W}_t - t$ killed on hitting 0 has density

$$g(x,y) = \frac{2}{\sigma^2}\,\frac{s(x\wedge y) - s(0)}{s'(y)} \qquad s(y) \equiv e^{2y/\sigma^2}$$

So we shall have for each y

$$V(x,y) = V(0,y) + \int_0^\infty \frac{2}{\sigma^2}\,\frac{s(x\wedge z) - s(0)}{s'(z)}\cdot\psi(z,y)\,dz$$

Can we use our knowledge of ψ, $\psi(z,y) = \{(1-R)(\rho V(z,y) - ry V_y(z,y))/R\}^{R/(R-1)}$? From this representation, we obtain

$$V_x(0,y) = \frac{2s'(0)}{\sigma^2}\int_0^\infty \frac{\psi(z,y)}{s'(z)}\,dz = (1+\epsilon) V_y(0,y)$$

(v) We can obtain a similar representation in the other direction. Indeed, since we have

$$(1-R)\{\rho V(z,y) - ry V_y(z,y)\} R^{-1} = \psi(z,y)^{(R-1)/R},$$

we have

$$\frac{\partial}{\partial y}\left[y^{-\rho/r} V(z,y)\,(1-R)\right] = -\frac{y^{-1-\rho/r}}{r}\,\psi(z,y)^{(R-1)/R}\cdot R$$

so

$$\boxed{V(z,y) = y^{\rho/r}\int_y^\infty \frac{t^{-1-\rho/r}}{r}\,\psi(z,t)^{(R-1)/R}\,\frac{dt}{1-R}\cdot R}$$

How can we be sure of the boundary conditions?

If $R \in (0,1)$, and we consider the problem where we just consume from bank account, we have optimum $\propto y^{1-R}$, provided the well-posed condition $\rho > r(1-R)$ holds (if not, the sup is unbounded, and it's easy to see that the same will be true of the problem we actually face). Thus the value is $O(y^{1-R}) = o(y^{\rho/r})$.

If $R > 1$, by considering strategies which immediately move wealth into x and then consume from x that $V(x,y) \uparrow 0$ as $y \uparrow \infty$.

(vi) Can we even solve the problem in closed form assuming $y_0 = 0$? The HJB equation for the value f^n now is just

$$\sup_c \left[U(c) - \rho v + \tfrac{1}{2}\sigma^2 v'' - v')c \right] = 0$$

$$\therefore \quad \frac{R}{1-R} \left(v' - \tfrac{1}{2}\sigma^2 v'' \right)^{1-1/R} = \rho v, \quad \text{or again} \quad \boxed{v' - \tfrac{1}{2}\sigma^2 v'' = \left(\frac{\rho(1-R)}{R} v \right)^{R/R-1}}$$

Writing $v' \equiv p$, $v'' = p\frac{dp}{dv}$, this becomes the first-order equation

$$\boxed{p - \frac{\sigma^2}{2} p \frac{dp}{dv} = \left(\frac{\rho(1-R)}{R} v \right)^{R/R-1}}$$

Interesting questions etc.

1) Arising in discussions with Auke Plantinga at Groningen. If an agent may invest in a stock for which

$$dS_t/S_t \equiv dX_t = \sigma dW_t + \mu dt,$$

and μ is fixed but random, $\mu \sim N(\mu_0, \Sigma_0)$, what is the optimal investment strategy for max $E\int_0^\infty e^{-\rho t} U(c_t)\,dt$, for max $E\,U(W_T)$, $U'(x) = x^{-R}$ . . . ?
This relates to earlier work of Lakner, and what I did with Stewart.

2) Theo Dijkstra asks this. Suppose we're working in discrete time term structure model, and we want to have always

$$P(n, n+j) = \exp\Big[-\sum_{r=0}^{K} \xi_n^{(r)} f_j^{(r)} \Big]$$ for certain stochastic factors

$\xi_n^{(r)}$ and fixed vectors $f^{(r)}$. What sorts of models can these be?

3) Forward-starting call options effectively give a payoff $E\,|M_2 - M_1|$,
where M is a martingale (the price process). If the laws of M_1 and M_2 are given, what's the joint law which maximizes $E(|M_2 - M_1|)$, or $E\,\varphi(M_2 - M_1)$ more generally?