A discrete-time version of the liquidity problem (8/3/98)

(i) Let's consider the situation of an agent investing and consuming in a world with one risky asset $dS = S(\sigma dW + \alpha dt)$ and one riskless asset with rate of return r. He aims to achieve

$$V(w) = \sup E\left(\int_0^\infty e^{-\rho t} U(c_t)\,dt \;\middle|\; w_0 = w\right)$$

where $U(x) \equiv x^{1-R}/(1-R)$. If he were allowed to continuously adjust his consumption and portfolio, the solution is the well-known Merton solution:

- $V(w) \equiv V_M(w) = \dfrac{a_M}{1-R}\, w^{1-R}$, $\quad a_M^{1/R} \equiv \dfrac{R}{\rho + (R-1)\{r + (\alpha-r)^2/2R\sigma^2\}}$
- proportion $\theta = \dfrac{\alpha - r}{\sigma^2 R}$ of wealth is held in risky asset at all times
- Consume at rate $a_M^{-1/R} \cdot w$ at all times.

However, suppose the agent is only allowed to adjust his consumption rate c and portfolio at the times kh, where $h>0$ is known. What is the best thing for him to do, and how does it relate to the solution of the Merton problem?

(ii) By scaling, it is clear that the value function again has the form

$$(1) \qquad V(w) = \frac{a}{1-R}\, w^{1-R}$$

for some $a = a(h) > 0$, and the Bellman equation gives

$$(2) \qquad V(w) = \sup\left\{ U(c)\,\tilde{h} + a e^{-\rho h} (w - ch)^{1-R}\, E\, \frac{(pZ + q e^{rh})^{1-R}}{1-R} \right\}$$

where $\tilde{h} \equiv \int_0^h e^{-\rho u}\,du \equiv (1-e^{-\rho h})/\rho$, and the sup is taken over $c>0$, and $p \in [0,1]$. Clearly, we would never choose to invest a proportion $p \notin [0,1]$ in the risky asset. We are using the notation $Z = \exp(\sigma W_h + \mu h)$, $q \equiv 1-p$, $\mu \equiv \alpha - \sigma^2/2$. If we were able to compute

$$(3) \qquad \boxed{\kappa \equiv \sup_{0\le p\le 1} E\, \frac{(pe^{Z} + q e^{rh})^{1-R}}{1-R}}$$

then the Bellman equation becomes

$$(4) \qquad \frac{a}{1-R} = \sup_{y>0} \left\{ \tilde{h}\, \frac{y^{1-R}}{1-R} + a e^{-\rho h} \kappa (1-yh)^{1-R} \right\}$$

Differentiating with respect to y and setting equal to zero leads us to

$$\tilde{h}\, y^{-R} = (1-R) a \kappa e^{-\rho h} (1-yh)^{-R} h$$

so that
$$\frac{1-yh}{y} = \beta \quad , \quad \beta^R = h(1-R) a \kappa e^{-\rho h} / \tilde{h}$$

and
$$y = \frac{1}{h+\beta} .$$

Plugging back into the Bellman equation leads to

$$a = \frac{\tilde{h}}{h} (h+\beta)^R$$

which allows us to solve for $a^{1/R}$:

$$(5) \qquad \boxed{a^{1/R} = \frac{h \cdot (\tilde{h}/h)^{1/R}}{1 - (\kappa(1-R) e^{-\rho h})^{1/R}}}$$

So everything depends on finding out about κ. A sensible measure of the difference between the Merton problem and this discrete version is the efficiency, defined as

$$(6) \qquad \theta = \left(\frac{a}{a_M}\right)^{1/(1-R)} = \left(\frac{a^{1/R}}{a_M^{1/R}}\right)^{R/(1-R)} .$$

The maximal expected utility using the discrete game with initial wealth 1 is the same as the maximal expected utility using the Merton game and initial wealth θ.

(iii) The special case $\alpha \leq r$. This situation is somewhat anomalous, so let's dispose of it first. The Merton rule would say that you should sell the share short, but in the discrete game you can't do that. Observe that

$$\sup E\, U(p e^Z + e^{rh}) \leq \sup U(p e^{\alpha h} + q e^{rh}) \qquad \text{(Jensen)}$$
$$\leq U(e^{rh}) \qquad (\alpha \leq r)$$

so for the discrete player, best is to put everything into riskless asset and ignore the share. So for this case,

$$\kappa = \frac{e^{r(1-R)h}}{1-R} ,$$

and we obtain the following asymptotic for the efficiency θ:

$$\boxed{\theta = \left\{ \frac{\rho + (R-1)\left\{ r + \frac{(\alpha-r)^2}{2R\sigma^2} \right\}}{\rho + (R-1)r} \right\}^{R/(1-R)} - \frac{rh}{2} \left\{ 1 - \frac{(1-R)(\alpha-r)^2}{2R\sigma^2(\rho+(R-1)r)} \right\}^{R/(1-R)} + O(h^2)}$$

$$(7) \quad \boxed{= \left(1 - \frac{rh}{2}\right) \left\{ \frac{\rho + (R-1)(r + (\alpha-r)^2/2R\sigma^2)}{\rho + (R-1)r} \right\}^{R/(1-R)} + O(h^2)}$$

(iv) Having dealt with this, we'll now assume $\alpha > r$, and analyse the asymptotics of κ. Writing

$$X \equiv p\,(e^{\sigma W_h + \mu h} - 1) + q\,(e^{rh} - 1),$$

we have

$$\kappa \equiv \sup_{p \in [0,1]} E\, \frac{(1+X)^{1-R}}{1-R}.$$

By the binomial theorem / Taylor's theorem,

$$\frac{(1+X)^{1-R}}{1-R} = \frac{1}{1-R} + X - \frac{R}{2}X^2 + \frac{R(1+R)}{6}X^3\,(1+\lambda X)^{-R-2} \qquad \text{for some } \lambda \in (0,1)$$

Let's estimate the final term's expected value;

$$E\,|X^3\,(1+\lambda X)^{-R-2}| \leq (EX^6)^{\frac{1}{2}} \left(E\,(1+\lambda X)^{-2(R+2)}\right)^{\frac{1}{2}}$$

Now
$$\begin{aligned} E\,(1+\lambda X)^{-2(R+2)} &\leq E\left[(1+\lambda X)^{-2(R+2)} : X \geq 0\right] + E\left[(1+\lambda X)^{-2(R+2)} ; X<0\right] \\ &\leq 1 + E\,\left(pe^{\sigma W_h + \mu h} + qe^{rh}\right)^{-2(R+2)} \\ &\leq 1 + p\,E\,e^{-(\sigma W_h + \mu h)(2R+4)} + q\,e^{-(2R+4)rh} \\ &\leq c, \end{aligned}$$

a constant, for all $0 < h \leq 1$, say. On the other hand,

$$E\,X^6 = 15\,p^6\sigma^6 h^3 + O(h^4)$$

so we certainly have

$$E\,X^6 \leq 30\,\sigma^6 h^3$$

for small enough h.

In conclusion, for some constant, we shall have

$$\boxed{E \left| \frac{R(1+R)}{6} X^3 (1+\lambda X)^{-R-2} \right| \leq C \cdot h^{3/2}}$$

for all small enough h..

How does the main part compare?

Case 1: $\alpha - r \leq \sigma^2 R$

We are looking for the value of p for which

$$E\left(\frac{1}{1-R} + X - \frac{R}{2} X^2 \right)$$

is maximised. We find that the optimising p is of the form

$$p = \frac{\alpha - r}{\sigma^2 R} - h \frac{(\alpha - r)}{2R\sigma^4} \left\{ (\alpha - r)^2 + \sigma^2 (2rR + 3\alpha - r) + \sigma^4 \right\} + O(h^2),$$

which, for h small, will always be less than $\frac{\alpha - r}{\sigma^2 R} \leq 1$, so we can choose the maximising p, since it will be in $[0,1]$. We find that

$$\max_p E\left(\frac{1}{1-R} + X - \frac{R}{2} X^2 \right) = \frac{1}{1-R} + h\left(r + \frac{(\alpha - r)^2}{2\sigma^2 R} \right) + O(h^2)$$

For the efficiency θ we obtain

$$\theta = 1 - \frac{h}{4R\sigma^2} \frac{4r\sigma^4 R^2 (\rho + (R-1)r) + 2R\sigma^2 t^2 (\rho + \sigma^2 R + 2r(R^2 + 2R - 1)) + 4R^2\sigma^2 t^3 + (R+1)(3R-1)t^4}{2\sigma^2 R(\rho + (R-1)r) + (R-1)t^2} + o(h), \tag{8}$$

where $t \equiv (\alpha - r)$. NEVERTHELESS, THIS IS WRONG: SEE p 7

Case 2: $\alpha - r > \sigma^2 R$ In this situation, the best thing to do is choose $p = 1$, and this gives us an expansion for the efficiency

$$\theta = \left\{ \frac{\rho + (R-1)\left(r + (\alpha - r)^2 / 2\sigma^2 R\right)}{\rho + (R-1)(\alpha - \sigma^2 R/2)} \right\}^{R/(1-R)} \tag{9}$$

$$\left(1 - \frac{h}{4} \cdot \frac{\sigma^4 R^2 (R^2 - 2R - 1) + \sigma^2 (2\rho R - 4\alpha R(1+R^2)) + 4\alpha^2 (1-R) - 4\alpha\rho}{\sigma^2 R(R-1) - 2\rho + 2\alpha(1-R)} \right) + o(h)$$

(V) A natural variant of the problem arises when the times at which the portfolio and consumption rates may be adjusted form an independent Poisson process of intensity $\lambda \equiv 1/h$. In this case, we may not consume at a fixed rate until the first time τ_1, so let's suppose that the rule is to consume at a rate proportional to cash holding, at rate γx_t, say. Then the value function is again of the form $V(w) = a\, w^{1-R}/(1-R)$, where

$$(10) \quad a\frac{w^{1-R}}{1-R} = \sup\Big[E\int_0^{\tau_1} e^{-\rho s}\, U(c_s)\,ds + E\, e^{-\rho\tau_1} \frac{a}{1-R}\big(y_0 S_{\tau_1} + x_0\, e^{(r-\gamma)\tau_1}\big)^{1-R} \Big]$$

$$= \sup \int_0^\infty e^{-(\rho+\lambda)s} \frac{\left(\gamma x_0 e^{(r-\gamma)s}\right)^{1-R}}{1-R}\,ds + \frac{a}{1-R}\, E\, e^{-(\rho+(R-1)(r-\gamma))\tau_1}\left(y_0 S_{\tau_1} e^{(\gamma-r)\tau_1} + x_0\right)^{1-R}$$

so if we abbreviate $\beta \equiv \rho+\lambda + (1+R)(r-\gamma)$ and divide by w^{1-R}, then if $x_0/y_0 = p/(1-p)$ we shall have

$$\frac{a}{1-R} = \sup\Big[\frac{(\gamma p)^{1-R}}{1-R}\,\beta^{-1} + \frac{a}{1-R}\,\frac{\lambda}{\beta}\, E\,\big((1-p)\, S_T\, e^{(\gamma-r)T} + p\big)^{1-R}\Big]$$

where $T \sim \exp(\beta)$ independent of W. Thus if $k = \alpha - \frac{1}{2}\sigma^2 + \gamma - r$, we have

$$(11) \quad \frac{a}{1-R} = \sup_{\gamma,\, 0\le p\le 1} \Big[\frac{(\gamma p)^{1-R}}{1-R}\,\beta^{-1} + \frac{a\lambda}{(1-R)\beta}\, E\,\big((1-p)\, e^{\sigma W_T + kT} + p\big)^{1-R}\Big].$$

We can proceed in two ways. The first (as in WN XIV, pp 46–47) is to use the known density

$$(12) \quad \frac{\beta}{(k^2+2\beta\sigma^2)^{1/2}} \exp\Big[\frac{ky}{\sigma^2} - \frac{|y|}{\sigma^2}(k^2+2\beta\sigma^2)^{1/2} \Big]$$

for $\sigma W_T + kT$, and do some numerics. The other route is to approximate using the quadratic approximation to the utility; if $X \equiv (1-p)\{\exp(\sigma W_T + kT) - 1\}$, we have

$$E\,\frac{(1+X)^{1-R}}{1-R} \doteq \frac{1}{1-R} + EX - \frac{R}{2}\,EX^2$$

$$= \frac{1}{1-R} + \frac{(k+\frac{1}{2}\sigma^2)(1-p)}{\beta - k - \frac{1}{2}\sigma^2} - \frac{R}{2}(1-p)^2\, \frac{\beta\sigma^2 + 2k^2 + 3k\sigma^2 + \sigma^4}{(\beta - 2k - 2\sigma^2)(\beta - k - \frac{1}{2}\sigma^2)}.$$

However, we can expect to have to use higher-order terms in the expansion. For this, we use the BDG inequalities for any $m \ge 2$

$$(13) \quad \left(E\,\big|e^{\sigma W_t - \sigma^2 t/2} - 1\big|^m\right)^{1/m} \le c_m \left(E\left(\int_0^t \sigma^2 e^{2\sigma W_s - \sigma^2 s}\,ds\right)^{m/2}\right)^{1/m}$$

$$\le c_m\, t^{1/2} \left(E\int_0^t \sigma^m e^{m\sigma W_s - m\sigma^2 s/2}\,\frac{ds}{t}\right)^{1/m}$$

$\left[e^{x}-1 \leq xe^{x}\right] \longrightarrow$

$$= c_m t^{\frac12} \left\{ \frac{e^{m(m-1)\sigma^2 t/2} - 1}{m(m-1)\sigma^2 t/2} \right\}^{1/m}$$

$$\leq c_m t^{\frac12} \exp\left((m-1)\sigma^2 t/2\right).$$

Thus if we want to know $(1-R)^{-1} E(1+X)^{1-R}$ up to and including all terms of order $h^{N+1} \equiv \lambda^{-N-1}$, we shall need to use the approximation

$$(14) \qquad E\,\frac{(1+X)^{1-R}}{1-R} = E\left[\frac{1}{1-R} \sum_{j=0}^{2N+2} \frac{\Gamma(2-R)}{\Gamma(2-R-j)} \frac{X^j}{j!} + \frac{1}{1-R} \frac{\Gamma(2-R)}{\Gamma(2-R-2N-3)} (1+\theta X)^{-R-2N-2} \frac{X^{2N+3}}{(2N+3)!} \right]$$

We estimate the remainder term, since $(1+\theta X)^{-R-2N-2} \leq p^{-R-2N-2}$, and $(m \equiv 2N+3)$

$$(15) \qquad E\,|X|^m = E\,\left|(1-p)(e^{\sigma W_T + kT} - 1)\right|^m$$

$$= E\,(1-p)^m \left| e^{\sigma W_T + kT} - e^{kT + \frac12\sigma^2 T} + e^{kT+\frac12\sigma^2 T} - 1 \right|^m$$

$$\leq 2^m (1-p)^m \left\{ E\; e^{(k+\frac12\sigma^2)mT} \left|e^{\sigma W_T - \frac12\sigma^2 T} - 1\right|^m + E\,\left|e^{kT+\frac12\sigma^2 T} - 1\right|^m \right\}$$

$$\leq (2-2p)^m \left\{ \frac{c_m^m\; \beta\; \Gamma(1+\frac{m}{2})}{(\beta - m(k+\frac{\sigma^2}{2} + (m-1)\sigma^2/2)^{m/2+1}} + \frac{\Gamma(m+1)\,\beta(k+\sigma^2/2)^m}{\{\beta - m(k+\sigma^2/2)\}^{m+1}} \right\}$$

$$\leq \frac{\text{Const}}{(1+\lambda)^{m/2}} = \frac{\text{Const}}{(1+\lambda)^{N+3/2}} \quad \text{for all large enough } \lambda.$$

(vii) Our methodology now will be as follows. Take a Taylor expansion for p, γ

$$(16) \qquad p = \sum_{j=0}^{N+1} a_j h^j/j!, \qquad \gamma = \sum_{j=0}^{N+1} b_j h^j/j!$$

(order $N+1$ will certainly be sufficient) and now find out what the coefficients a_j, b_j must be, by finding where the $O(h^{N+1})$-approximation to thing we've maximising has <u>its</u> maximum.

(viii) Back to the example with fixed time steps as studied on pages 1–4. If we want to know the efficiency of the discrete rule relative to the Merton rule up to order h^N, then we have to know the optimum κ up to order h^{N+1}, since the expression on p2 for $a^{1/R}$ has a factor h in the numerator. In view of the estimate (13) we therefore need to use the expansion (14), i.e., up to and including the term in X^{2N+2}. Our strategy therefore is to take a partial Taylor expansion for the optimising p,

$$p = \sum_{j=0}^{N} a_j \, h^j / j!$$

discover the coefficients a_j by maximising the $O(h^{N+1})$-approximate-payoff, and then plug back in and get the expansion of the efficiency up to the term in h^N. <u>When we do this</u>, <u>assuming the Merton proportion is in</u> $(0,1)$, we obtain

(17) $$\theta = 1 + c_1 h + c_2 h^2 + o(h^2),$$

where $(t \equiv \alpha - r)$:

$$c_1 = -\frac{1}{4R\sigma^2} \cdot \frac{4R^2\sigma^4 r\left(\rho + r(R-1)\right) + 2R\sigma^2\left(\rho + 2r(R-1) + R\sigma^2\right)t^2 - 4R\sigma^2 t^3 + (1+R)t^4}{2R\sigma^2\left(\rho + r(R-1)\right) + (R-1)t^2}$$

$$c_2 = \xi \Big/ \left(96\, R^3 \sigma^4 \left(2R\sigma^2 (\rho + r(R-1)) + (R-1)t^2 \right)^2 \right),$$

$$\begin{aligned}
\xi = {} & 16 R^4 \sigma^8 \left(4Rr^2 - (\rho - r)^2\right)\left(\rho + r(R-1)\right)^2 \\
& + 16 R^3 \sigma^6 (\rho + (R-1)r)\left(\sigma^4 R^2 + 3R\sigma^2(\rho + 2rR - r) + 8r^2R^2 - 10Rr^2 + 7Rr\rho - \rho^2 R + 2(\rho - r)^2\right)t^2 \\
& + 32 R^3 \sigma^6 (\rho + (R-1)r)\left(R\sigma^2 (2R-1) - 6rR + 3(r - \rho)\right)t^3 \\
& + 4R^2\sigma^4\Big(\sigma^4 R^2 (2R+1) + \sigma^2\left(16R^2 r - 24R^3 r - 8\rho R - 30\rho R^2 + 8Rr\right) + 24R^3 r^2 + 30R^2 \rho r \\
& \qquad + 10\rho^2 R - 12 r\rho - 6Rr\rho - 30R^2 r^2 - \rho^2 R^2 + 6r^2 + 6\rho^2 \Big) t^4 \\
& + 16 R^2 \sigma^4 \left(R\sigma^2 (2R^2 - 3R - 2) + 6(\rho + (R-1)r) + 3\rho R \right) t^5 \\
& + 4\sigma^2 R \left(R\sigma^2 (19 + 17R - 18R^2) + 10r + 8R^3 r - 8Rr - 10rR^2 - 10\rho + 5\rho R(R-1) \right) t^6 \\
& + 24 R\sigma^2 (1+R)(2R-3)t^7 + (1+R)(4R^2 - 21R + 23)\, t^8
\end{aligned}$$

If the Merton proportion is negative, then the expression (7) is correct.
<u>If the Merton proportion is greater than 1</u>, then we have

(18) $$\theta = c_0 + c_1 h + c_2 h^2 + o(h^2)$$

where: $$C_0 = \left\{ \frac{R}{a_M^{1/R}\left(\rho + (R-1)\alpha - \frac{\sigma^2}{2} R(R-1)\right)} \right\}^{R/(1-R)}$$

$$C_1 = C_0 \left(\frac{R\sigma^2}{4} - \frac{\alpha}{2} \right)$$

$$C_2 = \frac{C_0}{96R} \left\{ R^2\sigma^4(4R-1) + 4R\sigma^2(\alpha - \rho - 4\alpha R) - 4(\rho-\alpha)^2 + 16R\alpha^2 \right\}$$

(ix) Let's return to the Poisson version of the problem, pages 5 and 6. By truncating the binomial expansion, we find ourselves looking at the function

$$\varphi(a,\gamma,q,h) = \frac{(\gamma(1-q))^{1-R}\;h}{1+\theta_1 h + (1-R)\gamma h} + \frac{a\lambda}{\beta} \sum_{j=0}^{M} \frac{\Gamma(2-R)}{j!\,\Gamma(2-R-j)} q^j \sum_{r=0}^{j} \binom{j}{r} \frac{(-1)^{j-r}\;\beta}{\beta - rk - \frac{1}{2}r^2\sigma^2} - a$$

where $\beta \equiv h^{-1} + \theta_1 + (1-R)\gamma$, $\theta_1 \equiv \rho + r(R-1)$;

$$= \frac{(\gamma(1-q))^{1-R}\;h}{1+\theta_1 h + (1-R)\gamma h} + a \sum_{j=0}^{M} \frac{\Gamma(2-R)}{j!\,\Gamma(2-R-j)} q^j \sum_{r=0}^{j} \binom{j}{r} \frac{(-1)^{j-r}}{1+(\theta_1 + (1-R)\gamma - rk - \frac{1}{2}r^2\sigma^2)h} - a$$

and we require that $\varphi(a,\gamma,q,h) = 0$, $\frac{\partial\varphi}{\partial q}(a,\gamma,q,h) = 0$, $\frac{\partial\varphi}{\partial\gamma}(a,\gamma,q,h) = 0$

If we now try the expansions

$$a = \sum_{m\geq 0} a_m h^m, \quad q = \sum_{m\geq 0} b_m h^m, \quad \gamma = \sum_{m\geq 0} c_m h^m,$$

we obtain

$$\boxed{a_0 = a_M \equiv \left\{ \frac{R}{\rho + (R-1)(r + (\alpha-r)^2/2\sigma^2 R)} \right\}^R, \quad b_0 = \frac{\alpha - r}{\sigma^2 R}, \quad c_0 = \frac{a_M^{-1/R}}{1-b_0}}$$

as expected. Next we find that

$$a_1 = \frac{a_M^{1+1/R}}{2R^3\sigma^4} \cdot \frac{(\alpha-r)^2(R-1)}{2} \left(2R^2\sigma^4 + 2R(\rho + r(R+1) - 2\alpha)\sigma^2 + (R+1)(\alpha-r)^2 \right)$$

$$b_1 = \frac{r-\alpha}{\sigma^4 R^3} \left\{ \sigma^4 R^2 + \sigma^2 R(2rR + r - 3\alpha + 2\rho) + (R+1)(\alpha-r)^2 \right\}$$

$$c_1 = -\frac{(\alpha-r)\left\{ R^3\sigma^6((R-1)(\alpha+r)+2\rho) + 2R^2\sigma^4(((R-1)\alpha+\rho)^2 - (\alpha-r)(R^2(r+\alpha) + R(2\rho-\alpha) - r + \rho)) + 2R(\alpha-r)^2\sigma^2(R+1)\cdot((R-1)r+\rho) + \frac{1}{2}(R^2-1)(\alpha-r)^4 \right\}}{2R^3\sigma^2\left(\sigma^4 R^2 - 2(\alpha-r)R\sigma^2 + (\alpha-r)^2\right)}$$

An optimal investment/consumption problem with uncertainty (26/3/98)

(i) Suppose we have the standard dynamics for the share

$$dS_t = S_t \sigma (dW_t + \lambda dt)$$

but that λ is not known with certainty; at time 0, we know only that $\lambda \sim N(\lambda_0; v_0)$. The interest rate r and volatility σ are known constants, and the agent is now going to try to

$$\max \; E\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \;\middle|\; w_0 = w \right]$$

where $U(c) = c^{1-R}/(1-R)$ and $R > 1$ in order for the problem to be well posed. If the agent's policies have to be adapted to the observation filtration, what should he do?

(ii) The first thing is to obtain the filtering equations for $X_t \equiv W_t + \lambda t$. The likelihood of an observed path if λ is the true value will be $(X_0 = 0)$

$$\exp\left(\lambda X_t - \tfrac{1}{2}\lambda^2 t \right)$$

and so the posterior density for λ given the observed path will be

$$\propto \exp\left[\lambda X_t - \tfrac{1}{2}\lambda^2 t - (\lambda - \lambda_0)^2 / 2v_0 \right]$$

$$\propto \exp\left[-\frac{\lambda^2}{2v_t} + \lambda (X_t + \lambda_0/v_0) \right] \qquad \frac{1}{v_t} = t + \frac{1}{v_0} = \frac{v_0 t + 1}{v_0}$$

so that the distribution of λ given $\mathcal{G}_t \equiv \sigma(\{X_u : u \leq t\})$ is

$$N\left(v_t \left(X_t + \frac{\lambda_0}{v_0}\right), v_t \right) = N\left(\frac{v_0 X_t + \lambda_0}{1 + v_0 t}, \frac{v_0}{1 + t v_0} \right)$$

Hence in the observation filtration

$$\boxed{dX_t = d\hat{W}_t + \frac{v_0 X_t + \lambda_0}{1 + v_0 t}\, dt}$$

Notice that

$$d \frac{X_t}{1 + v_0 t} = \frac{d\hat{W}_t}{1 + v_0 t} + \frac{\lambda_0}{(1 + v_0 t)^2}\, dt$$

So we get

$$\boxed{\frac{X_t}{1 + v_0 t} = \hat{W}(A_t) + \lambda_0 A_t}, \qquad A_t \equiv \int_0^t \frac{ds}{(1 + v_0 s)^2} = \frac{1}{v_0}\left(1 - \frac{1}{1 + v_0 t}\right)$$

or again

$$(1 - v_0 \tau) X\left(\frac{\tau}{1 - v_0 \tau} \right) = \hat{W}_\tau + \lambda_0 \tau \qquad 0 \leq \tau < v_0^{-1}$$

It's clear from this that $t^{-1} X_t \to v_0 \hat{W}(1/v_0) + \lambda_0$, which has the correct distribution!

(iii) What martingale gives us the change of measure which makes the asset price process (when discounted) into a martingale?

$$\int_{-\infty}^{\infty} \exp\left(-\frac{a}{2}x^2 - bx - c\right) dx = \sqrt{\frac{2\pi}{a}}\ \exp\left(-c + \frac{b^2}{2a}\right)$$

If P_0 denotes Wiener measure, we have after a few calculations that

$$\left.\frac{dP}{dP_0}\right|_{\mathcal{F}_t} = (1+v_0 t)^{-1/2} \exp\left[\frac{1}{2}\frac{(v_0 X_t + \lambda)^2}{v_0(1+v_0 t)} - \frac{\lambda^2}{2v_0} \right]$$

and hence the change-of-measure martingale which makes X into a BM with drift r/σ will be

$$Z_t = (1+v_0 t)^{1/2} \exp\left[-\frac{1}{2}\frac{(v_0 X_t + \lambda_0)^2}{v_0(1+v_0 t)} + \frac{\lambda^2}{2v_0} + \frac{r}{\sigma}X_t - \frac{1}{2}\left(\frac{r}{\sigma}\right)^2 t \right]$$

$$= (1+v_0 t)^{1/2} \exp\left[-\frac{1}{2}\frac{v_0}{1+v_0 t}X_t^2 + \left(\frac{r}{\sigma} - \frac{\lambda_0}{1+v_0 t}\right)X_t + \frac{t}{2}\left(\frac{\lambda_0^2}{1+v_0 t} - \frac{r^2}{\sigma^2}\right) \right]$$

and in the limit as $v_0 \downarrow 0$ this thing behaves correctly.

(iv) The state-price density $\zeta_t = e^{-rt} Z_t$ is defined in terms of this, and the optimal consumption satisfies

$$e^{\rho t} c_t^{-R} = \lambda \zeta_t$$

for some $\lambda > 0$ chosen to make the budget condition hold:

$$w_0 = E\int_0^\infty \zeta_t c_t \, dt = E_* \int_0^\infty e^{-rt} c_t \, dt$$

$$= \lambda^{-1/R} E\int_0^\infty (e^{-rt} Z_t)^{1-1/R} e^{-\rho t/R} \, dt$$

$$\equiv \lambda^{-1/R} \varphi(\lambda_0, v_0),$$

where the function φ remains to be made more explicit. As for the payoff of the problem, this is

$$E\int_0^\infty \frac{c_t^{1-R}}{1-R} e^{-\rho t} \, dt = \frac{\lambda^{1-1/R}}{1-R} \int_0^\infty \zeta_t^{1-1/R} e^{-\rho t/R} \, dt$$

$$= \frac{w_0 \lambda}{1-R} = \frac{w_0^{1-R}}{1-R} \varphi(\lambda_0, v_0)^R.$$

How about wealth at intermediate times? We have

$$w_t = E_t^* \int_t^\infty e^{-r(s-t)} c_s \, ds$$

$$= E_t \int_t^\infty \frac{Z_s}{Z_t} e^{-r(s-t)} \cdot \lambda^{-1/R} e^{-(\rho-r)s/R} Z_s^{-1/R} \, ds$$

$$= \lambda^{-1/R} e^{-(\rho-r)t/R} Z_t^{-1/R} E_t \int_t^\infty (\zeta_s/\zeta_t)^{1-1/R} e^{-\rho(s-t)/R} \, ds$$

$$= \lambda^{-1/R} e^{-\rho t/R} \zeta_t^{-1/R} \varphi(\lambda_t, v_t) = c_t \varphi(\lambda_t, v_t).$$

For the problem studied by Kalkur, of maximising $E\,U(w_T)$, the solution is

$$\max E\,U(w_T) = U(w_0)\left\{E\,\zeta_T^{1-1/R}\right\}^R$$

As for the function $\varphi(\lambda_0, v_0)$, we know that $X_t \sim N(\lambda_0 t, (1+v_0 t)t)$, hence we can evaluate

$$E\, Z_t^{1-1/R} = E \exp\left\{ -\frac{1}{2}\frac{v_0}{1+v_0 t}(1-\tfrac{1}{R})X_t^2 + \left(\frac{r}{\sigma} - \frac{\lambda_0}{1+v_0 t}\right)(1-\tfrac{1}{R})X_t + \frac{t}{2}(1-\tfrac{1}{R})\left(\frac{\lambda_0^2}{1+v_0 t} - \frac{r^2}{\sigma^2}\right)\right\}(1+v_0 t)^{(R-1)/2R}$$

$$= (1+cv_0 t)^{-1/2}(1+v_0 t)^{(R-1)/2R} \exp\left\{ - \frac{(\lambda_0 - r/\sigma)^2\, c\, t}{2R(1+cv_0 t)} \right\} \qquad \left(c \equiv 1 - \tfrac{1}{R}\right)$$

after some calculation.

Hence

$$\boxed{\varphi(\lambda_0, v_0) = \int_0^\infty (1+cv_0 t)^{-1/2}(1+v_0 t)^{(R-1)/2R} \exp\left\{ - \frac{(\lambda_0 - r/\sigma)^2 c\, t}{2R(1+cv_0 t)} - \frac{\rho + r(R-1)}{R} t \right\} dt}$$

As $v_0 \to 0$, this agrees with what it has to be.

(v) How does this compare with the agent who is actually told the value of the random drift? We have $\alpha \sim N(\sigma\lambda_0, \sigma^2 v_0)$ so that the maximum expected utility for <u>this</u> agent (who will in fact be following the Merton rule) will be

$$\frac{w_0^{1-R}}{1-R} \int_{-\infty}^{\infty} \frac{e^{-(\alpha - \sigma\lambda_0)^2/2\sigma^2 v_0}}{\sqrt{2\pi\sigma^2 v_0}} \left\{ \frac{R}{\rho + (R-1)r + (\alpha - r)^2/2R\sigma^2} \right\}^R d\alpha$$

It looks like we cannot do this in closed form.

(vi) How about the optimal investment strategy? The value function is $w_0^{1-R}(1-R)^{-1}\varphi(\lambda_0, v_0)^R$, so when we do an Itô expansion of the (super) martingale

$$\int_0^t e^{-\rho s} U(c_s)\, ds + e^{-\rho t} U(w_t)\, \varphi(\hat\lambda_t, v_t)^R \qquad \hat\lambda_t \equiv E[\lambda_0 | \mathcal{G}_t] = \frac{v_0 X_t + \lambda_0}{1+v_0 t}$$

the terms involving the portfolio process θ are

$$e^{-\rho t}\varphi(\hat\lambda_t, v_t)^R (1-R)\left\{ -R\frac{\sigma^2\theta^2}{2w^2} + \frac{\theta\sigma}{w}\left(\hat\lambda - \frac{r}{\sigma}\right)\right\} U(w)$$

<u>CARE!!</u> <u>This is wrong</u> – <u>there are terms involving θ from the covariation of</u> $U(w)$ <u>and</u> $\varphi(\hat\lambda_t, v_t)$!!

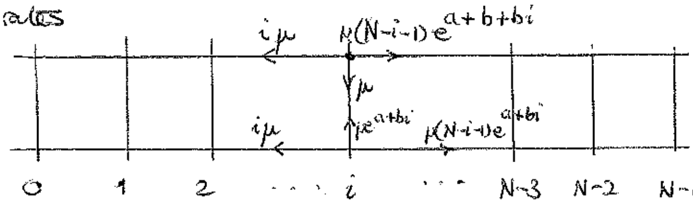
Optimising this over θ gives

$$\theta_t^* = \frac{\sigma\hat\lambda_t - r}{\sigma^2 R} w_t$$

i.e., the Merton proportion, using the estimated value of α in place of the exact value.

Doing the Ising model More Honestly (31/3/98)

Of principal interest is the expectation $(\pi, f_i R_\beta f_j)$ where we may suppose that $f_i = e_i$, $f_j = e_j$. So it's about computing $P[x_i(0) = 1 = x_j(T)]$ for $T \sim \exp(\beta)$. We take a statespace $\{0,1\} \times \{0,1,\dots,N-1\}$ with jump rates as shown in the diagram.



If we write out the states as a sequence

$(0,0), (0,1), \dots, (0,N-1), (1,0), (1,1), \dots, (1,N-1)$

then the Q-matrix is $2N \times 2N$ and is partitioned as

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The invariant law is given by

$$\pi(0,j) = c.\exp\left(aj + \tfrac{1}{2} b j(j-1)\right)\binom{N-1}{j}$$

$$\pi(1,j) = c\exp\left(a(j+1) + \tfrac{1}{2} b j(j+1)\right)\binom{N-1}{j}.$$

Hence
$$\pi(0,i)\; q((0,i),(1,i)) = c.\binom{N-1}{i}\mu\exp\{a(i+1) + \tfrac{1}{2} b\, i(i+1)\} = \mu\,\pi(1,i)$$

$$\pi(0,i)\; q((0,i),(0,i-1)) = c\,(N-1)\binom{N-2}{i-1}\mu\exp\{ai + \tfrac{1}{2} b\, i(i-1)\} = \mu(N-i)e^{a+b(i-1)}\pi(0,i-1)$$

$$\pi(1,i)\; q((1,i),(1,i-1)) = c\,(N-1)\binom{N-2}{i-1}\mu\exp\{a(1+i) + \tfrac{1}{2} b i(1+i)\} = \mu(N-i)e^{a+bi}\pi(1,i-1)$$

Thus the matrix $S \equiv MQ$ is symmetric, where M is the diagonal matrix of the invariant law. If
$$y_i = 0 \quad (i=0,\dots,N-1); \quad = 1 \quad (i=N,\dots,2N-1)$$
$$z_i = i/(N-1) \quad (i=0,\dots,N-1); \quad = (i-N)/(N-1) \quad (i=N,\dots,2N-1)$$

then

$$P[x_i(0) = 1 = x_j(T)] = \begin{cases} y^T M\beta(\beta - Q)^{-1} z & (i \neq j) \\ y^T M\beta(\beta-Q)^{-1}\, y & (i=j) \end{cases}$$

$$= \begin{cases} y^T M\, \beta(\beta M - S)^{-1} M z & (i \neq j) \\ y^T M\, \beta(\beta M-S)^{-1} M y & (i=j) \end{cases}$$

Law of the maximum of discretely-sampled Brownian motion (1/4/98)

(i) Suppose we see $B(jh)$, $j=0,1,\dots,N$, for some fixed $h>0$: what can we say of the law of $\bar{B}_{Nh} \equiv \sup_{1\le j\le N} B(jh)$?

Introduce an independent exponential r.v. T of rate λ, and try to find out the law of the sup of B_{jh}, $jh \le T$. We see a geometric number of steps, with prob $p = 1-e^{-\lambda h}$ of success. If we set $\bar{X} \equiv \sup\{B_{jh} : jh \le T\}$ then by the standard Wiener–Hopf factorisation

$$E\, e^{i\theta \bar{X}}\; E\, e^{i\theta \underline{X}} = \frac{p}{1-q\varphi(\theta)} \qquad \left[\varphi(\theta) = e^{-\theta^2 h/2}\right]$$

$$= p \exp\Big[+\sum_{n\ge1} \tfrac{1}{n} q^n \varphi(\theta)^n\Big]$$

$$= \exp\Big[-\sum_{n\ge1} (1-\varphi(\theta)^n)\, q^n/n\Big]$$

so from Spitzer's identity we deduce that

$$E\, e^{i\theta\bar{X}} = \exp\Big[-\sum_{n\ge1} q^n/n \int_0^\infty (1-e^{i\theta x})\, P(B_{nh}\in dx)\Big]$$

$$= \exp\Big[-\sum_{n\ge1} \frac{q^n}{n}\Big\{ \tfrac12 (1-e^{-\theta^2 nh/2}) - i\int_0^\infty e^{-x^2/2nh} \sin\theta x \,\frac{dx}{\sqrt{2\pi nh}}\Big\}\Big]$$

The imaginary part can be expressed in various ways, but all involve integrals which are not obtainable in closed form.

(ii) How about the mean? This is

$$E\bar{X} = \sum_{n\ge1} \frac{q^n}{n} \int_0^\infty e^{-x^2/2nh}\, . \,x\, \frac{dx}{\sqrt{2\pi nh}}$$

$$= \sum_{n\ge1} \frac{e^{-n\lambda h}}{n} \sqrt{\frac{nh}{2\pi}}\,.$$

So if $\mu(t) \equiv E\big[\sup_{jh\le t} B_{jh}\big]$ we learn from this that

$$\Delta\mu(nh) = \left(\frac{h}{2\pi n}\right)^{1/2}$$

and so if we have $h = 1/N$

$$E\big[\sup_{j\le N} B_{j/N}\big] = \sum_{r=1}^N (2\pi r N)^{-1/2} = \frac{1}{N}\sum_{r=1}^N \left(\frac{r}{N}\right)^{-1/2} \frac{1}{\sqrt{2\pi}} \uparrow \sqrt{\frac{2}{\pi}} \text{ as } N\uparrow\infty.$$

The difference is

$$\frac{1}{\sqrt{2\pi}} \sum_{r=1}^N \Big\{ \int_{\frac{r-1}{N}}^{r/N} \frac{dx}{\sqrt{x}} - \frac{1}{\sqrt{rN}}\Big\} = \frac{1}{\sqrt{2\pi}}\sum_{r=1}^N \Big\{ 2\Big(\sqrt{\tfrac{r}{N}} - \sqrt{\tfrac{r-1}{N}}\Big) - \frac{1}{\sqrt{rN}}\Big\}$$

The claim is that this is behaving like $c/\sqrt{N}$, for some constant c to be determined. To see this, multiply by $\sqrt{N}$ to get

$$\frac{1}{\sqrt{2\pi}} \sum_{r=1}^{N} \left(2(\sqrt{r} - \sqrt{r-1}) - \frac{1}{\sqrt{r}} \right) = \frac{1}{\sqrt{2\pi}} \sum_{r=1}^{N} \left(\frac{2}{\sqrt{r}+\sqrt{r-1}} - \frac{1}{\sqrt{r}} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{r=1}^{N} \frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r}\,(\sqrt{r}+\sqrt{r-1})}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{r=1}^{N} \frac{1}{\sqrt{r}\,(\sqrt{r}+\sqrt{r-1})^2}$$

$$\to \frac{1}{\sqrt{2\pi}} \sum_{r=1}^{\infty} \frac{1}{\sqrt{r}\,(\sqrt{r}+\sqrt{r-1})^2} = 0.5825971579$$

$$= -\zeta(\tfrac{1}{2})/\sqrt{2\pi}$$

as in Broadie-Glasserman-Kou!

More detail on Skorohod embedding (1/4/98)

If we do the embedding by picking random barriers $a < 0 < b$ according to the law $c \,.\, F_+(db)\, F_-(da)\, (b-a)$ $\quad (c \equiv 1/EX^+ \equiv 1/EX^-)$, we have that conditional on having chosen barriers (a, b)

$$E\left[e^{-\frac{1}{2}\theta^2 \tau} \,;\, B_\tau = b \right] = \frac{-\sinh \theta a}{\sinh \theta (b-a)}$$

so we can differentiate (or use Maple to differentiate...!) to obtain

$$E[\tau : B_\tau = b] = -\frac{ab}{3} \, \frac{b-2a}{b-a}$$

$$E[\tau : B_\tau = a] = -\frac{ab}{3} \, \frac{2b-a}{b-a}$$

Hence

$$E[\tau \mid B_\tau = b] = \int_{-\infty}^{0} F_-(da)\, c \,.\, \frac{-ab}{3}(b-2a) = \frac{b}{3} \left\{ b + 2c \int_{-\infty}^{0} a^2 F_-(da) \right\}$$

$$E[\tau \mid B_\tau = a] = \frac{-a}{3} \left\{ -a + 2c \int_{0-}^{\infty} b^2 F_+(db) \right\}$$

Approximating the price of a European option (2/4/98)

(i) Damien Lamberton has been looking at the discrete approximation of option prices by taking some IID r.v.s $X_1, X_2, \dots$ and approximating a Brownian motion by $n^{-\frac{1}{2}} \sum_{j=1}^{[nt]} X_j$. To be specific, let's suppose we realise the n^{th} approximation from a fixed Brownian motion B by Skorokhod-embedding at the times $\tau_j^{(n)}$, $\tau^{(n)}$, where we do the independent randomisation trick. We have that $\tau_1^{(n)} \stackrel{\mathcal{D}}{=} n^{-1} \tau_1^{(1)}$. Let's write τ_j in place of $\tau_j^{(n)}$ if the particular choice of n is clear, and let's write $T_n \equiv \tau_n^{(n)}$.

We'll assume that $EX_i = 0$, $EX_i^2 = 1$, and consider firstly the case of a European option with payoff $f(B_1)$, where f is assumed to have bounded continuous derivatives of all orders up to order 2.

Then

$$
\begin{aligned}
|E\, f(B_1) - E f(S_n/\sqrt{n})| &= |E f(B_1) - E\, f(B(\tau_n))| \\
&= \left| E \int_1^{\tau_n} \tfrac{1}{2} f''(B_u)\,du \right| \\
&\le \tfrac{1}{2} \|f''\|_\infty \quad E\, |\tau_n - 1|
\end{aligned}
$$

If we also assume X_1 has finite fourth moment then $\tau_1^{(n)}$ has finite second moment, and we get the bound

$$
|E\, f(B_1) - E\, f(S_n/\sqrt{n})| \le \tfrac{1}{2} \|f''\|_\infty \; \frac{1}{\sqrt{n}} \; (\mathrm{var}(\tau_1^{(1)}))^{\frac{1}{2}}
$$

and this $O(n^{-1/2})$ estimate appears to be as good as we can hope for in general. This is because

$$
\begin{aligned}
& E\left[\int_1^{\tau_n} \tfrac{1}{2} f''(B_u)\,du - (\tau_n - 1)\tfrac{1}{2} f''(B_{\tau_n})\right] \\
&= E\left[\tfrac{1}{2}\int_1^{\tau_n} (f''(B_u) - f''(B_{\tau_n}))\,du\right] \\
&= E\left[-\int_1^{\tau_n} ds \left\{\int_s^{\tau_n} \tfrac{1}{2} f'''(B_u)\,dB_u + \tfrac{1}{4}\int_s^{\tau_n} f^{(4)}(B_u)\,du\right.\;;\; \tau_n > 1\right] \\
&\quad - E\left[\int_{\tau_n}^1 ds \left(\int_{\tau_n}^s \tfrac{1}{2} f'''(B_u)\,dB_u + \int_{\tau_n}^s \tfrac{1}{4} f^{(4)}(B_u)\,du\right) \;;\; \tau_n \le 1\right] \\
&= -\tfrac{1}{4}\; E\left[\int_1^{\tau_n} ds \int_s^{\tau_n} f^{(4)}(B_u)\,du;\; \tau_n > 1\right] - \tfrac{1}{4}\, E\left[\int_{\tau_n}^1 ds \int_{\tau_n}^s f^{(4)}(B_u)\,du;\; \tau_n \le 1\right] \\
&\le \frac{\text{Const}}{n}
\end{aligned}
$$

if we also had $f \in C_b^4$. Thus a better rate of convergence than $1/\sqrt{n}$ would require that $E(\tau_n - 1) f''(B_{\tau_n})$ tends to zero faster than $\sqrt{n}$. However, we know that

$$\left(\sqrt{n}(\tau_n - 1),\ B_{\tau_n}\right) \xrightarrow{\mathcal{D}} N\left(0, \begin{pmatrix} \mathrm{var}(\tau_1^2) & \mathrm{cov}(\tau_1, B(\tau_1)) \\ \mathrm{cov}(\tau_1, B(\tau_1)) & 1 \end{pmatrix}\right)$$

so that typically $E(\tau_n - 1) f''(B_{\tau_n})$ will be $O(1/\sqrt{n})$, UNLESS

$$E\ \tau_1 B(\tau_1) = 0.$$

From the third Hermite polynomial, $E\ \tau_1 B(\tau_1) = \frac{1}{3} E[B(\tau_1)^3]$, so the situation is that we can really only look for $o(1/\sqrt{n})$ convergence if

$$E X_1^3 = 0.$$

(ii) Here's a totally different approach. Define the kernels P_h and $\tilde{P}_h$ by

$$P_h f(x) \equiv E\ f(x + \sqrt{h}\, X_1), \qquad \tilde{P}_h f(x) = E\ f(x + \sqrt{h}\, B_1)$$

Now for C^4 functions f with $f^{(4)}$ bounded, we shall have

$$P_h f(x) = E\left[f(x) + \sqrt{h}\, X_1 f'(x) + \tfrac{1}{2} h X_1^2 f''(x) + \tfrac{1}{6} h^{3/2} X_1^3 f'''(x) + \tfrac{1}{24} h^2 X_1^4 f^{(4)}(x + \theta\sqrt{h} X)\right]$$

so that (assuming $EX_1^4 < \infty$) we get (assuming $EX_1 = 0$, $EX_1^2 = 1$)

$$\left| P_h f(x) - f(x) - \tfrac{1}{2} h f''(x) - \tfrac{1}{6} h^{3/2} f'''(x) E X_1^3 \right| \leq \text{const.}\, h^2 \qquad \forall x.$$

Hence (provided $EX_1^3 = EB_1^3$) it's immediate that

$$\left| P_h f(x) - \tilde{P}_h f(x) \right| \leq c.\, h^2.$$

Thus if $\varepsilon_k = \| P_h^k f - \tilde{P}_h^k f \|_\infty$ we have

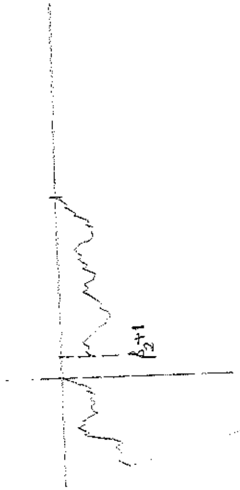
$$\varepsilon_{k+1} = \sup_x \left| P_h^{k+1} f(x) - \tilde{P}_h^{k+1} f(x) \right|$$

$$\leq \sup_x \left| P_h \tilde{P}_h^k f(x) - P_h P_h^k f(x) \right| + \sup_x \left| (P_h - \tilde{P}_h) \tilde{P}_h^k f(x) \right|$$

$$\leq \varepsilon_k + c\, h^2$$

since the bound on the fourth derivative of f holds also for the fourth derivative of $\tilde{P}_h^k f$. Hence $\varepsilon_{k+1} \leq c.(k+1)\, h^2$, and so

$$\boxed{\left| E\ f\left(\tfrac{S_n}{\sqrt{n}}\right) - E f(B_1) \right| \leq \tfrac{c}{n}}$$



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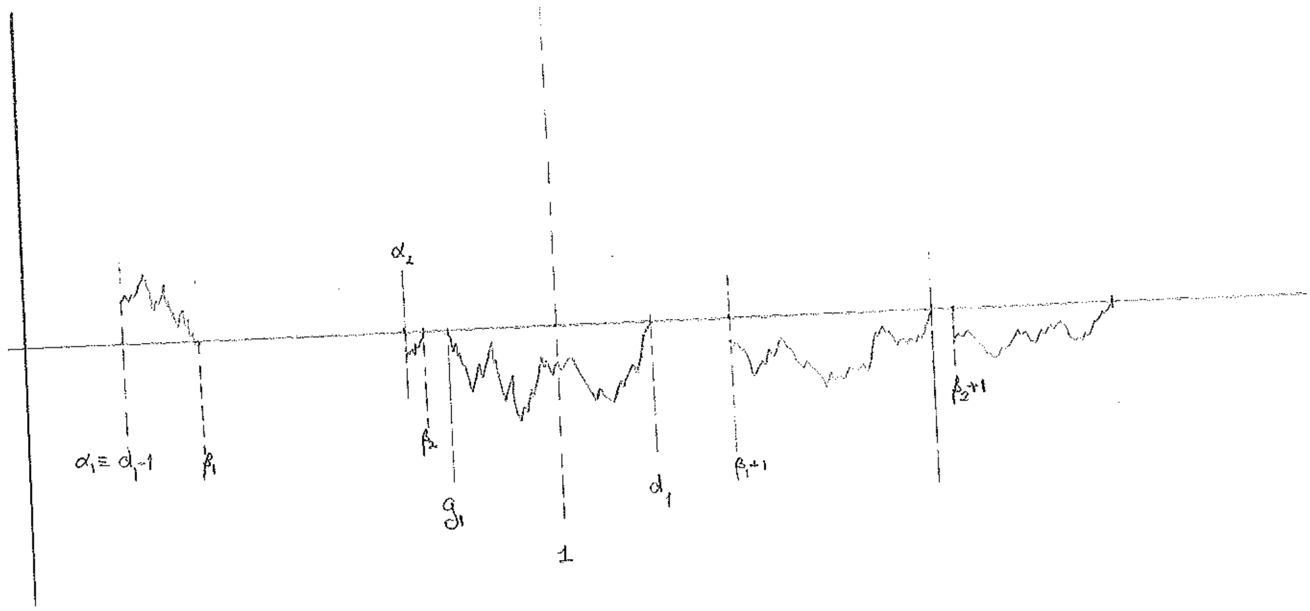
zero faster than $\sqrt{n}$. However, we know

$\mathrm{cov}(\tau_1, B(\tau_1))$))

ESS

the situation is that



α_2
$\alpha_1 \equiv d_1 - 1$
β_1
β_2
g_1
1
d_1
β_1+1
β_2+1

Law of a Brownian stopping time (27/4/98)

(i) Let $\alpha \equiv \inf\{t : B_t = 0,\ B_{1+t} > 0\}$. What's the law of α (or, equivalently the law of the stopping time $1+\alpha$)?

Let's collect a few pieces of information first. If $g_t \equiv \sup\{u < t : B_u = 0\}$, and $d_t \equiv \inf\{u > t : B_u = 0\}$ then we shall have

$$P(g_u \in ds,\ d_u \in dt) = \frac{dt\, ds}{2\pi\, (t-s)^{3/2}\, s^{1/2}},$$

$$P(g_u \in ds) = \frac{ds}{\pi\,(s(u-s))^{1/2}},$$

$$P(d_u \in dt) = \frac{\sqrt{u}\; dt}{\pi t \sqrt{t-u}}.$$

Next, if we have a Brownian bridge X from $(0,0)$ to $(a,0)$, and some time $u \in (0,a)$, then the law of $d_u^a \equiv \inf\{s > u : X_s = 0\}$ is given by

$$P(d_u^a \in dt)/dt = \left\{\frac{a u}{(a-t)(t-u)}\right\}^{\frac{1}{2}} \frac{1}{\pi t} \qquad (u < t < a)$$

(ii) Closed-form solution to the problem appears unlikely. However, we can make some sort of approximation. Firstly, note that if $B_1 > 0$, then $\alpha = 0$. Also, if $d_1 - g_1 \geq 1$, we have that α cannot be $< d_1$, and we have a renewal epoch at d_1. Thus as a first approximation we might want to take

$$\begin{aligned}
\varphi(\lambda) &\equiv E\left[e^{-\lambda\alpha}\right] \\
&\doteq \tfrac{1}{2} + \tfrac{1}{2}\int_0^1 ds \int_{1+s}^{\infty} \frac{dt}{2\pi}\, e^{-\lambda t}\, \varphi(\lambda)\, (t-s)^{-3/2} s^{-1/2} \\
&\quad + \tfrac{1}{2}\int_0^1 \frac{ds}{\sqrt{s}} \int_1^{1+s} \frac{dt}{2\pi (t-s)^{3/2}} \int_{t-1}^{s} \frac{dv}{\pi v} \left\{\frac{s(t-1)}{(s-v)(v-t+1)}\right\}^{\frac{1}{2}} e^{-\lambda v}\,\varphi(\lambda),
\end{aligned}$$

which amounts to assuming that after $\beta_1 \equiv \inf\{t > d_1 - 1 ;\ B_t = 0\}$ we get a renewal (so all the information we've got concerning the path in the future is conveniently forgotten!). We get

$$2\varphi(\lambda) \doteq 1 + \varphi(\lambda)\sqrt{\frac{2}{\lambda}}\left(\Phi(\sqrt{2\lambda}) - \tfrac{1}{2}\right)\left(\sqrt{\frac{2}{\pi}}\, e^{-\lambda} - 2\sqrt{2\lambda}\, \bar{\Phi}(\sqrt{2\lambda})\right)$$

$$+\, \varphi(\lambda) \int_0^1 ds \int_1^{1+s} \frac{dt}{2\pi (t-s)^{3/2}} \int_{t-1}^{s} \frac{dv\, e^{-\lambda v}}{\pi v} \left\{\frac{t-1}{(s-v)(v-t+1)}\right\}^{\frac{1}{2}}.$$

$$\frac{\mu(dx)}{dx} = \int_0^\infty \nu_+(d\alpha)\, e^{-\alpha x}\, I_{\{x>0\}} + \int_0^\infty \nu_-(d\alpha)\, e^{\alpha x}\, I_{\{x<0\}}$$

For Brownian case, $\mu(dx) = \dfrac{dx}{2\sqrt{2\pi x^3}}$ for $x > 0$

$$\nu_+(d\alpha) = \frac{1}{\pi\sqrt{2}} \cdot \sqrt{\alpha}\, d\alpha$$

Some more thoughts on Wiener-Hopf (11/5/98)

(i) We return to the situation of a one-dimensional BM, measure m on $\mathbb{R}$,

$$\varphi_t^+ = \int_{(0,\infty)} L_t(x)\, m(dx) \quad , \quad \varphi_t^- = \int_{(-\infty,0]} L_t(x)\, m(dx)$$

and subordinators $Z_t^\pm = \varphi^\pm(\gamma_t)$, where $\gamma_t = \inf\{u : L_u(0) > t\}$. DW has been considering the situation where $m|_{(-\infty,0]}$ is a finite collection of point masses, one of which is zero. In this case, we have

$$P^0(\tau_0 > 0) = 1,$$

where $\tau_x \equiv \inf\{t : Z_t > x\}$ $(x > 0)$, with $Z_t \equiv Z_t^+ - Z_t^-$, and if $\sigma \equiv \inf\{t > \tau_0 : Z_t < 0\}$ DW's result is that if the BM starts according to some dist^{n} μ^* concentrated on the atoms of m in $(-\infty, 0]$, then (if $m|_{\mathbb{R}^+} = \text{Leb}$)

$$P^{\mu^*}(B(\tilde{\sigma}) \in dx, \tilde{\sigma} < \infty) = \mu^*(dx) \cdot \tfrac{1}{2}$$

where $\tilde{\sigma} \equiv \inf\{t > \gamma(\tau_0) : \varphi(t) < 0\}$. How can we use classical results of WH theory to understand this?

(ii) Let T be $\exp(\eta)$ independent of Z, $\bar{Z}_t \equiv \sup_{s \le t} Z_s$, $\underline{Z}_t \equiv \inf_{s \le t} Z_s$, and let

$$(1) \quad \begin{cases} \Phi_\eta^+(s) \equiv E \exp s \bar{Z}_T & (\operatorname{Re} s \le 0) \\ \Phi_\eta^-(s) \equiv E \exp s \underline{Z}_T & (\operatorname{Re} s \ge 0) \\ E \exp s Z_t \equiv \exp t\psi(s) & (\operatorname{Re} s = 0) \end{cases}$$

Let's observe that the Lévy measure of Z has a completely monotone density, so that

$$\psi(s) = \int_0^\infty \frac{\nu_+(d\alpha)}{\alpha} \frac{s}{\alpha - s} + \int_0^\infty \frac{\nu_-(d\alpha)}{\alpha} \frac{-s}{\alpha + s}$$

$$\equiv \psi_+(s) + \psi_-(s).$$

In the case of $m|_{\mathbb{R}^+} = \text{Leb}$, we have of course $\psi_+(s) = -(-\frac{s}{2})^{1/2}$. If $m|_{(-\infty,0]}$ is finite atomic, the process in $\mathbb{R}^-$ is a finite Markov chain with Q-matrix Q; then it is not hard to prove that

$$(2) \quad \psi_-(s) = (r_{00}(s))^{-1} = \{(s-Q)^{-1}(0,0)\}^{-1}.$$

Now the random variables $\bar{Z}_T$, $-\underline{Z}_T$ are mixtures of exponentials:

$$(3) \quad \Phi_\eta^+(s) = \int_{(0,\infty]} \mu_\eta^+(d\alpha) \frac{\alpha}{\alpha - s} \quad , \quad \Phi_\eta^-(s) = \int_{(0,\infty)} \mu_\eta^-(d\alpha) \frac{\alpha}{\alpha + s}$$

where $\mu^+_\eta(\{\infty\}) > 0$ because $\mathring{P}(\tau_0 > 0) = 1$. We have the result that for $\lambda, v > 0$

(4) $$\int_0^\infty e^{-\lambda x} E \exp(-\eta \tau_x - v(X(\tau_x) - x))\, dx = \frac{1}{\lambda - v}\left\{1 - \frac{\Psi^+_\eta(-\lambda)}{\Psi^+_\eta(-v)}\right\}$$

as reported in NHB's survey paper, p708. This can easily be proved by considering the probability that $\bar{Z}_T$ exceeds $Y_1 + Y_2$, $Y_1 \sim \exp(\lambda)$, $Y_2 \sim \exp(v)$. Hence combining we get

$$\int_0^\infty e^{-\lambda x} E \exp(-\eta \tau_x - v\{X_{\tau_x} - x\})\, dx = \frac{\displaystyle\int_{(0,\infty)} \frac{\alpha\, \mu^+_\eta(d\alpha)}{(\alpha+\lambda)(\alpha+v)}}{\displaystyle\int_{(0,\infty]} \frac{\alpha\, \mu^+_\eta(d\alpha)}{\alpha + v}}$$

and multiplying by λ and letting $\lambda \to \infty$ leads to

(5) $$\boxed{E \exp\{-\eta \tau_0 - v X(\tau_0)\} = 1 - \frac{\mu^+_\eta(\{\infty\})}{\displaystyle\int_{(0,\infty]} \frac{\alpha\, \mu^+_\eta(d\alpha)}{\alpha + v}}} = 1 - \frac{\mu^+_\eta(\{\infty\})}{\Psi^+_\eta(-v)}$$

$$= \int_{[0,\infty)} m_{\eta,0}(dt)\, \frac{t}{t+v}$$

the latter using Pick function representation technology: $X(\tau_0)$ has a mixture-of-exponentials law.

(iii) Thus we can find an expression for

$$P[\sigma < T] = \int_{[0,\infty)} m_{\eta,0}(d\alpha) \left\{1 - E\, e^{-\alpha |\bar{Z}_T|}\right\}$$

$$= \int_{[0,\infty)} m_{\eta,0}(d\alpha) \left\{1 - \int_{(0,\infty)} \mu^-_\eta(d\lambda) \frac{\lambda}{\lambda + \alpha}\right\}$$

(6) $$= \int_{[0,\infty)} m_{\eta,0}(d\alpha) \int_{(0,\infty)} \mu^-_\eta(d\lambda) \frac{\alpha}{\lambda + \alpha}$$

As we let $\eta \downarrow 0$, there's a meaningful limit for $m_{\eta,0}$ and for μ^-_η.

(iv) We can use the identity of my AIHP paper, which says (since Z is FV)

(7) $$E\, e^{-\lambda \bar{Z}_T} = \eta \left\{ \eta + \int_{[-\infty,0]} P(\underline{Z}_T \in dy) \int_{-y}^\infty \mu(dx) (1 - e^{-\lambda(x+y)}) \right\}^{-1}.$$

From this,

$$\frac{1}{\Psi^+_\eta(-v)} = 1 + \eta^{-1} \int_{-\infty}^0 P(\underline{Z}_T \in dy) \int_{-y}^\infty \mu(dx)\left(1 - e^{-v(x+y)}\right)$$

$$= 1 + \eta^{-1} \int_{(0,\infty)} \mu^-_\eta(d\alpha) \int_0^\infty \alpha e^{-\alpha y}\, dy \int_y^\infty \mu(dx)\left(1 - e^{-v(x-y)}\right)$$

(8) $$= 1 + \eta^{-1} \int_{(0,\infty)} \mu^-_\eta(d\alpha)\, \frac{1}{\sqrt{2}}\, \frac{\sqrt{\alpha v}}{\sqrt{\alpha} + \sqrt{v}}$$ after some calculations.

Hence

$$P[\sigma < T] = \int_{(0,\infty)} \mu^-_\eta(dv) \left\{ 1 - \mu^+_\eta(\{\infty\}) \left(1 + \eta^{-1} \int_{(0,\infty)} \mu^-_\eta(d\alpha) \frac{\sqrt{\alpha v/2}}{\sqrt{\alpha} + \sqrt{v}}\right) \right\}$$

Notice that $\mu^+_\eta(\{\infty\}) = P(\underline{Z}_T = 0) = P(\tau_0 > T) = E(1 - e^{-\eta\tau_0}) \sim \eta E\tau_0$ as $\eta \downarrow 0$, so it appears that as we let $\eta \downarrow 0$ we obtain

(9) $$\boxed{P[\sigma < \infty] = 1 - E\tau_0 \int_0^\infty \mu^-_0(d\lambda) \int_0^\infty \mu^-_0(d\alpha) \frac{\sqrt{\alpha\lambda/2}}{\sqrt{\alpha} + \sqrt{\lambda}}}$$

(iv) This assumes $E\tau_0 < \infty$ ... But we can derive this from (8) as follows. Letting $v \to \infty$, we obtain

$$\frac{1}{P(\tau_0 > T)} = 1 + \eta^{-1} \int_{(0,\infty)} \mu^-_\eta(d\alpha) \sqrt{\frac{\alpha}{2}}.$$

Multiplying by η gives us

$$\frac{\eta}{E(1 - e^{-\eta\tau_0})} = \eta + \int_{(0,\infty)} \mu^-_\eta(d\alpha)\, \frac{1}{\sqrt{2}} \int_0^\infty \alpha e^{-\alpha t}\, \frac{dt}{\sqrt{t}} \cdot \frac{1}{\Gamma(1/2)}$$

$$= \eta + \frac{1}{\sqrt{2}\,\Gamma(1/2)}\, E\left[\,|\underline{Z}_T|^{-1/2}\right].$$

As $\eta \downarrow 0$, LHS $\downarrow 1/E\tau_0$, RHS $\downarrow \frac{1}{\sqrt{2\pi}} E\left[|\underline{Z}_\infty|^{-1/2}\right]$. So we conclude

(10) $$\boxed{\frac{1}{E\tau_0} = \frac{1}{\sqrt{2\pi}} E\left[|\underline{Z}_\infty|^{-1/2}\right] = \int_{(0,\infty)} \mu^-_0(d\alpha) \sqrt{\frac{\alpha}{2}}.}$$

(v) Contrasting with David's result, $\boxed{P^0[\sigma < \infty] > \tfrac{1}{2}.}$ Why? If X, X' are indept r.v.s such that X^2, X'^2 have law μ^-_0, then $P^0[\sigma < \infty] > \frac{1}{2}$ iff $\frac{1}{\sqrt{2}} E\tau_0 . E\left(\frac{XX'}{X+X'}\right) < \frac{1}{2}$ iff

$$4E\frac{XX'}{X+X'} < \frac{2\sqrt{2}}{E\tau_0} = 2EX = E(X+X'),$$ and this last inequality is easy to prove.

Question for Emily (16/5/98)

(i) We make a discrete approximation to a share $\exp\{\sigma W_t + (\alpha - \sigma^2/2)t\}$ and bond e^{rt} by fixing Δt and then matching moments:

$$\begin{cases} p\, e^{\delta} + q\, e^{-\delta} = e^{\alpha \Delta t} \\ p\, e^{2\delta} + q\, e^{-2\delta} = e^{2\alpha\Delta t + \sigma^2 \Delta t} \end{cases} \qquad (A \equiv e^{\alpha\Delta t},\ k \equiv e^{\sigma^2 \Delta t}.)$$

solved by

$$\boxed{\begin{aligned} e^{\delta} &= \frac{1 + A^2 k + \{(1+A^2k)^2 - 4A^2\}^{\frac{1}{2}}}{2A} \\ p &= \frac{Aa - 1}{a^2 - 1}, \qquad a \equiv e^{\delta} \end{aligned}}$$

as is well known.

(ii) By taking M steps at once, we may suppose that the share price jumps from 1 to $\theta_i \equiv a^{2i-M}$ with probability $p_i \equiv \binom{M}{i} p^i (1-p)^{M-i}$ $(i = 0, \dots, M)$. The bond meantime grows to $\exp(Mr\Delta t) \equiv \rho$.

Suppose that the agent aims to maximise expected utility of terminal wealth, where $U(x) = x^{1-R}/(1-R)$, and the amount to be invested in the share has to be decided one time before. Thus if $V_n(w,x)$ denotes the value function with n steps to go and an amount x committed to the share, we have the Bellman equation

$$\begin{cases} V_{n+1}(w,x) = \max\limits_{y} \sum\limits_{i=0}^{M} p_i\, V_n\big(w\rho + x(\theta_i - \rho), y\big) & (n \geq 0) \\ V_0(w,x) \equiv U(w). \end{cases}$$

Evidently, by scaling we have $V_n(w,x) = w^{1-R}\, v_n(x/w)$, so the Bellman equation for the v_n becomes more simply

$$\boxed{\begin{aligned} v_{n+1}(t) &= \max_{z} \sum_{i=0}^{M} p_i \big(\rho + t(\theta_i - \rho)\big)^{1-R} v_n\Big(\frac{z}{\rho + t(\theta_i - \rho)}\Big) \\ v_0(t) &\equiv 1/(1-R) \end{aligned}}$$

(iii) It's clear that we must always have

$$\boxed{b_0 \equiv \frac{\rho}{\rho - \theta_M} < \frac{x}{w} < \frac{\rho}{\rho - \theta_0} \equiv b_1}$$

or else there's positive probability of disaster. This therefore constrains what choices can be

If we let $\tilde{V}_n(w,x)$ be max. expected utility if at time n the discounted wealth is w, and discounted cash comm. to share is x (so actual values are $\rho^n w$, $\rho^n x$) then we have

$$\tilde{V}_n(w,x) = \max_y \sum p_i \tilde{V}_{n+1}\left(w + x\left(\frac{\theta_i}{\rho} - 1\right), y\theta_i\right) \quad , V_{N-1}(w,x) = \sum p_i \, U\left(w + x\left(\frac{\theta_i}{\rho} - 1\right)\right) \cdot \rho^{N(1-R)}$$

$$\tilde{v}_n(t) = \max_z \sum p_i \left(1 + t\left(\frac{\theta_i}{\rho} - 1\right)\right)^{1-R} \tilde{v}_{n+1}\left(\frac{z\theta_i}{1 + t\left(\frac{\theta_i}{\rho} - 1\right)}\right)$$

$$\tilde{v}_{N-1}(t) = \sum p_i \, U\left(1 + t\left(\frac{\theta_i}{\rho} - 1\right)\right) \cdot \rho^{N(1-R)}$$

made in advance; we must choose y so that

$$lo < \frac{y}{w'} < hi$$

where $w' = \rho w + x(\theta_i - \rho)$ for some i. Hence we must have for all i

$$lo\left\{\rho + \frac{x}{w}(\theta_i - \rho)\right\} < \frac{y}{w} < hi\left\{\rho + \frac{x}{w}(\theta_i - \rho)\right\}$$

(iv) However, if we want to make comparisons for which there will be monotone value, we have to do the M-step aggregation differently. Instead of supposing that an amount ξ invested in the share gets inflated to $\xi \cdot X_1 X_2 \dots X_M$, where the X_j are IID, we have to consider what would happen if the amount ξ were set aside at the start of each of the M steps. If w_j is wealth after j steps, we have

$$w_j = (w_{j-1} - \xi)\, b + \xi X_j \qquad\qquad b = e^{r\Delta t}$$

$$\therefore \quad w_j - \xi = (w_{j-1} - \xi)\, b + \xi (X_j - 1)$$

and if $Z_j \equiv (w_j - \xi) b^{-j}$ we shall have

$$Z_j = Z_{j-1} + b^{-j}\, \xi (X_j - 1)$$

from which $Z_j = Z_0 + \sum_{i=1}^{j} b^{-i} \xi (X_i - 1)$ and so

$$\boxed{w_j - \xi = b^j \left\{ w_0 - \xi + \sum_{i=1}^{j} \xi (X_i - 1) b^{-i} \right\}}$$

This requires us to keep the distⁿ for the 2^M atoms if we want law of w_M.

(v) <u>Alternatively</u> (and preferably) we consider an investor who precommits the <u>number</u> of shares to be held. Thus if the state of the investor is (as before) (w, x), where w is wealth, and x is cash value of commitment to shares, if the investor now decides how many shares to buy next time, and that number of shares is now worth y, then next period the cash value to be held in shares is $y\theta_i$ if θ_i was return. So the DP equations change to

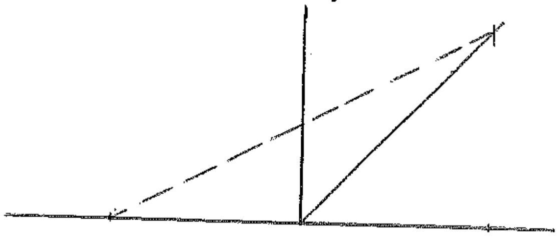
$$\boxed{\begin{aligned} V_{n+1}(w, x) &= \max_y \sum_{i=0}^{M} p_i\, V_n\big(w\rho + x(\theta_i - \rho),\, y\theta_i\big) \\ v_{n+1}(t) &= \max \sum_{i=0}^{M} p_i\, \big(\rho + t(\theta_i - \rho)\big)^{1-R} v_n\left(\frac{z\,\theta_i}{\rho + t(\theta_i - \rho)}\right) \end{aligned}}$$

$\left(t \equiv \frac{x}{w}\right)$
$(z \equiv y/w)$

This way when we aggregate from M-steps-at-once to 2M-steps-at-once, we really must get monotonicity.

Different moments of embedding times: an example (3/6/98)

Damien asks about different moments of embedding times in different forms of the Skorohod embedding. Here's an example (where we embed $U(-1,1)$ two ways) which shows this can happen.



(i) If we take the AY embedding the barycentre function is $(1+x)/2$.

In general, if we are waiting for the first time T that $|B_t| > f(L_t)$, the Laplace transform of the law can be easily shown to be (using excursion theory)

$$E\, e^{-\lambda T} = \int_0^\infty \exp\left[-\int_0^\ell \theta \coth \theta f(x)\, dx\right] \frac{d\ell}{f(\ell)} \cdot \frac{\theta f(\ell)}{\sinh \theta f(\ell)}$$

$$= \int_0^\infty \exp\left(-\int_0^\ell \theta \coth \theta f(x)\, dx\right) \frac{d\ell}{\sinh \theta f(\ell)}.$$

In this example, $f(\ell) = (1-\ell)^+$, so a few calculations give us

$$E\, e^{-\lambda T} = \frac{\theta}{\sinh\theta}$$

which is the LT of the law of the first passage time to level 1 for a $BES^0(3)$ process. This is actually obvious when you know the 2M-X property of Brownian motion! Using Maple to bust out some moments, we get

$$\tfrac{1}{2} E\, T^2 = 7/90.$$

(ii) If we choose the interval $[a,b]$, $a<0<b$, at random according to the density $(b-a)\, I_{\{a<0<b\}}\, da\, db$, and then wait for B to exit $[a,b]$, we embed the $U(-1,1)$ law. A bit of Maple applied to this gives

$$\tfrac{1}{2} E\, T^2 = 13/90.$$

(iii) There's a literature on minimal Skorohod embedding times.

(iv) I conjectured that if we wish to embed a law F for which $\int x\, F(dx) = 0 = \int x^3 F(dx)$, then it should be possible to find an embedding s.t. (τ, B_τ) are independent. But this is false, as the law $P(X=1)=\tfrac{1}{2}$, $P(X=3)=\tfrac{1}{10}$, $P(X=-2)=\tfrac{2}{5}$ demonstrates.

When we're in the situation of Lebesgue measure in $\mathbb{R}^+$, then $\nu^+(d\beta) = \sqrt{\beta}\,d\beta/\pi\sqrt{2}$, and

$$H(\lambda) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{\sqrt{\alpha\lambda}\ \mu_0^-(d\alpha)}{\sqrt{\alpha} + \sqrt{\lambda}}$$

More on completely monotone Lévy processes (4/6/98)

(i) Using the notation of pp 18–20, under the assumption that Z is a CM Lévy process such that 0 is regular for $(-\infty, 0)$ but not for $(0,\infty)$ and $Z_t \to \infty$ $(t \to \infty)$, we have from the WH factorisation and my AIHP identity that for $\mathrm{Re}(s) = 0$

$$(11)\quad \boxed{\begin{aligned} \lim_{\eta \downarrow 0} \frac{\eta}{\Psi^+_\eta(s)} &= -\psi(s)\,\Psi^-_0(s) = \int_{-\infty}^0 P(Z_\infty \in dy) \int_{-y}^\infty \{1 - e^{s(x+y)}\}\,\mu(dx) \\ &= \int_0^\infty \mu^-_0(d\alpha) \int_0^\infty \frac{\nu_+(d\beta)}{\beta}\,\frac{\alpha}{\alpha+\beta}\,\frac{-s}{\beta - s} \equiv H(-s) \end{aligned}}$$

for short. Thus the overshoot formula (4) has a nice limit as $\eta \downarrow 0$:

$$(12)\quad \int_0^\infty e^{-\lambda x}\, E\, e^{-v(Z(\tau_x) - x)}\, dx = \frac{1}{\lambda - v}\left\{1 - \frac{H(v)}{H(\lambda)}\right\} \qquad (\mathrm{Re}\,\lambda \geq 0,\ \mathrm{Re}\, v \geq 0)$$

and correspondingly for $\mathbb{R}^-$ ($\tau_x \equiv \inf\{u : Z_u < x\}$ for $x < 0$):

$$(13)\quad \int_{-\infty}^0 e^{\lambda x}\, E\left[e^{v(Z(\tau_x) - x)} : \tau_x < \infty\right] dx = \frac{1}{\lambda - v}\left\{1 - \frac{\Psi^-_0(\lambda)}{\Psi^-_0(v)}\right\} \qquad (\mathrm{Re}\,\lambda \geq 0,\ \mathrm{Re}\, v \geq 0).$$

(ii) The plan is now the following. From (12), we shall be able to represent the law of the first overshoot of $\xi_\lambda \sim \exp(\lambda)$ explicitly as a mixture of exponentials. So if we were to start at $-\xi_\lambda$, we can write down the law of first entry to $\mathbb{R}^+$ as a mixture of exponentials, and then we can find an expression for the law of Z the next time (if ever) it enters $\mathbb{R}^-$.

Indeed, (12) can be re-expressed as

$$(14)\quad \frac{1}{H(\lambda)} \cdot \int_0^\infty \mu^-_0(d\alpha) \int_0^\infty \nu_+(d\beta)\, \frac{\alpha}{\alpha+\beta} \cdot \frac{1}{\lambda+\beta} \cdot \frac{1}{v+\beta}$$

so that the law of the overshoot of ξ_λ has density

$$(15)\quad \frac{\lambda}{H(\lambda)} \int_0^\infty \mu^-_0(d\alpha) \int_0^\infty \nu_+(d\beta)\, \frac{\alpha}{(\alpha+\beta)(\lambda+\beta)}\, e^{-\beta x}\, dx$$

which is clearly a mixture of exponential distributions with mixing measure

$$(16)\quad \begin{aligned} F_\lambda(d\beta) &\equiv \frac{\lambda}{H(\lambda)} \left\{\int_0^\infty \frac{\alpha\, \mu^-_0(d\alpha)}{\alpha+\beta}\right\} \frac{\nu_+(d\beta)}{\beta(\lambda+\beta)} \\ &= \frac{\lambda}{H(\lambda)} \cdot E\, e^{\beta Z_\infty} \cdot \frac{\nu_+(d\beta)}{\beta(\lambda+\beta)} \\ &= \frac{\lambda}{H(\lambda)}\, \Psi^-_0(\beta)\, \frac{\nu_+(d\beta)}{\beta(\lambda+\beta)}. \end{aligned}$$

The undershoot of $-\xi_\beta$ has (from (13)) Laplace transform

$$(17) \qquad \frac{\beta}{\Psi_0^-(v)} \int \frac{\mu_0^-(dz)\, z}{(v+z)(\beta+z)}$$

so that if we started at $-\xi_\lambda$, the LT of the depth below 0 when Z first returns to $\mathbb{R}^-$ will be

$$(18) \qquad \frac{\lambda}{H(\lambda)} \int_0^\infty \frac{\Psi_0^-(\beta)\, \gamma_+(d\beta)}{(\lambda+\beta)} \int \frac{\mu_0^-(dz)\, z}{(\beta+z)(v+z)} \cdot \frac{1}{\Psi_0^-(v)}$$

We seek a magic mixing distr^n $G_-(d\lambda)$ with the property that

$$(19) \qquad \boxed{\int_0^\infty \frac{G_-(d\lambda)\,.\,\lambda}{H(\lambda)} \int_0^\infty \frac{\Psi_0^-(\beta)\, \gamma_+(d\beta)}{(\lambda+\beta)} \int_0^\infty \frac{\mu_0^-(dz)\, z}{(\beta+z)(v+z)} \cdot \frac{1}{\Psi_0^-(v)} = a \int_0^\infty \frac{G_-(d\lambda)\, \lambda}{\lambda+v}}$$

for some constant a.

(iii) Let's observe that we can use the AIHP identity in the other direction. We already have that for some measure m

$$\lim_{\eta \downarrow 0} \frac{1}{\eta} E e^{-v Z_\tau} = \int_0^\infty \frac{m(ds)}{s+v} = \frac{1}{H(v)}$$

so using the AIHP identity on Z_τ gives

$$E e^{v Z_\infty} = \Psi_0^-(v) = \left[\lim \eta^{-1} \int_0^\infty P(Z_\tau \in dy) \int_{-\infty}^{-y} (1 - e^{v(x+y)})\, \mu(dx) + 1 + cv \right]^{-1}$$

$$= \left[\int_0^\infty m(ds) \int_0^\infty e^{-sy}\, dy \int_0^\infty \gamma_-(d\beta) \int_{-\infty}^{-y} e^{\beta x} (1 - e^{v(x+y)})\, dx + 1 + cv \right]^{-1}$$

$$= \left[\int_0^\infty m(ds) \int_0^\infty \gamma_-(d\beta) \int_0^\infty e^{-sy}\, dy \int_y^\infty e^{-\beta x} (1 - e^{-v(x-y)})\, dx + 1 + cv \right]^{-1}$$

$$= \left[\int_0^\infty m(ds) \int_0^\infty \gamma_-(d\beta)\, \frac{1}{\beta+s}\, \frac{v}{\beta(\beta+v)} + 1 + cv \right]^{-1}$$

$$(20) \qquad = \left[\int_0^\infty \gamma_-(d\beta)\, \frac{1}{H(\beta)}\, \frac{v}{\beta(v+\beta)} + 1 + cv \right]^{-1} \qquad \left[c = \left(\int_0^\infty \alpha\, \mu_0^-(d\alpha) \right)^{-1} \right]$$

(iv) However, much of this carries through without the CM assumptions. If we suppose that $-Z_\infty$ has density f, and μ has density ρ, without much loss of generality, we may write

$$(21)\quad \boxed{H(-s) \equiv \lim_{\eta\downarrow 0} \frac{\eta}{\Psi^+_\eta(s)} = \int_0^\infty f(y)\,dy \int_y^\infty \left(1-e^{s(x-y)}\right)\rho(x)\,dx,}$$

from which

$$(22)\quad \frac{H(\lambda)-H(v)}{\lambda - v} = \int_0^\infty e^{-vs}\left\{\int_0^\infty f(y)\,dy \int_s^\infty \rho(y+t)\,e^{-\lambda(t-s)}\,dt\right\} ds \equiv \int_0^\infty e^{-vs}\,h(\lambda,s)\,ds.$$

Thus we may invert the overshoot formula for $\mathbb{R}^+$ with respect to v; if we start at $-\xi_\lambda$ the place where we first enter $\mathbb{R}^+$ has density

$$(23)\quad \frac{\lambda\, h(\lambda,x)}{H(\lambda)}\,dx.$$

On the other hand, if we take the undershoot formula (13)

$$\int_{-\infty}^0 e^{\lambda x}\,E\left[e^{v(Z(\tau_x)-x)};\ \tau_x<\infty\right]dx = \frac{1}{\Psi_0^-(v)}\int_0^\infty f(y)\left\{\int_0^y e^{-\lambda z - v(y-z)}\,dz\right\}dy$$

$$= \frac{1}{\Psi_0^-(v)}\int_0^\infty e^{-\lambda z}\left\{\int_z^\infty f(y)\,e^{-v(y-z)}\,dy\right\}dz$$

so that the law of the undershoot of $x<0$ has LT

$$(24)\quad \frac{1}{\Psi_0^-(v)}\int_{-x}^\infty f(y)\,e^{-v(y+x)}\,dy = \frac{1}{\Psi_0^-(v)}\int_0^\infty f(t-x)\,e^{-vt}\,dt.$$

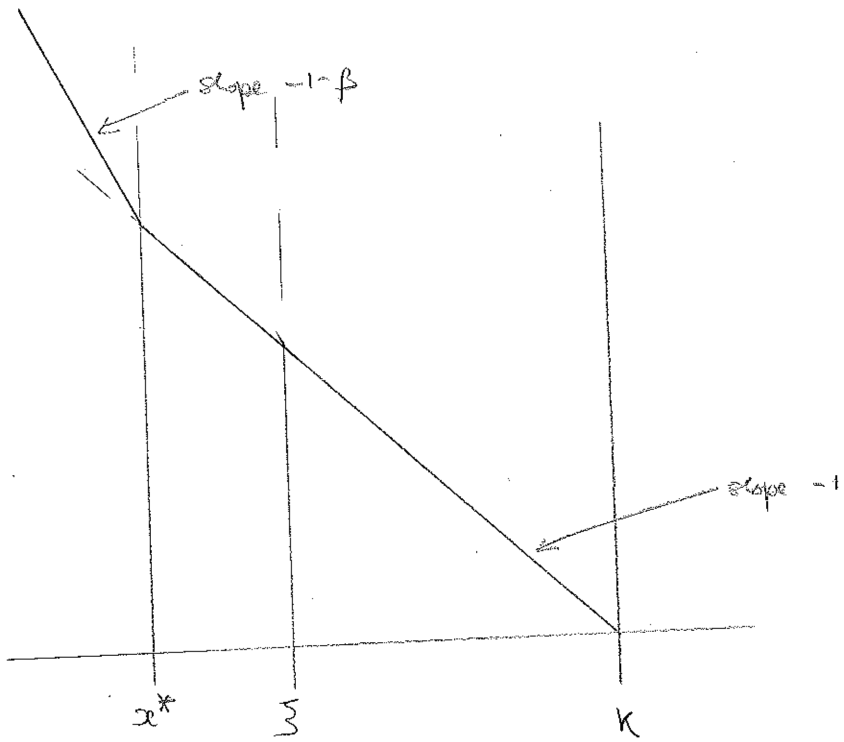
Combining (23) and (24) shows that if we start at $-\xi_\lambda$, the first return to $\mathbb{R}^-$ has Laplace transform

$$\boxed{v \mapsto \frac{1}{\Psi_0^-(v)}\int_0^\infty h(\lambda,x)\int_0^\infty f(t+x)\,e^{-vt}\,dt\,dx\,.\ \lambda/H(\lambda)}$$

(v) It appears that the expression for $1/\Psi_0^-(v)$ as a LT is likely to be quite important. Reverting to the CM assumptions, we have a representation

$$\boxed{\begin{aligned}\frac{1}{\Psi_0^-(v)} &= 1 + bv + \int_0^\infty m(dt)\left(\frac{1}{t} - \frac{1}{t+v}\right)\\ &= 1 + bv + v\int_0^\infty e^{-vy}\left\{\int e^{-ty}\,\frac{m(dt)}{t}\right\}dy\end{aligned}}$$

where $b \equiv 1/\lim_{v\to\infty} v\,\Psi_0^-(v) = 1/\int_0^\infty \alpha\,\mu_0^-(d\alpha) \geq 0$, and m is some non-negative measure. In fact, $\boxed{b=0}$ as we see from the WH factorisation (11), $s\,\Psi_0^-(s) = -H(-s)\{s/\psi(s)\}$, since for $s=i\theta$ we have $\frac{1}{s}\psi(s)\to 0$, and $H(-i\theta)$ doesn't tend to zero as $\theta\to\infty$. This simplifies the story a little.



Payoff at K is

$$\beta(K-x^*) - (1+\beta)(K-\xi) = -\beta x^* - K + \xi(1+\beta) = 0$$

Robust hedging of an up-and-in put (8/6/98)

Suppose we have a martingale $(M_t)_{0\le t\le 1}$ whose law at time 1 is given, in the form of the call price $C(K) \equiv E\,(M_1-K)^+$ for each K. M_0 can be general; Define $\bar M_t \equiv \sup_{s\le t} M_s$. Suppose the barrier ξ is $> M_0$.

A knock-in put with strike K and barrier ξ has payoff $(K-M_1)^+ I_{\{\bar M_1 \ge \xi\}}$. Can we find a universal upper bound for the price of this thing, and hence a robust hedge?

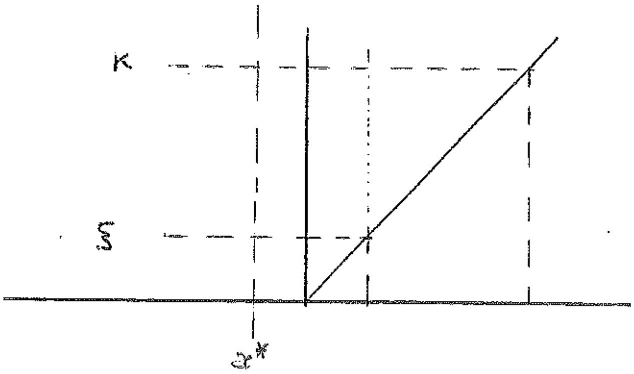
The Lagrangian form of the problem is

$$\max_{\rho\ge 0} \int_{-\infty}^{\infty} dx \int_0^{\infty} dy \;\; \rho(x,y)\;\Lambda(x,y)$$

where ρ is the joint density of $(M_1, \bar M_1)$ and

$$\Lambda(x,y) \equiv (K-x)^+ I_{\{y\ge\xi\}} - \alpha(x) - \int_0^y (x-a)\,\theta(da)$$

in terms of the unknown multipliers α and θ.



Case I: $M_0 < \xi < K$

Thinking how Λ varies with y, it's clear that θ has to be concentrated on ξ, putting mass δ there, say. The jump in Λ as y crosses ξ is

$$(K-x)^+ - \delta(x-\xi) \;=\; K+\delta\xi - (1+\delta)x \qquad \text{for } x<K$$

If we take $\delta = -1-\beta$, $\beta>0$, we get that the jump in Λ is $\beta x + K-\xi-\beta\xi$ which is zero if

$$x = x^* \equiv \xi - \frac{K-\xi}{\beta}$$

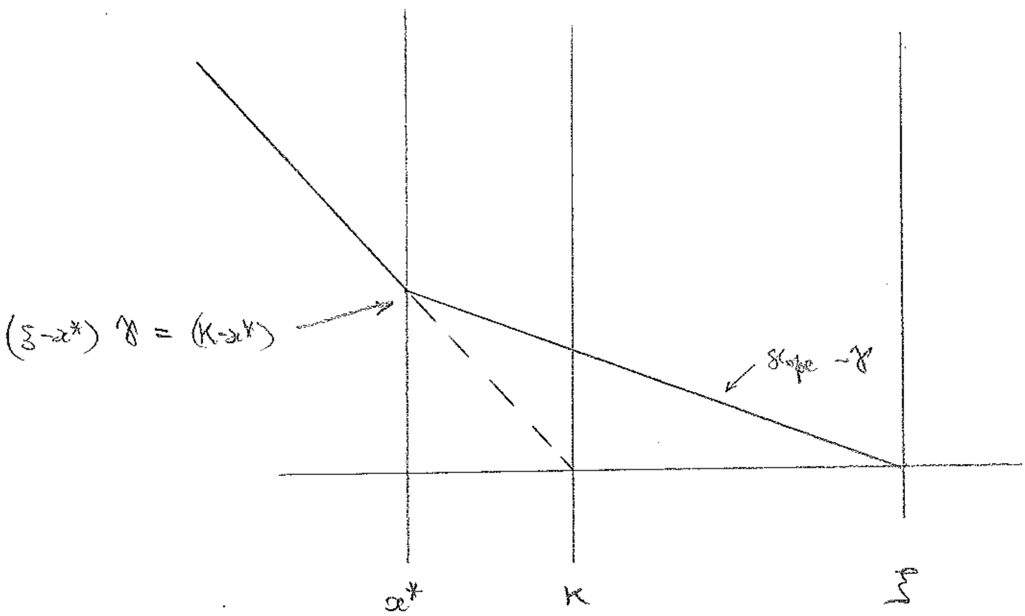
and the jump is positive to the right of x^*, negative to the left of x^*. So if we were to use

$$\alpha(x) \;=\; \beta\,(x-x^*)^+ + (x-K)^+ \qquad \left[\beta = \frac{K-\xi}{\xi-x^*}\right]$$

we've made Λ which is ≤ 0 everywhere, and $=0$ in $[x^*,\infty)\times[\xi,\infty)\;\cup\;(-\infty,x^*]\times(-\infty,\xi]$.

If we took $x^* \equiv b^{-1}(\xi)$ we can build a joint distⁿ concentrated on the set where $\Lambda=0$ (by AY embedding), and this is therefore optimal.

From the dual point of view, the strategy we are considering is to hold β calls of strike x^*, to sell forward $1+\beta$ shares when price hits ξ, and to hold 1 call of strike K. As is easily seen, this hedging strategy superreplicates the up-and-in put. If we consider the cost of this strategy, it is

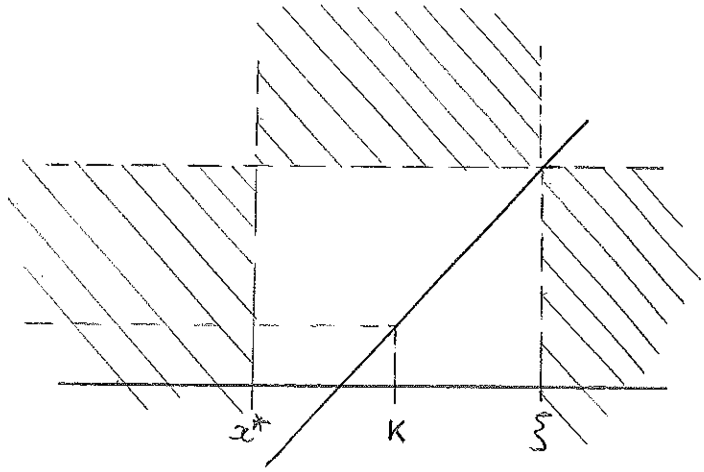


$(\xi - x^*)\gamma = (K - x^*)$
slope $-\gamma$
x^*
K
ξ

$$C(K) + \beta\, C(x^*) = C(K) + \beta\, C\left(\xi - \frac{K-\xi}{\beta}\right)$$

which we attempt to minimise over β, or, equivalently, x^*. This happens when $x^* = b^{-1}(\xi)$, and then

$$\boxed{\beta = \frac{K-\xi}{\xi - b^{-1}(\xi)}}.$$



Case II: $K < \xi$.

Once again, θ can only charge the point ξ, so suppose it puts weight $-\gamma \in (-1,0)$ at ξ. Then the jump in Λ as y crosses ξ is

$$(K-x)^+ + \gamma(x-\xi)$$

which is zero at $x = x^* \equiv (K-\gamma\xi)/(1-\gamma) < K$, positive to the left of x^*, negative to the right. So if we choose

$$\alpha(x) = (1-\gamma)(x^*-x)^+ + \gamma\,(x-\xi)^+$$

we have that Λ is non positive, and negative in the shaded region. Thus if we could find a feasible law ρ which was concentrated on the unshaded region, it would be optimal.

If F is the law of M_1, and we sweep all of the mass in $(-\infty, x^*] \cup [\xi, \infty)$ onto ξ, we should have mean M_0:

$$M_0 = \int_{(x^*,\xi]} x\, F(dx) + \xi\left\{F(x^*) + \bar{F}(\xi)\right\}$$

$$= \int_{x^*}^{\infty} (x-x^*)\, F(dx) - \int_{\xi}^{\infty} (x-\xi)\, F(dx) + x^* \bar{F}(x^*) - \xi \bar{F}(\xi) + \xi\left(F(x^*) + \bar{F}(\xi)\right)$$

$$= C(x^*) - C(\xi) + x^* \bar{F}(x^*) + \xi\, F(x^*)$$

$$= c(x^*) - c(\xi) - x^* c'(x^*) + \xi + \xi\, c'(x^*)$$

Is there a solution? A unique solution?

Perhaps easiest way to think of this is in ~~terms~~ of the dual problem. We have a family of strategies which can be described as

"hold γ calls strike ξ, $1-\gamma$ puts strike x^* $\quad (x^* \equiv (K-\gamma\xi)/(1-\gamma))$ and sell forward γ shares when price reaches ξ"

and these always super replicate the knock-in put, generating surplus if the share ends up outside (x^*,ξ) before ξ is hit, or if it ends up in (x^*,ξ) after ξ is hit.

If we now consider the cost of such a strategy it will be

$$\gamma\, C(\xi) \;+ (1-\gamma)\left\{ C(x^*) + x^* - M_0 \right\}$$

$$= \frac{K - x^*}{\xi - x^*} C(\xi) + \frac{\xi - K}{\xi - x^*}\left\{ x^* + C(x^*) - M_0 \right\}$$

$$= \frac{\xi - K}{\xi - x^*}\left\{ C(x^*) + x^* + \frac{K - x^*}{\xi - K} C(\xi) - M_0 \right\} \qquad (*)$$

Now the function in $\{\ \}$ is a convex function of x^*, so the price of the super-replicating portfolio has a unique minimum, where

$$0 = \frac{1}{\xi - x} + \frac{C'(x) + 1 - C(\xi)/(\xi - K)}{C(x) + x + \frac{K - x}{\xi - K} C(\xi) - M_0}$$

This reduces to the condition on the preceding page for feasibility. However, if the minimising x is not less than K, we have to think again.

Observe that the function $(*)$ which we seek to minimise can be expressed as

$$\frac{\xi - K}{\xi - x}\left\{ C(x) + \xi - C(\xi) - M_0 \right\} - (\xi - K) + C(\xi)$$

so if we consider the slope of the log of the first bit we get

$$\frac{1}{\xi - x} + \frac{C'(x)}{C(x) + \xi - C(\xi) - M_0}$$

and if $x = M_0$, this is $\dfrac{1}{\xi - M_0} - \dfrac{P(M_1 > M_0)}{\xi + C(M_0) - C(\xi) - M_0} > 0$. Thus we always have that the minimising x^* must be $< M_0$. So the only way we can have the minimising $x \geq K$ is when K is itself $< M_0$.

In this case, then, the optimal policy is clear; we choose the embedding on the previous page where $M_1 \in [x^*, \xi]$ iff $\bar{M}_1 \leq \xi$, and then simply hold one put with strike K. This super-replicates with no surplus.

Pricing a 'knock-down' option (29/6/98)

(i) The idea here is that the log Brownian share $S_t = S_0 \exp\{\sigma W_t + (r - \frac{1}{2}\sigma^2)t\}$ is observed at the times $\delta, 2\delta, \dots, N\delta \equiv T$ and we have a payoff

$$(K - S_T)^+ \quad I_{\{ S(j\delta)/S(j\delta - \delta) \in A_j \quad \forall j = 1, \dots, N \}}$$

where typically A_j would be some interval. What would be the price of this?

(ii) Let $X_t \equiv \sigma W_t + \mu t$, $\mu \equiv r - \sigma^2/2$. If we set

$$\varphi_j(s) \equiv E\left[\exp sX(\delta) \; ; \quad X(\delta) \in \Gamma_j \right],$$

then

$$\varphi(s) \equiv \prod_{j=1}^{N} \varphi_j(s) = E\left[\exp\{ s \log(S_T/S_0) \} \; ; \quad X(j\delta) - X((j-1)\delta) \in \Gamma_j \quad \forall j = 1, \dots, N \right]$$

and we may recover the price of the option from this. Indeed, we have

$$E\left[(K - S_T)^+ \; : \; X(j\delta) - X(j\delta - \delta) \in \Gamma_j \right]$$

$$= \lim_{\varepsilon \downarrow 0} E\left[(K - e^{X_T})^+ e^{\varepsilon X_T} \; : \; \Delta_j X \in \Gamma_j \right] \qquad \left(\Delta_j X \equiv X_{j\delta} - X_{j\delta - \delta} \right)$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int \varphi(i\theta) \overline{g_\varepsilon(i\theta)} \, d\theta$$

where $g_\varepsilon(i\theta) \equiv \int e^{i\theta x} (K - e^x)^+ e^{\varepsilon x} dx = K e^{(i\theta + \varepsilon)\log K} \left\{ \frac{1}{i\theta + \varepsilon} - \frac{1}{1 + i\theta + \varepsilon} \right\}$ is the FT of the (approximate) payoff function. Thus the price is

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi(z) \, g_0(-z + \varepsilon) \, dz \qquad \left[g_0(z) \equiv K e^{z \log K} / z(1+z) \right]$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \varphi(z) \, g_0(-z + \varepsilon) \, dz \qquad \text{for any } c < 0$$

$$= \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \varphi(z) \, g_0(-z) \, dz$$

Cunning choice of c (as in the saddlepoint method) may be valuable, but in principle a numerical integration should work OK. The CF φ can't be expected to go to zero very rapidly, so there may be slow convergence of the Fourier integral.

What we <u>can</u> do is evaluate the terms φ_j more explicitly? Not obviously.

Perhaps better than this is to compute the first few moments of each $\Delta_j X$ conditional on the event $\Delta_j X \in \Gamma_j$, and then do a CLT approximation; the price can be expressed as

$$\int_{-\infty}^{\log K} e^y \, P[X_T \leq y; F] \, dy = \int_0^K P(S_T \leq x; F) \, dx$$

$$F \equiv \bigcap_{j=1}^{N} F_j$$

$$F_j = \{\Delta_j X \in \Gamma_j\}$$

$$= P(F) \int_{-\infty}^{K} e^y \, P[X_T \leq y \mid F] \, dy$$

Thus if, for example, all the Γ_j are the same, we should be able to use some of the ideas from Petrov to approximate the CDF $P(X_T \leq \cdot \mid F)$ very well.

Why sharing of insurance risk may be worthwhile (17/7/98)

(i) An insurance company has to meet total claims X, against which they have a reserve K of capital, where
$$P[X > K] = \varepsilon.$$
The (small) parameter ε is either imposed by regulatory requirements, or by the insurance company's own service standards.

Now suppose the company is contemplating taking on an additional claim Y; what is the minimum premium they will require to achieve that same service level? Clearly, they must charge at least k, where
$$P[X+Y > K+k] = \varepsilon.$$
Thus if X has dist^n f^n F, Y has dist^n f^n G, we'll need
$$\boxed{\varepsilon = \int_0^\infty G(dy)\ \bar{F}(K+k-y) = \int_0^\infty F(dx)\ \bar{G}(K+k-x)}$$

(ii) Example Suppose $X \sim N(\mu, \sigma^2)$, $Y \sim N(b, v)$ {This is a little unrealistic in that it allows negative claims, but it's still worth doing.} If $P[N(0,1) > \xi] = \varepsilon$, we must have
$$K = \mu + \xi\sigma, \quad K+k = \mu + b + \xi\sqrt{\sigma^2 + v}$$

Thus the extra premium
$$k = \xi\{\sqrt{\sigma^2+v} - \sigma\} + b = \frac{\xi v}{\sqrt{\sigma^2+v}+\sigma} + b$$
depends not only on the claim Y, but also on the existing portfolio X.

(iii) If now we were to consider several insurers, the i^{th} having a portfolio of claims $X_i \sim N(\mu_i, \sigma_i^2)$, and suppose that insurer i will pay out $p_i Y$, $\sum p_i = 1$, then the additional cost for insurer i will be
$$p_i b + \frac{\xi p_i^2 v}{\sqrt{\sigma_i^2 + p_i^2 v} + \sigma_i}$$

so we shall have to charge a total premium of
$$b + \xi \sum_{i=1}^n \left\{ \sqrt{\sigma_i^2 + p_i^2 v} - \sigma_i \right\}.$$

If we try to minimise this subject to the constraint $\sum p_i = 1$, we get the optimal risk-sharing:
$$\boxed{p_i \propto \sigma_i}$$

One thing to note here is that in order to offer the client the same degree of protection, we have to know that with probability at least $1-\varepsilon$ all of the insurers can pay up, so we'd need to use $\bar{\Phi}^{-1}(\varepsilon/n)$ in place of $\xi \equiv \bar{\Phi}^{-1}(\varepsilon)$. Since $\bar{\Phi}$ drops so rapidly, this shouldn't make a huge difference

(iv) Example $X \sim H_a$, $Y \sim H_b \equiv \inf\{t : W_t = b\}$, The capital required to protect against X is $c.a^2$ for some constant c, then if we take Y on as well, we have to have additional

$$k = (b^2+2ab).c$$

– so it's cheaper to set up a separate company!! This example is a bit weird, as a claim dist^n with infinite mean is probably too risky to be insuring.

(v) Example Suppose X, Y have Pareto distributions

$$P(X>x) = \left(\frac{a}{a+x}\right)^{\theta}, \quad P(Y>y) = \left(\frac{b}{b+y}\right)^{\nu}$$

Now suppose that $\theta = n+\alpha$, $\alpha \in (0,1)$, so that X has moments of all orders up to n inclusive. Then a result of Bingham and Doney (Thm 8.1.6 in BGT) says that

[DEAD END: see next page]

$$(-1)^{n+1} E\left[e^{-sX} - \sum_{r=0}^{n} (-sX)^r/r!\right] \sim s^{\theta}\, \ell(1/s) \qquad (s \downarrow 0)$$

$$\Leftrightarrow$$

$$\bar{F}(x) \sim \frac{(-1)^n}{\Gamma(1-\theta)}\, x^{-\theta}\, \ell(x) \qquad (x \to \infty)$$

For the Pareto law of X, $\ell(x) \equiv \Gamma(1-\theta)(-1)^n a^{\theta}$, so we have

$$\boxed{E\, e^{-sX} \sim \sum_{r=0}^{n} \frac{(-s)^r}{r!} EX^r - s^{\theta} a^{\theta}\, \Gamma(1-\theta)}$$

If $\theta < \nu$, then we have

$$E\, e^{-s(X+Y)} \sim \sum_{r=0}^{n} \frac{(-s)^r}{r!} E\left[(X+Y)^r\right] - s^{\theta} a^{\theta}\, \Gamma(1-\theta).$$

If $\theta = \nu$, then

$$E\, e^{-s(X+Y)} \sim \sum_{r=0}^{n} \frac{(-s)^r}{r!} E\left[(X+Y)^r\right] - s^{\theta}(a^{\theta}+b^{\theta})\, \Gamma(1-\theta).$$

If $\nu = m+\beta < \theta$, then

$$E\, e^{-s(X+Y)} \sim \sum_{r=0}^{m} \frac{(-s)^r}{r!} E(X+Y)^r - s^{\nu} b^{\nu}\, \Gamma(1-\nu)$$

The moral of this then is that the tail of $X+Y$ is the same as the fatter of the two tails. Nevertheless, this is too crude to help us decide how much extra we need to take on the additional risk.

So if we introduce the (incomplete beta ?) function

$$B(\alpha,\beta\,;\,x) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

for $\alpha,\beta>0$, $x\in[0,1]$, we have $B(\alpha,\beta\,;\,1) = \{\Gamma(\alpha+\beta)/\Gamma(\alpha)\Gamma(\beta)\}^{-1} \equiv B(\alpha,\beta)$. In terms of this,

$$P(X+Y>x) = \left(\frac{b}{b+x}\right)^{\nu} + \int_0^x \frac{\nu b^{\nu}}{(b+y)^{\nu+1}}\left(\frac{a}{a+x-y}\right)^{\theta} dy$$

$$= \left(\frac{b}{b+x}\right)^{\nu} + \nu\left(\frac{a}{a+b+x}\right)^{\theta}\left(\frac{b}{a+b+x}\right)^{\nu}\left\{B\left(-\nu,1-\theta\,;\,\frac{b+x}{a+b+x}\right) - B\left(-\nu,1-\theta\,;\,\frac{b}{a+b+x}\right)\right\}$$

This seems to be about the best we can do. We can approximate the incomplete beta f^n by a binomial expansion: for $0<\xi<\eta\leq\frac{1}{2}$ we have

$$\boxed{\begin{aligned}\int_\xi^\eta t^{\alpha-1}(1-t)^{\beta-1}\,dt &= \int_\xi^\eta t^{\alpha-1}\left\{\sum_{r=0}^{n}\frac{(-t)^r}{r!}\frac{\Gamma(\beta)}{\Gamma(\beta-r)} + \frac{(-t)^{n+1}}{(n+1)!}\frac{\Gamma(\beta)}{\Gamma(\beta-n-1)}(1-\tilde{t})^{\beta-2-n}\right\}dt \quad (0<\tilde{t}<t)\\ &= \sum_{r=0}^{n}\frac{(-1)^r\Gamma(\beta)}{r!\,\Gamma(\beta-r)}\left(\frac{\eta^{\alpha+r}-\xi^{\alpha+r}}{\alpha+r}\right) + R_n\end{aligned}}$$

The remainder term R_n can be bounded by (assuming $n+2>\beta$)

$$|R_n| \leq \frac{2^{n+2-\beta}\,\Gamma(\beta)}{(n+1)!\,\Gamma(\beta-n-1)}\;\frac{\eta^{n+\alpha+1}-\xi^{n+\alpha+1}}{n+\alpha+1}.$$

This could be useful in a numerical analysis

(vi) <u>What's the magnitude of the improvement in the Gaussian situation?</u> Returning to the situation of (ii)–(iii), the optimal choice of p_i is $c\sigma_i$, where $c=(\sum_1^n\sigma_j)^{-1}$. The total cost if company i goes it alone will be

$$b + \xi\left(\sqrt{\sigma_i^2+v} - \sigma_i\right) \simeq b + \sigma_i\xi\cdot\frac{v}{2\sigma_i^2} = b + \xi\frac{v}{2\sigma_i}$$

if we may assume that $v \ll \sigma_i^2$. But if the insurers share out the risk, the cost will be

$$b + \tilde{\xi}\,\frac{1}{c}\left(\sqrt{1+c^2v}-1\right) \simeq b + \tilde{\xi}\,\frac{cv}{2} = b + \tilde{\xi}\,\frac{v}{2\sum\sigma_j}$$

So however it's done, you have to charge a premium of b, plus an additional 'safety cost' which is

$$\xi \cdot \frac{v}{2\sigma_i} \quad \text{singly}, \qquad \xi_n \frac{v}{2\sum \sigma_j} \quad \text{collectively}$$

Thus if we had $\sigma_i = \sigma$ for all i, $\frac{v}{\sigma^2} = 0.05$, the safety cost would be ($\varepsilon = 0.01$)

$0.0581575\ \sigma$	for one insurer
$0.0321975\ \sigma$	for two insurers
$0.01758125\ \sigma$	for four insurers

(vii) <u>What happens in general if Y doesn't have fat tails?</u> We're trying to find k so that

$$\varepsilon = \int G(dy)\, \bar{F}(K+k-y) = \bar{F}(K)$$

so

$$0 = \int G(dy) \left\{ \bar{F}(K+k-y) - \bar{F}(K) \right\}$$

So if F has density f, we can do a Taylor expansion to give

$$0 \simeq \int G(dy) \left\{ \sum_{r=1}^{n} f^{(r-1)}(K)\, (k-y)^r / r! \right\}$$

Taking just two terms in this gives

$$0 = f(K)\,(k - EY) + \tfrac{1}{2} f'(K) \left\{ \operatorname{var}(Y) + (k - EY)^2 \right\}$$

If we may suppose $(k-EY)^2$ small compared to $\operatorname{var}(Y)$, we'd have approximately

$$\boxed{k \doteq EY - \frac{f'(K)}{2f(K)} \operatorname{var}(Y)}$$

– only makes sense if $f'(K) < 0$
... all in all, it doesn't look too plausible!

(viii) <u>One-period optimisation problem.</u>

If the claims taken on amount to X_π and bring in premiums $\beta\pi$, and the company can invest its initial wealth w in <u>either</u> a risky asset which returns random amount Z, <u>or</u> in safe bank account returning R, the aim is to choose the amount x to put into the risky asset so as to

$$\max_{x,\pi} E\left[xZ + (w - x + \beta\pi) R - X_\pi \right]$$

subject to $P\left[xZ + (w - x + \beta\pi) R < X_\pi \right] \leq \varepsilon$. The crucial thing for this problem is therefore

$$\varphi(x, r) \equiv P\left[\; x(Z-R) + wR < X_r - \beta R r \;\right]$$

since we're concerned with the greatest convex minorant of this function. Nevertheless, this is probably going to be an ugly hack in any specific example. Conceivably the simplest way to obtain numerical values of φ would be by simulation!

(ix) Continuous-time optimisation problem.

Taking a simple problem first, we'll assume that there's a single line of business, and that the insurance company engages in it at level θ_t at time t (so the total claims less premiums by time t is $\tilde{Z}_t = Z\left(\int_0^t \theta_u \, du\right)$, where Z is a spectrally positive Lévy pr) The level of business may not exceed $b(w_t)$, where w_t is the value of the insurance company at time t. The function b is increasing; it embodies the regulatory constraint. The assets are invested in some log-Brownian process with return $\sigma dW_t + \alpha dt$. Thus the wealth equation is

$$dw_t = w_t(\sigma dW_t + \alpha dt) - d\tilde{Z}_t - \delta_t dt$$

where δ_t is dividend stream. If the firm tries to maximise

$$E\left[\int_0^\zeta e^{-rt} \delta_t \, dt - K e^{-r\zeta}\right]$$

where $K \geq 0$ is the bankruptcy penalty and $\zeta \equiv \inf\{t : w_t < 0\}$ is bankruptcy time. [CARE: mumbo-jumbo about risk-neutral valuation...] The value function

$$V(w) \equiv \max E\left[\int_0^\zeta e^{-rs} \delta_s \, ds - K e^{-r\zeta} \;\Big|\; w_0 = w\right]$$

therefore will satisfy

$$\boxed{\sup_{\substack{0 \leq \theta \leq b(w) \\ \delta \geq 0}} \Big[-rV(w) + \tfrac{1}{2}\sigma^2 w^2 V''(w) + \alpha w V'(w) - \delta V'(w) + \theta \Big\{ \beta V'(w) + \int_0^\infty (V(w-y) - V(w))\, \mu(dy) \Big\} + \delta \Big] = 0}$$

where β is the rate of premium income, μ the Lévy measure of claim. We also have

$$\boxed{V(w) = -K \quad (w < 0)}$$

Complementary slackness $\Rightarrow$ $\boxed{V'(w) \geq 1}$ everywhere. It's clear that V should be non-decreasing, $V(0) = 0$, and b should be non-decreasing. Numerical sol^n ok?

If we set $\mathcal{G}V(w) \equiv \beta V'(w) + \int_0^\infty \mu(dy)\{V(w-y) - V(w)\}$, then the system we have to solve is

$$\boxed{\max\left\{ \tfrac{1}{2}\sigma^2 w^2 V''(w) + \alpha w V'(w) - rV(w) + b(w)\,\mathcal{G}V(w)^+,\ 1 - V'(w) \right\} = 0,} \qquad V(w) = -K \quad (w<0)$$

Let's note that for any $\tilde{V}$ satisfying this, $\int_0^t e^{-rs}\delta_s\,ds + e^{-rt}\tilde{V}(w_t)$ is a supermartingale, so we conclude that

$$\tilde{V}(w) \geq E\left[\int_0^T e^{-rs}\delta_s\,ds - Ke^{-rT} \,\middle|\, w_0 = w \right]$$

whatever δ and θ may be used. So we can consider the problem to be

$$\boxed{\begin{array}{l} \min \quad V(1) \\ \text{subject to} \\ \max\left\{ \tfrac{1}{2}\sigma^2 w^2 V''(w) + \alpha w V'(w) - rV(w) + b(w)\,\mathcal{G}V(w)^+,\ 1 - V'(w) \right\} = 0 \end{array}}$$

(x) Discretisation. Let's fix $h>0$ small to be the step of the discretisation, and then we discretise the distribution μ onto $h\mathbb{Z}^+$. One way to do this would be to set

$$m_k = h^{-1}\left\{ \int_{(k-1)h}^{kh} \bar{F}(x)\,dx - \int_{kh}^{kh+h} \bar{F}(x)\,dx \right\} \qquad (k \geq 1)$$

which keeps the mean of the distribution the same.

We now seek a solution vector $(v_k)_{k\geq 1}$ (v_k approximating $V(kh)$) with the properties that $v_0 = 0$, $v_k \geq 0$, and some auxiliary vector $(z_k)_{k\geq 1}$, $z_k \geq 0$ for all k, such that the constraints

$$z \geq Gv + a$$

$$Mv + Bz \leq 0$$

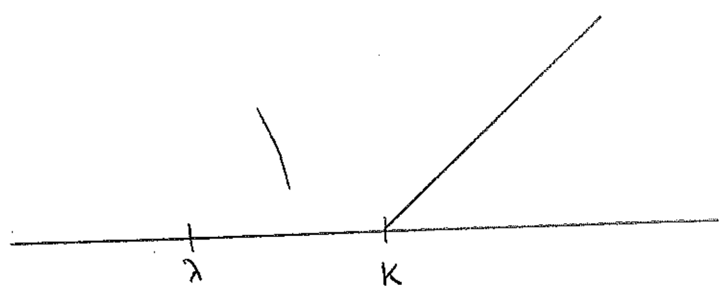
$$Dv \geq 1$$

are satisfied (where G is the matrix analogue of the operator $\mathcal{G}$, $a_k \equiv -K \sum_{j>k} m_j$, B is the diagonal matrix $\mathrm{diag}(b(jh))$, D is the matrix analogue of differentiation, and M the matrix analogue of $\tfrac{1}{2}\sigma^2 w^2 D^2 + \alpha w D - r$), and so as to minimise the objective function

$$v_1$$

Easy. This is a linear program. If we were to have N points of discretisation, we would then have $2N$ variables, and $3N$ inequality constraints. This is well within the capabilities of a smart LP, even for N of the order of thousands.

Note that the same approach works for multiple lines of business too.


λ
K

Robust hedging of the various barrier-options (30/7/98)

(i) If we return to the situation of p 27, we can work out the cheapest super-replicating hedge and associated extremal martingale for a variety of single-barrier options. We'll just work with "up-and-..." options, because a down-and-out call on X is the same as an up-and-out put on $-X$, for example. Assume the 'up' barrier λ is above X_0.

(ii) Let's suppose the payout at time 1 is $\varphi(X_1)$ depending on whether $\bar{X}_1 \equiv \sup_{t\leq 1} X_t$ is greater than λ (up-and-in) or less than λ (up-and-out). Assume that φ is some convex function. If $\rho(x,y)$ is to be the joint density of $(X_1, \bar{X}_1)$, then the Lagrangian for the problem is

$$\Lambda(x,y) = \begin{cases} \varphi(x)\, I_{\{y>\lambda\}} - \alpha(x) - \int_0^y (x-a)\,\theta(da) & \text{(up-and-in)} \\ \varphi(x)\, I_{\{y\leq\lambda\}} - \alpha(x) - \int_0^y (x-a)\,\theta(da) & \text{(up-and-out)}. \end{cases}$$

Now it's clear that θ must be a mass δ at λ and perhaps also k at zero, so we have

$$\Lambda(x,y) = \begin{cases} \big(\varphi(x) - \delta(x-\lambda)\big) I_{\{y>\lambda\}} - \alpha(x) & \text{(up-and-in)} \\ \big(\varphi(x) + \delta(x-\lambda)\big) I_{\{y\leq\lambda\}} - \alpha(x) - \delta(x-\lambda) & \text{(up-and-out)} \end{cases}$$

If we set $A_\pm = \{x : \pm\beta(x) > 0\}$, where $\beta(x) = \varphi(x) - \delta(x-\lambda)$ for up-and-in, $\varphi(x)+\delta(x-\lambda)$ for up-and-out, then the set A_- is convex. For simplicity, let's assume that $\operatorname{supp}(X_1)$ is unbounded.

(iii) Up-and-in examples For these, the extremal measure will be concentrated on

$$\big(A_+ \times (\lambda,\infty)\big) \cup \big(A_- \times (-\infty,\lambda]\big).$$

Since we assume that $\operatorname{supp}(X_1)$ is not bounded above, it has to be that $A_- = (x_*, x^*)$, where $x^* < \infty$, $x_* \geq -\infty$. The super-replication is achieved by holding a portfolio of calls which at time 1 is worth

$$\{\varphi(X_1) - \delta(X_1-\lambda)\}^+$$

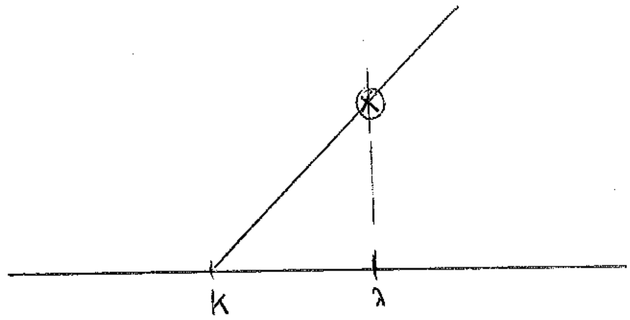
and the objective is to select δ to minimize the cost of this.

This appears to be difficult to do in general, but just let's do the call and the put to see what happens there.

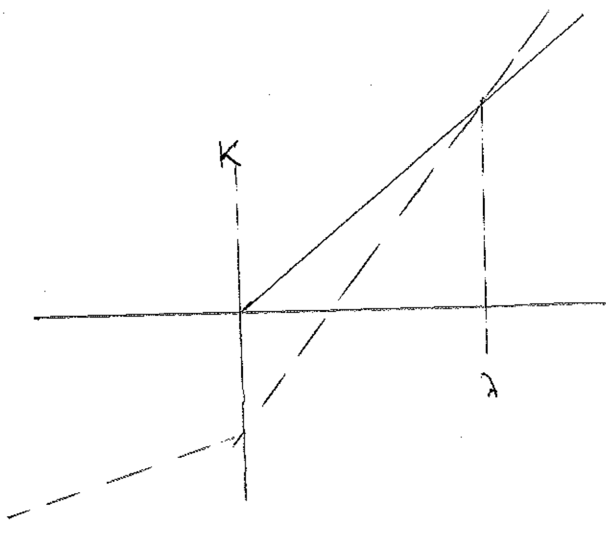
Up-and-in call (a) $K \geq \lambda$. In this case we'd clearly never use $\delta < 0$ (it's always cheaper just to hold the call), so we're really only bothered with the case where $\delta > 0$, and then $A_- = (\lambda, \xi)$ for some $\xi > K$. In fact,

$$\delta = \frac{K-\xi}{\lambda-\xi} \in (0,1)$$

... but it's obvious here that holding the call is cheapest super-replication !!



Up-and-out call:



(b) $K<\lambda$. In this case, it's clear we're only interested in $\delta=-t<0$, which results in

$$\left(\varphi(x)-\delta(x-\lambda)\right)^+ = (1+t)\,(x-\xi)^+ \qquad \xi = \frac{\lambda t+K}{1+t}, \quad t = \frac{\xi-K}{\lambda-\xi}$$

The cost of this is $\frac{\lambda-K}{\lambda-\xi}\,c(\xi)$, which is minimised at $\xi = b'(\lambda)$ if $b'(\lambda)\geq K$, else by taking $\xi=K$.

Up-and-in put   We did this on pp 27–29. If $K\geq\lambda$, we use $\delta = -(K-x^*)/(\lambda-x^*)\leq -1$ where $x^* = b'(\lambda)$. If $K<\lambda$, the optimal embedding is the chequerboard embedding.

(iv) Up-and-out examples*. In this situation, we should replace the original payoff $\varphi(x)$ by $\varphi(x)\,I_{\{x<\lambda\}}$. This loses convexity, but it doesn't really matter. The Lagrangian is now

$$\Lambda = \varphi(x)\,I_{\{y<\lambda\}} - \alpha(x) - \delta(x-\lambda)\,I_{\{y\geq\lambda\}}$$

and so

$$\sup_y \Lambda = \begin{cases} -\alpha(x) + \max\{\varphi(x),\ \delta(\lambda-x)\} & \text{if } x<\lambda \\ -\alpha(x) - \delta\,(x-\lambda) & \text{if } x\geq\lambda \end{cases}$$

which gives

$$\alpha(x) = I_{\{x<\lambda\}}\max\{\varphi(x), \delta(\lambda-x)\} + I_{\{x\geq\lambda\}}\,\delta(\lambda-x)$$

and the aim is to choose δ to minimise the cost of this. In general, hard to do much, but now we can do puts and calls.

Up-and-out call   Wlog $K<\lambda$, and the point where $\varphi(x) = (x-K)^+ I_{\{x<\lambda\}}$ crosses $\delta(\lambda-x)$ is

$$t = \frac{\lambda\delta+K}{1+\delta}, \qquad \delta = \frac{t-K}{\lambda-t},$$

assuming $\delta\geq 0$ (it's clearly more expensive to take $\delta<0$!) The cost of our super-replication is the cost of

$$\alpha(X_1) = \delta(\lambda-X_1) + (1+\delta)(X_1-t)^+ I_{\{X_1<\lambda\}}$$

which costs

$$\delta\lambda + (1+\delta)\left\{c(t)-c(\lambda)+(\lambda-t)c'(\lambda)\right\} - \delta X_0$$

$$= -\lambda + \frac{\lambda-K}{\lambda-t}\left\{c(t)-c(\lambda)+\lambda+(\lambda-t)c'(\lambda)\right\} - \frac{t-K}{\lambda-t}X_0$$

$$= -\lambda + (\lambda-K)c'(\lambda) + \frac{\lambda-K}{\lambda-t}\left\{c(t)-c(\lambda)+\lambda\right\} - \frac{t-K}{\lambda-t}X_0 .$$

Minimising this over $t\in[K,\lambda]$ is exactly the same problem as we considered at (*) on

* NB: we are of course insisting that the martingale is continuous, else we optimise by using a one-jump martingale!!

page 29; there is a unique t^* which minimises without any constraint, and if $t^* \leq K$ then the optimal super-replication is to use the contingent claim $(x-K)^+ I_{\{x<\lambda\}}$. The chequerboard embedding achieves the bound. If, on the other hand, $t^* \in (K, \lambda)$, then the optimal is to have

$$\alpha(x) = \delta^*(\lambda - x) + I_{\{x<\lambda\}}(1+\delta^*)(x-t^*)^+ \qquad \left[\delta^* = (t^*-K)/(\lambda - t^*)\right]$$

and buy forward δ shares if ever the max exceeds λ. Again, for the chequerboard embedding, there is no surplus.

Up-and-out put (a) $K \leq \lambda$. Here we have

$$\alpha(x) = \max\{\varphi(x), \delta(\lambda-x)\} I_{\{x<\lambda\}} - \delta(x-\lambda) I_{\{x \geq \lambda\}}$$

$$= \delta(\lambda - x) + (1-\delta)(t-x)^+$$

If we use $\delta \in [0,1)$, and have $t = \frac{K-\delta\lambda}{1-\delta}$ $\left(\delta = \frac{K-t}{\lambda - t}\right)$. The cost this time is

$$-\delta X_0 + \delta\lambda + (1-\delta)(t + c(t) - X_0) = \lambda + (1-\delta)\{t - \lambda + c(t)\} - X_0$$

$$= \lambda + \frac{\lambda - K}{\lambda - t}\{t - \lambda + c(t)\} - X_0$$

$$= K + \frac{\lambda - K}{\lambda - t} c(t) - X_0$$

So we'd take $t = b^{-1}(\lambda)$ provided $b^{-1}(\lambda) \leq K$, else we use $t = K$, $\delta = 0$, $\alpha(x) = (K-x)^+$. Either way, it's the AY embedding which makes the extremal martingale.

(b) $K > \lambda$. This time, it's only worth considering $\delta \geq 1$, and the crossover is at $t = (\delta\lambda - K)/(\delta - 1) < \lambda$. Then

$$\alpha(x) = \delta(\lambda - x) + (\delta - 1)(x-t)^+ I_{\{x<\lambda\}}$$

and the cost is

$$\delta\lambda + (\delta - 1)\left[c(t) - c(\lambda) + (\lambda - t)c'(\lambda)\right] - \delta X_0$$

$$= \lambda + \frac{K-\lambda}{\lambda - t}\{c(t) - c(\lambda) + \lambda + (\lambda - t)c'(\lambda) - X_0\} - X_0$$

$$= \lambda + (K-\lambda)c'(\lambda) + \frac{K-\lambda}{\lambda - t}\{c(t) - c(\lambda) + \lambda - X_0\} - X_0$$

The minimisation we have to carry out is exactly the minimisation for the chequerboard embedding: if t^* is the value where the unrestricted min is achieved, then we simply take the chequerboard embedding with this value of t^*, and the super-replication is done by holding the cc

$$\alpha(X_1) = \delta^*(\lambda - X_1) + (\delta^* - 1)(X_1 - t^*)^+ I_{\{X_1 < \lambda\}}$$

and buying δ^* shares forward if ever X reaches λ.

Robust hedging of a two-time-point Asian option (1/8/98)

(i) Let's suppose that $(X_t)_{t\geq 0}$ is a non-negative martingale, and that the laws of X_1 and X_2 are known, but otherwise nothing is known about the dist^n of X. As a first step toward pricing an Asian option, let's consider the put which pays

$$(K - X_1 - X_2)^+$$

at time 2; what is the most expensive this can be, what is a super-replicating strategy?

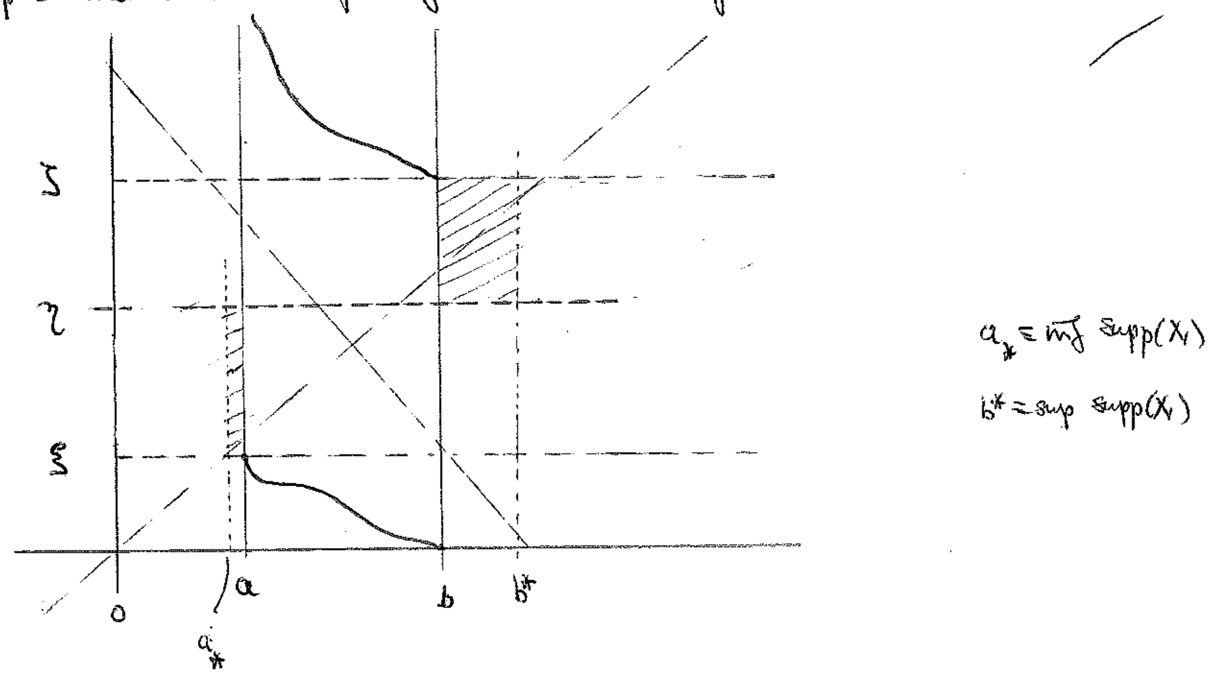
(ii) The Lagrangian form of the problem is

$$\max_{\rho \geq 0} \int_0^\infty dx \int_0^\infty dy \ \Lambda(x,y)\, \rho(x,y)$$

where

$$\Lambda(x,y) = (K-x-y)^+ - \alpha(x) - \beta(y) - \theta(x)(y-x).$$

Numerical examples show that the form of the extremal joint distribution is as follows:

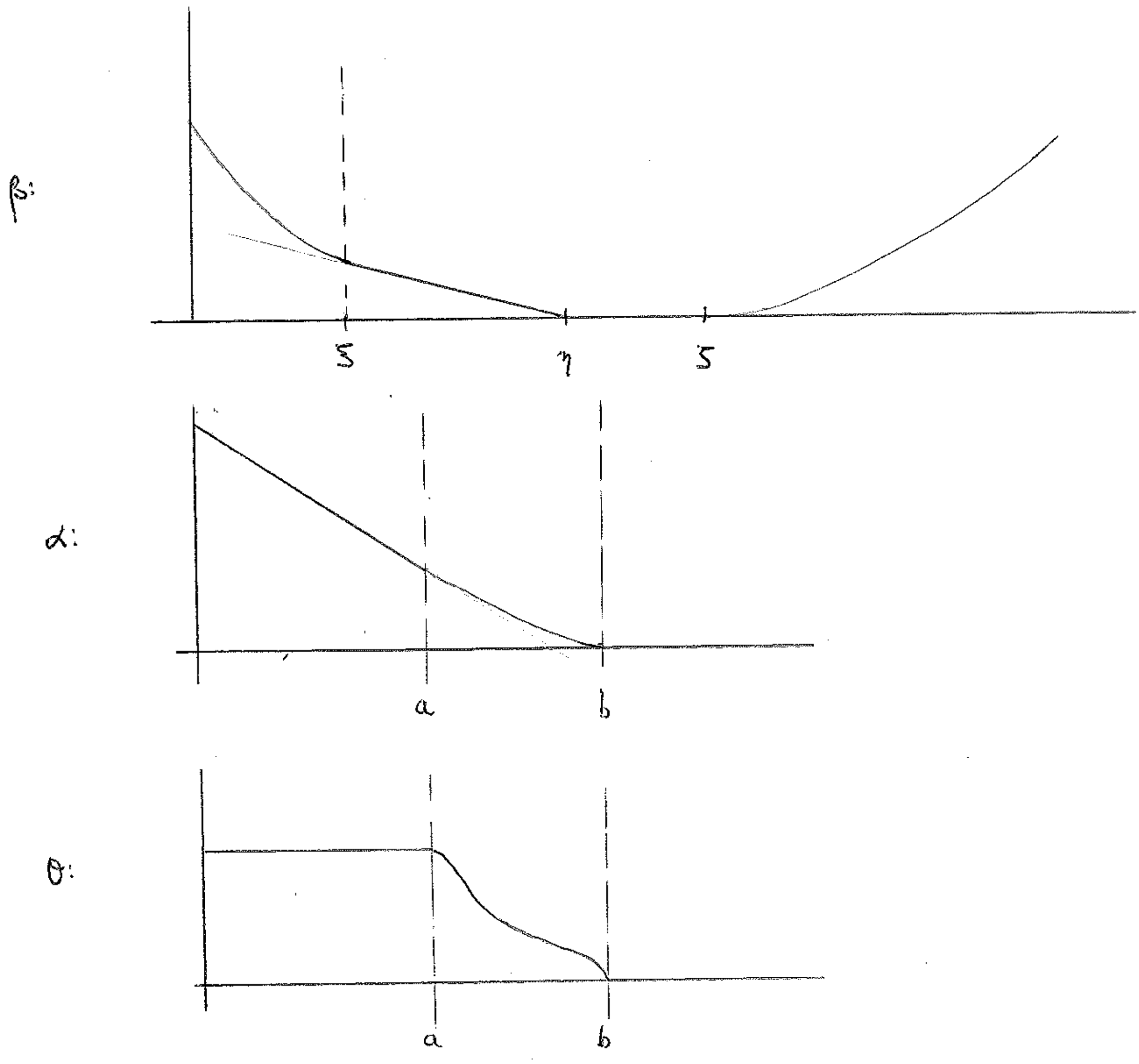


The law is concentrated on two rectangles $[a_*, a] \times [\xi, \eta]$ and $[b, b^*] \times [\eta, \zeta]$ together with two decreasing functions from $[a, b]$ to $[0, \xi]$ and $[\zeta, \infty)$.

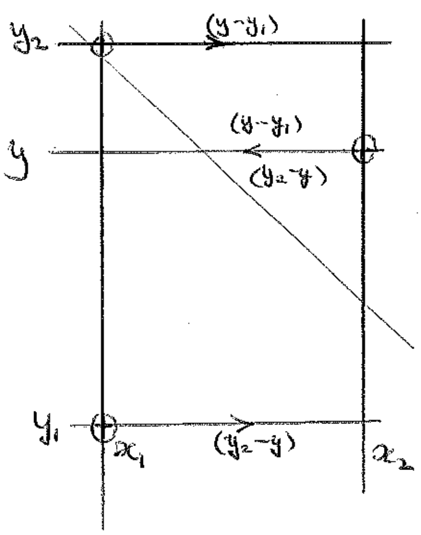
It also appears from the numerics that we can describe the form of the various multiplier functions. We observe that α, β, θ appear continuous and non-negative, α, β are convex, α is zero in $[b, \infty)$, β is zero in $[\eta, \zeta]$ and piecewise linear in $[\xi, \zeta]$, α is linear in $[0, a]$, and both α and θ are decreasing, θ const in $[0, a]$, 0 in $[b, \infty)$.

These various properties are illustrated in the pictures on the next page.

[<u>NB</u> the solution <u>cannot</u> have this form if the laws of X_1, X_2 have unbounded support!]



(iii) We can investigate various configurations of the masses which would be impossible for the optimum, thus learning a lot about where the optimal configuration puts weight.
The essential idea is to consider perturbations of the form:



which keeps row and column totals the same, without affecting the conditional mean of Y/X. Think of the total mass (y_2-y_1) shifted left as being made up of $y-y_1$ from the upper rectangle and y_2-y from the lower. The moves in the upper rectangle don't decrease the payoff and may strictly increase it, and the other way for the lower rectangle; it's now a question of deciding which change predominates.

We could hack through the analysis, but a few examples done on Maple give the following conclusions:

(2) For $x < \xi$, $x > -b+K$, $K-\xi \leq y \leq K-x$

$$\text{gain} = -(y-a)(\cancel{K}-y-x) + (b-y)(\xi-x)$$

$$= x(2y-a-b) + y^2 - y(K+a+\xi) + aK + b\xi$$

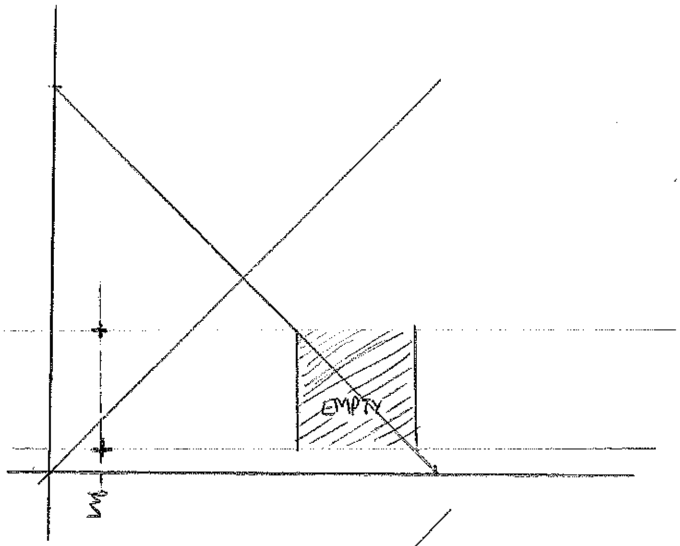
If $K-x < y \leq b$, gain is positive, if $y < K-\xi$, gain is negative

(1) $\xi \in (0, K/2)$, $b \in (\xi, K-\xi)$

Here, the region

$$(K-b, K-a) \times (a, b)$$

will be empty



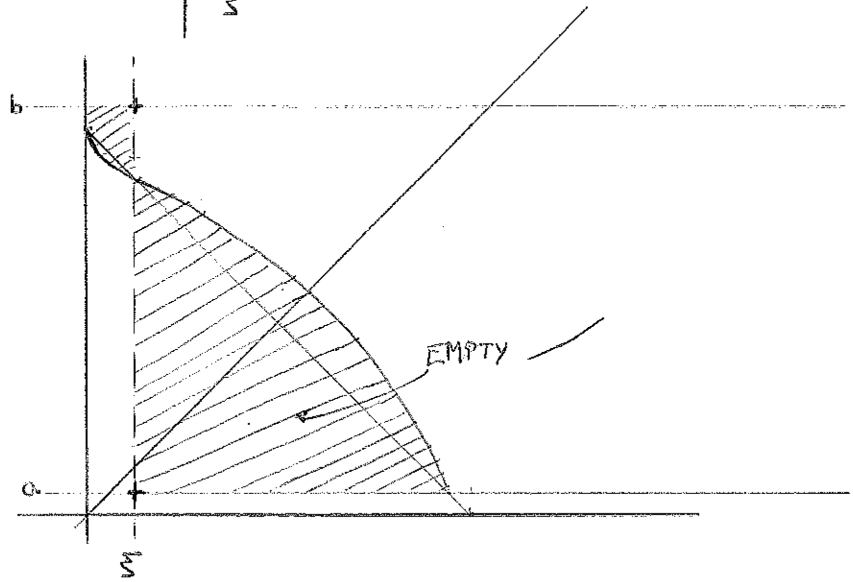
(2) $\xi \in (0, K/2)$ $b > K-\xi$

This is more complicated.

If we were to have some mass at (x,y) with $x > \xi$, $y \in (a,b)$, then gains are possible with the above perturbation iff

$$y < \frac{(b-a)K + a\xi - bx}{b-a-(x-\xi)}$$

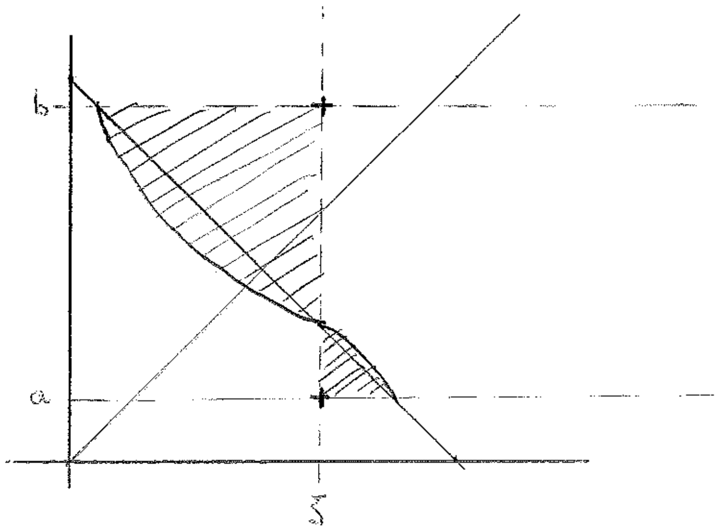
The largest x-value for which gains are possible is $K-a$. When $x=\xi$, $y=K-\xi$



(3) for $\xi \in (K/2, K)$, $a < K-\xi$

A similar story to case 2:

This picture applies also for $b < \xi$, should it be of any interest

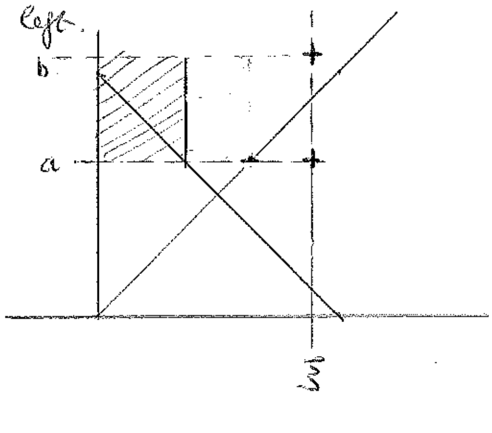


Clearly, if the bottom LH corner of the rectangle of perturbation is outside $\{x+y<K\}$, there's no change when we perturb to the right. But we can also consider perturbing to the left.

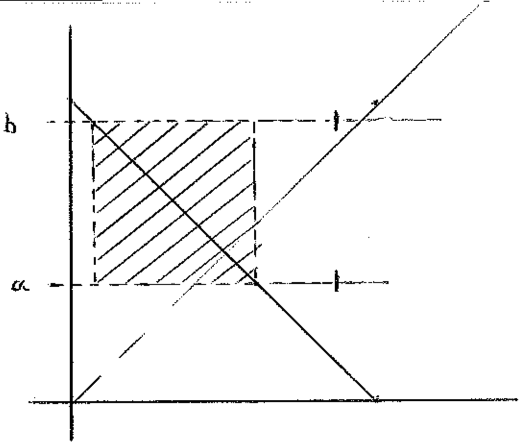
(4) For $a > K-\xi$, $b > \xi$, $a < \xi$, the only other interesting picture we get this dead zone:

Nothing in

$$(0, K-a) \times (a,b), \text{ if } b \geq K,$$



and if $b < K$ the picture is:



no mass in

$(K-b, K-a) \times (a,b)$

We can now begin to piece together the appearance of the optimal joint law. To begin with, define

$$x_0 \equiv \inf \{ t < K/2 : \pi(\{(x,y): x+y \geq K, x < t\}) > 0 \}.$$

If in fact $x_0 = K/2$, we would have to have no mass in $(0,K/2)\times(K/2,\infty)$, by (1). Indeed, if there was mass at (x,y), $x+y<K$, $y > K/2 > x$, then the square $(K-y, K-x) \times (x,y)$ would be empty, and there would be no mass above the diagonal in the x-range $(K-y, K/2)$. This couldn't happen if constraints are satisfied. We note further that there can be no mass in $(K/2,\infty)\times(0,K/2)$ else, by (3), there would be an empty neighbourhood of $(K/2, K/2)$, again a contradiction. To summarise then, $x_0 = K/2 \Rightarrow F_1(K/2) = F_2(K/2)$ and the the solution is trivial; hold a 1-put strike $K/2$ and a 2-put strike $K/2$.

Discarding this trivial case we henceforth <u>assume</u> $x_0 < K/2$.

The region $\{(x,y): x+y>K, x<x_0\}$ is empty, but in fact the region $(0,x_0)\times(K-x_0,\infty)$ must also be empty, by the above argument based on (1). The region shaded is thus excluded.

Now let

$$y_1 \equiv \inf \{ t > x_0 : \pi((0,x_0)\times(t,\infty)) = 0 \} \leq K - x_0 .$$

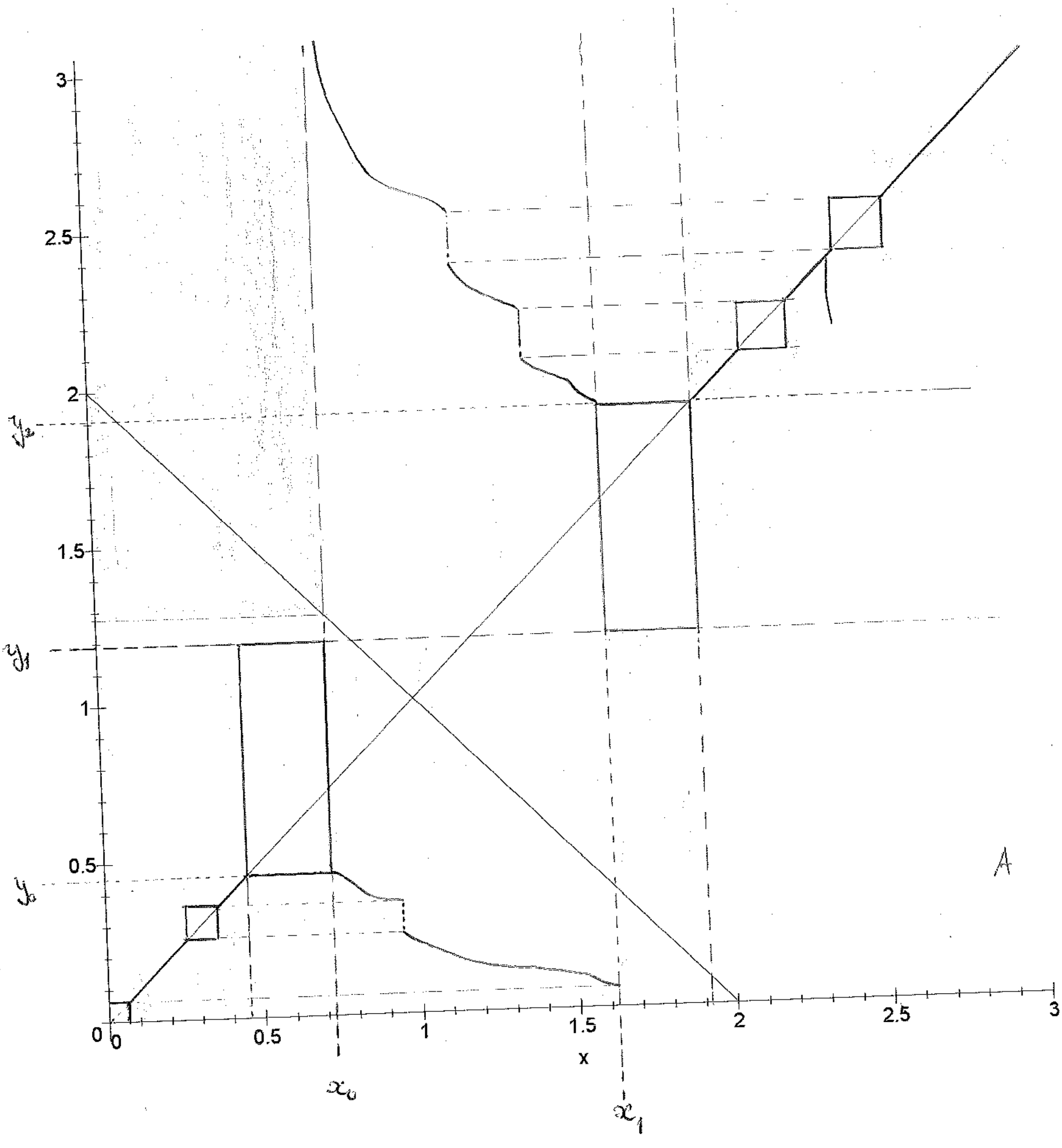
Thus the region is empty, the little triangle using (2) and the fact that there's mass above $x+y=K$ at x_0, and mass below $y=x$ at x_0.

Next define

$$x_1 \equiv \inf \{ t > K/2 : \pi(\{(x,y): x \geq y, x+y \geq K, x < t\}) > 0 \} \leq K.$$

There are two non-increasing functions $\eta_0, \eta_1 : (x_0, x_1) \to \mathbb{R}^+$ such that the mass of π in the range $x_0 < x < x_1$ lies on the graphs of η_0, η_1. This follows from (2). Let $y_0 = \eta_0(x_0)$, $y_2 = \eta_1(y_1)$. Then the region is empty except for those graphs.

Next, we see that the region $(x_1, y_2)\times(y_2,\infty)$ must be empty, because if not we



The general picture for the optimal distribution.

could move mass to the right from the graph of η_1 while moving mass left from above and below the diagonal. This would leave the payoff unchanged, but would leave a configuration violating (3) and therefore capable of improvement. The same argument applies to the regions $(y_0, x_0) \times (0, y_0)$ and $(0, y_0) \times (y_0, y_1)$, so that the region shaded is empty. Likewise $(y_2, \infty) \times (y_1, y_2)$.

Next, the triangle where $x+y \leq K$, $x \geq x_1$ must be empty, else by (3) we have a contradiction of the definition of x_1; excluded.

Next, the region $x+y \geq K$, $x \geq x_1$, $y \leq y_1$ is excluded, by considering two cases separately. First, if $x_1 > K - x_0$, then (4) applies: the left side of the excluded square is $< x_0$, and the lower edge is below y_1. But we may move mass around in the rectangle $(y_0, x_0) \times (y_0, y_1)$ without altering the payoff, and, in particular, we could shift some mass into the top right corner, and then could strictly improve the payoff if there were mass in $A \equiv \{(x,y) : x+y \geq K,\ x \geq x_1,\ y \leq y_1\}$.

In the second case, $x_1 \leq K - x_0$ and therefore a fortiori $x_1 \leq K - y_0$. Using (1), this tells us that $A \cap \{(x,y) : x_1 < x < K - y_0\}$. If there were mass in A, it would therefore have to be to the right of $K - y_0$, and now we use (4); the excluded rectangle certainly takes at least a strip off the top of $(y_0, x_0) \times (y_0, y_1)$ – but that strip has to contain mass by definition of y_1. So the conclusion is that A is empty.

To understand the rest, observe that if for some $x \in (0, y_0)$ the law $\pi(y | x)$ is not concentrated at x, then if x is in the range of η_0 it is possible to make a perturbation to a configuration from which improvement is possible. So the behaviour in $(0, y_0)^2$ must be as shown, and similarly in $(y_2, \infty)^2$.

(iv) The general shape of the optimal joint law is now clear, but how are we to determine x_0, x_1, y_0, y_1, y_2 from the given marginals?

If we consider the trajectories

$$t \longmapsto (\xi_i(t), \eta_i(t)) \equiv \left(F_i(t), \int_0^t s\, F_i(ds)\right)$$

then these are increasing convex functions of t, going from $(0,0)$ to $(1, \mu)$, where $\mu = EX_1$. We can alternatively think of them as

$$\eta_i(t) \equiv \varphi_i(F_i(t))$$

where $\varphi_i(x) \equiv \int_0^{F_i^{-1}(x)} s\, F_i(ds)$. One set of conditions which must be satisfied by x_0, x_1, y_0, y_1, y_2 is that

$$\begin{cases} \left(\xi_1(x_0)-\xi_1(y_0),\ \eta_1(x_0)-\eta_1(y_0)\right) = \left(\xi_2(y_1)-\xi_2(y_0),\ \eta_2(y_1)-\eta_2(y_0)\right) \\ \left(\xi_1(y_2)-\xi_1(x_1),\ \eta_1(y_2)-\eta_1(x_1)\right) = \left(\xi_2(y_2)-\xi_2(y_1),\ \eta_2(y_2)-\eta_2(y_1)\right) \end{cases}$$

One interesting observation here is the following. The function φ_i is convex for each i, and (omitting the subscript i for clarity) the dual function is

$$a \mapsto \sup_t \{at - \varphi(t)\} = a F(a) - \varphi(F(a)) = \int (a-x)^+ F(dx)$$

since the sup is attained where $\varphi'(t) = a$, that is $F^{-1}(t) = a$. Thus if P_i is the put price function for each i, we see that

$$\boxed{P_i \text{ is the convex dual of } \varphi_i \quad , i = 1, 2}$$

That gives $\varphi_i(t) = \sup_a \{at - P_i(a)\}$, so in particular we must have $\varphi_2 \leq \varphi_1$.

Maximising $E\,|M_2 - M_1|$ (11/9/98)

(i) We may consider a range of options of the form $|M_2 - M_1|^\gamma$, where the laws of M_1 and M_2 are fixed; it appears that the behaviour for all $\gamma < 2$ is similar, in that the extremal law looks the same. Likewise the behaviour for all $\gamma > 2$ is similar, and the case $\gamma = 2$ is of course one where the payoff is the same however you embed.

(ii) The case $\gamma = 1$ seems to be perhaps the one of greatest practical significance. Numerical examples (and the perturbation argument) suggest that there are functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$, increasing, such that $\alpha(x) \le x \le \beta(x)$ $\forall x$, and the optimal joint law is concentrated on the graphs of α, β.

Can we find out the form of α, β from the laws F_1, F_2 of M_1, M_2?

(iii) If we consider the Lagrangian formulation, we have

$$\Lambda = |x - y| - \psi(x) - \theta(y) + \lambda(x)(y - x)$$

in terms of the unknown multiplies ψ, θ, λ. We need $\sup_y \Lambda = 0$ for each x. If we assume that θ is convex, with dual function

$$g(t) \equiv \sup_y \{ty - \theta(y)\} \qquad \left[g'(t) = (\theta')^{-1}(t)\right]$$

then for each x the sup of Λ over y will be attained in two places, where

$$\theta'(y) = 1 + \lambda(x) \;,\quad \lambda(x) - 1.$$

So if φ is inverse to θ', we have that the max is attained at $\varphi(\lambda(x) \pm 1)$, which two points must be either side of x. The two local maxima must have the same value:

$$\varphi(\lambda_x + 1) - x - \theta(\varphi(\lambda_x + 1)) + \lambda(x)\,\varphi(\lambda_x + 1) - x\lambda_x$$

$$= g(1 + \lambda_x) - x - x\lambda_x$$

$$= -\varphi(\lambda_x - 1) + x - \theta(\varphi(\lambda_x - 1)) + \lambda_x\,\varphi(\lambda_x - 1) - x\lambda_x$$

$$= g(\lambda_x - 1) + x - x\lambda_x .$$

Thus we shall have to have

$$\boxed{g(\lambda_x + 1) - g(\lambda_x - 1) = 2x}$$

We also conclude that

$$\boxed{\psi(x) = g(1 + \lambda_x) - x - x\lambda_x = g(\lambda_x - 1) - x(\lambda_x - 1)} \qquad (\ge -\theta(x), \text{ if it helps})$$

Various identities follow from these:

$$\lambda'_x = 2\{\varphi(\lambda_x+1) - \varphi(\lambda_x-1)\}^{-1} > 0;$$

$$\psi(x) = \tfrac{1}{2}\{g(\lambda_x+1) + g(\lambda_x-1) - 2x\lambda_x\}$$

so that

$$\psi'(x) = -\lambda_x + \lambda'_x\left\{\frac{\varphi(\lambda_x+1)+\varphi(\lambda_x-1)}{2} - x\right\}$$

$$= \frac{1}{\varphi(\lambda_x+1)-\varphi(\lambda_x-1)}\left\{\varphi(\lambda_x+1)+\varphi(\lambda_x-1) - 2x - \lambda_x(\varphi(\lambda_x+1)-\varphi(\lambda_x-1))\right\}$$

$$= \frac{(\lambda_x+1)\varphi(\lambda_x-1) - (\lambda_x-1)\varphi(\lambda_x+1) - 2x}{\varphi(\lambda_x+1) - \varphi(\lambda_x-1)}$$

But since φ is increasing, and $\int_{\lambda_x-1}^{\lambda_x+1} \varphi(t)\,\frac{dt}{2} = x$, it must be that $\varphi(\lambda_x+1) > x > \varphi(\lambda_x-1)$ and hence

$$\boxed{\psi'(x) < 0.}$$

[By adding an affine function to θ if need be, we may assume $\theta'(0) = \theta(0) = 0$, so that $\varphi(0)=0$, and $g(0)=0$. Let's assume this done.]

(iv) The cost of the superreplicating strategy will be

$$\int \psi(x)\, F_1(dx) + \int \theta(y)\, F_2(dy)$$

and by minimising this expression we obtain optimal rule. For the second part, we have

$$\int \theta(y)\, F_2(dy) = \int_0^\infty \left(\int_0^y \theta'(t)\,dt\right) F_2(dy) - \int_{-\infty}^0 \left(\int_y^0 \theta'(t)\,dt\right) F_2(dy)$$

$$= \int_0^\infty \theta'(t)\,\bar F_2(t)\,dt - \int_{-\infty}^0 \theta'(t)\,F_2(t)\,dt$$

$$= \int_0^\infty u\,\bar F_2(\varphi(u))\,\varphi'(u)\,du - \int_{-\infty}^0 u\,F_2(\varphi(u))\,\varphi'(u)\,du$$

$t = \varphi(u)$
$dt = \varphi'(u)du$

Thus if we define the function $h(x) = \int_x^\infty \bar F_2(t)\,dt \quad (x \geq 0)\; ; \; = \int_{-\infty}^x F_2(t)\,dt \quad (x \leq 0)$

we have

$$\boxed{\int \theta(y)\, F_2(dy) = -\int_0^\infty u\; d(h(\varphi(u))) - \int_{-\infty}^0 u\; d\,h(\varphi(u)) = \int_{-\infty}^\infty h(\varphi(u))\,du}$$

if the evaluation between limits is zero.

Questions

1) Different filtrations / prob^{ys} influencing agents?
2) Liquidity at times of crisis?
3) What are implications of a tick data model for economic fundamentals?