Computing first-passage distributions of certain additive functionals (31/8/97)

Suppose that $X_t = \sigma W_t + \mu t$, $S_t = \exp(X_t)$ and for some deterministic positive function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ we have a time-inhomogeneous additive functional

$$A_t \equiv \int_0^t \alpha_u S_u \, du.$$

How might we compute $P(S_T \in dx,\ A_T \leq a)$? Even computing $P(A_T \leq a)$ is not easy - it's a generalisation of the law of the payoff of an Asian option, and time-inhomogeneity destroys any of the exponential randomisation tricks...

We can modify the method of R+Shi to give a numerical scheme. Define (with λ fixed)

$$v(t,x) \equiv E\left[S_T^\lambda ; \int_t^T \alpha_u S_u \, du \leq x \,\middle|\, S_t = 1 \right].$$

Then we have the representation

$$\begin{aligned} M_t &\equiv E\left[S_T^\lambda : \int_0^T \alpha_u S_u \, du \leq a \,\middle|\, \mathcal{F}_t \right] \\ &= E\left[S_T^\lambda : \int_t^T \alpha_u S_u du \leq a - \int_0^t \alpha_u S_u du \,\middle|\, \mathcal{F}_t \right] \\ &= S_t^\lambda \, v\left(t,\ S_t^{-1}\left(a - \int_0^t \alpha_u S_u du\right)\right) \end{aligned}$$

If we set $\xi_t \equiv S_t^{-1}(a - A_t)$, then $d\xi_t = \xi_t\left(-\sigma dW_t + (-\mu + \tfrac{1}{2}\sigma^2) dt\right) - \alpha_t dt$, and

$$dM_t \doteq S_t^\lambda \left[v.(\mu\lambda + \tfrac{1}{2}\sigma^2\lambda^2) + \dot{v} + v'\left((\tfrac{1}{2}\sigma^2 - \mu)\xi - \alpha\right) + \tfrac{1}{2}\sigma^2\xi^2 v'' - \sigma^2\lambda\xi v' \right] dt$$

From this we conclude that v solves the PDE

$$\boxed{\begin{gathered} \dot{v} + \tfrac{1}{2}\sigma^2\xi^2 v'' + \left[(\tfrac{1}{2}\sigma^2 - \mu)\xi - \alpha(t) - \sigma^2\lambda\xi\right] v' + \lambda(\mu + \tfrac{1}{2}\sigma^2\lambda) v = 0 \\ v(t,0) = 0 \quad \forall\ 0 \leq t < T \\ v(T,x) = 1 \quad \forall\ x \geq 0 \end{gathered}}$$

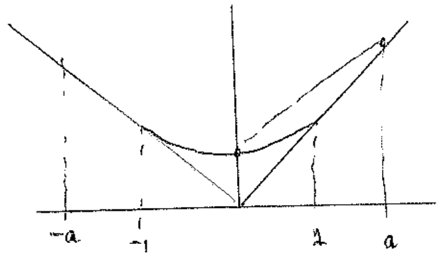
This should be a reasonably simple task for method-of-lines, provided we can get around the problems at $\xi = 0$, where the c/e of v'' vanishes.

If we had a lower bound on α, for example, we should be able to cut off at $\xi = \varepsilon$, since the PDE is just a Cauchy problem for a different diffusion!

One explicit example of the maximum maximum (1/9/97)

We'll take $M_1 \sim U[-1,1]$ and M_2 symmetrically distributed on $\{-a, 0, a\}$. Can we now determine how to get stochastically largest value of $\bar{M}_2$?

(i) N+S for the law of M_2 to be embeddable is that $E\,|M_2 - x| \geq E\,|M_1 - x|$ for all x. Can show $E\,|M_1 - x| = \frac{1}{2}(1+x^2)$ for $|x| \leq 1$,

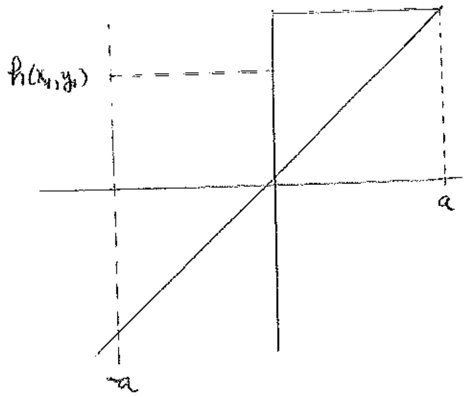


so the smallest mass we can put on a $(a>1)$ is

$$\tfrac{1}{2}\left\{1 - \frac{a-\frac{1}{2}}{a}\right\} = \frac{1}{4a},$$

biggest mass is $\frac{1}{2}$.

(ii) At time 1, we observe $M_1 = x_1$, $\bar{M}_1 = y_1$ and then choose a law to embed up to time 2, using Azema-Yor. The law is completely specified by the height $h(x_1, y_1)$ chosen for the barycentre on $(-a, 0)$. Note that $h(x_1, y_1) \geq x_1 \vee 0$.



We then have

$$P(\bar{M}_{12} > b) = \frac{a + x_1}{a + b \wedge h(x_1, y_1)} \cdot \frac{b \wedge h(x_1, y_1)}{b}$$

for all $x_1 \leq b < a$.

(iii) Let $\psi(x_1, y_1)$ be the joint density of $(M_1, \bar{M}_1)$, so the constraints are

$$\begin{cases} \int_0^1 \psi(x_1, y_1)\, dy_1 = \frac{1}{2} & \forall x_1 \in (-1,1) \\[1ex] \int_{-1}^1 dx_1 \int_v^1 (v - x_1)\, \psi(x_1, y_1)\, dy_1 = 0 & \forall v \in (0,1) \\[1ex] \int_{-1}^1 dx_1 \int_0^1 dy_1\, \psi(x_1, y_1)\, \frac{a + x_1}{a + h(x_1, y_1)} \cdot \frac{h(x_1, y_1)}{a} = P(M_2 = a) \end{cases}$$

If φ is positive increasing, $\varphi(0) = 0$, the Lagrangian form of the problem $\max E\,\varphi(\bar{M}_2)$ subject to these constraints is

$$\int_0^a \varphi'(b)\, db \int_{-1}^1 dx_1 \int_0^1 dy_1\, \psi(x_1, y_1) \left\{ I_{\{y_1 > b\}} + I_{\{y_1 \leq b\}} \frac{a + x_1}{a + b \wedge h(x_1, y_1)} \; \frac{b \wedge h(x_1, y_1)}{b} \right\}$$

$$- \int_{-1}^1 \alpha(x_1)\, dx_1 \int_0^1 \psi(x_1, y_1)\, dy_1 - \int_0^1 \theta_v\, dv \int_{-1}^1 dx_1 \int_v^1 dy_1\, (v - x_1)\, \psi(x_1, y_1)$$

$$- \lambda \int_{-1}^1 dx_1 \int_0^1 dy_1\, \psi(x_1, y_1)\, \frac{a + x_1}{a + h(x_1, y_1)} \cdot \frac{h(x_1, y_1)}{a}$$

Computing first pas

One explicit

Mori

1.

2

Also for $y_1 \le y^*$, $\Lambda = (a+x)\int_0^{y^*} \frac{\phi(v)}{a+v}\,dv - d(x)$

$$= \int_{\ast}^{1} dx_1 \int_0^1 dy_1 \; \psi(x_1,y_1) \Big[\int_0^a \varphi'(b) \Big\{ I_{\{y_1 \geq b\}} + I_{\{y_1 < b\}} \frac{a+x_1}{a+b\wedge h} \cdot \frac{b\wedge h}{b} \Big\} db - \alpha(x_1) - \int_0^{y_1} (v-x_1)\theta_v \, dv$$
$$- \lambda \frac{a+x_1}{a+h} \frac{h}{a} \Big]$$

writing h as an abbreviation of $h(x_1,y_1)$. The set of pairs (x_1,y_1) where the sup of $[\cdots]$ is attained will be the set of points charged by the law of $(M_1,\bar{M}_1)$. (the sup being over h, of course). If we write Λ for the expression in $[\cdots]$, the derivative of this wrto h is

$$a\int_0^a \varphi'(b)\, I_{\{y_1<b\}} I_{\{h<b\}} \frac{a+x_1}{(a+h)^2} \frac{db}{b} - \lambda \frac{a+x_1}{(a+h)^2}.$$

$$= \frac{a+x_1}{(a+h)^2}\Big[-\lambda + a\int_{y_1\vee h}^a \varphi'(b)\frac{db}{b} \Big]$$

Thus if y^* has the property $a\int_{y^*}^a \varphi'(b)\frac{db}{b} = \lambda$, the best choice is

$$\boxed{\begin{aligned} h(x_1,y_1) &= y^* \quad \text{if } y_1 \leq y^* \\ &= x_1 \vee 0 \quad \text{if } y_1 > y^* \end{aligned}}$$

(iv) What can we learn about φ from this? For $y_1 \leq y^*$,

$$\Lambda = \varphi(y_1) + \int_{y_1}^{y^*} \varphi'(b) \frac{a+x_1}{a+b} db - \alpha(x_1) - \int_0^{y_1} (v-x_1)\theta_v \, dv$$

and for $y_1 > y^*$,

$$\Lambda = \varphi(y_1) + \int_{y_1}^a \varphi'(b) \frac{a+x_1}{a+x_1^+}\cdot\frac{x_1^+}{b} db - \alpha(x_1) - \int_0^{y_1}(v-x_1)\theta_v\,dv - \lambda\frac{x_1+a}{x_1^++a}\cdot\frac{x_1^+}{a}$$

$$= \varphi(y_1) + x_1^+\Big(-\int_{y^*}^{y_1}\varphi'(b)\frac{db}{b}\Big) - \alpha(x_1) - \int_0^{y_1}(v-x_1)\theta_v\,dv.$$

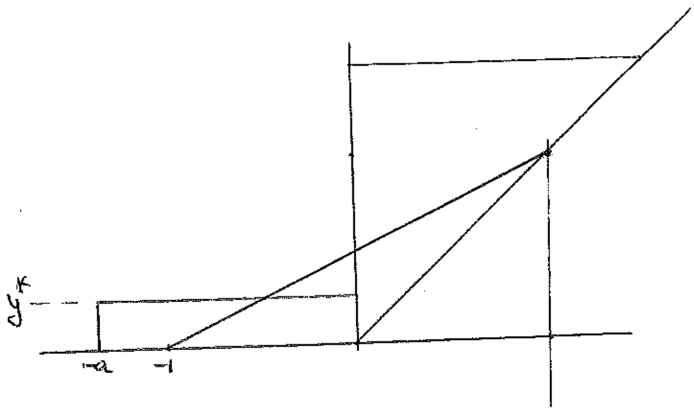
Fixing x_1, we get

$$\frac{\partial\Lambda}{\partial y_1} = (y_1-x_1)\Big\{\frac{\varphi'(y_1)}{a+y_1} - \theta(y_1)\Big\} \qquad (y_1<y^*)$$

$$= (y_1-x_1)\Big\{\frac{\varphi'(y_1)}{y_1} - \theta(y_1)\Big\} - \frac{x_1^-}{y_1}\varphi'(y_1) \qquad (y_1>y^*)$$

The UI property of the mg gives us $\boxed{\varphi'(y_1) = \theta(y_1)(a+y_1) \quad \forall y_1<y^*,}$ and

$$\boxed{\Lambda = \int_0^{y^*} \frac{\varphi'(v)\,dv}{a+v}(a+x_1) - \alpha(x_1) \qquad (y_1\leq y^*)}$$



If we run til hit $y = (x+1)/2$ and then if $\bar{M}_1 < y^*$ we AY til hit the barycentre f^n defined to be y^* in $(-a, 0)$

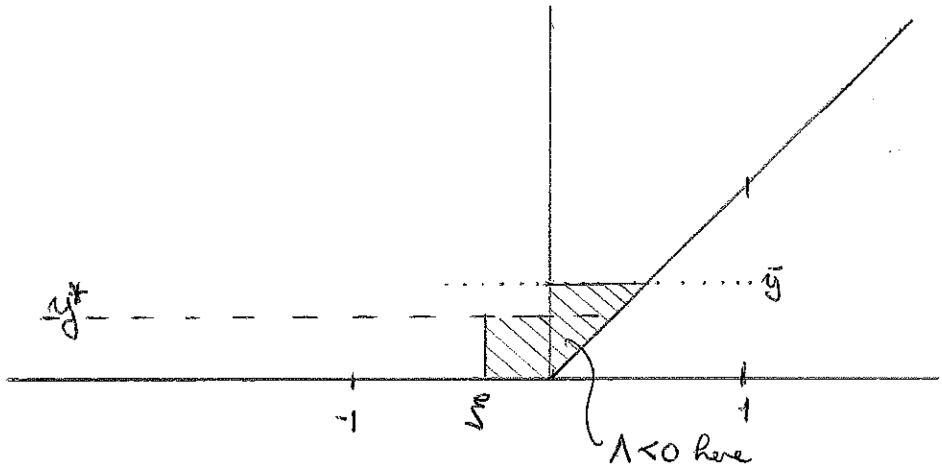
else run til hit $\{-a, 0, a\}$

then $$P(M_2 = -a) = \int_{-1}^{2y^*-1} \frac{dt}{2} \, \frac{y^* - t}{y^* + a} + \int_{2y^*-1}^{0} \frac{dt}{2} \, \frac{-t}{a} = \frac{y + a + 4y^2(y + a - 1)}{4a(y+a)} \qquad (y \equiv y^*)$$

If we had prob^y p of $M_2 = -a$, the barycentre $\rho \equiv b_2(-1) = ap/(1-p)$.

plotting $\rho - y$ for a range of a, y values always produces a non-negative value;

see u/ma/s/maslcgr/Maple/maxmax1.mws.



The picture emerging is this. There is some $\xi \in (-1, 0)$, and $\bar{y} > y^*$ such that in the shaded region $\Lambda < 0$, and there is some function $f : [\xi, 0] \to [y^*, \bar{y}]$ which we might assume is strictly increasing with inverse g, such that $\sup_{x_1} \Lambda(x_1, y_1)$ is achieved for $y_1 \in (y^*, \bar{y})$ at the point $g(y_1)$, and likewise the sup over y_1 is achieved at $f(x_1)$ for $x_1 \in (\xi, 0)$.

Now for $y_1 > y^*$ we have

$$\Lambda = x_1^+ \Big(- \int_{y^*}^{y_1} \varphi'(b) \frac{db}{b} \Big) - \alpha(x_1) + (a + x_1) \int_0^{y^*} \frac{\varphi'(v)}{a+v} \, dv - \int_{y^*}^{y_1} (v - x_1) \, \theta_v \, dv$$
$$+ \int_{y^*}^{y_1} \varphi'(v) \, dv,$$

and if for $x_1 \in (\xi, 0)$ the sup is attained at $f(x_1)$, we learn that

$$\boxed{\theta(y_1) = \frac{\varphi'(y_1)}{y_1 - g(y_1)}, \qquad y^* < y_1 < \bar{y}}$$

and so

$$\alpha(x_1) = (a + x_1) \int_0^{y^*} \frac{\varphi'(v)}{a+v} \, dv + \int_{y^*}^{f(x_1)} \frac{x_1 - g(v)}{v - g(v)} \, \varphi'(v) \, dv \qquad (\xi \leq x_1 \leq 0)$$

which is convex.

In the region $y > \bar{y}$ we have $\theta(y) = \varphi'(y)/y$ and

$$\Lambda = x_1^+ \Big(- \int_{y^*}^{y_1} \varphi'(b) \frac{db}{b} \Big) - \alpha(x_1) + (a + x_1) \int_0^{y^*} \frac{\varphi'(v)}{a+v} \, dv + \int_{y^*}^{\bar{y}} \frac{x_1 - g(v)}{v - g(v)} \, \varphi'(v) \, dv + x_1 \int_{\bar{y}}^{y_1} \varphi'(b) \frac{db}{b}.$$

Thus if we take

$$\alpha(x_1) = (a + x_1) \int_0^{y^*} \frac{\varphi'(v)}{a+v} \, dv + \int_{y^*}^{\bar{y}} \frac{x_1 - g(v)}{v - g(v)} \, \varphi'(v) \, dv - x_1 \int_{y^*}^{\bar{y}} \varphi'(v) \frac{dv}{v}$$

for $x_1 \geq 0$, we find that $\Lambda \equiv 0$ in $x_1 \geq 0$, $y_1 \geq \bar{y}$.

The fact that we're trying to make the law $U[-1, 1]$ determines quite a lot about the solution, but not uniquely.

[PS: The law of M_2 is uniquely specified by the height p of $b_2(\cdot)$ in the interval $(-1, 0)$. Do we have that $p - y^*$ is of constant sign? No; a Maple plot shows this.] Correction

Moving-average knockouts (9/9/97)

Let $S_t \equiv \exp(X_t) \equiv \exp(\sigma W_t + \mu t)$ be a standard log-Brownian share, and suppose we want to compute the price of (say) a European call option which gets knocked out if

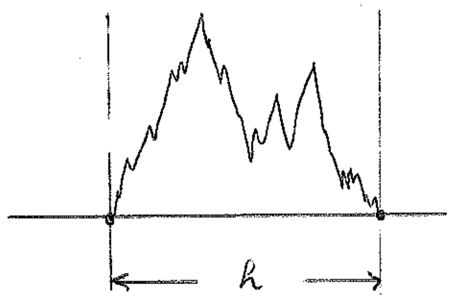
$$A_n \equiv \frac{1}{K+1} \sum_{j=0}^{K} S((n-j)\delta) > a$$

for some $n \leq N \equiv T\delta^{-1}$. It appears impossible to solve this in analytic closed form, but various approximations can be tried.

(i) Approximation I Approximate the maximal value of the moving average

$\tilde{\alpha}_t \equiv \frac{1}{h}\int_{t-h}^{t} S_u\, du$ by $\sup \exp\left[\frac{1}{h}\int_{t-h}^{t} X_u\, du\right]$ and compare with $\sup_t S_t$.

Assuming the max of the MA $\alpha_t \equiv \frac{1}{h}\int_{t-h}^{t} X_u\, du$ happens near the max of X, we shall see something like a Brownian excursion of duration h. The difference between the average of this excursion and its sup should be the discrepancy between the standard up-and-out option and this MAKO option; a crude attempt at pricing just takes a standard up-and-out option with the barrier displaced upward by the appropriate amount.



If ξ_t^h is the standard Brownian excursion of duration h, we have

$$\xi_t^h \overset{\mathcal{D}}{=} \sqrt{h}\, \xi^1\left(\frac{t}{h}\right)$$

by Brownian scaling and

$$\sup_{t \leq h} \xi_t^h - \frac{1}{h}\int_0^h \xi_s^h\, ds \overset{\mathcal{D}}{=} \sqrt{h}\left\{ \sup_{t \leq 1} \xi_t^1 - \int_0^1 \xi_s^1\, ds \right\}$$

$$\overset{\mathcal{D}}{=} \sqrt{h} \int_0^1 \xi_s^1\, ds \equiv \sqrt{h}\, Z,$$

say. The law of Z is not easy to work with, but we can obtain a couple of moments using the representation

$$\xi_t^1 = (1-t)\, R\left(\frac{t}{1-t}\right)$$

where R is a BES(3) process started at 0. Thus we have

$$E\,\xi^1_t = (1-t)\sqrt{\frac{8}{\pi}}\cdot\sqrt{\frac{t}{1-t}}$$

and

$$E\left(\int_0^1 \xi^1_s\,ds\right) = \sqrt{\frac{8}{\pi}}\int_0^1\sqrt{t(1-t)}\,dt = \sqrt{\pi/8}\,.$$

For the second moments, we need to evaluate for $s>0$, $t>0$

$$E\,R_s R_{s+t} = \int_0^\infty dx\int_0^\infty dy\; xy.\;\frac{2x^2}{\sqrt{2\pi s^3}}\,e^{-x^2/2s}\;.\;\frac{y}{x}\left\{e^{-(x-y)^2/2t}-e^{-(x+y)^2/2t}\right\}\frac{1}{\sqrt{2\pi t}}$$

$$= \int_0^\infty dx\int_0^\infty dy\;\frac{x^2y^2}{\pi\sqrt{s^3t}}\,e^{-x^2/2s}\left\{e^{-(x-y)^2/2t}-e^{-(x+y)^2/2t}\right\}$$

$$= \sum_{n\ge0}\int_0^\infty dx\int_0^\infty dy\;\frac{x^2y^2}{\pi\sqrt{s^3t}}\exp\left(-\frac{x^2}{2\tau}-\frac{y^2}{2t}\right).\,2\,.\,\frac{(xy/t)^{2n+1}}{(2n+1)!}\qquad\left[\frac{1}{\tau}\equiv\frac{s+t}{st}\right]$$

$$= \frac{2}{\pi\sqrt{s^3t}}\sum_{n\ge0}\frac{t^{-2n-1}}{(2n+1)!}\,(2\tau)^{n+1}\,(2t)^{n+1}\;\tau t\;((n+1)!)^2$$

$$= \frac{2\tau t^2}{\pi\sqrt{s^3t}}\sum_{n\ge0}\left(\frac{\tau}{t}\right)^{n+1}(n+1).\,8.\,\frac{(n+1)!\;\Gamma(\frac12)}{\Gamma(n+3/2)}$$

$$= \frac{2\tau t^2}{\pi\sqrt{s^3t}}\sum_{n\ge0}\left(\frac{s}{s+t}\right)^{n+1}(8n+8)\,\frac{(n+1)!}{\Gamma(n+3/2)}\;\Gamma(\tfrac12)\,.$$

This looks pretty hard to get further with analytically, though probably we could make progress numerically.

So as a first approximation, we could try an up-and-out European call with the barrier shifted from b to $b\exp(\sqrt{\pi h/8})$.

Approximation 2. Working in discrete time, let $x_n = X(n\delta)$, and try to find the transition density

$$P\left[x_{n+1} \in dy,\ A_{n+1} \leq a \mid x_n = x,\ A_m \leq a\ \forall m \leq n \right].$$

If we could find this exactly, then we could price the knock-out option in the usual way. The transition density won't be amenable to some analytic closed form.

As a crude first-shot, could try computing instead

$$P\left[x_{n+1} \in dy,\ \alpha_{n+1} \leq \log a \mid x_n = x,\ \alpha_m \leq \log a \quad \forall m \leq n \right],$$

where $\alpha_n = \frac{1}{K+1} \sum_{j=0}^{K} x_{n-j}$. This introduces errors into both numerator and denominator in the conditional probability, but they are in the same direction, so should cancel to some extent.

This reduces to reasonably simple computations with multivariate Gaussians.

Lagrangian approach to the max max problem (9/9/97)

Suppose we've been told the joint law of $(M_1, \bar{M}_1)$ (where M is a continuous martingale started at 0), and the laws of M_2; how then do we construct the martingale between times 1 and 2 to make $E\,\varphi(\bar{M}_2)$ as large as possible subject to these conditions? Here, $\varphi(0)=0$, φ is C^1 and strictly increasing, $\bar{M}_t \equiv \sup_{s\le t} M_s$. We also demand that M is UI, of course.

(i) If $\rho(x_1, y_1, x_2, y_2)$ is the joint density of the variables $(M_1, \bar{M}_1, M_2, \bar{M}_{12})$, where $\bar{M}_{12} = \sup_{1\le t\le 2} M_t$, then the constraints to be satisfied are

$$\iint dx_2\, dy_2\, \rho(x_1,y_1,x_2,y_2) = \text{given joint density } f_1(x_1, y_1) \qquad \forall x_1, y_1$$

$$\iiint dx_1\, dy_1\, dy_2\, \rho(x_1,y_1,x_2,y_2) = f_2(x_2), \quad \text{given density of } M_2 \qquad \forall x_2$$

$$\int_a^\infty dy_2 \int dx_2\; (a-x_2)\,\rho(x_1,y_1,x_2,y_2) = 0 \qquad \forall x_1, y_1, \quad \forall a \ge x_1,$$

the last being the condition for $\bar{M}_{12}$ to be the max of mg with terminal r.v. M_2. The Lagrangian form of the problem is to obtain

$$\sup_{\rho\ge 0} \iiiint dx_1\, dy_1\, dx_2\, dy_2\; \rho(x_1,y_1,x_2,y_2)\; \Lambda(x_1,y_1,x_2,y_2)$$

where

$$\boxed{\Lambda \equiv \varphi(y_1 \vee y_2) - \alpha(x_1,y_1) - \beta(x_2) - \int_{x_1}^{y_2} \theta(a, x_1, y_1)(a - x_2)\, da}$$

If we can choose the multipliers α, β, θ in such a way that

(a) $\displaystyle\sup_{x_i, y_i} \Lambda = 0$

(b) some ρ satisfying the constraints is concentrated on the set where $\Lambda = 0$,

then we're finished.

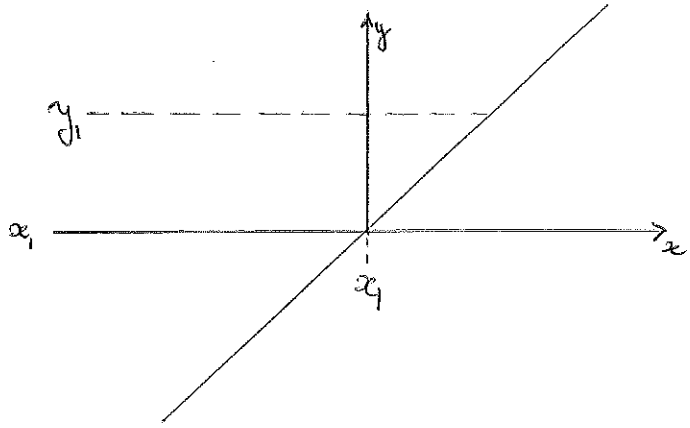
(ii) Let's begin by making the following assumptions about the multipliers:

$$\boxed{\beta(\cdot) \text{ is convex}, \qquad \theta \ge 0}$$

These assumptions will be justified if they work!

So let's fix x_1, y_1 and consider the maximisation over x_2, y_2 (which we'll write x, y for brevity). We're considering

$$\varphi(y, \vee y) - \beta(x) - \int_{x_1}^{y} \theta(a)\,(a-x)\,da \qquad (y \geq x_1,\ y \geq x)$$



If we assume that β is strictly convex, then for each $y \geq x_1$, there is a unique x maximising the above expression. More precisely, if we define

$$\tilde{\beta}(\lambda) \equiv \sup\left[-\beta(x) + \lambda x\right]$$

and $I(\lambda) \equiv (\beta')^{-1}(\lambda)$, then the maximising value of x will be

$$I\left(\int_{x_1}^{y} \theta_a\,da\right) \wedge y.$$

(iii) It turns out to be easier to describe the solution in terms of some increasing function $\eta : \mathbb{R} \to \mathbb{R}^+$, such that $\eta(x) > x \quad \forall x$. Let ξ denote the inverse function to η. The interpretation of η is that <u>in the region</u> $y > y_1$, <u>the law of $(M_2, \bar{M}_{12})$ will be concentrated on the graph of η, and that we use the same η, whatever (x_1, y_1).</u>

If the law is concentrated on the graph of η, we have to have for $y > y_1$

$$\varphi'(y) - \theta(y, x_1, y_1)(y - x) = 0$$

at $y = \eta(x)$, which tells us about θ:

$$\boxed{\theta(y, x_1, y_1) = \frac{\varphi'(y)}{y - \xi(y)} \qquad (y > y_1)}$$

This is obtained by fixing x and considering the maximisation over y. If we similarly fix y and consider maximising over x, we have that

$$-\beta'(x) + \int_{x_1}^{y} \theta_a\,da = 0$$

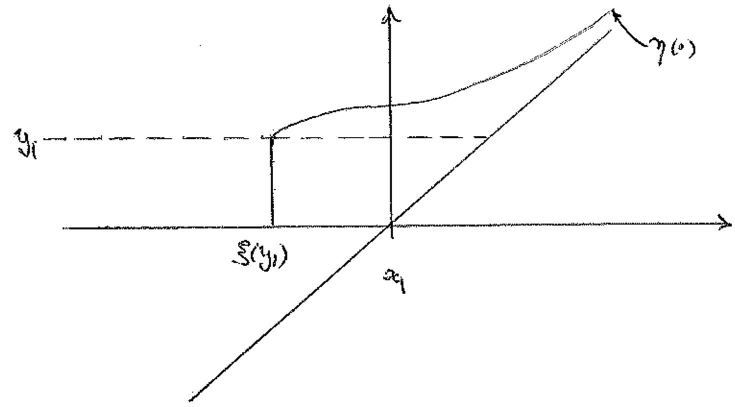
at $x = \xi(y)$, so that

$$\boxed{\beta''(x) = \frac{\eta'(x)\,\varphi'(\eta(x))}{\eta(x) - x}}$$

This tells us (in terms of the function η) what β must be (to within an unimportant linear function. We still need to specify the multiplier function θ in the region $y \in [x_1, y_1]$, which we do as follows.

If $\eta(x_1) \geq y_1$, then $\theta(y, x_1, y_1) \equiv 0$ for $y \in (x_1, y_1]$, and θ puts a point mass on x_1 of magnitude δ, where δ is chosen so that

$$\delta = \beta'(\xi(y_1))$$



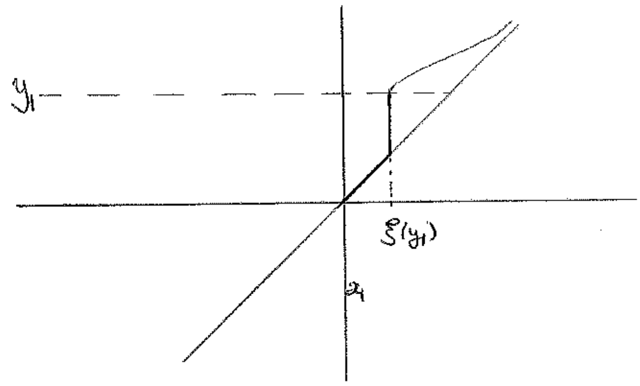
On the other hand, if $\eta(x_1) < y_1$, we set $\theta(y, x_1, y_1) \equiv 0$ for $y \in (\xi(y_1), y_1]$, and then in $[x_1, \xi(y_1)]$ we give θ a mass $\beta'(x_1)$ at x_1 and otherwise choose θ to give

$$\beta'(x) = \int_{x_1}^{x} \theta(da) \qquad \forall x \in [x_1, \xi(y_1)]$$

This ensures that the maximal value of Λ is attained all along the graph

$$\{(x,x) : x_1 \leq x \leq \xi(y_1)\} \cup \{(\xi(y_1), y) : \xi(y_1) \leq y \leq y_1\} \cup \{(x, \eta(x)) : x \geq \xi(y_1)\}$$

and the picture shows us what happens: if we're at x_1, y_1, where $\eta(x_1) < y_1$, the best thing is to stop at once!



[Aside: our example where M_2 had the three point distribution didn't quite fit this description, but we can easily see how it's included. Indeed, in that example, the increasing function η is a step function, so there are intervals where η' (and therefore β'') vanishes. If we choose θ as above, it turns out that $\theta = 0$ in $(x_1, y_1]$, and the maximal value of Λ is attained in all of $\{(x,y) : \alpha \leq x \leq \xi(y_1),\ x \vee x_1 \leq y \leq y_1\}$, where β'' vanishes in $(\alpha, \xi(y_1))$.]

To sum up, the optimal rule is to stop at

$$\boxed{\tau \equiv \inf\{t > 1 : M_t < \xi(\bar{M}_t)\}}$$

To complete the proof, we have to show how to choose ξ so as to make $M_2 \sim f_2$. The above argument tells us how to construct the Lagrange multipliers from which optimality follows.

Parisian-style knockouts (11/9/97)

(i) Let $X_t \equiv \sigma W_t + \mu t$ be log-price, which we want to see remaining in some interval $[a, b]$, where $a < X_0 = 0 < b$. Indeed, if

$$A_t = t - \int_0^t I_{[a,b]}(X_u)\,du \equiv \int_0^t \rho(X_u)\,du$$

the Parisian option knocks out once A reaches some level. How to determine the law of X_t on the event that knockout hasn't happened?

We have that

$$E^x\left[\int_0^\infty \exp(-\lambda t - \gamma A_t)\, f(X_t)\,dt\right] \equiv \varphi(x)$$

solves

$$(\lambda + \gamma\rho - \tfrac{1}{2}\sigma^2 D^2 - \mu D)\,\varphi = f$$

and from this we could in principle find the law of X_t on the event of no knock out, just by solving the de and inverting the two Laplace transforms... it's not so terrible numerically.

(ii) Fix $y \in (a, b)$, and let's proceed to compute φ, taking $f = \delta_y$. The solution can thus be expressed as

$$\varphi(x) = c_0 \exp\left[\alpha_0 (x-a)\right], \quad x \leq a; \qquad \left[\alpha_0 \equiv \frac{\sqrt{\mu^2 + (2\lambda+2\gamma)\sigma^2} - \mu}{\sigma^2}\right]$$

$$= c_1 \exp\left[\beta_0 (x-y)\right] + c_2 \exp\left[-\beta_1 (x-y)\right], \quad a \leq x \leq y;$$

$$\left[\begin{aligned} \beta_0 &= \frac{\sqrt{\mu^2 + 2\lambda\sigma^2} - \mu}{\sigma^2} \\ \beta_1 &= \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \end{aligned}\right]$$

$$= c_3 \exp\left[\beta_0 (x-y)\right] + c_4 \exp\left[-\beta_1 (x-y)\right], \quad y \leq x \leq b;$$

$$= c_5 \exp\left[-\alpha_1 (x-b)\right], \quad x \geq b \qquad \left[\alpha_1 = \frac{\mu + \sqrt{\mu^2 + (2\lambda+2\gamma)\sigma^2}}{\sigma^2}\right]$$

with the appropriate matching conditions:

$$c_0 = c_1 \exp(\beta_0(a-y)) + c_2 \exp(-\beta_1(a-y))$$

$$\alpha_0 c_0 = \beta_0 c_1 \exp(\beta_0(a-y)) - \beta_1 c_2 \exp(-\beta_1(a-y))$$

$$c_1 + c_2 = c_3 + c_4$$

$$\beta_0 c_1 - \beta_1 c_2 - \beta_0 c_3 + \beta_1 c_4 = 2/\sigma^2$$

$$c_5 = c_3 \exp(\beta_0(b-y)) + c_4 \exp(-\beta_1(b-y))$$

$$-\alpha_1 c_5 = \beta_0 c_3 \exp(\beta_0(b-y)) - \beta_1 c_4 \exp(-\beta_1(b-y))$$

We can solve this using Maple. We don't actually want the whole solution; we really would be quite happy just to know

$$\varphi(0) = c_1 e^{-\beta_0 y} + c_2 e^{\beta_1 y} \qquad \text{if } y>0$$

$$= c_3 e^{-\beta_0 y} + c_2 e^{\beta_1 y} \qquad \text{if } y<0$$

If $y>0$, then I get

$$\varphi(0) = \frac{1}{\theta}\Big\{ (\alpha_0+\beta_1)(\beta_1-\alpha_1)\, e^{\beta_1(2y-a-b)-\beta_0 y} + (\alpha_0+\beta_1)(\beta_0+\alpha_1)\, e^{\beta_1(y-a)+\beta_0(b-2y)}$$

$$+ (\beta_0+\alpha_1)(\beta_0-\alpha_0)\, e^{\beta_0(a+b-2y)+\beta_1 y} + (\alpha_0-\beta_0)(\alpha_1-\beta_1)\, e^{\beta_0(a-y)+\beta_1(2y-b)}\Big\}.$$

$$\Big\{(\alpha_0-\beta_0)(\alpha_1-\beta_1)\, e^{\beta_0(a-y)+\beta_1(y-b)} + (\alpha_0+\beta_1)(\alpha_1+\beta_0)\, e^{\beta_0(b-y)+\beta_1(y-a)}\Big\}^{-1}$$

Maximum maximum: the dual problem (15/9/97)

Returning to the treatment of pp 8–10, the Lagrangian which we maximise is in fact

$$\iiiint \rho(x_1,y_1,x_2,y_2)\ \Lambda(x_1,y_1,x_2,y_2) + \int dx_1 \int dy_1\ \alpha(x_1,y_1) f_1(x_1,y_1) + \int \beta(x_2) f_2(x_2)\,dx_2 ,$$

where the multipliers α, β are computed from the increasing function $\eta : \mathbb{R} \to \mathbb{R}^+$ by the recipe outlined there. In particular, we showed that we should have

$$\beta''(x) = \eta'(x)\varphi'(\eta(x)) / (\eta(x) - x),$$

so we may assume that $\beta'(-\infty) = \beta(-\infty) = 0$, and

$$\boxed{\beta'(x) = \int_{-\infty}^{x} \frac{\eta'(s)\varphi'(\eta(s))}{\eta(s) - s}\,ds, \qquad \beta(x) = \int_{-\infty}^{x} (x-s)\frac{\eta'(s)\varphi'(\eta(s))}{\eta(s) - s}\,ds .}$$

How do we find the multiplier α? For x_1, y_1 fixed, we would have to have

$$\sup_{x_2,y_2}\left\{ \varphi(y_1 \vee y_2) - \alpha(x_1,y_1) - \beta(x_2) - \int_{x_1}^{y_2} \theta(a,x_1,y_1)(a-x_2)\,da \right\} = 0$$

and this determines α. There are two cases to consider:

Case 1: $\xi(y_1) \leq x_1$.

In this case, θ does not charge $(x_1, y_1]$ but puts mass $\beta'(\xi(y_1))$ at x_1. The maximum of Λ will be attained in various places, among them $x_2 = \xi(y_1), y_2 = y_1$, so we learn that

$$\boxed{\alpha(x_1,y_1) = \varphi(y_1) - \beta(\xi(y_1)) - \beta'(\xi(y_1))(x_1 - \xi(y_1))}$$

Case 2: $\xi(y_1) > x_1$. This time, the maximum will be attained (with x_1, y_1 fixed) at (x_1, x_1) among other places. The mass which θ assigns to x_1 is then $\beta'(x_1)$, and we obtain

$$\boxed{\alpha(x_1,y_1) = \varphi(y_1) - \beta(x_1)}$$

Combining the two cases gives us

$$\boxed{\begin{aligned} \alpha(x_1,y_1) &= \varphi(y_1) - \beta(x_1 \wedge \xi(y_1)) - (x_1 - \xi(y_1))^+ \beta'(\xi(y_1)) \\ &= \varphi(y_1) - \int_{-\infty}^{x_1} \beta'(s \wedge \xi(y_1))\,ds. \end{aligned}}$$

The dual problem is therefore that of picking the function η so as to give

We also have

$$\int_{-\infty}^{\xi} dv\ h(v,y) = -\varphi'(y) \int_{-\infty}^{\xi} dx \left\{ f_2(x) - \bar{f}_1(x,y) \right\} + \varphi'(y) \int_{\xi}^{\infty} \frac{x-y}{y-\xi} \left\{ f_2(x) - \bar{f}_1(x,y) \right\} dx$$

$$\min \left\{ - \int dx_1 \int dy_1\ f_1(x_1,y_1) \int_{-\infty}^{x_1} \beta'(s \wedge \xi(y_1))\, ds + \int \beta(x_2) f_2(x_2)\, dx_2 \right| + E\varphi(\bar{M}_1)$$

$$= \min \left\{ - \int dx_1 \int dy_1\ f_1(x_1,y_1) \int_{-\infty}^{x_1} ds \int_{\infty}^{s \wedge \xi(y_1)} \frac{\eta'(z)\varphi'(\eta(z))}{\eta(z)-z}\, dz \right.$$

$$\left. + \int dx_2\ f_2(x_2) \int_{-\infty}^{x_2} (x_2 - z) \frac{\eta'(z)\varphi'(\eta(z))}{\eta(z)-z}\, dz \right\} + E\varphi(\bar{M}_1)$$

$$= \min \int_{-\infty}^{\infty} \frac{\eta'(z)\varphi'(\eta(z))}{\eta(z)-z}\, dz \left\{ \int_z^{\infty} (x-z) f_2(x)\, dx - \int_z^{\infty} dx_1 \int_{\eta(z)}^{\infty} dy_1\ (x_1-z) f_1(x_1,y_1) \right\} + E\varphi(\bar{M}_1)$$

$$= \min \int_0^{\infty} \frac{\varphi'(y)}{y-\xi(y)}\, dy \left\{ \int_{\xi(y)}^{\infty} (x-\xi(y)) f_2(x)\, dx - \int_{\xi(y)}^{\infty} dx_1 \int_y^{\infty} dy_1\ (x_1 - \xi(y)) f_1(x_1,y_1) \right\} + E\varphi(\bar{M}_1)$$

$$= \min \int_0^{\infty} \varphi'(y) \left[P(\bar{M}_1 \le y) + \int_{-\infty}^{\xi(y)} \left(\int_v^{\infty} (x-y)\{f_2(x) - \bar{f}_1(x,y)\}\, dx \right) \frac{dv}{(y-v)^2} \right] dy + E\varphi(\bar{M}_1)$$

So we're trying to pick increasing ξ to minimise $\left[\bar{f}_1(x,y) = \int_y^{\infty} f_1(x,s)\, ds \right]$

$$\boxed{\begin{aligned} &\int_0^{\infty} dy \int_{-\infty}^{\xi(y)} dv\ h(v,y), \qquad h(v,y) \equiv \frac{\varphi'(y)}{(y-v)^2} \int_v^{\infty} (x-y)\{f_2(x) - \bar{f}_1(x,y)\}\, dx \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \frac{\varphi'(y)}{(y-v)^2} \left\{ -y - \int_{-\infty}^{v} (x-y)\{f_2(x) - \bar{f}_1(x,y)\}\, dx \right\} \end{aligned}}$$

using the fact that f_1 is the joint density of $(M_1, \bar{M}_1)$ for the last equivalence.

Another view of the cost of liquidity (16/9/97)

Let's suppose an investor can choose between bank account with constant rate of interest r and a share with return $\sigma dW + \alpha dt$. If the investor aims to

$$\max E\, U(w_{\tau_1})$$

where $U(x) = x^{1-R}/(1-R)$, $\tau_1 \sim \exp(\lambda)$, and w is the wealth process obtained by splitting wealth at time 0 into π in bank, $1-\pi$ in risky asset, then the problem is

$$(1) \qquad \max_{0\le\pi\le1} E\left(\pi e^{r\tau_1} + (1-\pi)\exp\{\sigma W(\tau_1) + (\alpha - \tfrac12\sigma^2)\tau_1\}\right)^{1-R} / (1-R)$$

If we follow the Merton policy, we shall obtain from initial wealth 1

$$E\, U(\tau_1) = \int_0^\infty \frac{\lambda e^{-\lambda t}\,dt}{1-R} \exp\left\{-(R-1)\left(r + \frac{(\alpha-r)^2}{2\sigma^2 R}\right)t\right\}$$

$$(2) \qquad = (1-R)^{-1}\,\lambda\left\{\lambda + (R-1)\left(r + \frac{(\alpha-r)^2}{2\sigma^2 R}\right)\right\}^{-1},$$

and we should be comparing with this. Looking at (1), we see we need to understand

$$E\left[e^{r(1-R)\tau_1}\left(\pi + (1-\pi)\exp\{\sigma W(\tau_1) + (\alpha - r - \tfrac12\sigma^2)\tau_1\}\right)^{1-R}\right]$$

$$= \frac{\lambda}{\beta} E\left[\left(\pi + (1-\pi)\exp\{\sigma W_T + (\alpha - r - \tfrac12\sigma^2)T\}\right)^{1-R}\right] \qquad \left[\begin{array}{l}\beta \equiv \lambda - r(1-R), \\ T \sim \exp(\beta)\end{array}\right]$$

$$= \frac{\lambda}{\beta} E\left[\left(\pi + (1-\pi)\exp(\sigma W_T + \kappa T)\right)^{1-R}\right], \text{ say.} \qquad \left[\kappa \equiv \alpha - r - \tfrac12\sigma^2\right]$$

If we abbreviate $Y \equiv \exp(\sigma W_T + \kappa T)$, then the moments of Y are easy to find:

$$m_n \equiv E\, Y^n = \frac{\beta}{\beta - n\kappa - \frac12 n^2\sigma^2},$$

assuming the denominator is positive. Thus if $a \equiv \pi + (1-\pi)m_1$ we are considering

$$\frac{\lambda}{\beta} E\left[\left(a + (1-\pi)(Y - m_1)\right)^{1-R}\right] \equiv \frac{\lambda}{\beta} E\left[(a+Z)^{1-R}\right] \qquad \text{for short}$$

$$= \frac{\lambda}{\beta} E\left[\sum_{j=0}^{N-1} \frac{Z^j}{j!}\frac{\Gamma(2-R)}{\Gamma(2-R-j)} a^{1-R-j} + \frac{\Gamma(2-R)}{\Gamma(2-R-j)}\frac{Z^N}{N!}(a+\theta Z)^{1-R-N}\right]$$

by Taylor. The remainder term is at worst $\pi^{1-R-N}\dfrac{\Gamma(2-R)}{\Gamma(2-R-N)} E\left(Z^N/N!\right)\cdot\dfrac{\lambda}{\beta}$.

Let's take this up to $N=3$ to begin with. We have

$$E Z^2 = (1-\pi)^2 \frac{\beta(\beta\sigma^2 + (\kappa+\frac{1}{2}\sigma^2)^2)}{(\beta - 2\kappa - 2\sigma^2)(\beta - \kappa - \frac{1}{2}\sigma^2)^2}$$

and taking this up to the first 3 terms of the binomial expansion, we have to maximise

$$\frac{\lambda}{\beta}\left[a^{1-R} - \frac{R(1-R)}{2} a^{1-R-2} (1-\pi)^2 \frac{\beta(\beta\sigma^2 + (\kappa+\frac{1}{2}\sigma^2)^2)}{(\beta-2\kappa-2\sigma^2)(\beta-\kappa-\frac{1}{2}\sigma^2)^2} \right]$$

over choice of π. This can be done with Maple; it is solving a quadratic. We obtain an expression for the optimising π which expands as

$$1 - \frac{\alpha - r}{\sigma^2 R} + \frac{1}{\beta}(2\kappa+\sigma^2)\left\{(21\sigma^4 + 28\kappa\sigma^2 + 4\kappa^2)R - 3(2\kappa+\sigma^2)^2\right\} / 16\sigma^4 R^2$$

The snag with this is that it is always going to be the Merton proportion plus a correction — and we know that there have to be situations where this is simply incorrect — when the Merton proportion is not in [0, 1], for example!

Detecting a trend in tick data (24/4/97)

(i) Suppose that there's an underlying price process of the form

$$\xi_t = \int_0^t x_u \, du$$

where $dx_u = \sigma \, dW_u - \beta(x_u - \mu)\, dt$, but we only observe it at a certain discrete set of times (τ_i), independent of W, and the observations are noisy, so we see

$$Y_i = \xi(\tau_i) + \varepsilon_i \qquad (\varepsilon_i \text{ are IID } N(0, \sigma_\varepsilon^2)).$$

If $(x(\tau_n), \xi(\tau_n))^T \sim N(\mu_n, V_n)$ given $\mathcal{Y}_n$, how does this update?

Starting at (x_0, ξ_0) we have

$$\begin{cases} x_t = \mu + (x_0 - \mu) e^{-\beta t} + \sigma \int_0^t e^{-\beta(t-s)} \, dW_s \\ \xi_t = \xi_0 + \mu t + (x_0 - \mu) \dfrac{1 - e^{-\beta t}}{\beta} + \sigma \int_0^t \dfrac{1 - e^{-\beta(t-u)}}{\beta} \, dW_u \end{cases}$$

so that

$$E x_t = \mu + (x_0 - \mu) e^{-\beta t}, \qquad E \xi_t = \xi_0 + \mu t + (x_0 - \mu) \frac{1 - e^{-\beta t}}{\beta},$$

$$\operatorname{var}(x_t) = \sigma^2 \frac{1 - e^{-2\beta t}}{2\beta}, \qquad \operatorname{var}(\xi_t) = \frac{\sigma^2}{\beta^2} \left[t - \frac{2(1 - e^{-\beta t})}{\beta} + \frac{1 - e^{-2\beta t}}{2\beta} \right],$$

$$\operatorname{cov}(x_t, \xi_t) = \frac{\sigma^2}{2\beta^2} \left(1 - e^{-\beta t}\right)^2.$$

We may therefore write

$$\begin{pmatrix} x_t \\ \xi_t \end{pmatrix} = a(t) + x_0 \, b(t) + Z(t) + \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}$$

where

$$a(t) = \begin{pmatrix} \mu(1 - e^{-\beta t}) \\ \mu t - \mu \left(\dfrac{1 - e^{-\beta t}}{\beta} \right) \end{pmatrix}, \qquad b(t) = \begin{pmatrix} e^{-\beta t} \\ \dfrac{1 - e^{-\beta t}}{\beta} \end{pmatrix}$$

and

$$\operatorname{cov} Z(t) = \frac{\sigma^2}{2\beta^2} \begin{bmatrix} \beta(1 - e^{-2\beta t}) & (1 - e^{-\beta t})^2 \\ (1 - e^{-\beta t})^2 & 2t - 4\left(\dfrac{1 - e^{-\beta t}}{\beta}\right) + \dfrac{1 - e^{-2\beta t}}{\beta} \end{bmatrix} \equiv \Sigma(t)$$

$E Z_t = 0.$

Thus if we know how long we wait for the next signal, τ, say, we get

$$\mathcal{L}\left(\begin{pmatrix} x(\tau) \\ \xi(\tau) \end{pmatrix} \Big| \mathcal{Y}_0\right) \sim N\left(\mu_0^1 \, b(\tau) + \begin{pmatrix} 0 \\ \mu_0^2 \end{pmatrix} + a(\tau), \; \Sigma(\tau) + C(\tau) V_0 C(\tau)^T \right)$$

$$= N\left(a(\tau) + C(\tau)\mu_0 \,,\; \Sigma(\tau) + C(\tau) V_0 C(\tau)^T \right),$$

where $$C(\tau) \equiv \begin{pmatrix} e^{-\beta\tau} & 0 \\ \beta^{-1}(1-e^{-\beta\tau}) & 1 \end{pmatrix}.$$

If we write $V_{01}(\tau)$ for the covariance matrix $\Sigma(\tau) + C(\tau) V_0 C(\tau)^T$, then we shall have

$$\mathcal{L}\left(\begin{pmatrix} x(\tau) \\ \xi(\tau) \\ Y_1 \end{pmatrix} \Big| \mathcal{Y}_0\right) = N\left(\left(\begin{array}{c} a(\tau) + C(\tau)\mu_0 \\ \hline e_1^T(a(\tau)+C(\tau)\mu_0) \end{array}\right), \left(\begin{array}{c|c} V_{01}(\tau) & \begin{matrix} v_{01}^{x\xi}(\tau) \\ v_{01}^{\xi\xi}(\tau) \end{matrix} \\ \hline v_{01}^{x\xi}(\tau), v_{01}^{\xi\xi}(\tau) & v_{01}^{\xi\xi}(\tau) + \sigma_\varepsilon^2 \end{array}\right) \right)$$

where $v_{01}^{\xi\xi}(\tau)$ is the (ξ,ξ) entry of $V_{01}(\tau)$; abbreviate the vector $(v^{x\xi}, v^{\xi\xi})$ to v, and we get that

$$\boxed{\mathcal{L}\left(\begin{pmatrix} x(\tau) \\ \xi(\tau) \end{pmatrix} \Big| \mathcal{Y}_1\right) = N\left(a(\tau) + C(\tau)\mu_0 + v \cdot \frac{Y_1 - e_1^T(a(\tau)+C(\tau)\mu_0)}{v_{01}^{\xi\xi}(\tau) + \sigma_\varepsilon^2}, \; V_{01}(\tau) - \frac{v v^T}{v_{01}^{\xi\xi}(\tau) + \sigma_\varepsilon^2} \right)}$$

(ii) How are we going to estimate the parameters of the model? If we denote for $\lambda > 0$

$$m_n^j(\lambda) \equiv \sum_{r \le n} (\tau_n - \tau_r)^j \exp(-\lambda(\tau_n - \tau_r))$$

then it is easy to keep $m_n^0(\lambda)$, $m_n^1(\lambda)$, ... updated. Now consider

$$\eta_n(\lambda) \equiv \sum_{r \le n} (Y_n - Y_r) \exp(-\lambda(\tau_n - \tau_r)) \equiv Y_n \cdot m_n^0(\lambda) - \sum_{r \le n} Y_r \exp(-\lambda(\tau_n - \tau_r)),$$

which will also be easy to update. Assuming the underlying OU process is in equilibrium,

$$\boxed{E\,\eta_n(\lambda) = \mu \, m_n^1(\lambda),}$$

and this suggests a way to estimate μ. How about the other parameters of the problem? For notational simplicity, assume $n=0$, $t_j \equiv -\tau_{-j}$, $\xi_j \equiv \xi(0)-\xi(\tau_{-j})$, $\tau_n = 0$ and consider

$$\psi(\lambda) \equiv \sum_{j\geq 0} e^{-\lambda t_j}(Y_{-j}-Y_0)^2 = \sum_{j\geq 0} e^{-\lambda t_j}(\xi_j + \varepsilon_{-j} - \varepsilon)^2$$

so that

$$E\psi(\lambda) = \sum_{j\geq 1} 2\sigma_\varepsilon^2 e^{-\lambda t_j} + E\sum_{j\geq 0} e^{-\lambda t_j}\xi_j^2$$

$$= \sum_{j\geq 1} 2\sigma_\varepsilon^2 e^{-\lambda t_j} + \sum_{j\geq 0} e^{-\lambda t_j}\left\{\mu^2 t_j^2 + \frac{\sigma^2}{\beta^2} f(t_j)\right\}$$

where $f(t) \equiv t - 2(1-e^{-\beta t})\beta^{-1} + (1-e^{-2\beta t})/2\beta \doteq t - 3/2\beta$ for t large.

Hence if

$$\boxed{\psi_n(\lambda) \equiv \sum_{i\leq n} e^{-\lambda(\tau_n - \tau_i)}(Y_n - Y_i)^2}$$

we have

$$\boxed{E\,\psi_n(\lambda) \doteq 2\sigma_\varepsilon^2\{m_n^0(\lambda)-1\} + \mu^2 m_n^2(\lambda) + \frac{\sigma^2}{\beta^2}\left\{m_n^1(\lambda) - \frac{3}{2\beta} m_n^0(\lambda)\right\}.}$$

Perhaps a better approach is to use $\hat\mu_n(\lambda) \equiv \eta_n(\lambda)/m_n^1(\lambda)$ as our estimator of μ, and try to take that contribution out of ψ. Specifically, we consider

$$\tilde\psi(\lambda) \equiv \sum_{j\geq 0} e^{-\lambda t_j}\left(Y_{-j} - Y_0 - \hat\mu(\lambda).t_j\right)^2$$

$$= \sum_{j\geq 0} e^{-\lambda t_j}\left(\xi_j + \varepsilon_{-j} - \varepsilon_0 - \frac{t_j}{m^1(\lambda)}\cdot\sum_{i\geq 0} e^{-\lambda t_i}(\xi_i + \varepsilon_i - \varepsilon_0)\right)^2$$

$$= \sum_{j\geq 0} e^{-\lambda t_j}\left(\xi_j - \frac{t_j}{m^1(\lambda)}\sum_{i\geq 0} e^{-\lambda t_i}\xi_i\right)^2 + \sum_{j\geq 0} e^{-\lambda t_j}\left(\varepsilon_{-j} - \varepsilon_0 - \frac{t_j}{m^1(\lambda)}\sum_{i\geq 0}(\varepsilon_{-i}-\varepsilon_0)e^{-\lambda t_i}\right)^2$$

+ zero-mean r.v.

More on the maximum maximum (30/9/97)

Recall that our aim is to find some increasing function $\xi : (0,\infty) \to \mathbb{R}$ such that $\xi(y) \le y$ and

$$\boxed{\int_0^\infty \frac{\varphi'(y)\,dy}{y-\xi(y)} \int_{\xi(y)}^\infty (x-\xi(y))\, g(x,y)\,dx \;\equiv \Phi(\xi)}$$

is minimised, where $g(x,y) \equiv f_2(x) - \bar{f}_1(x,y) \equiv f_2(x) - \int_y^\infty f_1(x,s)\,ds$. We may also write alternatively

$$\boxed{\Phi(\xi) = \int_0^\infty dy \int_{-\infty}^{\xi(y)} h(v,y)\,dv,} \qquad h(v,y) \equiv \frac{\varphi'(y)}{(y-v)^2}\int_v^\infty (x-y)\, g(x,y)\,dx.$$

(i) If we were to take the first form of Φ (or the second) and do an unconstrained minimisation w.r.t. $\xi(y)$ for each y, we'd have

$$\frac{\partial}{\partial \xi}\left[\frac{1}{y-\xi}\int_\xi^\infty (x-\xi)\,g(x,y)\,dx\right] = \frac{1}{(y-\xi)^2}\int_\xi^\infty (x-y)\,g(x,y)\,dx$$

so we shall be looking among the values ξ for which

$$\boxed{\int_\xi^\infty (x-y)\,g(x,y)\,dx = 0.}$$

Are there any? As $\xi \uparrow y$, we get

$$\int_y^\infty (x-y)\,g(x,y)\,dx = E(M_2-y)^+ - E\left((M_1-y)^+ : \bar{M}_1 \ge y\right)$$
$$= E(M_2-y)^+ - E(M_1-y)^+ \quad \ge 0$$

and as $\xi \to -\infty$ we get limit $-y < 0$, so there will be zeros (or at least places where the f^n jumps across 0, which is as good). Notice that the derivative is <0 for ξ large negative, and ≥ 0 at y, so we don't get the inf at $-\infty$, or at y (except in the degenerate case $E(M_2-y)^+ = E[(M_1-y)^+]$, which implies that $\{M_{1+t} : t \ge 0\}$ stops as soon as it hits y).

(ii) Another remark is that for v fixed

$$\frac{\partial}{\partial y}\int_v^\infty \frac{x-y}{y-v}\, g(x,y)\,dx = -\int_v^\infty \frac{x-v}{(y-v)^2}\, g(x,y)\,dx + \int_v^\infty \frac{x-y}{y-v}\, f_1(x,y)\,dx$$

$$= \frac{-1}{(y-v)^2}\left\{E\left((M_2-v)^+\right) - E\left((M_1-v)^+ ; \bar{M}_1 \ge y\right)\right\} - \int_v^y \frac{y-x}{y-v} f_1(x,y)\,dx$$

and this is always ≤ 0 as before, and <0 if $y > v$, in fact. Thus for fixed v,

$\frac{1}{\varphi'(y)}(y-v)\, h(v,y)$ <u>is strictly decreasing</u>

and so $h(v,y)$ is positive below some curve, and negative above it.

(iii) (21/10/97). We are trying to characterise the $\xi = \xi(y) \equiv \xi_y$ which minimises

$$\frac{1}{y-\xi}\; c(\xi,y) \;\equiv\; \frac{1}{y-\xi}\{c_2(\xi) - c_1(\xi,y)\},$$

where

$$c_2(\xi) \equiv \int_\xi^\infty (x-\xi)\, f_2(x)\,dx = \int_\xi^\infty P(M_2 \geq v)\,dv$$

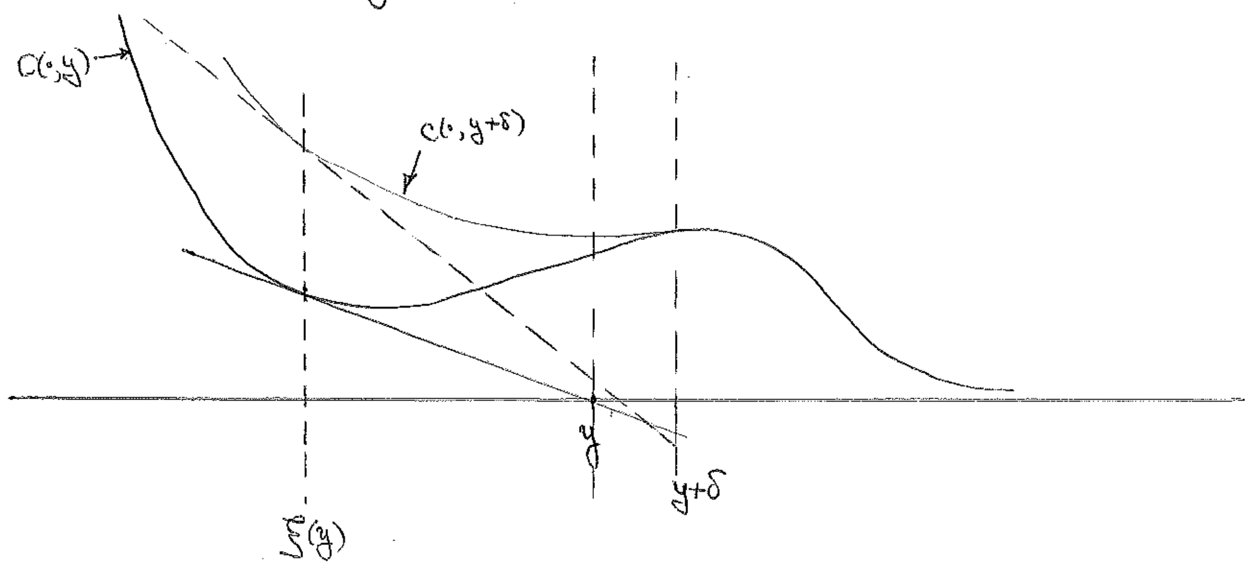
$$c_1(\xi,y) \equiv \int_\xi^\infty (x-\xi)\, \tilde{f}_1(x,y)\,dx = \int_\xi^\infty P(M_1 \geq v,\ \bar{M}_1 \geq y)\,dv$$

Notice that $c_2(\cdot)$ and $c_1(\cdot,y)$ are positive, convex, decreasing, $c(\xi,\cdot)$ is increasing for each ξ, and $0 \leq c_2(\xi) - c_1(\xi,0) \leq c_2(\xi)$ $\forall \xi$. Notice also that for all $\delta > 0$

$$c(\xi, y+\delta) - c(\xi,y) = \int_\xi^\infty P(M_1 \geq v,\ y \leq \bar{M}_1 < y+\delta)\,dv$$

$$= \int_\xi^{y+\delta} P(M_1 \geq v,\ y \leq \bar{M}_1 < y+\delta)\,dv$$

is positive convex decreasing.



Suppose we've found $\xi(y)$, so that

$$x \longmapsto c(\xi_y, y) + (x-\xi_y)\, c'(\xi_y, y) \equiv \ell_1(x)$$

is a supporting tangent to $c(\cdot,y)$ at ξ_y. Since we also know that

$$x \mapsto c(\xi_y, y+\delta) - c(\xi_y, y) + (x-\xi_y)\{c'(\xi_y, y+\delta) - c'(\xi_y, y)\} \equiv \ell_2(x)$$

is a supporting tangent at ξ_y to the convex function $c(\cdot, y+\delta) - c(\cdot, y)$, we deduce that

$$x \mapsto \ell_1(x) + \ell_2(x)$$

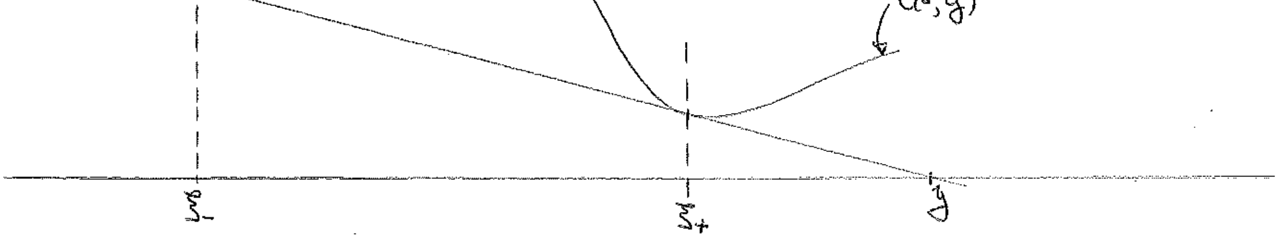
is a supporting tangent to $c(\cdot, y+\delta)$ at ξ_y. Now observe that

$$\ell_2(y+\delta) = \int_{\xi_y}^{y+\delta} P(M_1 \geq v,\ y \leq \bar{M}_1 < y+\delta)\, dv - (y+\delta-\xi_y) P(M_1 \geq \xi_y,\ y \leq \bar{M}_1 < y+\delta)$$

$$\leq 0$$

so that the supporting tangent is $\leq \ell_1(y+\delta) < 0$ at $y+\delta$. Hence immediately $\xi(y+\delta) \geq \xi(y)$.

(iv) The picture now becomes somewhat clearer. The function ξ is increasing, but may have jump intervals. Suppose (ξ_-, ξ_+) is one of these, occurring at value y. If $\xi(\bar{M}_1) < M_1$, we continue in an AY fashion, whereas if $M_1 < \xi(\bar{M}_1)$, we would generally stop immediately. An exception would be if $M_1 \in (\xi_-, \xi_+)$, $\bar{M}_1 > y$; we would then continue so as to embed the law f_2 in the interval (ξ_-, ξ_+) (which could not be achieved by the AY stop). Is this possible?



We have $c'(\xi_-, y) = c'(\xi_+, y)$, which implies that $P[M_2 \in [\xi_-, \xi_+)] = P[M_1 \in [\xi_-, \xi_+),\ \bar{M}_1 \geq y]$ so the probability to be accounted for agrees. The n+s condition for embeddability is that

$$\int_x^{\xi_+} (z-x)\, f_2(z)\, dz \geq \int_x^{\xi_+} (z-x)\, \bar{f}_1(z, y)\, dz \qquad \forall\, x \in (\xi_-, \xi_+)$$

or again that

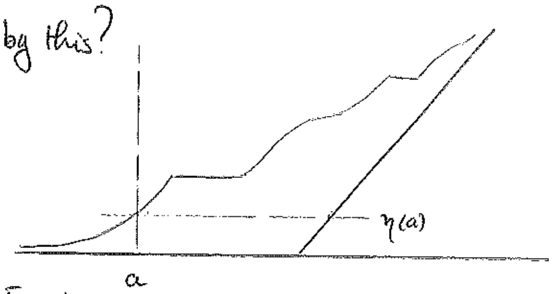
$$0 \leq \int_x^{\xi_+} (z-x)\, g(z,y)\, dz = c(x,y) - \int_{\xi_+}^{\infty} (z-x)\, g(z,y)\, dz$$

$$= c(x,y) - (\xi_+ - x) \int_{\xi_+}^{\infty} g(z,y)\, dz - c(\xi_+, y)$$

$$= c(x,y) - c(\xi_+, y) + (\xi_+ - x)\, c'(\xi_+, y)$$

which is clearly non-negative, so we can embed $\mathcal{L}(M_2)$ in (ξ_-, ξ_+).

(V) Have we actually embedded the correct law of M_2 by this?
For any point of increase a of η_1 we get



$$P(M_2 > a) = \int_a^{\infty} \bar{f}_1(x, \eta(a))\, dx + \int_0^{\eta(a)} dy \int_{\xi(y)}^{y} dx\, f_1(x,y)\, P(\bar{M}_2 > \eta(a) \mid M_1 = x, \bar{M}_1 = y)$$

$$= \int_a^{\infty} f_2(x)\, dx - \int_a^{\infty} dx\, g(x, \eta(a)) + \int_0^{\eta(a)} dy \int_{\xi(y)}^{y} dx\, f_1(x,y)\, \frac{x - \xi_y}{y - \xi_y} \exp\left[-\rho(\eta(a)) + \rho(y)\right]$$

where $\rho(t) \equiv \int_0^t \frac{dl}{l - \xi(l)}$. Thus it is N+S to prove for all $y \geq 0$ that

$$\boxed{e^{\rho(y)} \int_{\xi(y)}^{\infty} g(x,y)\, dx = \int_0^y dv \int_{\xi(v)}^{v} dx\, f_1(x,v)\, \frac{x - \xi_v}{v - \xi_v}\, e^{\rho(v)}}$$

As $y \downarrow 0$, both sides tend to 0. The LHS does not have a discontinuity at discontinuities of ξ. Now we have

$$0 = \int_{\xi_t}^{\infty} (x-t)\, g(x,t)\, dx = \int_{\xi_t}^{\infty} (x - \xi_t)\, g(x,t)\, dx - (t - \xi_t) \int_{\xi_t}^{\infty} g(x,t)\, dx,$$

$$\int_{\xi_t}^{\infty} g(x,t)\, dx = -c'(\xi_t, t) = \frac{c(\xi_t, t)}{t - \xi_t}$$

Differentiating $\int_{\xi_t}^{\infty} (x-t)\, g(x,t)\, dx$ gives us

$$0 = \frac{d\xi^c(t)}{dt} (t - \xi_t)\, g(\xi_t, t) - \int_{\xi_t}^{\infty} g(x,t)\, dx + \int_{\xi_t}^{\infty} (x-t)\, f_1(x,t)\, dx$$

and it is now routine to verify that the boxed equation does hold.

Maximum maximum with two time points (30/10/97)

(i) If we're told the law of M_1 and of M_2 how do we embed so as to maximise $E\,\varphi(\bar{M}_2)$, where φ is smooth, increasing, $\varphi(0)=0$? We have solved the similar problem where we were told $\mathcal{L}(M_1, \bar{M}_1)$, and whatever we did between 1 and 2 for that problem must match what we do between 1 and 2 for this problem, when we pick the optimal joint law for $(M_1, \bar{M}_1)$.

Rather than take that problem and try to optimise over the joint law subject to the law of M_1, it seems better to start again. We seek the joint density $\rho(x_1, y_1, x_2, y_2)$ to satisfy the constraints

$$\iiint \rho(x_1,y_1,x_2,y_2)\,dy_1\,dx_2\,dy_2 = f_1(x_1) \qquad \forall x_1$$

$$\iiint \rho(x_1,y_1,x_2,y_2)\,dx_1\,dy_1\,dy_2 = f_2(x_2) \qquad \forall x_2$$

$$\int_a^\infty dy_1 \int dx_1 \int dx_2 \int dy_2\,(a-x_1)\,\rho(x_1,y_1,x_2,y_2) = 0 \qquad \forall\ a \ge 0$$

$$\int_a^\infty dy_2 \int dx_2\,(a-x_2)\,\rho(x_1,y_1,x_2,y_2) = 0 \qquad \forall\ x_1, y_1, \quad a \ge x_1,$$

and to max $\iiiint \varphi(y_1 \vee y_2)\,\rho(x_1,y_1,x_2,y_2)\,dx_1\,dy_1\,dx_2\,dy_2$. The Lagrangian form is therefore

$$\iiiint \rho(x_1,y_1,x_2,y_2)\,\Lambda(x_1,y_1,x_2,y_2)\,dx_1\,dy_1\,dx_2\,dy_2 + \int \beta_1(x_1) f_1(x_1)\,dx_1 + \int \beta_2(x_2) f_2(x_2)\,dx_2$$

where

$$\boxed{\Lambda(x_1,y_1,x_2,y_2) = \varphi(y_1 \vee y_2) - \beta_1(x_1) - \beta_2(x_2) - \int_0^{y_1} \theta_1(a)(a-x_1)\,da - \int_{x_1}^{y_2} \theta_2(a;x_1,y_1)(a-x_2)\,da}$$

We achieve a proof by Lagrangian methods if we can provide multipliers $\beta_1, \beta_2, \theta_1, \theta_2$ such that $\Lambda \le 0$ everywhere, and some feasible ρ exists concentrated on the set where $\Lambda = 0$.

(ii) The form we found for what happens between 1 and 2 is determined by an increasing function η_2, in terms of which

$$\beta_2(x) = \int_{-\infty}^{x} (x-s)\,\frac{\eta_2'(s)\,\varphi'(\eta_2(s))}{\eta_2(s) - s}\,ds$$

and then for fixed x_1, y_1

$$\sup_{x_2, y_2} \left\{ \varphi(y_1 \vee y_2) - \beta_2(x_2) - \int_{x_1}^{y_2} \theta_2(a;x_1,y_1)(a-x_2)\,da \right\}$$

$$= \varphi(y_1) - \int_{-\infty}^{x_1} \beta_2'(s \wedge \xi_2(y_1))\,ds,$$

so the remaining maximisation over x_1, y_1 is the problem

$$\max_{x_1, y_1} \left\{ \varphi(y_1) - \int_{-\infty}^{x_1} \beta_2'(s \wedge \xi_2(y_1))\, ds - \beta_1(x_1) - \int_0^{y_1} \theta_1(a)(a - x_1)\, da \right\}$$

$$= \max_{x_1, t} \left\{ \varphi(\eta_2(t)) - \int_{-\infty}^{x_1} \beta_2'(s \wedge t)\, ds - \beta_1(x_1) - \int_0^{\eta_2(t)} \theta_1(a)(a - x_1)\, da \right\}.$$

Two cases arise.

<u>Case 1:</u> $\underline{t \equiv \xi_2(y_1) > x_1}$, when we have the expression

$$\varphi(y_1) - \beta_2(x_1) - \beta_1(x_1) - \int_0^{y_1} \theta_1(a)(a - x_1)\, da.$$

Differentiation wrt y_1 gives $\varphi'(y_1) = \theta_1(y_1)(y_1 - x_1)$, so this will hold for at most one $x_1 = \xi_1(y_1)$, and

$$\beta_1'(x_1) + \beta_2'(x_1) = \int_0^{y_1} \theta_1(a)\, da \qquad \text{when } x_1 = \xi_1(y_1)$$

So wherever there's mass in $\{(x_1, y_1) : x_1 < \xi_2(y_1)\}$ we shall have

$$\boxed{\begin{aligned} \theta_1(y_1) &= \frac{\varphi'(y_1)}{y_1 - \xi_1(y_1)} < \frac{\varphi'(y_1)}{y_1 - \xi_2(y_1)} \\ \beta_1'(x_1) &= \int_0^{\eta_1(x_1)} \theta_1(a)\, da - \beta_2'(x_1) \\ x_1 &= \xi_1(y_1) \end{aligned}}$$

In particular,

$$\beta_1'(x_1) = \int_0^{\eta_1(x_1)} \theta_1(a)\, da - \int_0^{\eta_2(x_1)} \frac{\varphi'(a)\, da}{a - \xi_2(a)}$$

$$= \int_{\eta_2(x_1)}^{\eta_1(x_1)} \frac{\varphi'(a)\, da}{a - \xi_1(a)} - \int_0^{\eta_2(x_1)} \left\{ \theta_1(a) - \frac{\varphi'(a)}{a - \xi_2(a)} \right\} da$$

<u>Case 2:</u> $\underline{t \equiv \xi_2(y_1) < x_1}$. In this region, we have

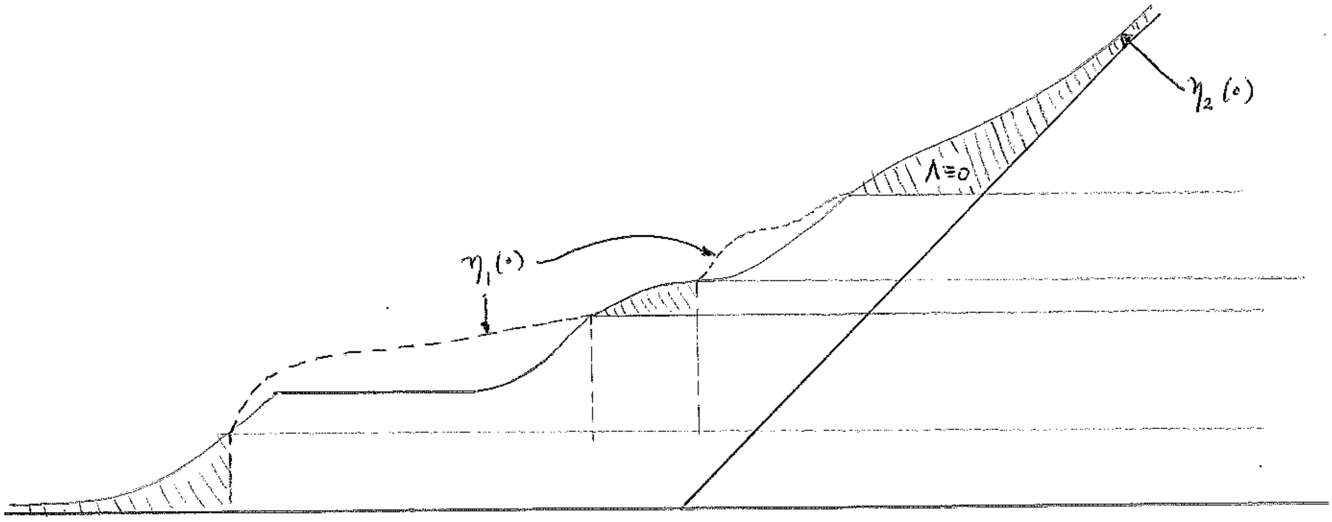
$$\frac{\partial \Lambda}{\partial t} = \eta_2'(t) \left\{ \varphi'(\eta_2(t)) - \theta_1(\eta_2(t))(\eta_2(t) - x_1) \right\} - \beta_2''(t)(x_1 - t)$$

$$= \eta_2'(t)(\eta_2(t) - x_1) \left\{ \frac{\varphi'(y_1)}{y_1 - \xi_2(y_1)} - \theta_1(y_1) \right\} \qquad [y_1 \equiv \eta_2(t)]$$

So if there is going to be mass in this region, it has to be that

$$\boxed{\theta_1(y_1) = \frac{\varphi'(y_1)}{y_1 - \xi_2(y_1)}}$$

and a picture starts to emerge:



In the shaded regions, $\Lambda \equiv 0$. We may think of η_1 as being continued through the shaded regions (as will be the case shortly when we go for the AY embedding), but the solution is indeterminate there. Outside the shaded regions, Λ is zero on the graph of η_1. We have for all $y > 0$

$$\boxed{\theta_1(y) = \frac{\varphi'(y)}{y - \xi_1(y) \wedge \xi_2(y)}}$$

and in the shaded regions $\partial \Lambda / \partial x_1 \equiv 0$, so we conclude that here

$$\begin{aligned}
\beta_1'(x_1) &= -\beta_2'(\xi_2(y_1)) + \int_0^{y_1} \theta_1(a)\, da \\
&= -\int_0^{y_1} \varphi'(a) \left\{ \frac{1}{a - \xi_2(a)} - \frac{1}{a - \xi_1(a) \wedge \xi_2(a)} \right\} da \\
&= -\int_0^{y_1} \frac{(\xi_2(a) - \xi_1(a))^+ \, da\, \varphi'(a)}{(a - \xi_1(a))(a - \xi_2(a))}
\end{aligned}$$

while on the curve $\eta_1(\cdot)$ we get

$$\begin{aligned}
\beta_1'(x_1) &= -\beta_2'(x_1) + \int_0^{\eta_1(x_1)} \theta_1(a)\, da \\
&= \int_{\eta_2(x_1)}^{\eta_1(x_1)} \frac{\varphi'(a)\, da}{a - \xi_1(a)} - \int_0^{\eta_2(x_1)} \frac{(\xi_2(a) - \xi_1(a))^+ \varphi'(a)\, da}{(a - \xi_1(a))(a - \xi_2(a))}
\end{aligned}$$

Once we've picked our curve η_1 for which a feasible embedding is possible, this tells us how we must make the Lagrange multipliers for the Lagrangian sufficiency proof.

So it <u>is</u> optimal to use AY up to time 1, in which case $\eta_1 = b_1$, though other optimal rules are possible.

If we assume AY embedding up to time 1, we have

$$C_1(\xi, y) \equiv \int_\xi^\infty P(M_1 \geq v,\ \bar{M}_1 \geq y)\, dv$$

$$= \int_\xi^\infty P(M_1 \geq v \vee b_1^{-1}(y))\, dv$$

Notice that for any other embedding

$$\tilde{C}_1(\xi, y) \equiv \int_\xi^\infty \tilde{P}(M_1 \geq v,\ \bar{M}_1 \geq y)\, dv$$

$$\leq \begin{cases} \int_\xi^\infty P(M_1 \geq v)\, dv = C_1(\xi, y) & \text{if } \xi \geq b_1^{-1}(y) \\ \int_{b_1^{-1}(y)}^\infty P(M_1 \geq v)\, dv + \int_\xi^{b_1^{-1}(y)} P(\bar{M}_1 \geq y)\, dv = C_1(\xi, y) & \text{if } \xi < b_1^{-1}(y) \end{cases}$$

So that

$$\tilde{C}_1(\xi, y) \leq C_1(\xi, y) \qquad \forall \xi,\ \forall y$$

which confirms directly that AY is optimal up to time 1.

(iii) It doesn't appear that we can make the specification of ξ_2 any more explicit. We can interpret the price in terms of a portfolio of calls, though. We have for all x that

$$\boxed{\beta_1'(x) + \beta_2'(x) = \int_0^{\eta_{12}(x)} \theta_1(a)\, da}$$

$\eta_{12}(x) \equiv \eta_1(x) \vee \eta_2(x)$, inverse to $\xi_{12}(y) \equiv \xi_1(y) \wedge \xi_2(y)$.

So that the final maximised price is

$$\int \beta_1(x) f_1(x)\, dx + \int \beta_2(x) f_2(x)\, dx$$

$$= \int (\beta_1'(x) + \beta_2'(x))\, P(M_1 \geq x)\, dx + \int \beta_2'(x) \{ P(M_2 \geq x) - P(M_1 \geq x) \} dx$$

$$= \int_0^\infty \frac{\varphi'(a)\, da}{a - \xi_{12}(a)} C_1(\xi_{12}(a)) + \int_0^\infty \frac{\varphi'(a)\, da}{a - \xi_2(a)} \{ C_2(\xi_2(a)) - C_1(\xi_2(a)) \}$$

where $C_j(z) = E(M_j - z)^+$ is the call price function. We can think of this as the cost of a portfolio formed by

buying $\dfrac{\varphi'(a)\, da}{a - \xi_{12}(a)}$	calls of strike $\xi_{12}(a)$,	expiry	1
buying $\dfrac{\varphi'(a)\, da}{a - \xi_2(a)}$	calls of strike $\xi_2(a)$,	expiry	2
Selling $\dfrac{\varphi'(a)\, da}{a - \xi_2(a)}$	calls of strike $\xi_2(a)$,	expiry	1

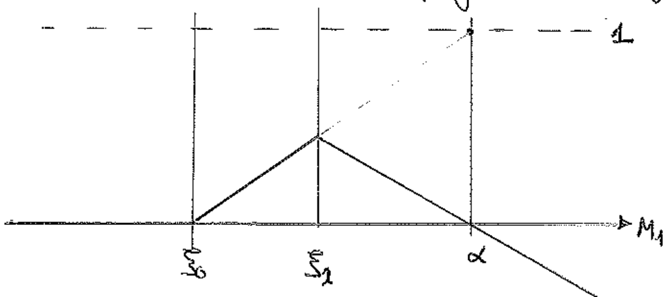
(iv) How can we show that this will superhedge? If we take $\varphi(y) \equiv I_{\{y \geq \alpha\}}$ then we have to show that the portfolio of

$\frac{1}{\alpha - \xi_0}$ 1-calls of strike ξ_0 $(\equiv \xi_{12}(\alpha))$

$\frac{1}{\alpha - \xi_2}$ 2-calls of strike ξ_2 $(\equiv \xi_2(\alpha))$

$\frac{-1}{\alpha - \xi_2}$ 1-calls of strike ξ_2

can be handled so as to generate at least $I_{\{\bar{M}_2 \geq \alpha\}}$. The argument here is David's, rephrased. The value of the 1-calls as a f^n of M_1 is displayed (assuming $\xi_0 < \xi_2$ - it's zero else)



If $\bar{M}_1 \geq \alpha$, sell $\frac{1}{\alpha - \xi_0}$ shares forward at the instant M_1 reaches α, which generates

$$\frac{1}{\alpha - \xi_0}(\alpha - M_1) \quad \text{at time } 1$$

Together with the 1-calls of strike ξ_0 this guarantees at least 1 at time 0, and the 2-calls of strike ξ_2 certainly cancel out the shorted 1-calls at time 1, so you have certainly 1 at time 1 if $\bar{M}_1 \geq \alpha$.

If on the other hand $\bar{M}_1 < \alpha$, then the 1-calls are worth ≥ 0 at time 1. So now just run on with the 2-calls, selling $(\alpha - \xi_2)^{-1}$ shares forward the instant M. first touches α (if at all). This certainly guarantees 1 if $\bar{M}_2 \geq \alpha > \bar{M}_1$, and a non-negative amount if $\bar{M}_2 < \alpha$.

(v) In the context of quite general φ, we can describe the handling of the optimal portfolio as follows; as $\bar{M}$ rises across level y, you lock in the gain by selling

- the 1-calls of strike $\xi_1(y)$ if $\xi_1(y) < \xi_2(y)$, and $t \leq 1$
- the 2-calls of strike $\xi_2(y)$ if $\xi_1(y) \geq \xi_2(y)$ or $t > 1$

For any y such that $\xi_1(y) < \xi_2(y)$, the 2-calls (with strike $\xi_2(y)$) are used to cancel out the shorted 1-calls at time 1 if y has been reached by time 1.

(vi) Fixing $\varphi(y) \equiv I_{\{y \geq a\}}$ for now, we are able to interpret the criterion of p.14:

$$P(\bar{M}_1 \geq a) + \inf \frac{C_2(\xi) - C_1(\xi, a)}{a - \xi}$$

in the following way. If we have some portfolio which super-replicates $I_{\{\bar{M}_1 \geq a\}}$, we can for any $\xi < a$ construct a portfolio super-replicating $I_{\{\bar{M}_1 < a \leq \bar{M}_2\}}$ by

buying $\frac{1}{a-\xi}$ 2-calls of strike ξ

selling $\frac{1}{a-\xi}$ 1-calls of strike ξ, knocked in when $\bar{M}$ reaches a.

If $\bar{M}_1 \geq a$, then the knock-in calls have value at time 1, but this can be at least cancelled by the 2-calls. If $\bar{M}_1 < a$, the knock-in calls are worthless, and the 2-calls can be used to give at least 1 if $\bar{M}_2 \geq a$. If we can construct the martingale for $t \in (1, 2]$ so as to generate no excess, this should be optimal, provided we can show that the law of M_2 is correct.

$$\cancel{E\,\varphi(M_1,\bar M_1) = E\left[\varphi(M_1,\bar M_1);\ M_1 \le b_1^{-1}(\bar M_1)\right] + E\left[\varphi(M_1,\bar M_1);\ M_1 > b_1^{-1}(\bar M_1)\right]}$$

$$\cancel{\le E\left[\varphi(b_1^{-1}(\bar M_1),\bar M_1);\ M_1 \le b_1^{-1}(\bar M_1)\right] + E\left[\varphi(M_1, b_1(M_1));\ M_1 > b_1^{-1}(\bar M_1)\right]}$$

(*)

$$-E\,\varphi(M_1,\bar M_1) = \int \mu(dt)\, E\left[c(M_1,t);\ \bar M_1 \le t\right]$$

$$\ge \int \mu(dt)\, E\left[c(M_1,t);\ M_1 \le b_1^{-1}(\bar M_1),\ \bar M_1 \le t\right]$$

$$\ge \int \mu(dt)\, E\left[c(b_1^{-1}(\bar M_1),t);\ \bar M_1 \le t\right]$$

But for each t, $y \mapsto c(b_1^{-1}(y),t)\, I_{\{y \le t\}}$ is decreasing, so we want stochastically largest $\bar M_1$.

The maximum maximum: ~~getting~~ closer (13/11/97)

The problem is to max $E\,\varphi(\bar{M}_n)$, where $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, $\varphi(0)=0$, and M is a martingale (without loss of generality continuous), $\bar{M}_t \equiv \sup_{u \le t} M_u$, and $\mathcal{L}(M_j)$ is constrained to be some given law μ_j, where the potentials of the laws μ_j are increasing (so that the laws can be embedded sequentially in a martingale)

We shall

(i) give a construction of a continuous martingale X with $X_0=0$, $\mathcal{L}(X_j)=\mu_j$ $\forall j$;

(ii) specify a portfolio of calls of different strikes and expiries;

(iii) prove that the portfolio of calls will always permit super-replication of $I_{\{\bar{M}_t \ge \alpha\}}$ where $\alpha > 0$ is fixed;

(iv) prove that for the martingale X the superreplication is achieved without any excess.

This proves that the martingale X is an optimal solution to the problem max $P(\bar{M}_n \ge \alpha)$ subject to $\mathcal{L}(M_j)=\mu_j$ $(j=0,\dots,n)$. Since the same martingale works for every α, we conclude that actually X solves max $E\,\varphi(\bar{M}_n)$ s.t. $\mathcal{L}(M_j)=\mu_j$ $\forall j \le n$.

An interesting feature of the solution is that if you increase n to $n+1$, the portfolio you would choose may change considerably, but the martingale X constructed will actually be the same up to time n.

FOR NOW, WE ASSUME THAT ALL OF THE η_i ARE STRICTLY INCREASING.

(i) The construction. The construction is in terms of functions $\eta_j: \mathbb{R} \to \mathbb{R}^+$ which are increasing, with right-continuous inverses ξ_j. We have $\eta_j(x) \ge x^+$ for all j. To start with, we take $\eta_1 = b_1$, the barycentre function of μ_1, and the martingale up to time 1 is a Brownian motion run up to the Azema-Yor stopping time $\tau_1 = \inf\{t: \bar{B}_t > b_1(B_t)\}$.

Having determined the embedding up to time $n-1$, we define it up to time n by the following steps

(a) For each $y>0$, find the $\xi \equiv \xi_n(y)$ which minimises

$$\frac{c_n(\xi) - c_{n-1}(\xi, y)}{y-\xi}$$

where $c_n(\xi) \equiv E(M_n - \xi)^+$, the price of an n-call with strike ξ, and

$$c_{n-1}(\xi, y) \equiv E\left[(X_{n-1}-\xi)^+ \,;\, \bar{X}_{n-1} \ge y\right]$$

is the price of an $(n-1)$-call of strike ξ, knocked in at level y. Notice that $c_n(\xi)$ depends only on μ_n, whereas $c_{n-1}(\xi, y)$ depends on the particular embedding.

(b) Construct $(X_t)_{n-1 \le t \le n}$ as follows. If $\bar{X}_{n-1} \geq \eta_n(X_{n-1})$ then you ~~stop~~ still for $t \in [n-1, n]$, unless X_{n-1} is in a flat interval of η_n, in which case you allow the martingale to move around in the flat interval to achieve the correct law of X_n in that interval.

If on the other hand $\bar{X}_{n-1} < \eta_n(X_{n-1})$ you Azema-Yor until you hit the curve η_n; if we consider taking a Brownian motion with $B_{n-1} = X_{n-1}$, $\bar{B}_{n-1} = \bar{X}_{n-1}$, we would run it until $\tau_n \equiv \inf\{ t > \tau_{n-1} : \bar{B}_t \geq \eta_n(B_t) \}$.

This then specifies the joint law of $(X_n, \bar{X}_n)$ in a recursive fashion; as we've seen, it actually does embed each of the laws μ_j correctly.

(ii) The portfolio. Fix some $\alpha > 0$; we shall specify a portfolio of calls which when correctly handled will guarantee a wealth at time n of at least $I_{\{\bar{M}_n \geq \alpha\}}$.

Firstly, the portfolio contains $1/(\alpha - \xi_n(\alpha))$ n-calls of strike $\xi_n(\alpha)$.

Next, take $i_1 \equiv \sup\{ j < n : \xi_j(\alpha) < \xi_n(\alpha) \}$. If there is no such j, the portfolio contains nothing more, but if there is, we include in the portfolio

$$\frac{-1}{\alpha - \xi_n(\alpha)} \quad i_1\text{-calls of strike } \xi_n(\alpha)$$

$$\frac{1}{\alpha - \xi_{i_1}(\alpha)} \quad i_1\text{-calls of strike } \xi_{i_1}(\alpha)$$

Proceed inductively to find $i_{k+1} \equiv \sup\{ j < i_k : \xi_j(\alpha) < \xi_{i_k}(\alpha) \}$ and, if there is such a j, to include in the portfolio

$$\frac{-1}{\alpha - \xi_{i_k}(\alpha)} \quad i_{k+1}\text{-calls of strike } \xi_{i_k}(\alpha)$$

$$\frac{1}{\alpha - \xi_{i_{k+1}}(\alpha)} \quad i_{k+1}\text{-calls of strike } \xi_{i_{k+1}}(\alpha),$$

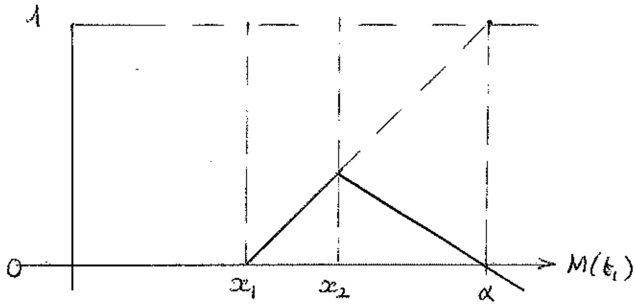
Eventually this procedure stops, and we have a bunch of calls of m different strikes, $x_1 < x_2 < \cdots < x_m \equiv \xi_n(\alpha)$, with expiries $t_1 < t_2 < \cdots < t_m \equiv n$; we are holding

$$\boxed{\begin{array}{ll} \dfrac{1}{\alpha - x_r} \quad t_r\text{-calls with strike } x_r, & r = 1, \dots, m \\[2ex] \dfrac{-1}{\alpha - x_r} \quad t_{r-1}\text{-calls with strike } x_r, & r = 2, \dots, m \end{array}}$$

(iii) <u>The super-replication</u>. Now we show how to trade in the calls selected so as to guarantee at least $I_{\{\bar{M}_n \geq \alpha\}}$ at time n.

Firstly, let the martingale run up to time t_1, but if ever it reaches α you immediately sell the t_1-calls for what they are worth (which has to be at least $\alpha - x_1$ each) and thereby generate 1 unit of wealth. If $\bar{M}_{t_1} \geq \alpha$, you reach time t_1 with at least 1 in your pocket, and at time t_1 you liquidate your position in the remaining calls, which will generate at least zero.

On the other hand, if $\bar{M}_{t_1} < \alpha$, the net value of the t_1-calls when you get to time t_1 is plotted as a function of $M(t_1)$ here, and it's clear that it is non-negative in the interval $(-\infty, \alpha]$, in which we know $M(t_1)$ lies. Thus



the t_1-calls net out to give a non-negative payment, and you are left with calls of expiry t_2 and beyond. Now continue in like fashion to run ahead to time t_2, immediately selling the t_2-calls if ever $\bar{M}$ gets to α. Proceeding thus will guarantee at least $1_{\{\bar{M}_n \geq \alpha\}}$.

(iv) <u>The bound is sharp</u>. To prove this, we have to know that all the places where excess might be made it will be zero for the embedding defined at (i).

If $\bar{M}(t_1) \geq \alpha$, when the t_1-calls were sold we already know that M will not fall below x_1 before time t_1, so the value of the call has to be $M(\tau) - x_1 = \alpha - x_1$. Likewise, the calls of expiry t_2, strike x_2 cancel the t_1-calls of strike x_2, because we know that between t_1 and t_2 the martingale will not cross level x_2. Indeed, either $M(t_1) \leq \xi_{i_1+1}(\bar{M}(t_1))$, <u>so that the martingale doesn't move in</u> (t_1, t_1+1), or else $M(t_1) > \xi_{i_1+1}(\bar{M}(t_1))$ in which case M remains above $\xi_{i_1+1}(\alpha) \geq x_2$ during (t_1, t_1+1). Roll argument forward.

If $\bar{M}(t_1) < \alpha$, then $M(t_1) < \xi_{i_1}(\alpha) = x_1$, so the t_1-calls of strikes x_1 and x_2 are all worth nothing, and no excess is generated here.

Liquidity costs: some heuristics. (25/11/97)

(i) We have constant rate of interest $r \geq 0$ and share with return $S^{-1}dS = \sigma\, dW + \mu\, dt$. The state-price density ζ solves

$$d\zeta_t = \zeta_t \{ -r dt - \tilde\mu\, dW_t \}, \quad \zeta_0 = 1 \qquad (\tilde\mu \equiv (\mu - r)/\sigma)$$

We are considering a number of problems:

(I) Basic problem

$$V_I(x_0) = \sup \left\{ E\left[U(x_0 e^{rT} + X_T)\right] : X \in \mathcal{X} \right\}$$

where $\mathcal{X}$ is the class of wealth processes which can be generated from zero initial wealth. We have explicitly

$$e^{-rt} X_t = \int_0^t e^{-rs} \theta_s \sigma\, d\tilde W_s,$$

where $\tilde W_t \equiv W_t + \tilde\mu t$. We also get from the FOC that

$$U'(x_0 e^{rT} + X_T^I) = \lambda \zeta_T = \lambda \exp\left[-\tilde\mu W_T - (r + \tfrac{1}{2}\tilde\mu^2) T \right]$$

for some λ, fixed by the requirement that

$$x_0 = E\left[\zeta_T I(\lambda \zeta_T) \right] \qquad (I \circ U' = \text{identity.})$$

Let θ^I be the optimal portfolio, used to generate optimal gains-from-trade X_T^I. Since

$$e^{rT} x_0 + X_T^I = f_I(\tilde W_T) \equiv I\left(\lambda \exp\left(-\tilde\mu \tilde W_T + (\tfrac{1}{2}\tilde\mu^2 - r)T \right) \right),$$

we can be somewhat more explicit;

$$\boxed{\theta_t^I = e^{-rT} (P_{T-t} \nabla f_I)(\tilde W_t) e^{rt} \sigma^{-1}}$$

(II) Problem with delay and paying out a contingent claim. This time we consider the problem

$$V_{II}(x) = \sup \left\{ E\, U(x e^{rT} + X_T - \varepsilon Y) : X \in \mathcal{X}^\delta \right\}$$

where Y is a bounded non-negative contingent claim,

$$Y_T e^{-rT} \equiv a + \int_0^T \sigma e^{-rs} \theta_s^Y d\tilde W_s$$

and $\mathcal{X}^\delta$ is the class of wealth processes generated by portfolios which are $(\mathcal{F}_{t-\delta})$-adapted. Now if x is close to x_0 and ε, δ are both small, we can expect the optimal solution for this problem to be close to that for the first problem. So we shall express the payoff for problem (II) as

$$g(x) = \left\{ K - S_0 \exp\left(\sigma x + (r - \tfrac{1}{2}\sigma^2)T\right) \right\}^+$$

$$E\,U(xe^{rT}+X_T-\varepsilon Y) = E\ U\left(x_0e^{rT}+X^I_T + \left(X_T - X^I_T - \varepsilon Y + (x-x_0)e^{rT}\right)\right)$$

$$\doteq E\Big[U(x_0e^{rT}+X^I_T) + U'(x_0e^{rT}+X^I_T)\Big\{\int_0^T e^{r(T-s)}\sigma(\theta_s-\theta^I_s-\varepsilon\theta^Y_s)\,d\tilde W_s + (x-x_0-\varepsilon a)e^{rT}\Big\}$$

$$+\tfrac12 U''(x_0e^{rT}+X^I_T)\Big\{\int_0^T e^{r(T-s)}\sigma(\theta_s-\theta^I_s-\varepsilon\theta^Y_s)\,d\tilde W_s + (x-x_0-\varepsilon a)e^{rT}\Big\}^2\Big] + O(\varepsilon^3,\delta^3).$$

$$= V_I(x_0) + \lambda e^{-rT}\tilde E\Big[\int_0^T e^{r(T-s)}\sigma(\theta_s-\theta^I_s-\varepsilon\theta^Y_s)\,d\tilde W_s + (x-x_0-\varepsilon a)e^{rT}\Big]$$

$$+\tfrac12\lambda e^{-rT}\tilde E\Big[\frac{U''}{U'}(w^*_T)\Big\{\int_0^T e^{r(T-s)}\sigma(\theta_s-\theta^I_s-\varepsilon\theta^Y_s)\,d\tilde W_s + (x-x_0-\varepsilon a)e^{rT}\Big\}^2\Big] + O(\varepsilon^3,\delta^3)$$

$$= V_I(x_0) + \lambda\ (x-x_0-\varepsilon a)$$

$$+\tfrac12\lambda e^{-rT}\tilde E\Big[\frac{U''}{U'}(w^*_T)\Big(\int_0^T e^{r(T-s)}\sigma(\theta_s-\theta^I_s-\varepsilon\theta^Y_s)\,d\tilde W_s + (x-x_0-\varepsilon a)e^{rT}\Big)^2\Big] + O(\varepsilon^3,\delta^3)$$

where $w^*_T \equiv x_0e^{rT}+X^I_T$ is optimal wealth for problem (I).

(ii) Special case : $\boxed{U(x) = -\exp(-\gamma x)}$ In this example, we have

$$\gamma\exp(-\gamma w^*_T) = \lambda\zeta_T = \lambda\exp\left[-\tilde\mu\tilde W_T + (\tfrac12\tilde\mu^2 - r)T\right]$$

so that

$$w^*_T = -\frac1\gamma\left\{\log\frac{\lambda}{\gamma} - \tilde\mu\tilde W_T + (\tfrac12\tilde\mu^2-r)T\right\}$$

Taking $\boxed{x_0=0}$ wlog, we deduce that we need

$$\boxed{\lambda = \gamma\ \exp(r-\tfrac12\tilde\mu^2)T}$$

and $\boxed{w^*_T = \tilde\mu\gamma^{-1}\tilde W_T}$, from which $\boxed{\theta^I_t = e^{r(t-T)}\tilde\mu/(\gamma\sigma),}$ which is deterministic.

Let's concentrate on the situation now where $S_0>0$, and Y is a European put option with strike K. If we realise that

$$Y = (K-S_T)^+ \equiv g(\tilde W_T)$$

we can write the portfolio process

$$\boxed{\theta^Y_t = \sigma^{-1}e^{r(t-T)}(P_{T-t}\nabla g)(\tilde W_t)}$$

and the BS price

$$\boxed{a = Ke^{-rT}\Phi\left(\frac{\log(K/S_0)-(r-\frac12\sigma^2)T}{\sigma\sqrt T}\right) - S_0\Phi\left(\frac{\log(K/S_0)-(r+\frac12\sigma^2)T}{\sigma\sqrt T}\right)}$$

quite explicitly. We maximise expected utility by taking

$$\theta_s = \theta_s^I + \tilde{E}\left[\varepsilon\,\theta_s^Y \mid \mathcal{F}_{s-\delta}\right]$$

where $\mathcal{F}_u = \{\emptyset, \Omega\}$ for $u \le 0$. The value of the problem B (to second order)

$$V_I(0) + \lambda(x - \varepsilon a) - \tfrac{1}{2}\gamma\lambda e^{-rT}\varepsilon^2\left\{e^{2rT}(x-\varepsilon a)^2 + \tilde{E}\int_0^T\left\{(P_{T-s}\nabla g)(\tilde{W}_s) - (P_{(T-s+\delta)\wedge T}\nabla g)(\tilde{W}_{(s-\delta)^+})\right\}^2 ds\right\}$$

$$= V_I(0) + \lambda\left[x - \varepsilon a - \frac{\gamma\varepsilon^2}{2}e^{-rT}\left\{(x-\varepsilon a)^2 e^{2rT} + \tilde{E}\int_{T-\delta}^T (P_{T-s}\nabla g)(\tilde{W}_s)^2 ds - \delta(P_T\nabla g)(\tilde{W}_0)^2\right\}\right]$$

We have $\nabla g(\tilde{W}_T) = -\sigma S_T I_{\{S_T < K\}}$, so

$$\boxed{(P_{T-t}\nabla g)(\tilde{W}_t) = -\sigma S_t e^{r(T-t)}\,\Phi\left(\frac{\log(K/S_t) - (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)} \qquad (\tau \equiv T-t)$$

We can simplify a bit:

$$\tilde{E}\left(\int_{T-\delta}^T (P_{T-s}\nabla g)(\tilde{W}_s)^2 ds\right) = \tilde{E}\left[g(\tilde{W}_T)^2 - P_\delta g(\tilde{W}_{T-\delta})^2\right],$$

for example, but this remains about as simple as we could get.

Hence we have for the marginal price

$$\boxed{\frac{1}{\varepsilon}x \sim a + \varepsilon\frac{\gamma}{2}e^{-rT}\left(\tilde{E}\left\{g(\tilde{W}_T)^2 - P_\delta g(\tilde{W}_{T-\delta})^2\right\} - \delta(P_T\nabla g)(0)^2\right)}$$

(iii) Other utility functions appear to be hopeless, but we could do the discretised version of this, where riskless saving increases 1 to R over one period, and the wealth equation is

$$w_n = R\,w_{n-1} + \theta_n\left(\frac{S_n}{S_{n-1}} - R\right)$$

$$\therefore \quad R^{-n}w_n = R^{-(n-1)}w_{n-1} + \theta_n R^{-n}\left(\frac{S_n}{S_{n-1}} - R\right),$$

where $S_n/S_{n-1} - R \equiv \tilde{Z}_n$ is a $\tilde{P}$-mg difference sequence.

(iv) But CARA utility is <u>not</u> a very interesting example; the optimal investment is deterministic, so we see no cost of liquidity at all! The same holds in discrete time too.

So what happens with CRRA utility $U(x) = x^{1-R}/(1-R)$?

$$w_t^* = w_0^* \exp\left[\frac{\tilde{\mu}}{R}W_t + \left(r + \frac{\tilde{\mu}^2}{R} - \frac{1}{2}\frac{\tilde{\mu}^2}{R^2}\right)t\right], \qquad \theta_t^I = \frac{\tilde{\mu}}{\sigma R}w_t^*.$$

$$= w_0^* \exp\left[\frac{\tilde{\mu}}{R}\tilde{W}_t + \left(r - \tfrac{1}{2}\tilde{\mu}^2/R^2\right)t\right]$$

We can set up the equations in discrete time for this, and exploit scaling. If we let $V_n(\theta, w, S)$ be the value at the start of day n if we are committed to holding θ shares this day, w is our current wealth measured in shares, and S is current stock price, we then have by scaling

$$V_n(\theta, w, S) = S^{1-R}\, V_n(\theta, w, 1)$$

and

$$V_n(\theta, w, S) = \max_{\varphi} \left[p\, V_{n+1}(\varphi, \theta + (w-\theta)\rho e^{-\delta}, Se^{\delta}) + q\, V_{n+1}(\varphi, \theta + (w-\theta)\rho e^{\delta}, Se^{-\delta}) \right]$$

where 1 invested in the bank grows to ρ, and 1 invested in the share grows to e^{δ} with prob^y p, and $e^{-\delta}$ with prob^y $q = (1-p)$. Hence

$$\boxed{V_n(\theta, w, 1) = \max_{\varphi} \left[p\, e^{\delta(1-R)} V_{n+1}(\varphi, \theta + (w-\theta)\rho e^{-\delta}) + q\, e^{-\delta(1-R)} V_{n+1}(\varphi, \theta + (w-\theta)\rho e^{\delta}) \right]}$$

Notice that we always require for no bankruptcy that

$$(w-\theta)\rho + \theta e^{-\delta} \geq 0 \qquad \text{if } \theta \geq 0$$

$$(w-\theta)\rho + \theta e^{\delta} \geq 0 \qquad \text{if } \theta \leq 0$$

so we need

$$\boxed{\frac{-w}{e^{\delta} - \rho} \leq \theta \leq \frac{w}{\rho - e^{-\delta}}}\,.$$

But by scaling again, we have $V_n(\lambda\theta, \lambda w, 1) = \lambda^{1-R} V_n(\theta, w, 1)$, so we may write $v_n(x) \equiv V_n(x, 1, 1)$ and simplify the dynamic-programming equations to

$$\boxed{v_n(x) = \max_{y} \left[p\, e^{\delta(1-R)} v_{n+1}\left(\frac{y}{x + (1-x)\rho e^{-\delta}} \right) + q\, e^{-\delta(1-R)} v_{n+1}\left(\frac{y}{x + (1-x)\rho e^{\delta}} \right) \right]}$$

The argument of $v_n(\cdot)$ must always lie in the range

$$\left[\frac{-1}{e^{\delta} - \rho}\,,\ \frac{1}{\rho - e^{-\delta}} \right]$$

Stochastic intensities: an example from physics (28/11/97)

We have a continuous-time chain on statespace $I = \{0,1\}^N$. Any site in state 1 is excited, and drops back to state 0 at rate $\mu > 0$. Let's suppose that the invariant distribution is

$$\pi(x; N, \alpha, \beta) = \pi(x) \propto \exp\left[\tfrac{1}{2}\beta \sum_{i \neq j}\sum x_i x_j + \alpha \sum_i x_i\right]$$

where α, β are constants. This gives for the jump intensities

$$x \mapsto x' \quad \text{at rate} \quad \frac{\pi(x')}{\pi(x)} \cdot \mu = \mu \exp\left[\alpha + \beta \sum_i x_i\right],$$

where $x'_i = x_i \ \forall i \neq b$, $x'_b = 1$, $x_b = 0$. If we just consider $\nu(t) \equiv \sum x_i(t)$, then ν is a birth-and-death chain on $\{0, 1, \dots, N\}$, with intensities

$$k \mapsto k+1 \qquad \text{rate } (N-k)\,\mu \exp(\alpha + \beta k)$$

$$k \mapsto k-1 \qquad \text{rate } k\mu$$

so that in equilibrium

$$P[\nu_0 = k] = c \cdot \binom{N}{k} \exp\left[\alpha k + \tfrac{1}{2}\beta\, k(k-1)\right].$$

How about some moments? If we were to take $f_i(x) = c_i + \lambda_i I_{\{x_i = 1\}}$, we need to compute

$$P(x_i(0) = 1) = \sum_{k=0}^{N} \frac{k}{N} \cdot \binom{N}{k} \exp\left[\alpha k + \tfrac{1}{2}\beta\, k(k-1)\right] \Big/ \sum_{j=0}^{N} \binom{N}{j} \exp\left[\alpha j + \tfrac{1}{2}\beta\, j(j-1)\right]$$

$$= \sum_{r=0}^{N-1} \binom{N-1}{r} \exp\left[\alpha r + \alpha + \tfrac{1}{2}\beta\, r(r+1)\right] \Big/ \sum_{j=0}^{N} \binom{N}{j} \exp\left(\alpha j + \tfrac{1}{2}\beta\, j(j-1)\right)$$

For small N, we might work with exactly this expression. For larger N, we have

$$P(\nu_0 = k) \sim \text{const} \cdot \exp\left[\alpha k + \tfrac{1}{2}\beta\, k(k-1) - (k+\tfrac{1}{2}) \log \tfrac{k}{N} - (N-k+\tfrac{1}{2}) \log(1 - \tfrac{k}{N})\right]$$

$$= \text{const} \quad \exp\left[N\left\{\alpha \frac{k}{N} + \tfrac{1}{2}\beta N \cdot \frac{k}{N}\frac{k-1}{N} - \frac{k+\frac{1}{2}}{N} \log \frac{k}{N} - \left(1 - \frac{k+\frac{1}{2}}{N}\right) \log\left(1 - \frac{k}{N}\right)\right\}\right]$$

so the proportion of excited sites must be effectively the value of k/N which maximises $\{\dots\}$; to find this, we must solve

$$\frac{1-x}{x} = \exp(-\alpha - \beta N x)$$

If $\beta N < 0$, there's exactly one root, but if $\beta N > 0$ there can be 1, or 3.

To compute $E\, \eta_0^{ij}$, we need to be able to calculate

$$P(x_j(T)=1 \mid x_i(0)=1)$$

for $T \sim \exp(\theta)$ independent of the process. For $i \neq j$,

$$P(x_j(T)=1 \mid x_i(0)=1) = \frac{\theta}{\theta+\mu} P(x_j(T)=1 \mid x_i(s)=1 \ \forall s \leq T) + \frac{\mu}{\theta+\mu} P(x_j(T)=1 \mid x_i(0)=0)$$

If we now write

$$p(N,\alpha,\beta) \equiv P(x_i(0)=1) = \frac{\sum_{r=0}^{N-1} \binom{N-1}{r} \exp\left[\alpha r + \alpha + \frac{1}{2}\beta r(r+1)\right]}{\sum_{k=0}^{N} \binom{N}{k} \exp\left[\alpha k + \frac{1}{2}\beta k(k-1)\right]}$$

we know that

$$\begin{cases} p(N,\alpha,\beta) = P(x_j(T)=1 \mid x_i(0)=1)\, p(N,\alpha,\beta) + P(x_j(T)=1 \mid x_i(0)=0)\,(1-p(N,\alpha,\beta)) \\ P(x_j(T)=1 \mid x_i(0)=1) = \frac{\theta}{\theta+\mu}\, p(N-1,\alpha+\beta,\beta) + \frac{\mu}{\theta+\mu} P(x_j(T)=1 \mid x_i(0)=0) \end{cases}$$

from which we can compute $P(x_j(T)=1 \mid x_i(0)=1)$. If need be, we may use the above approximation to $p(N,\alpha,\beta)$.

Computing higher-order expressions seems pretty tough, though.

We get

$$\boxed{p(x_j(T)=1 \mid x_i(0)=1) = \frac{\mu\, p(N,\alpha,\beta) + \theta\, p(N-1,\alpha+\beta,\beta)(1-p(N,\alpha,\beta))}{\mu + \theta(1-p(N,\alpha,\beta))}}$$

For $i=j$, we could just replace $p(N-1,\ldots)$ by 1.

Do note, however, that this argument is a little approximate; if we start with $x_i=1$, the law of the process at the first time τ that $x_i=0$ does not have to be the invariant law conditioned by $x_i=0$.

Bayesian updating of mean + variance (4/12/97)

Suppose we observe IID Gaussian variables $Z_1, Z_2, \ldots$ with common mean α and common variance $v \equiv 1/w$, where we assume that (α, w) have the initial joint density

$$f_0(\alpha, w) = c_0 \exp\left[-\tfrac{1}{2} k_0 w (\alpha - a_0)^2 - \rho_0 w - \theta/w - \gamma_0 \log w \right]$$

where k_0, ρ_0, θ and γ_0 are positive constants, and c_0 is the appropriate normalising constant,

$$c_0 = \sqrt{\frac{k_0}{2\pi}} \cdot 2 \left(\frac{\theta}{\rho_0}\right)^{(2\gamma+3)/4} K_{\gamma+3/2}\left(2\sqrt{\rho_0 \theta}\right).$$

If we assume inductively that the posterior density $f_n(\alpha, w)$ given $Z_1, \ldots, Z_n$ has the similar form

$$\boxed{f_n(\alpha, w) = c_n \exp\left[-\tfrac{1}{2} k_n w (\alpha - a_n)^2 - \rho_n w - \frac{\theta}{w} - \gamma_n \log w \right]}$$

we get that the joint density of Z_{n+1}, α, w is

$$c_n \exp\left[-\tfrac{1}{2} w (z - \alpha)^2 - \tfrac{1}{2} k_n w (\alpha - a_n)^2 - \rho_n w - \frac{\theta}{w} - \gamma_n \log w \right] (2\pi/w)^{-\frac{1}{2}}$$

$$= c_n \exp\left[-\tfrac{1}{2} w (1 + k_n)\left(\alpha - \frac{z + k_n a_n}{1 + k_n}\right)^2 - \tfrac{1}{2} w \frac{k_n (z - a_n)^2}{1 + k_n} - \rho_n w - \frac{\theta}{w} - (\gamma_n - \tfrac{1}{2}) \log w \right] \sqrt{2\pi}$$

after some rearrangement. Hence we obtain the updating rules

$$\boxed{\begin{aligned}
k_{n+1} &= 1 + k_n \\
a_{n+1} &= \frac{z + k_n a_n}{1 + k_n} \\
\rho_{n+1} &= \rho_n + \frac{k_n (z - a_n)^2}{2(1 + k_n)} \\
c_{n+1} &= \sqrt{\frac{k_{n+1}}{2\pi}} \; 2 \left(\frac{\theta}{\rho_{n+1}}\right)^{(2\gamma_{n+1}+3)/4} K_{\gamma_{n+1} + \frac{3}{2}}\left(2\sqrt{\rho_{n+1} \theta}\right) \\
\gamma_{n+1} &= \gamma_n - \tfrac{1}{2}
\end{aligned}}$$

Some Gaussian term-structure things (31/12/97)

Reading Mantion's thesis, I find it helpful to express some of the things he's doing in my own terms. He sets up a model where domestic and foreign bonds have deterministic volatility:

$$\frac{dP^d(t,T)}{P^d(t,T)} = r^d(t)\,dt + \Sigma^d(t,T)\,dW_t^d$$

$$\frac{dP^f(t,T)}{P^f(t,T)} = b(t)\,dt + \Sigma^f(t,T)\,dW_t^f$$

$$dS_t/S_t = b^s(t)\,dt + \Sigma^s(t)\,dW_t^s$$

and S_t is the price of 1 unit of foreign currency in domestic. W^d, W^f, W^s are BMs in the domestic risk neutral measure, and Σ^d, Σ^f, Σ^s are all deterministic. From this we conclude that

$$\exp\left(-\int_0^t r^d(s)\,ds\right) P^d(t,T) = \exp\left[\int_0^t \Sigma^d(s,T)\,dW_s^d - \tfrac{1}{2}\int_0^t \Sigma^d(s,T)^2\,ds\right] P^d(0,T)$$

thus

$$\boxed{\int_0^T r^d(s)\,ds = -\int_0^T \Sigma^d(s,T)\,dW_s^d - \log P^d(0,T) + \tfrac{1}{2}\int_0^T \Sigma^d(s,T)^2\,ds}$$

We notice also that $S_t \exp\left(\int_0^t r^f(s)\,ds\right)$ is a traded asset in the home country*, so we conclude that

$$S_t\, \beta_t^d / \beta_t^f \quad \text{is a mg} \qquad \left[\beta_t^k \equiv \exp\left(-\int_0^t r^k(u)\,du\right)\right]$$

whence

$$\boxed{dS_t = S_t\left[\Sigma^S(t)\,dW_t^S + \{r^d(t) - r^f(t)\}\,dt\right]}$$

We also have to have $S_t P^f(t,T)$ is a domestic traded asset, and

$$d\left(S_t P^f(t,T)\right) = S_t P^f(t,T)\left[\Sigma^f(t,T)\,dW_t^f + \Sigma_t^S\,dW_t^S + \{b(t) + r^d(t) - r^f(t) + \rho_{fs}\Sigma^f(t,T)\Sigma_t^S\}\,dt\right]$$

so we conclude that

$$\boxed{b(t) = r^f(t) - \rho_{fs}\,\Sigma^f(t,T)\,\Sigma_t^S}$$

---

* It's the domestic price of a foreign savings account

Now we can also exploit the fact that $\beta^d_t S_t P^f(t,T)$ is a log-Gaussian martingale:

$$\beta^d_T \, S_T = S_0 \, P^f(0,T) \exp\Big[\int_0^T \{\Sigma^f(t,T)\, dW^f_t + \Sigma^S(t)\, dW^S_t\} - \tfrac{1}{2} \int_0^T \{\Sigma^f(t,T)^2 + 2\rho_{fS}\,\Sigma^f(t,T)\Sigma^S(t) + \Sigma^S(t)^2\}\, dt \Big]$$

This gives us that

$$S_T = S_0 \, P^f(0,T) \exp\Big[\int_0^T \{ \Sigma^f(t,T)\, dW^f_t + \Sigma^S(t)\, dW^S_t - \Sigma^d(t,T)\, dW^d_t \} - \tfrac{1}{2}\int_0^T (\Sigma^f(t,T)^2 + 2\rho_{fS}\, \Sigma^f(t,T)\Sigma^S(t) + \Sigma^S(t)^2 - \Sigma^d(t,T)^2)\, dt \Big] / P^d(0,T)$$

and likewise

$$\beta^d_T S_T / \beta^f_T = \exp\Big[\int_0^T \Sigma^S(t)\, dW^S_t - \tfrac{1}{2} \int_0^T \Sigma^S(t)^2 dt \Big] S_0$$

Hence

$$\boxed{\beta^f_T = P^f(0,T) \exp\Big[\int_0^T \Sigma^f(t,T)\, dW^f_t - \tfrac{1}{2} \int_0^T \{\Sigma^f(t,T)^2 + 2\rho_{fS}\, \Sigma^f(t,T)\Sigma^S(t)\}\, dt \Big]}$$

($dW^f_t - \rho_{fS}\,\Sigma^S(t)\,dt$ is a foreign-risk-neutral BM)

It's a simple exercise to show that if $X = e^{\xi}$, $Y = e^{\eta}$ are log-Gaussian with mean 1, and $\mathrm{cov}(\xi,\eta) = V$, then for positive a, b

$$E\big[(aX - bY)^+\big] = E\big[Y\,(a e^{\xi-\eta} - b)^+\big] = E\,(a e^{\gamma} - b)^+$$

where $\mathrm{var}\,\gamma = \mathrm{var}(\xi - \eta)$, and $E\, e^{\gamma} = 1$.

$f^*(t) \equiv \sup\{-f(\theta) + t\theta\}$

$\tilde{U}(\lambda) \equiv \sup\{U(x) - x\lambda\}$

An optimal investment problem with liquidity effects (8/1/98)

Let's suppose that the base price S_t of some share follows $dS_t = S_t(\sigma dW_t + \mu dt)$ and at time t investor holds $\tilde{\xi}_t$ in cash and η_t shares. If the investor tries to change η at rate θ, he will have to pay $f(\theta)S_t$ for each share, where we suppose $f: \mathbb{R} \to \mathbb{R}$ is convex increasing, $f(0)=0$, $f'(0)=1$. Thus the dynamics are

$$\begin{cases} d\eta_t = \theta_t \, dt \\ d\tilde{\xi}_t = \{ r\tilde{\xi}_t - c_t - S_t f(\theta_t) \} dt \end{cases}$$

The objective is to

$$\max E\left[\int_0^\infty e^{-\rho t} U(c_t) dt \right]$$

where $U(x) = x^{1-R}/(1-R)$. Let's write $c_t \equiv \gamma_t S_t$, so that the objective is to

$$\max E\left[\int_0^\infty S_0^{1-R} \exp\{ \sigma(1-R)W_t + (\mu - \sigma^2/2)(1-R)t \} e^{-\rho t} U(\gamma_t) \, dt \right]$$

Defining $\xi_t \equiv \tilde{\xi}_t / S_t$, we obtain the dynamics

$$\boxed{\begin{aligned} d\eta_t &= \theta_t \, dt \\ d\xi_t &= \{ r\xi_t - \gamma_t - f(\theta_t) \} dt - \xi_t \{ \sigma dW_t + (\mu - \sigma^2) dt \} \end{aligned}}$$

Thus if we define

$$v(\xi, \eta) \equiv \max E\left[\int_0^\infty \exp\left[\sigma(1-R)W_t + (\mu - \sigma^2/2)(1-R)t\right] U(\gamma_t) e^{-\rho t} dt \,\Big|\, \xi_0 = \xi, \eta_0 = \eta \right]$$

we have in the usual way

$$\boxed{\int_0^t S_v^{1-R} U(\gamma_v) e^{-\rho v} dv + e^{-\rho t} S_t^{1-R} v(\xi_t, \eta_t) \quad \text{is a supermartingale}}$$

Hence the HJB equations:

$$\boxed{\sup_{\gamma, \theta} \left\{ U(\gamma) - \rho v + (1-R)(\mu - \tfrac{\sigma^2 R}{2}) v + v_\xi \left((r - \mu + R\sigma^2)\xi - \gamma - f(\theta) \right) + \tfrac{1}{2}\sigma^2 \xi^2 v_{\xi\xi} + \theta v_\eta \right\} = 0}$$

whence

$$\boxed{\tilde{U}(v_\xi) - (\rho + (\tfrac{\sigma^2 R}{2} - \mu)(1-R)) v + \xi v_\xi (r - \mu + R\sigma^2) + \tfrac{1}{2}\sigma^2 \xi^2 v_{\xi\xi} + v_\xi \cdot f^*(v_\eta / v_\xi) = 0}$$

Even the case $\sigma = 0$ is non-trivial !? We have

$$d\xi_t = (r-\mu)\xi_t\,dt - (\gamma_t + f(\theta_t))\,dt$$

so $\xi_t = e^{(r-\mu)t}\left(\xi_0 - \int_0^t e^{(\mu-r)s}(\gamma_s + f(\theta_s))\,ds\right)$, so that the problem becomes

$$\boxed{\begin{array}{ll} \max & \int_0^\infty \exp((\mu(1-R)-\rho)t)\, U(\gamma_t)\,dt \\ \text{subject to} & \int_0^\infty \theta_t\,dt = -\eta_0 \\ & \int_0^\infty e^{(\mu-r)t}(\gamma_t + f(\theta_t))\,dt = \xi_0 \end{array}}$$

However, we see that if $\mu \neq r$, the problem is ill-posed; take $\theta = 0$ for a very long time, and choose γ = constant. Then ξ converges to the equilibrium value $-\gamma/(\mu-r)$. After a long time, take γ to be something nearer 0 and continue ... we can this way make arbitrarily large wealth. If $\mu = r$, you delay moving wealth over, and do it infinitesimally slowly. The overall effect is as if $\eta_0 = 0$, $\xi_0 + \eta_0$ is initial value of bank account.

Approximate higher moments for tickdata model from physics (14/1/98)

(i) We have $I = \{0,1\}^N$, $x = (x_1, \dots, x_N) \in I$ is interpreted as $x_i = 1 \Leftrightarrow$ share i is excited, $x_i = 0 \Leftrightarrow$ share i is quiet. Excited shares become quiet at rate μ, and invariant dist^{n} is

$$\pi(x) \propto \exp\left[\tfrac{1}{2} b \sum_{i \neq j}\sum x_i x_j + a \sum_i x_i \right]$$

We can compute exactly

$$\pi(N,a,b) \equiv P(x_i(0)=1) = \frac{\sum_{k=0}^{N-1} \binom{N-1}{k} \exp\{ak + a + \tfrac{1}{2} b\, k(k+1)\}}{\sum_{k=0}^{N} \binom{N}{k} \exp\{ak + \tfrac{1}{2} b\, k(k-1)\}}$$

$$\equiv e^{a} \quad h(N-1, a+b, b) / h(N,a,b),$$

with $h(N,a,b) \equiv \sum_0^N \binom{N}{k} \exp\{ak + \tfrac{1}{2} b\, k(k-1)\}$. Now let's introduce

$$\psi_{ij}(\theta; N,a,b) \equiv P[x_j(\tau_1) = 1 / x_i(0) = 1], \quad \varphi_{ij}(\theta; N,a,b) = P[x_j(\tau_1) = 1 / x_i(0) = 0],$$

$$\psi_{ijk}(\theta,\lambda; N,a,b) \equiv P[x_j(\tau_1) = 1, x_k(\tau_2) = 1 \mid x_i(0) = 1], \quad \varphi_{ijk}(\theta,\lambda; N,a,b) = P[x_j(\tau_1) = 1 = x_k(\tau_2) \mid x_i(0) = 0]$$

where $\tau_1 \sim \exp(\theta)$, $\tau_2 - \tau_1 \sim \exp(\lambda)$. We have already that

$$\begin{cases} \psi_{ij} \doteq \dfrac{\theta}{\theta+\mu}\left\{\pi(N-1, a+b, b)(1-\delta_{ij}) + \delta_{ij}\right\} + \dfrac{\mu}{\theta+\mu}\varphi_{ij} \\[2ex] \pi(N,a,b) = \pi(N,a,b)\,\psi_{ij} + (1-\pi(N,a,b))\,\varphi_{ij} \end{cases}$$

which allows us to deduce ψ_{ij}, φ_{ij}, approximately. We also have the equations

$$\begin{cases} \psi_{ijk} \doteq \dfrac{\theta}{\theta+\mu}\cdot\dfrac{\lambda}{\lambda+\mu}\cdot\left\{\pi(N-1,a+b,a)\,\psi_{jk}(\lambda+\mu; N-1, a+b, b)(1-\delta_{ij})(1-\delta_{ik}) + \{\delta_{ij}(1-\delta_{ik}) + \delta_{ik}(1-\delta_{ij})\}\pi(N-1,a+b,b) + \delta_{ij}\delta_{ik}\right\} \\[2ex] \qquad + \dfrac{\theta}{\theta+\mu}\cdot\dfrac{\mu}{\lambda+\mu}\cdot\left\{\delta_{ij} + (1-\delta_{ij})\pi(N-1,a+b,b)\right\}\varphi_{ik}(\lambda; N,a,b) \\[2ex] \qquad + \dfrac{\mu}{\theta+\mu}\varphi_{ijk}, \\[2ex] \pi(N,a,b)\,\psi_{ijk} + (1-\pi(N,a,b))\,\varphi_{ijk} = \pi(N,a,b)\,\psi_{jk}(\lambda; N,a,b) \end{cases}$$

and solving these will give ψ_{ijk}, φ_{ijk}.

The last stage we need to cover is calculating

$$\psi_{ijkl}(\theta,\lambda,\rho; N,a,b) = P[x_j(\tau_1) = 1 = x_k(\tau_2) = x_l(\tau_3) \mid x_i(0) = 1]$$

$$\varphi_{ijkl}(\theta,\lambda,\rho; N,a,b) = P[\quad\text{———————}\quad \mid x_i(0) = 0\ ]$$

The equations this time are

$$\psi_{ijkl} \doteqdot \frac{\theta}{\theta+\mu}\cdot\frac{\lambda}{\lambda+\mu}\cdot\frac{\rho}{\rho+\mu}\Big\{(1-\delta_{ij})(1-\delta_{ik})(1-\delta_{il})\,\psi_{jkl}(\lambda+\mu,\rho+\mu;N-1,a+b,b)\,\pi(N-1,a+b,b)$$

$$+\ \delta_{ij}(1-\delta_{ik})(1-\delta_{il})\,\pi(N-1,a+b,b)\,\psi_{kl}(\rho+\mu;N-1,a+b,b)$$

$$+\ (1-\delta_{ij})\,\delta_{ik}\,(1-\delta_{il})\,\pi(N-1,a+b,b)\,\tilde{\psi}_{jl}(\lambda+\mu,\rho+\mu;N-1,a+b,b)$$

$$+\ (1-\delta_{ij})(1-\delta_{ik})\,\delta_{il}\,\pi(N-1,a+b,b)\,\psi_{jk}(\lambda+\mu;N-1,a+b,b)$$

$$+\Big\{(1-\delta_{ij})\,\delta_{ik}\,\delta_{il}+\delta_{ij}\,(1-\delta_{ik})\,\delta_{il}+\delta_{ij}\,\delta_{ik}\,(1-\delta_{il})\Big\}\,\pi(N-1,a+b,b)$$

$$+\ \delta_{ij}\,\delta_{ik}\,\delta_{il}\ \Big\}\ +$$

$$+\ \frac{\theta}{\theta+\mu}\cdot\frac{\lambda}{\lambda+\mu}\cdot\frac{\mu}{\rho+\mu}\Big\{\delta_{ij}\delta_{ik}+\big(\delta_{ij}(1-\delta_{ik})+\delta_{ik}(1-\delta_{ij})\big)\pi(N-1,a+b,b)+(1-\delta_{ij})(1-\delta_{ik})\,\pi(N-1,a+b,b)\,\psi_{jk}(\lambda+\mu,N-1,a+b,b)\Big\}\cdot$$

$$\varphi_{il}(\rho,N,a,b)$$

$$+\ \frac{\theta}{\theta+\mu}\cdot\frac{\mu}{\lambda+\mu}\Big\{\delta_{ij}+(1-\delta_{ij})\,\pi(N-1,a+b,b)\Big\}\,\varphi_{ikl}(\lambda,\rho;N,a,b)\ +\ \frac{\mu}{\theta+\mu}\,\varphi_{ijkl}(\theta,\lambda,\rho;N,a,b),$$

$$\pi(N,a,b)\,\psi_{ijkl}(\theta,\lambda,\rho;N,a,b)+(1-\pi(N,a,b))\,\varphi_{ijkl}(\theta,\lambda,\rho;N,a,b)=\pi(N,a,b)\,\psi_{jkl}(\lambda,\rho;N,a,b)$$

where $\tilde{\psi}_{jl}$ is obtained by solving the same equations as for ψ_{jl}, except that $\frac{\theta}{\theta+\mu}$ becomes $\frac{\theta_1}{\theta_1+\mu}\cdot\frac{\theta_2}{\theta_2+\mu}$ and $\frac{\mu}{\theta+\mu}$ becomes $1-\frac{\theta_1}{\theta_1+\mu}\cdot\frac{\theta_2}{\theta_2+\mu}$.

(ii) How about the expressions $(\pi, f_i R_\alpha f_j)$ etc? If we express

$$f_i = u(i) + v(i)\, I_{\{x_i=1\}} \equiv u(i) + v(i)\, e_i \quad , \quad \text{say}$$

then we obtain

$$g1(i,j,\beta) \equiv (\pi, f_i R_\beta f_j)$$

$$= (\pi, (u_i + v_i e_i) R_\beta (u_j + v_j e_j))$$

$$= \frac{u_j}{\beta}\,(u_i + v_i\,\pi(N,a,b)) + v_j\left(\frac{u_i}{\beta}\,\pi(N,a,b) + \frac{v_i}{\beta}\,\pi(N,a,b)\,\psi_{ij}(\beta;N,a,b)\right),$$

and

$$g2(i,j,k,\alpha,\beta) \equiv (\pi, f_i R_\alpha f_j R_\beta f_k)$$

$$= \frac{u_k}{\beta}\, g1(i,j,\alpha) + v_k\,(\pi, (u_i+v_i e_i) R_\alpha (u_j+v_j e_j) R_\beta e_k)$$

$$= \frac{u_k}{\beta}\, g1(i,j,\alpha) + v_k\Big[\frac{u_i u_j}{\alpha\beta}\,\pi(N,a,b) + \frac{u_i v_j}{\alpha\beta}\,\pi(N,a,b)\,\psi_{jk}(\beta;N,a,b)$$

$$+ \frac{u_j v_i}{\alpha-\beta}\,\pi(N,a,b)\left\{ \frac{\psi_{ik}(\beta;N,a,b)}{\beta} - \frac{\psi_{ik}(\alpha;N,a,b)}{\alpha}\right\}$$

$$+ \frac{v_i v_j}{\alpha\beta}\,\pi(N,a,b)\,\psi_{ijk}(\alpha,\beta;N,a,b)\Big]$$

and

$$g3(i,j,k,\ell,\alpha,\beta,\lambda) \equiv (\pi, f_i R_\alpha f_j R_\beta f_k R_\lambda f_\ell)$$

$$= \frac{u_\ell}{\lambda}\, g2(i,j,k,\alpha,\beta) + v_\ell\,(\pi, (u_i+v_i e_i) R_\alpha (u_j+v_j e_j) R_\beta (u_k+v_k e_k) R_\lambda e_\ell)$$

$$= \frac{u_\ell}{\lambda}\, g2(i,j,k,\alpha,\beta) + \frac{v_\ell u_i}{\alpha}\, g2(j,k,\ell,\beta,\lambda)$$

$$+ v_\ell v_i\,(\pi, e_i R_\alpha (u_j+v_j e_j) R_\beta (u_k+v_k e_k) R_\lambda e_\ell)$$

$$= \frac{u_\ell}{\lambda}\, g2(i,j,k,\alpha,\beta) + \frac{v_\ell u_i}{\alpha}\, g2(j,k,\ell,\beta,\lambda)$$

$$+ v_\ell v_i\Big[u_k u_j\,\pi(N,a,b)\left\{ \frac{\psi_{i\ell}(\alpha)/\alpha}{(\beta-\alpha)(\lambda-\alpha)} + \frac{\psi_{i\ell}(\beta)/\beta}{(\alpha-\beta)(\lambda-\beta)} + \frac{\psi_{i\ell}(\lambda)/\lambda}{(\alpha-\lambda)(\beta-\lambda)}\right\}$$

$$+ \frac{u_k v_j}{\beta-\lambda}\,\pi(N,a,b)\left(\frac{\psi_{ij\ell}(\alpha,\lambda)}{\alpha\lambda} - \frac{\psi_{ij\ell}(\alpha,\beta)}{\alpha\beta}\right)$$

$$+ \frac{u_j v_k}{\alpha-\beta}\,\pi(N,a,b)\left(\frac{\psi_{ik\ell}(\beta,\lambda)}{\beta\lambda} - \frac{\psi_{ik\ell}(\alpha,\lambda)}{\alpha\lambda}\right)$$

$$+ v_j v_k\,\pi(N,a,b)\,\frac{\psi_{ijk\ell}(\alpha,\beta,\lambda)}{\alpha\beta\lambda}\Big]$$

where the supplementary arguments N,a,b for ψ's have been omitted throughout for brevity.

Discretely-sampled barrier options: how to correct (4/2/98)

If $\log(S_t/S_0) = \sigma W_t + \mu t$, and we have a down-and-out call option with strike K, expiry T, knock-out barrier at $S_0 e^c$, then the time-zero value of the option is

$$e^{-rT} E\left[(S_T - K)^+ ; \inf_{u \le T} S_u \ge S_0 e^c \right] \qquad [\mu \equiv r - \tfrac{1}{2}\sigma^2]$$

However, suppose we only monitor discretely, at the times $jT/N \equiv j\delta$; then the option is worth

$$e^{-rT} E\left[(S_T - K)^+ : \inf X(j\delta) \ge c \right]$$

where $X_t \equiv \sigma W_t + \mu t$. If we knew enough about the random variable

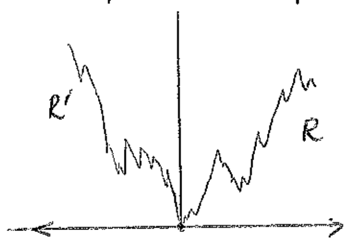
$$Y \equiv P\left[\inf_{0 \le j \le N} X(j\delta) \ge c \mid S_T, \underline{X}_T \right] \qquad \left[\underline{X}_t \equiv \inf_{u \le t} X_u \right]$$

then the price could be expressed as

$$e^{-rT} E\left[(S_T - K)^+ \, Y \right]$$

and evaluated.

However, for h small we have an approximation to Y based on path decomposition. Near the minimum, we see effectively two independent back-to-back BES(3) processes R and R', and assuming that $\inf_{j \le N} X(j\delta)$ is attained at one of the ends of the interval where X attains its minimum, we have approximately



$$\inf_{j \le N} X(j\delta) - \underline{X}_T \doteq R(hU) \wedge R'(h(1-U)) \equiv \xi_h$$

where U is $U[0,1]$ independent of the rest, and $h \equiv \sigma^2 \delta$. Now

$$P[\xi_h > x] \equiv \bar{F}_h(x) = \int_0^1 du \int_x^\infty dy \int_x^\infty dz \, \frac{2y^2 e^{-y^2/2hu}}{\sqrt{2\pi h^3 u^3}} \cdot \frac{2z^2 e^{-z^2/2h(1-u)}}{\sqrt{2\pi h^3 (1-u)^3}}$$

$$= \frac{1}{h} \int_x^\infty dy \int_x^\infty dz \, 4yz \, \frac{(y+z) e^{-(y+z)^2/2h}}{\sqrt{2\pi h^3}}$$

whence

$$\boxed{P\left[\frac{\xi_h}{\sqrt{h}} > v \right] \equiv \bar{F}(v) = \int_v^\infty dy \int_v^\infty dz \, \frac{4yz \cdot \exp\{-(y+z)^2/2\} \cdot (y+z)}{\sqrt{2\pi}}}$$

Hence we may approximate the price by

$$e^{-rT}\, E\left[(S_T-K)^+ \ \bar{F}\left(\frac{c-X_T}{\sqrt{h}}\right) : X_T>c \right]$$

$$= e^{-rT} \int_{-\infty}^{0} dy \int_{y\vee c}^{\infty} dx \ (S_0 e^x - K)^+ \ \bar{F}\left(\frac{c-y}{\sqrt{h}}\right) \ \frac{2(x-2y)}{\sqrt{2\pi\tau^3}} \exp\left[-\frac{(x-2y)^2}{2\tau} + ax - \tfrac{1}{2}a^2\tau\right]$$

where $\tau \equiv \sigma^2 T$, $a \equiv \mu\sigma^{-2}$. This exceeds the price of the continuously-monitored barrier call by

$$e^{-rT} \int_{-\infty}^{c} dy \int_{c}^{\infty} dx \ (S_0 e^x - K)^+ \ \bar{F}\left(\frac{c-y}{\sqrt{h}}\right) \ \frac{2(x-2y)}{\sqrt{2\pi\tau^3}} \exp\left[-\frac{(x-2y)^2}{2\tau} + ax - \tfrac{1}{2}a^2\tau\right].$$

We can simplify the y-integral somewhat - it is (assuming, of course, that $x \geq c$)

$$\int_{-\infty}^{c} dy \ \bar{F}\left(\frac{c-y}{\sqrt{h}}\right) \ \frac{2(x-2y)}{\tau} \exp\left[-\frac{(x-2y)^2}{2\tau}\right]$$

$$\left(\frac{c-y}{\sqrt{h}} = v\right)$$

$$(*) = \int_0^\infty \sqrt{h}\, dv \ \bar{F}(v) \ \frac{2}{\tau}(x-2c+2v\sqrt{h}) \exp\left[-(x-2c+2v\sqrt{h})^2/2\tau\right]$$

$$\sim \sqrt{h} \int_0^\infty \bar{F}(v) \ \frac{2(x-2c)}{\tau} \exp\left\{-(x-2c)^2/2\tau\right\} dv \qquad \text{as } h \downarrow 0$$

$$= \sqrt{h} \ \frac{2(x-2c)}{\tau} \exp\left\{-(x-2c)^2/2\tau\right\} \int_0^\infty dv \int_v^\infty dy \int_v^\infty dz \ \frac{4yz(y+z)\exp(-(y+z)^2/2)}{\sqrt{2\pi}}$$

$$= \sqrt{h} \ \frac{2(x-2c)}{\tau} \exp\left\{-(x-2c)^2/2\tau\right\} \cdot \frac{5}{3\sqrt{2\pi}}.$$

Hence the correction is

$$\frac{5\sqrt{h}}{3\sqrt{2\pi}} \int_c^\infty e^{-rT}(S_0 e^x - K)^+ \ \frac{2(x-2c)}{\tau} \exp\left[-\frac{(x-2c)^2}{2\tau} + ax - \tfrac{1}{2}a^2\tau\right] dx$$

$$= \frac{5}{3}\left(\frac{h}{2\pi}\right)^{\frac{1}{2}} e^{-rT-\frac{1}{2}a^2\tau} \int_{c\vee \log K/S_0}^{\infty} (S_0 e^x - K) \ 2e^{ax} \ d\left(-e^{-(x-2c)^2/2\tau}\right)$$

$$= \frac{5}{3}\left(\frac{k}{2\pi}\right)^{\frac{1}{2}} e^{-rT-\frac{1}{2}a^2\tau}\left[2\left(S_0 e^{b}-K\right) e^{-(b-2c)^2/2\tau+ab} + 2\int_b^\infty e^{-(x-2c)^2/2\tau}\left\{S_0 e^{(1+a)x}(1+a) - Kae^{ax}\right\}dx\right]$$

$$b \equiv cv\log(K/S_0)$$

$$= \frac{10}{3}\sqrt{\frac{k}{2\pi}}\, e^{-rT-\frac{1}{2}a^2\tau}\left\{\left(S_0 e^{b}-K\right)e^{-(b-2c)^2/2\tau+ab} + S_0(1+a)\exp\left(\frac{\tau(1+a)^2}{2}+2c(1+a)\right)\bar{\Phi}\left(\frac{b-2c-(1+a)\tau}{\sqrt{\tau}}\right)\sqrt{2\pi\tau}\right.$$

$$\left. - Ka\exp\left(\frac{\tau a^2}{2}+2ac\right)\bar{\Phi}\left(\frac{b-2c-a\tau}{\sqrt{\tau}}\right)\sqrt{2\pi\tau}\right\}$$

This appears to be not very good (BGK is much closer to true value). One thing we could do is to take higher terms in the expansion of (*). Using Maple, I get that (*) is

$$\frac{\sqrt{k}}{\sqrt{2\pi}}\, e^{-(x-2c)^2/2\tau}\left\{\frac{10}{3}(x-2c) + \sqrt{k}\,\frac{\left(\tau-(x-2c)^2\right)}{\tau^2}\,\frac{9}{8}\sqrt{2\pi} + k\cdot\frac{28}{15}\cdot\frac{(x-2c)\left((x-2c)^2-3\tau\right)}{\tau^3}\right\}$$

Another approach is to use the approximation to the law of the overshoot from Rogers + Satchell. The key thing to highest order is $\int \bar{F}(v)dv = E(\text{overshoot}) = \sqrt{2\pi}\left(\frac{1}{4} - \frac{\sqrt{2}-1}{\epsilon}\right) \simeq .4536$, from that paper. The second moment $\int 2v\bar{F}(v)dv$ is also there; it's $\frac{1}{12}\{1+3\pi/4\}$.

Coupling and weak convergence for discrete-time Markov processes (17/2/98)

(i) Consider a discrete-time Markov process with Polish statespace S, and transition kernel P which satisfies the Harris conditions:

(H1) There is some set A s.t. for every $x \in S$

$$P^x(\tau_A < \infty) > 0 \qquad (\tau_A \equiv \inf\{n : X_n \in A\})$$

(H2) There is a probability measure ρ concentrated on a set B such that for some $\varepsilon > 0$

$$P(x, C) \geq \varepsilon \rho(C) \qquad \forall x \in A, \quad \forall C \subseteq B.$$

Then from the theory of Harris chains (see, for example, Durrett "Probability: Theory and Examples" pp 282 – 290) if we can prove that $E^\rho(\tau_A) < \infty$ then there exists an invariant distribution, and if $P^x(\tau_A < \infty) = 1$ for all x, and the chain is aperiodic, then we get convergence to the invariant distribution.

To verify these conditions will usually require some specific information about the process.

(ii) With more structure, we can get more. Consider an example of the kind studied by Ole Barndorff-Nielsen[+] where

$$Y_n \sim N(0, Z_n)$$

$$Z_n \sim F(\varphi(Y_{n-1}^2 + \cdots + Y_{n-k}^2))$$

where φ is concave and increasing, and $F(\theta)$ is the law of the first passage time to level θ of a Brownian motion with fixed variance and fixed positive drift.

It's clear that if we start off with $Y_{-k} = Y_{-k+1} = \cdots = Y_0 = 0$, and assuming $\varphi(0) > 0$, then the laws of $Y_1^2, Y_2^2, \ldots$ are stochastically increasing. Next

$$\begin{aligned} E\, Y_n^2 &= c\, E\, \varphi(Y_{n-1}^2 + \cdots + Y_{n-k}^2) \\ &\leq c\, \varphi(E\, Y_{n-1}^2 + \cdots + E\, Y_{n-k}^2) \\ &\leq c\, \varphi(k . E\, Y_{n-1}^2) \end{aligned}$$

So if the sequence b_n is defined by $b_1 = EY_1^2$,

$$b_n = c\, \varphi(k\, b_{n-1}) \qquad (n \geq 2)$$

then the b_n increase, and provided $x = c\,\varphi(kx)$ has a (necessarily unique) fixed point b^*, then $b_n \leq b^*$ for all n, and $EY_n^2 \leq b^*$ for all n. Thus the

[+] Scand. J. Statistics $\underset{\sim}{24}$ 1 – 14, 1997. 'Normal inverse Gaussian distributions and stochastic volatility modelling'. (pp 9–10) laws of Y_n^2 converge weakly to a non-degenerate limit law.

If we are given any values $\tilde{Y}_{-k}, \tilde{Y}_{-k+1}, \dots, \tilde{Y}_0$ we can build a sequence $(\tilde{Y}_n^2)_{n\geq 0}$ and a sequence $(Y_n^2)_{n\geq 0}$ – ~~starting~~ ing with $Y_{-k} = Y_{-k+1} = \dots = Y_0 = 0$ – such that for all n

$$\tilde{Y}_n^2 \geq Y_n^2 \quad \text{a.s.}$$

Indeed, if we have $\tilde{Y}_j^2 \geq Y_j^2 \ \forall j \leq n-1$, we can ensure that $\tilde{Z} \geq Z$, and then pick the pair $(\tilde{Y}_n, Y_n)$ to be equal with maximal probability

$$1 - 2\bar{\Phi}\left(\sqrt{\frac{t^*}{\tilde{Z}_n}}\right) + 2\bar{\Phi}\left(\sqrt{\frac{t^*}{Z_n}}\right), \qquad t^* = \frac{Z_n \tilde{Z}_n}{\tilde{Z}_n - Z_n}\log(\tilde{Z}_n/Z_n)$$

and otherwise $\tilde{Y}_n^2 > Y_n^2$.

We now have to prove that $\tilde{Y}_n^2 - Y_n^2 \to 0$ a.s., if possible.

(iii) It turns out that it is simpler to use the Harris chain structure. If we consider the discrete-time Markov process

$$X_n \equiv (\tilde{Y}_{nk}^2, \tilde{Y}_{nk-1}^2, \dots, \tilde{Y}_{nk-k+1}^2)$$

and the event $A_n = \{\|X_n\| \leq 1\}$, then by Lévy's Borel-Cantelli lemma $\{\sum I_{A_n} < \infty\} = \{\sum P(A_n|\mathcal{F}_{n-1}) < \infty\}$ a.s. But $P(A_n|\mathcal{F}_{n-1}) = \varphi(X_{n-1})$ for some function φ which is strictly positive and continuous. Thus $\sum P(A_n|\mathcal{F}_{n-1}) < \infty$ $\Rightarrow \|X_n\| \to \infty$, and if this happened with positive probability we would have a contradiction of the L^1 boundedness of the X_n. Hence $P(\|X_n\| \leq 1 \text{ i.o.}) = 1$. Given X_n, the distribution of X_{n+1} has a positive continuous density, so we may take $C = A = \{x : \|x\| \leq 1\}$ in the definition of a Harris chain, with ρ to be uniform distribution over C. We've proved that the Harris chain is recurrent, but is it positive recurrent? Yes; the limit distribution of the Y's provides an invariant probability distribution, and we can use Theorem III.11.2 of Lindvall's book.

Assuming $\varphi(p) \sim \varphi(0) + \text{const}\ p^{1-R}$ for small p, and φ is C^2 at 1, we seek the behaviour of $\varphi(p) - \varphi(0)$

Simple liquidity effects model again (2/3/98)

(i) We return to the situation of WN XIV, p 39, where the holdings x_t in the bank and y_t in the share are governed by

$$dx_t = rx_t\,dt - c_t\,dt + \theta_t\,dN_t$$
$$dy_t = y_t(\sigma\,dW_t + \alpha\,dt) - \theta_t\,dN_t$$

and the objective is to

$$\max E\left[\int_0^\infty e^{-\rho t}\,U(c_t)\,dt \;\middle|\; x_0 = x,\, y_0 = y\right] \equiv V(x,y), \qquad U(x) \equiv x^{1-R}/(1-R)$$

Here, N is a standard Poisson process of rate λ. We can rephrase the problem as

$$\max E\left[\int_0^{\tau_1} \exp(-\rho t)\,U(c_t)\,dt + e^{-\rho\tau_1}\,a\,U(x_{\tau_1}, y_{\tau_1}) \;\middle|\; x_0 = x,\, y_0 = y\right]$$

where the constant a needs to be determined (this is v. similar to WN XIV p46, except we don't presume the behaviour in $(0,\tau_1)$). The resulting HJB will be

$$\max_{c\geq 0}\left[U(c) - \rho V + (rx-c)V_x + \tfrac{1}{2}\sigma^2 y^2 V_{yy} + \alpha y V_y + \lambda\{a\,U(x+y) - V(x,y)\}\right] = 0$$

Naturally, we exploit scaling: $V(x,y) = (x+y)^{1-R}\,\varphi(x/x+y) \equiv (x+y)^{1-R}\,\varphi(p)$, and get

$$\boxed{\begin{aligned}&\left(\frac{R}{1-R}\right)\left\{(1-R)\varphi + (1-p)\varphi'\right\}^{1-1/R} - \rho\varphi + rp\{(1-R)\varphi + (1-p)\varphi'\} + \alpha(1-p)\{(1-R)\varphi - p\varphi'\}\\ &+ \tfrac{1}{2}\sigma^2(1-p)^2\{-R\,\varphi(1-R) + 2Rp\varphi' + p^2\varphi''\} + \lambda\left(\frac{a}{1-R} - \varphi\right) = 0\end{aligned}}$$

(This is exactly what we got on p 43 of WN XIV, in fact ...)

(ii) If we now assume for concreteness that $\boxed{0 < R < 1}$ and for finiteness that

$$\theta \equiv \lambda + \rho - (1-R)\left(\alpha - \tfrac{\sigma^2}{2}R\right) > 0$$

then we know that the optimal φ will be non-negative. It seems likely that it will also be unimodal; and we can derive boundary conditions at 0 and 1. Indeed, it's easy to see that

$$V(0,y) = y^{1-R}\,\bar{\varphi}\,\lambda/\theta \qquad \left(\bar{\varphi} \equiv \sup_t \varphi(t)\right)$$

whence

$$\boxed{\varphi(0) = \lambda\bar{\varphi}/\theta}$$

Likewise, by considering the initial condition $y = 0$ we conclude that

$$\boxed{\varphi(1) = \gamma^{1-R}/(1-R),}$$

where γ solves

$$\gamma^{-R}\Big[\rho+\lambda-r(1-R)-\gamma R\Big] = \lambda\bar{\varphi}(1-R)$$

(iii) Another possible approach is to note that the optimal c_t is of the form $c_t = w_t f(p_t)$, where $w_t \equiv x_t + y_t$. Thus the proportion $p_t \equiv x_t/(x_t+y_t)$ is an autonomous diffusion

$$dp_t = -\sigma p_t(1-p_t)\,dW_t + (1-p_t)\{rp_t - \alpha p_t + \sigma^2 p_t(1-p_t) - f(p_t)\}dt + \theta\frac{dN_t}{w_t}$$

apart from the jumps. (this is from p40, WN XIV). Thus

$$d\log\frac{p}{1-p} = -\sigma dW + \Big[r-\alpha+\sigma^2(1-p) - \frac{f(p)}{p}\Big]dt - \frac{\sigma^2}{2}(1-2p)\,dt + d\,\text{jump}$$

$$= -\sigma dW + \Big[r-\alpha+\tfrac{1}{2}\sigma^2 - \frac{f(p)}{p}\Big]dt + d\,\text{jump}.$$

The wealth process satisfies

$$dw_t = w_t\Big[rp_t + \alpha(1-p_t) - f(p_t)\Big]dt + \sigma(1-p_t)\,dW_t \cdot w_t$$

$\therefore$

$$w_t = w_0\exp\Big[\int_0^t \sigma(1-p_s)dW_s + \int_0^t\{(r-\alpha)p_s - f(p_s) + \alpha - \tfrac{1}{2}\sigma^2(1-p_s)^2\}ds\Big]$$

Hence the objective is

$$E\int_0^\infty \exp\{-\rho t + (1-R)\{\int_0^t\sigma(1-p_s)dW_s + \int_0^t((r-\alpha)p_s - f(p_s) + \alpha - \tfrac{1}{2}\sigma^2(1-p_s)^2)ds\}\}\, f(p_t)^{1-R}\,dt/(1-R)$$

$$= E^*\int_0^\infty \exp\Big[-\int_0^t\{\rho + (1-R)\big(f(p_s) - \alpha - (r-\alpha)p_s + \tfrac{1}{2}\sigma^2 R(1-p_s)^2\big)\}ds\Big]\, f(p_t)^{1-R}\frac{dt}{1-R}$$

where P^* is the measure equivalent to P such that $dW = dW^* + \sigma(1-R)(1-p_t)dt$, with W^* a P^*-BM. Thus when we take into account the jumps as well, we get

$$\varphi(p) = E^*\Big(\int_0^\infty \exp\{-\lambda t - \int_0^t\psi(p_s)ds\}\Big[\frac{f(p_t)^{1-R}}{1-R} + \lambda\varphi(p^*)\Big]dt \;\Big|\; p_0 = p\Big)$$

where $\psi(p) \equiv \rho + (1-R)\big(f(p) - \alpha - (r-\alpha)p + \tfrac{1}{2}\sigma^2 R(1-p)^2\big)$. We conclude that φ satisfies

$$\Big\{\lambda + \psi(p) - \tfrac{1}{2}\sigma^2 p^2(1-p)^2 D^2 - (1-p)\{(r-\alpha)p + \sigma^2 p(1-p)R - f(p)\}D\Big\}\varphi = \frac{f(p)^{1-R}}{1-R} + \lambda\varphi(p^*),$$

which is the same as we had before.

(iv) Another approach would be the following. The ODE has problems at 0 and 1, yet it seems that the proportion p is unlikely ever to reach those points! Thus we could fix some $\varepsilon > 0$, and bound the value on two sides as follows:

(a) If the proportion reaches $\{\varepsilon, 1-\varepsilon\}$, jump immediately to p^*;

(b) If the proportion reaches $\{\varepsilon, 1-\varepsilon\}$, throw away all the shares and consume only from the bank account.

The problem (a) provides an upper bound, the problem (b) provides a lower bound. Note that in (a) there may be distortion of the problem in that there would be an incentive to rush to ε in order to get back to p^*. But rushing to ε would involve excessive early consumption and depletion of capital, so hopefully this effect will be small. In case (b) we find that the boundary conditions are

$$\varphi(\varepsilon) = \varepsilon^{1-R} \frac{1}{1-R} \left\{ \frac{\rho - r(1-R)}{R} \right\}^{-R}$$

$$\varphi(1-\varepsilon) = (1-\varepsilon)^{1-R} \frac{1}{1-R} \left\{ \frac{\rho - r(1-R)}{R} \right\}^{-R}$$

(v) Let's concentrate on the case $0 < R < 1$. If we take φ and expand as a Taylor series about 1, we have that

$$a_0 \equiv \varphi(1) = \gamma^{-R}/(1-R)$$

$$a_1 = \varphi'(1) = (\alpha - r)\,\gamma^{-R} / \{\alpha - \lambda - \rho - rR + R\gamma\}$$

What makes perhaps more sense is to use the fact that we know $\varphi(0) = \lambda\kappa/\theta$ (reverting to older notation $\kappa = \bar{\varphi}$) and that $\varphi(p) \sim \varphi(0) + \text{const}\; p^{1-R}$ for $p \downarrow 0$ to rewrite

$$\varphi(p) = \varphi(0) + p^{1-R} g(p)$$

from which we obtain the de for g:

$$\frac{R}{1-R}\left(\frac{(1-R)\lambda\kappa}{\theta} + p^{-R}(1-R)g + (1-p)p^{1-R}g'\right)^{1-1/R} - \rho\left(\frac{\lambda\kappa}{\theta} + p^{1-R}g\right) + rp\left(\frac{1-R}{\theta}\lambda\kappa + p^{-R}(1-R)g + (1-p)p^{1-R}g'\right)$$

$$+\alpha(1-p)\left(\frac{(1-R)\lambda\kappa}{\theta} - p^{2-R}g'\right) + \tfrac{1}{2}\sigma^2(1-p)^2\left[\frac{(1-R)^2\lambda\kappa}{\theta} - \frac{(1-R)\lambda\kappa}{\theta} + p^{3-R}g'' + 2p^{2-R}g'\right]$$

$$+ \lambda\left\{\kappa\left(1 - \frac{\lambda}{\theta}\right) - p^{1-R}g\right\} = 0.$$

For small p, we see in effect

$$p^{1-R}\left\{\frac{R}{1-R}\left((1-R)g(0)\right)^{1-1/R} - \rho g(0) + r(1-R)g(0) - \lambda g(0)\right\} = 0$$

So we decide that $g(0)$ must solve

$$\frac{R}{1-R}\left\{(1-R)\,g(0)\right\}^{1-1/R} = g(0)\left\{\rho - r(1-R) + \lambda\right\},$$

whence

$$\boxed{g(0) = \left\{\frac{R}{\rho+\lambda-r(1-R)}\right\}^{R}(1-R)^{-1}}$$