Modelling the impact of shocks on prices (3/10/99)

1) Here's a continuous-time model where we could do something. Suppose agent j receives endowment stream $\xi_j(\cdot)$ $(j=1,\dots,J)$ and that the K risky assets deliver dividend streams $\delta_k(\cdot)$, $(k=1,\dots,K)$. The wealth equation of agent j is

$$dw_j(t) = \{r_t w_j(t) - c_j(t) + \xi_j(t)\}dt + \theta_j(t)\cdot\{S_t^{-1}dS_t + \delta_t\,dt - r_t \mathbf{1}\,dt\}$$

and he has objective

$$E\left[\int_0^\infty e^{-\rho_j t}\, U_j(c_j(t))\,dt\right],$$

say. If we assume that the endowment processes are independent of the dividends, then by writing $\tilde c_j(t) = c_j(t) - \xi_j(t)$, the objective becomes

$$E\int_0^\infty e^{-\rho_j t}\, U_j(\tilde c_j(t) + \xi_j(t))\,dt \equiv E\int_0^\infty e^{-\rho_j t}\,\tilde U_j(\tilde c_j(t))\,dt.$$

This is now a conventional problem, where

$$e^{-\rho_j t}\,\tilde U_j'(\tilde c_j(t)) = \lambda_j \zeta_t$$

for a state-price density process ζ which therefore solves

$$\sum_j \tilde c_j(t) = \sum_k \delta_t^k = \sum_j \tilde I_j(\lambda_j e^{\rho_j t}\zeta_t).$$

2) We could similarly consider the objective

$$E\left[\int_0^T f_j(t)\, U_j(c_t)\,dt + v_j(w_T)\right].$$

Let's be somewhat more explicit. Let's suppose that $U_j(x) = -\exp(-\alpha_j x)$, $v_j(x) = -B_j e^{-\beta_j x}$ and that ξ_j is a compound Poisson process with Lévy exponent ψ_j. Then

$$f_j(t)\,\tilde U_j(c) = f_j(t)\, E\, -\exp(-\alpha_j(c + \xi_j(t)))$$

$$= -f_j(t)\exp[-\alpha_j c - \alpha_j\xi_j(0) + \psi_j(-\alpha_j)t]$$

$$= -\exp[-\alpha_j c - \alpha_j \xi_j(0) - (\rho_j - \psi_j(-\alpha_j))t] \qquad \text{if } f_j(t) = e^{-\rho_j t}$$

Thus

$$\boxed{\begin{aligned} \tilde c_j(t) &= -\frac{1}{\alpha_j}\log\left[\frac{\lambda_j\zeta_t}{\alpha_j}\right] - \xi_j(0) - (\rho_j - \psi_j(-\alpha_j))\,t/\alpha_j \\ w_j(T) &= -\frac{1}{\beta_j}\log\left[\frac{\eta_j\zeta_T}{B_j\,\beta_j}\right]\end{aligned}}$$

Now let's assume that the dividend process is an OU process:

$$d\delta = \sigma dW + \kappa(\mu - \delta)\,dt$$

where κ is a scalar, so that if $\Delta \equiv 1\cdot\delta$ is the total dividend stream, then Δ is a 1-dim'l OU process

$$d\Delta = 1\cdot\sigma dW + \kappa(1\cdot\mu - \Delta)\,dt$$

Market clearing tells us

$$\Delta(t) = \sum_j \tilde{c}_j(t) = \left(\sum -\tfrac{1}{\alpha_j}\right)\log\zeta_t + \sum_j\left\{\tfrac{1}{\alpha_j}\log(\alpha_j/\lambda_j) - \xi(0)\right\} + \sum_j\left(\psi_j(-\alpha_j) - \rho_j\right)t/\alpha_j$$

$$\equiv a_0\log\zeta_t + a_1 + a_2 t$$

which gives

$$\boxed{\zeta_t = \exp\left[(\Delta_t - a_1 - a_2 t)/a_0\right]}$$

This gives quite a simple picture:

$$d\zeta_t/\zeta_t = \frac{d\Delta_t - a_2\,dt}{a_0} + \frac{1}{2}a_0^{-2}\,|\sigma\cdot 1|^2\,dt$$

$$= \frac{1\cdot\sigma dW_t}{a_0} + \left\{\frac{\kappa(1\cdot\mu - \Delta_t)}{a_0} - \frac{a_2}{a_0} + \frac{1}{2}\frac{|\sigma\cdot 1|^2}{a_0^2}\right\}dt$$

Therefore the risk-neutral measure P^* has the property that

$$\boxed{dW - \frac{\sigma 1}{a_0}\,dt = dW^* \quad \text{is a } P^*\text{-BM}}$$

and the interest rate process is

$$\boxed{r_t = \frac{a_2}{a_0} - \frac{\kappa}{a_0}(1\cdot\mu - \Delta_t) - \frac{1}{2}\frac{|\sigma\cdot 1|^2}{a_0^2}}$$

which is just a Vasicek model. Notice that the dynamics of r, and P^*, don't depend on the λ_j. <u>This is bad news for the modelling, because this means that the prices of the shares won't change if there's a jump in one of the agent's' endowment streams.</u>

3) But all is not lost. Let's forget completely about ε_j (so they are all $\equiv 0$) and instead think that the effect of the shock is a sudden unforseen (and unforseeable) change in μ. Thus if we can compute equilibrium prices, we can determine the effect of the shock just by examining how they depend on μ. Since the state-price density doesn't depend on the λ_j, we can value the shares; at time 0, the ith share is priced at

$$\frac{1}{\zeta_0} E_0\left[\int_0^\infty \zeta_t\, \delta_i(t)\,dt\right] = \zeta_0^{-1} E_0 \int_0^\infty \exp\left\{\frac{\Delta_t - a_2 t}{a_0}\right\} \delta_i(t)\,dt$$

taking $a_1 = 0$, since it cancels out anyway. Now given $\mathcal{F}_0$, $\delta(t) \sim N\left(\mu + e^{-\kappa t}(\delta_0 - \mu),\ \frac{1-e^{-2\kappa t}}{2\kappa}\sigma\sigma^T\right)$ so

$$E_0 \exp(-\theta\cdot\delta(t)) = \exp\left\{-\theta\cdot(\mu + e^{-\kappa t}(\delta_0-\mu)) + \tfrac{1}{2}\,\frac{1-e^{-2\kappa t}}{2\kappa}\,|\sigma^T\theta|^2\right\}$$

and therefore

$$E_0\left[\exp(-\theta\cdot\delta(t))\,\delta_i(t)\right] = \left\{\mu_i + e^{-\kappa t}(\delta_i(0)-\mu_i) - \frac{1-e^{-2\kappa t}}{2\kappa}(\sigma\sigma^T\theta)_i\right\}\exp(\cdots).$$

This gives an expression for the price of the i^{th} share in this model:

$$\boxed{e^{-\Delta_0/a_0}\int_0^\infty \left\{\mu_i + e^{-\kappa t}(\delta_i(0)-\mu_i) + \frac{1-e^{-2\kappa t}}{2\kappa\, a_0}(\sigma\sigma^T 1)_i\right\}\exp\left[+a_0^{-1}(\mu + e^{-\kappa t}(\delta(0)-\mu))\cdot 1 - a_2 t/a_0 + \tfrac{1}{2}\,\frac{1-e^{-2\kappa t}}{2\kappa a_0^2}\,|\sigma^T 1|^2\right] dt}$$

$\left[\, a_0 \equiv -\sum_j \alpha_j^{-1},\quad a_2 \equiv -\sum_j \rho_j/\alpha_j \,\right]$

Notice (i) that the integral is convergent (ii) Since both of a_0, a_2 are <0, it seems impossible to decide whether this is increasing in the components of μ.

If we knew that a_2/a_0 were small (so the ρ_i are small - the agents are patient) the price of the ith share is approximately

$$\frac{a_0}{a_2}\exp\left(-\Delta_0/a_0 + a_0^{-1}\,\mu\cdot 1 + \frac{1}{4\kappa a_0^2}|\sigma^T 1|^2\right)\left(\mu_i + \frac{1}{2\kappa a_0}(\sigma\sigma^T 1)_i\right).$$

The j^{th} agent consumes $c_j(t) = -\frac{1}{\alpha_j}\log(\lambda_j e^{\rho_j t}\zeta_t/\alpha_j)$, so his wealth at time t will be

$$w_j(t) = \zeta_t^{-1} E_t\left[\int_t^\infty c_j(s)\,\zeta_s\,ds\right]$$

$$= \zeta_t^{-1} E_t \int_t^\infty -\frac{1}{\alpha_j}\left\{\log\frac{\lambda_j}{\alpha_j} + \rho_j s + \frac{\Delta_s - a_2 s}{a_0}\right\}\exp\left(\frac{\Delta_s - a_2 s}{a_0}\right) ds$$

$$= -\frac{1}{\alpha_j\zeta_t}\cdot\int_t^\infty \left\{\log\frac{\lambda_j}{\alpha_j} + (\rho_j - a_2/a_0)s + \frac{1}{a_0}\left(1\cdot(\mu + e^{-\kappa(s-t)}(\delta(t)-\mu)) + \frac{1-e^{-2\kappa(s-t)}}{2\kappa a_0}|\sigma^T 1|^2\right)\right\}.$$
$$\exp\left[a_0^{-1}(\mu + e^{-\kappa(s-t)}(\delta(t)-\mu))\cdot 1 + \tfrac{1}{2}\,\frac{1-e^{-2\kappa(s-t)}}{2a_0^2\kappa}|\sigma^T 1|^2 - a_2 s/a_0\right] ds$$

4) Another thing we could try would be a CIR model for the dividends:

$$d\delta_i = \sigma\sqrt{\delta_i}\, dW^i + (b_i + (B\delta)_i)\, dt$$

where the W^i are independent, and $1^T B = -\gamma 1^T$ for some $\gamma > 0$. Assume also that $B_{ij} > 0$ for $i \neq j$, so we get a stationary process, and $1^T\delta \equiv \Delta$ solves $\boxed{d\Delta_t = \sigma\sqrt{\Delta_t}\, d\tilde{W}_t + (1\cdot b - \gamma\Delta_t)\, dt}$

- so it's a CIR process.

If $U_j(x) = x^{1-R}/(1-R)$ for all agents j, we find that

$$\Delta_t = \sum_j (\lambda_j e^{\rho_j t}\zeta_t)^{-1/R} \equiv \psi(t)\, \zeta_t^{-1/R} \qquad \boxed{\therefore\ \zeta_t = (\psi(t)/\Delta(t))^R}$$

Agent j's wealth gets priced at

$$w_j(t) = \zeta_t^{-1} E_t \int_t^\infty (\lambda_j e^{\rho_j s})^{-1/R}\, \zeta_s^{1-1/R}\, ds$$

<u>More on credit risk modelling</u> (26/10/99)

1) In the earlier notes on credit risk modelling (WN XVII, 39–42), we considered an extension of the model of Leland "Bond prices, yield spreads and optimal capital structure with default risk", to allow for jumps in the value of the firm. When computing the value of the firm, we have a term for the tax benefits of the form

$$\text{const.}\; E^y\left[1-e^{-rH_0}\right] = \text{const.}\; E^y\left[\int_0^{H_0} re^{-rs} I_{\{X_s\geq 0\}}\,ds\right]$$

However, it turns out in Leland's analysis to be better to allow tax deductibility only when the value of the firm's assets exceeds some threshold, so we are faced instead with computing

$$g(y) \equiv E^y\left[\int_0^{H_a} e^{-rs} I_{\{X_s\geq b\}}\,ds\right]$$

for some $b > a$ (or, more precisely, we want the derivative of this with respect to y at $a+$). Now we can re-express g according to whether $y \geq b$ or $y \leq b$:

<u>Case 1</u>: <u>$a \leq y \leq b$</u>. Since $E^y \exp(-rH_b) = \exp(-(b-y)\beta^*)$ where $\beta^* = \beta^*(r)$ solves $\psi(\beta) = r$, we have

$$g(y) = e^{-\beta^*(b-y)} E^b\left[\int_0^\infty I_{\{X_s\geq b\}} e^{-rs}\,ds\right]$$

$$- E^y\left[\exp(-rH_a + \beta^*(X(H_a)-b))\right] . E^b\left[\int_0^\infty I_{\{X_s\geq b\}} e^{-rs}\,ds\right]$$

Let's abbreviate

$$B \equiv E^0\left[\int_0^\infty e^{-rs} I_{\{X_s\geq 0\}}\,ds\right]$$

so that for $a \leq y \leq b$

$$g(y) = \left\{e^{-\beta^*(b-y)} - E^y\, e^{-rH_a+\beta^*(X(H_a)-b)}\right\} . B$$

<u>Case 2</u>: <u>$y \geq b$</u>. This time,

$$g(y) = E^y\left[\frac{1-e^{-rH_b}}{r}\right] + \left\{E^y\, e^{-rH_b+\beta^*(X(H_b)-b)} - E^y\, e^{-rH_a+\beta^*(X(H_a)-b)}\right\} . B$$

We can't expect anything explicitly for g, but we <u>can</u> get something perfectly reasonable for the LT of g.

2) First, though, let's make B more explicit. We have (Spitzer-Rogozin)

$$\psi_\lambda^+(s) = \exp\left[\int_0^\infty (e^{sx}-1)\int_0^\infty \frac{e^{-\lambda t}}{t} P(X_t\in dx)\,dt\right]$$

As $\frac{\partial}{\partial\lambda} \log \psi_\lambda^+(s) = -\int_0^\infty (e^{sx}-1) \int_0^\infty e^{-\lambda t} P(X_t \in dx)\, dt$

$$\to \int_0^\infty e^{-\lambda t} P(X_t \geq 0)\, dt \qquad \text{as } s \to -\infty$$

So we can obtain B, since in this case $\psi_\lambda^+(s) = \beta^*(\lambda)/(\beta^*(\lambda)-s)$, and $\frac{\partial\beta^*}{\partial\lambda} = 1/\psi'(\beta^*(\lambda))$

So

$$\frac{\partial}{\partial\lambda} \log \psi_\lambda^+(s) = \frac{1}{\psi'(\beta^*(\lambda))}\left\{\frac{1}{\beta^*(\lambda)} - \frac{1}{\beta^*(\lambda)-s}\right\} \to \frac{1}{\beta^*(\lambda)\,\psi'(\beta^*(\lambda))} \qquad \text{as } s \to -\infty.$$

Therefore $\boxed{B = \frac{1}{\beta^*(r)\,\psi'(\beta^*(r))}}\ \boxed{= r^{-1}\psi_r^-(\beta^*(r))}$ after some calculations.

3)

So now we can form the LT of g:

$$\int_0^\infty \mu e^{-\mu v} g(a+v)\,dv = \int_0^{b-a} \mu e^{-\mu v} e^{-\beta^*(b-a-v)}\,dv\; B - Be^{-\beta^*(b-a)}\int_{-\infty}^0 \mu e^{\mu x} E\left[e^{-rH_x+\beta^* X(H_x)}\right]dx$$

$$+ e^{-\mu(b-a)} \int_0^\infty \mu e^{-\mu z} E^z\left[\frac{1-e^{-rH_0}}{r}\right] dz$$

$$+ e^{-\mu(b-a)} B \cdot \int_{-\infty}^0 \mu e^{\mu x} E\left[e^{-rH_x+\beta^* X(H_x)}\right]dx$$

$$= \frac{\mu B e^{-\beta^*(b-a)}\left(e^{(\beta^*-\mu)(b-a)}-1\right)}{\beta^*-\mu} - Be^{-\beta^*(b-a)}\,\frac{\psi_r^-(\beta^*)-\psi_r^-(\mu)}{\psi_r^-(\beta^*)}\,\frac{\mu}{\mu-\beta^*}$$

$$+ \frac{e^{-\mu(b-a)}}{r}\psi_r^-(\mu) + e^{-\mu(b-a)} B \,\frac{\mu}{\mu-\beta^*}\,\frac{\psi_r^-(\beta^*)-\psi_r^-(\mu)}{\psi_r^-(\beta^*)}$$

$$= \frac{\mu B\left(e^{-\mu(b-a)}-e^{-\beta^*(b-a)}\right)}{\beta^*-\mu} + B\left\{e^{-\mu(b-a)}-e^{-\beta^*(b-a)}\right\}\frac{\psi_r^-(\beta^*)-\psi_r^-(\mu)}{\psi_r^-(\beta^*)}\cdot\frac{\mu}{\mu-\beta^*} + \frac{e^{-\mu(b-a)}\psi_r^-(\mu)}{r}$$

$$= \frac{\mu B}{\beta^*-\mu}\left\{e^{-\mu(b-a)}-e^{-\beta^*(b-a)}\right\}\frac{\psi_r^-(\mu)}{\psi_r^-(\beta^*)} + e^{-\mu(b-a)}\frac{\psi_r^-(\mu)}{r}$$

The derivative of g at $(a+)$ is therefore the limit as $\mu\to\infty$ of μ times this expression, viz

$$\boxed{Be^{-\beta^*(b-a)}\left\{\frac{2r}{\sigma^2\beta^*\,\psi_r^-(\beta^*)}\right\}}$$

Note : if $0 < b < x$, then the expression for v changes to

$$v = V_B e^{x} + rC \left\{ \frac{\varphi(x-b,r)}{r} + B E^{x-b}\left[e^{\beta^* X(H) - rH} \right] - B E^{x}\left[e^{\beta^* X(H) - rH}\right] e^{-\beta^* b} \right\}$$

$$- \alpha V_B \left\{ 1 - \gamma(x,1,r) \right\}$$

$$\int_0^\infty \mu e^{-\mu x} \varphi(x,\lambda)\, dx = \psi_\lambda^-(\mu)$$

$$\int_0^\infty \mu e^{-\mu x} \gamma(x,\beta,\lambda)\, dx = \frac{-\beta}{\mu-\beta} + \frac{\mu}{\mu-\beta} \frac{\psi_\lambda^-(\mu)}{\psi_\lambda^-(\beta)}$$

$$\therefore \int_0^\infty \mu e^{-\mu x} E^{x}\left[e^{\beta X(H_0) - \lambda H_0} \right] dx = \frac{\mu}{\mu-\beta} \left\{ 1 - \frac{\psi_\lambda^-(\mu)}{\psi_\lambda^-(\beta)} \right\}$$

4) Using the notation of Leland's paper, we have an expression for the value of the debt:

$$D = \frac{C+mP}{m+r}\left\{1 - E^x e^{-(m+r)H}\right\} + (1-\alpha)V_B\, E^x e^{X(H)-(m+r)H}$$

where $H \equiv \inf\{t : X_t < 0\}$, and $x = \log(V_0/V_B)$; and the value of the firm is then

$$v = V_0 + \tau C\, E^x\left[\int_0^H e^{-rs} I_{\{X_s \geq b\}}\, ds\right] - \alpha V_B\, E^x\left(e^{X(H)-rH}\right)$$

where $b = \log(V_T/V_B)$. If we write $\varphi(x,\lambda) \equiv 1 - E^x e^{-\lambda H}$, $\gamma(x,\beta,\lambda) = E^x\{1 - e^{\beta X(H) - \lambda H}\}$, we therefore have that for x small enough (ie. $0 < x < b$)

$$\begin{cases} D = \dfrac{C+mP}{m+r}\varphi(x, m+r) + (1-\alpha)V_B \,.\, \{1 - \gamma(x,1,m+r)\} \\[2ex] v = V_B e^x + \tau C\left\{e^{\beta^* x} - 1 + \gamma(x,\beta^*,r)\right\} B e^{-\beta^* b} - \alpha V_B \,.\, \{1 - \gamma(x,1,r)\} \end{cases}$$

As we've already seen, $\varphi(0,\lambda) = \gamma(0,\beta,\lambda) = 0$, and

$$\varphi'(0,\lambda) = \frac{2\lambda}{\sigma^2\beta^*(\lambda)}, \quad \gamma'(0,\beta,\lambda) = \varphi'(0,\lambda)/\psi_\lambda^-(\beta) - \beta.$$

The smooth pasting condition

$$\frac{\partial}{\partial V_0}(v - D)\bigg|_{V_0 = V_B} = 0$$

yields a characterisation of the optimal V_B

$$\boxed{V_B = \frac{\dfrac{C+mP}{m+r}\varphi'(0,m+r) - \dfrac{\tau C}{r} e^{-\beta^* b} \,.\, \dfrac{2r}{\sigma^2\beta^*}}{\alpha\gamma'(0,1,r) + (1-\alpha)\gamma'(0,1,m+r) + 1}} \qquad (\beta^* = \beta^*(r))$$

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(i) In the case of no jumps, this does not reduce to Leland's answer, because Leland's answer is not correct: the solution he gives in Appendix B at (28) is incorrect because we have $\delta > 0$ here.

(ii) V_B appears implicitly on the RHS also, in that $b = \log(V_T/V_B)$. Though it looks complicated, we can use this relation fairly simply, because if we fix V_B, for any choice of C the value of P is readily calculated (P appears in just one place, linearly). This remains true if we follow Leland in making V_T a (linear) function of C, and δ a linear function of C.

(iii) Note that if we are simply dealing with the original situation with no tax threshold (or, put another way, $V_T \le V_B$) we shall have the expression

$$\boxed{V_B = \frac{\frac{C+mP}{m+r}\,\varphi'(0,m+r) - \frac{\tau C}{r}\,\varphi'(0,r)}{1+\alpha\,\gamma'(0,1,r) + (1-\alpha)\,\gamma'(0,1,m+r)}}$$

since the expression for v simplifies to

$$v = V_B e^{x} + \frac{\tau C}{r}\,\varphi(x,r) - \alpha V_B\{1-\gamma(x,1,r)\}$$

(iv) For the record,

$$\int_0^\infty \mu e^{-\mu x} D(V_B e^x, V_B)\,dx = \frac{C+mP}{m+r}\,\psi^-_{m+r}(\mu) + (1-\alpha)V_B\,\frac{\mu}{\mu-1}\left\{1-\frac{\psi^-_{m+r}(\mu)}{\psi^-_{m+r}(1)}\right\},$$

$$\int_0^\infty \mu e^{-\mu x} v(V_B e^x, V_B)\,dx = \frac{\mu V_B}{\mu-1} - \alpha V_B\frac{\mu}{\mu-1}\left\{1-\frac{\psi^-_r(\mu)}{\psi^-_r(1)}\right\}$$
$$+\tau C\left[\frac{\mu B}{\beta^*-\mu}\left\{e^{-\mu b}-e^{-\beta^* b}\right\}\frac{\psi^-_r(\mu)}{\psi^-_r(\beta^*)} + e^{-\mu b}\,\frac{\psi^-_r(\mu)}{r}\right].$$

(v) For the various numerical calculations, we chose various values $V_B = \xi$, and then used the criterion of Leland $D(V_0,\xi) = P$ (with $V_0 = 100$) to express P in terms of C:

$$P = D(V_0,\xi) = \frac{C+mP}{m+r}\,\varphi(y,m+r) + (1-\alpha)\xi\{1-\gamma(y,1,m+r)\} \qquad y = \log(100/\xi)$$

whence
$$P = \frac{C\varphi(y,m+r) + (1-\alpha)(m+r)\xi\,(1-\gamma(y,1,m+r))}{m+r - m\,\varphi(y,m+r)}.$$

Now substitute this expression for P into the expression for V_B and find the value of C for which we get this equal to xi

Solved by $u(t,x,y) = U(x)\, v(t,y)^{\alpha}$ where $\alpha = \dfrac{R}{R+\rho^2(1-R)}$, and v solves →

$$\boxed{\dot{v} + \tfrac{1}{2}a^2 v_{yy} + \left\{ b + \frac{\rho a \theta (1-R)}{R} \right\} v_y + \frac{1-R}{\alpha}\left\{ r + \frac{\theta^2}{2R} \right\} v = 0} \qquad v(T,y) \equiv 1$$

A remarkable result of Thaleia Zariphopoulou (22/11/99)

1) In a preprint "Asset valuation with unhedgeable risks", Thaleia considers the situation of an investor who can invest in a bank account with constant rate of return r, or a share

(1) $$dS_t = S_t \{ \sigma(t,Y_t)\, dW_t + \mu(t,Y_t)\, dt \}$$

where

(2) $$dY_t = a(t,Y_t)\, dB_t + b(t,Y_t)\, dt, \qquad dB\, dW = \rho\, dt.$$

The aim of the investor is to $\max E\, U(X_T)$ with T fixed, and $U(x) = x^{1-R}/(1-R)$. She obtains the remarkable result that

$$u(t,x,y) = \sup E\big[U(X_T) \,\big|\, X_t = x, Y_t = y \big]$$

(3) $$= \frac{x^{1-R}}{1-R} \cdot v(t,y)^{R/\{R+\rho^2(1-R)\}}$$

where v solves a <u>linear PDE</u>

(4) $$\dot v + \tfrac12 a^2 v_{yy} + \Big[b + \rho a \frac{(1-R)}{R}\theta \Big] v_y + \frac{1-R}{R}\big(R+\rho^2(1-R)\big)\Big[r + \frac{\theta^2}{2R} \Big] v = 0$$

where $\theta \equiv (\mu - r)/\sigma$. Here, of course, we use X to denote the wealth process, which solves

$$dX_t = r X_t\, dt + X_t \pi_t\, \sigma(t,Y_t)\,(dW_t + \theta_t\, dt).$$

Thaleia proves this by writing down the HJB equation, whose solution can be expressed as $u(t,x,y) = U(x)\, V(t,y)$, and then proposing $V(t,y) = v(t,y)^\alpha$ for some exponent α, which we discover has to be $R(R+\rho^2(1-R))^{-1}$ for the equation to linearise. The verification is quite routine, but it doesn't really illuminate what's going on.

HJB is

$$\sup_{\xi} \Big[\dot u + \tfrac12 \sigma^2 \xi^2 u_{xx} + \rho \sigma \xi a\, u_{xy} + \tfrac12 a^2 u_{yy} + \sigma \xi \theta u_x + b u_y + r x u_x \Big] = 0$$

giving

$$\xi = -\frac{\sigma\theta u_x + \rho \sigma a\, u_{xy}}{\sigma^2 u_{xx}} = -\frac{\theta u_x + \rho a\, u_{xy}}{\sigma\, u_{xx}}$$

whence

(5) $$\boxed{\dot u - \frac12 \frac{(\theta u_x + \rho a\, u_{xy})^2}{u_{xx}} + \tfrac12 a^2 u_{yy} + b u_y + r x u_x = 0}$$

2) This PDE approach isn't very illuminating – what other route could we try? It seems pretty obvious for this problem that the form of π must be $\pi(t, Y_t)$, so that we shall have

$$dX_t/X_t = r\,dt + \pi(t,Y_t)\,\sigma(t,Y_t)\left(dW_t + \theta_t\,dt\right)$$

So

$$X_t = x \exp\left[\int_0^t \pi_s\sigma_s\,dW_s + \int_0^t \{r + \pi_s\sigma_s\theta_s - \tfrac{1}{2}\pi_s^2\sigma_s^2\}\,ds \right]$$

and thus the payoff will be

$$U(x)\,E\,\exp\left\{ (1-R)\int_0^T \pi_s\sigma_s\,dW_s + (1-R)\int_0^T \{r + \pi_s\sigma_s\theta_s - \tfrac{1}{2}\pi_s^2\sigma_s^2\}\,ds \right\}$$

If we write $dW = \rho\,dB + \rho'\,dB'$, where $\rho' = \sqrt{1-\rho^2}$, we have payoff

$$U(x)\;E\;\exp\left[(1-R)\int_0^T \pi_s\sigma_s\rho\,dB_s + (1-R)\int_0^T \{r + \pi_s\sigma_s\theta_s - \tfrac{1}{2}\pi_s^2\sigma_s^2 + \tfrac{1}{2}\rho'^2\pi_s^2\sigma_s^2(1-R)\}\,ds \right]$$

$$= U(x)\;E^\pi\;\exp\left[(1-R)\int_0^T \{r + \pi_s\sigma_s\theta_s - \tfrac{1}{2}\pi_s^2\sigma_s^2 R\}\,ds \right]$$

where under E^π, $dB^\pi = dB - (1-R)\pi_s\sigma_s\rho\,ds$ is a Brownian motion, and

$$dY_t = a(t,Y_t)\,dB^\pi_t + \{b(t,Y_t) + (1-R)\pi_t\sigma_t a_t\rho\}\,dt.$$

Thus if we set

$$\varphi(t,y) \equiv \sup_\pi E^\pi\left(\exp\left[(1-R)\int_t^T \{r + \pi_s\sigma_s\theta_s - \tfrac{1}{2}\sigma_s^2\pi_s^2 R\}\,ds \right] \Big| Y_t = y \right)$$

we would have

$$\exp\left\{ \int_0^t (1-R)\{\text{---}\}\,ds \right\} \varphi(t,Y_t)$$ is a superm., and a martingale under optimal control.

This gives us

$$\sup_\pi \left[\dot\varphi + \tfrac{1}{2}a^2\varphi_{yy} + (b + (1-R)\rho a\sigma\pi)\varphi_y + (1-R)\{r + \pi\sigma\theta - \tfrac{1}{2}\pi^2\sigma^2 R\}\varphi \right] = 0$$

Optimal π is $(\varphi\theta + \rho a\varphi_y)/\sigma R\varphi$ giving the PDE

$$(6)\qquad \boxed{\dot\varphi + \tfrac{1}{2}a^2\varphi_{yy} + b\varphi_y + (1-R)\frac{(\theta\varphi + \rho a\varphi_y)^2}{2R\varphi} + (1-R)r\varphi = 0}$$

which is the same as you get by the approach Thaleia used, i.e. the HJB equations, once you do the identification $u(t,x,y) = U(x)\,\varphi(t,y)$.

3) Another approach we could use is to characterise the dual problem. We know that the optimal X_T has the property that for some λ_x and some SPD ζ^x

$$U'(X_T) = \lambda_x \zeta^x_T .$$

Also, for any SPD ζ we have $x = E[\zeta_T X_T] = E[\zeta_T I(\lambda_x \zeta^x_T)]$ and so

$$E\left[U(I(\lambda_x \zeta^x_T))\right] \le E\left[U(I(\lambda\zeta_T)) + U'(I(\lambda\zeta_T))\{I(\lambda_x\zeta^x_T) - I(\lambda\zeta_T)\}\right]$$

$$= E\left[\tilde{U}(\lambda\zeta_T) + \lambda x\right]$$

Thus we shall have (regarding y_0 as fixed)

$$(7) \qquad \boxed{u(0,x,y_0) = \inf\{E\tilde{U}(\alpha\zeta_T) + \alpha x : \zeta \text{ a SPD}, \alpha \in \mathbb{R}\}}$$

This much is completely general, and doesn't need CRRA assumption.

Assuming now $U(x) = x^{1-R}/(1-R)$, $\tilde{U}(y) = R\,y^{1-1/R}/(1-R)$, we can represent ζ_t as

$$\zeta_t = \beta_t Z_t \equiv e^{-rt} \exp\left[\int_0^t (\nu_s dW'_s - \theta_s dW_s) - \tfrac{1}{2}\int_0^t (\nu_s^2 + \theta_s^2)\,ds\right]$$

where $dB = \rho\,dW + \rho' dW'$, W and W' independent BMs, and therefore with α fixed we want to find

$$\inf_\zeta E\,\tilde{U}(\alpha\zeta_T) = \inf \tilde{U}(\alpha\beta_T)\; E Z_T^{(R-1)/R}$$

Now introduce

$$\varphi(t,y) = \inf E\left[\tilde{U}(\alpha\beta_T)\exp\left[\{\int_t^T(\nu_s dW'_s - \theta_s dW_s) - \tfrac{1}{2}\int_t^T(\nu_s^2+\theta_s^2)ds\}(R-1)/R\right] \,\middle|\, Y_t = y\right]$$

$$\eta_t \equiv \exp\left\{\frac{R-1}{R}\int_0^t (\nu_s dW'_s - \theta_s dW_s) - \frac{R-1}{2R}\int_0^t (\nu_s^2+\theta_s^2)\,ds\right\}$$

so that

$$d\eta_t = \eta_t\left\{\frac{R-1}{R}\left(\nu_t dW'_t - \theta_t dW_t\right) - \frac{R-1}{2R^2}(\theta_t^2+\nu_t^2)\,dt\right\}$$

and $\varphi(t, Y_t)\eta_t$ is a submartingale, and a martingale under optimal control.

By Itô,

$$\dot\varphi + \tfrac{1}{2}a^2\varphi_{yy} + b\varphi_y - \frac{R-1}{2R^2}(\theta^2+\nu^2)\varphi + a\varphi_y \cdot \frac{R-1}{R}(\nu\rho' - \rho\theta) \ge 0$$

$$= \dot\varphi + \tfrac{1}{2}a^2\varphi_{yy} + b\varphi_y + \frac{R-1}{R}\left\{-\tfrac{1}{2}\nu^2\frac{\varphi}{R} + \rho' a\varphi_y \nu - \rho\theta a\varphi_y - \frac{\theta^2\varphi}{2R}\right\} \ge 0$$

Thus optimal choice of ν is $\boxed{\nu = R\rho' a\varphi_y/\varphi}$ and the PDE to solve is

$$\dot\varphi + \tfrac12 a^2 \varphi_{yy} + b\varphi_y + \frac{R-1}{R}\left\{ \frac{R}{\varphi}(\rho' a \varphi_y)^2 - \rho\theta a \varphi_y - \frac{\theta^2\varphi}{2R}\right\} = 0.$$

To solve this, we could try as before taking $\varphi = g^\gamma$, $\varphi_y = \gamma g^{\gamma-1} g_y$, $\varphi_{yy} = \gamma g^{\gamma-1} g_{yy} + \gamma(\gamma-1) g^{\gamma-2} g_y^2$, giving

$$\dot g + \tfrac12 a^2\left(g_{yy} + (\gamma-1) g_y^2/g\right) + b g_y + \frac{R-1}{R}\left\{ R(a\rho')^2 \frac{\gamma g_y^2}{g} - \rho a\theta g_y - \frac{\theta^2}{2R\gamma} g\right\} = 0$$

from which we see that the choice for γ will be

$$\gamma = \frac{1}{R + \rho^2(1-R)}$$

and the PDE will be the linear PDE

$$\boxed{\dot g + \tfrac12 a^2 g_{yy} + b g_y - \frac{R-1}{R}\rho a\theta g_y - \frac{(R-1)\theta^2}{2R^2\gamma} g = 0,}$$

with the boundary condition $g(T,y) = \tilde u(\lambda\beta_T)^{1/\gamma}$. The PDE we find for g is virtually the same as the PDE we got originally for v; indeed,

$$v(t,y) = \tilde u(\lambda\beta_T)^{-1/\gamma}\, g(t,y)\, \exp\{(T-t)r(1-R)/\alpha\} \qquad (\alpha \equiv R\gamma)$$

expresses one in terms of the other. A few calculations on the dual formulation confirms that the two approaches agree, but apparently give no new knowledge.

4) David Hobson suggested we try the exponential utility

$$U(x) = -c^{-1} e^{-cx}, \quad U'(x) = e^{-cx}, \quad \tilde U(y) = \frac{y}{c}(\log y - 1)$$

The dual problem requires us to find

$$\min E\, \tilde U(\lambda\zeta_T) = \min E\left[\frac{\lambda\zeta_T}{c}(\log\lambda\zeta_T - 1)\right]$$

$$= \min\left\{\frac{\lambda}{c} E\, \zeta_T \log\zeta_T\right\} + e^{-rT}\tilde U(\lambda)$$

So it's about $E[\zeta_T\log\zeta_T] = e^{-rT} E[Z_T(\log Z_T - rT)] = e^{-rT}\{E(Z_T\log Z_T) - rT\}$ where the change-of-measure martingale Z is given as before in the form

$$dZ_t = Z_t(-\theta_t dW_t + \nu_t dW_t'), \qquad Z_0 = 1.$$

Suppose now we define

$$\varphi(t,y) \equiv \inf_\nu E\left[(Z_T/Z_t)\log(Z_T/Z_t) \mid Y_t = y\right]$$

Then $Z_t\{\varphi(t,Y_t) + \log Z_t\}$ is a submartingale, and a martingale under optimal control.

Using this,

$$d\left(Z_t \left[\varphi(t,Y_t) + \int_0^t (\nu_u\, dW_u' - \theta_u\, dW_u - \tfrac{1}{2}(\theta_u^2+\nu_u^2)\,du \right] \right)$$

$$\doteq Z_t\, dt \left\{ \dot\varphi + \tfrac{1}{2}a^2\varphi_{yy} + b\varphi_y + a\varphi_y(\rho'\nu_t - \rho\theta_t) + \tfrac{1}{2}(\nu_t^2+\theta_t^2) \right\}$$

so that the optimal ν_t will be $-\rho' a \varphi_y$, and we find the PDE

$$\dot\varphi + \tfrac{1}{2}a^2(\varphi_{yy} - \rho'^2\varphi_y^2) + (b - a\rho\theta)\varphi_y + \tfrac{1}{2}\theta^2 = 0.$$

For this, we use the transformation $f = \exp\{-\rho'^2\varphi\}$, to obtain

$$\boxed{\dot f + \tfrac{1}{2}a^2 f_{yy} + (b - a\rho\theta) f_y - \tfrac{1}{2}(\rho'\theta)^2 f = 0}$$

once again a <u>linear</u> PDE!

In terms of this, the min over the SPD of $E\, \tilde U(\lambda\zeta_T)$ comes out to be

$$e^{-rT}\left[\tilde U(\lambda) + \frac{\lambda}{c}(\varphi(0,y_0) - rT) \right]$$

and for a given x when we minimise $\inf E\,\tilde U(\lambda\zeta_T) + \lambda x$ over λ we get

$$\boxed{e^{-\varphi(0,y_0)}\, U(xe^{rT})}$$

$$C_j(t) = \frac{\rho_j\ \theta_j(t_0)\ e^{-\rho_j(t-t_0)}}{\tilde{\gamma}(t_0, t)}\ \tilde{\delta}(t) \qquad \tilde{\gamma}(t_0,t) \equiv \sum_{i<J} \theta_i(t_0)\, \rho_i\, e^{-\rho_i(t-t_0)}$$

$$\theta_j(t) = \frac{\theta_j(t_0)\ e^{-\rho_j(t-t_0)}}{\psi(t_0,t)} \qquad \psi(t_0,t) \equiv \int_t^{\infty} \tilde{\gamma}(t_0,s)\, \varphi_s^{-1}\, ds$$

Back to the large investor (12/12/99)

(i) Let's consider what would happen if at some time t_0 agent J approaches the pool with a proposal that he should consume in future $(1-\varphi_t)\,\delta_t$, and that agent j should start at t_0 with $\theta_j(t_0)$, possibly different from $\theta_j(t_0-)$, $j=1,2,\dots,J-1$.

As previously, there are parameters $p_j = p_j(t_0)$ such that for $t \geq t_0$

$$p_j(t_0)\, U_j'(c_j(t)) \equiv p_j(t_0)\, e^{-\rho_j t}/c_j(t) = \zeta(t_0,t),$$

from which $\boxed{\zeta(t_0,t) = \tilde{\gamma}(t_0,t)/\tilde{\delta}(t)}$, with $\tilde{\gamma}(t_0,t) \equiv \sum_{j<J} p_j(t_0)\, e^{-\rho_j t}$, $\tilde{\delta}_t \equiv \delta_t\,\varphi_t$.

Now NPV of j's future consumption is (at time $t \geq t_0$)

$$E_t\left[\int_t^\infty \zeta(t_0,s)\, c_j(s)\, ds\right] / \zeta(t_0,t) = p_j(t_0)\, e^{-\rho_j t}/\rho_j\, \zeta(t_0,t),$$

and by taking the value of this at $t=t_0$, we see that we may wlog <u>define</u>

$$p_j(t_0) = \theta_j(t_0)\, \rho_j\, e^{\rho_j t_0}$$

so that at time $t \geq t_0$ the NPV of j's future consumption is simply $\theta_j(t_0)\, e^{-\rho_j(t-t_0)}/\zeta(t_0,t)$.

The share price at any time $t \geq t_0$ is

$$S_t = E_t\left[\int_t^\infty \zeta(t_0,s)\,\delta_s\, ds\right] / \zeta(t_0,t) \equiv \psi(t_0,t)/\zeta(t_0,t)$$

where

$$\psi(t_0,t) \equiv \int_t^\infty \tilde{\gamma}(t_0,s)\,\varphi_s^{-1}\, ds,$$

so the NPV of j's future consumption at t_0 is expressed differently as

$$\theta_j(t_0)\, S_{t_0} = \theta_j(t_0)\,\psi(t_0,t_0)/\zeta(t_0,t_0) = \theta_j(t_0)/\zeta(t_0,t_0)$$

from which

$$\boxed{\psi(t_0,t_0) = 1, \qquad S_{t_0} = 1/\zeta(t_0,t_0).}$$

As before, we can easily confirm the wealth equation $dw_j(t) \equiv d(\theta_j(t)\, S_t) = \theta_j(t)\,(dS_t + \delta_t\, dt) - c_j(t)\,dt$ so that the agent keeps all wealth in the share.

(ii) The payoff to agent j is

$$E_{t_0}\int_{t_0}^\infty e^{-\rho_j t}\log\left[p_j(t_0)\, e^{-\rho_j t}\,\tilde{\delta}(t)/\tilde{\gamma}(t_0,t)\right] dt$$

$$= E_{t_0}\int_{t_0}^\infty e^{-\rho_j t}\log\left\{\frac{\rho_j\,\theta_j(t_0)\, e^{-\rho_j(t-t_0)}\,\tilde{\delta}_t}{\sum_{i<J}\rho_i\,\theta_i(t_0)\, e^{-\rho_i(t-t_0)}}\right\} dt$$

$$\geq E_{t_0}\int_{t_0}^\infty e^{-\rho_j t}\log\left\{\frac{\rho_j\,\theta_j(t_0-)\, e^{-\rho_j(t-t_0)}\sum_{i\leq J}\theta_i(t_0-)\,\delta_t}{\sum_{i<J}\rho_i\,\theta_i(t_0-)\, e^{-\rho_i(t-t_0)}}\right\} dt$$

for feasibility. Thus the feasibility constraint for agent j simplifies to

$$\int_{t_0}^{\infty} e^{-\rho_j t} \log\left\{ \frac{\theta_j(t_0)\,\varphi(t)}{\sum_{i<J} \rho_i \theta_i(t_0)\, e^{-\rho_i(t-t_0)}} \cdot \frac{\sum_{i<J} \rho_i \theta_i(t_0-)\, e^{-\rho_i(t-t_0)}}{\theta_j(t_0-) \sum_{i<J} \theta_i(t_0-)} \right\} dt \geq 0$$

We also have the <u>equality</u> constraint

$$1 = \psi(t_0, t_0) = \int_{t_0}^{\infty} \frac{\tilde{\gamma}(t_0, s)}{\varphi_s}\, ds = \int_{t_0}^{\infty} \frac{\sum_{j<J} \theta_j(t_0)\, \rho_j\, e^{-\rho_j(t-t_0)}}{\varphi_t}\, dt$$

and J's objective is to

$$\max \int_{t_0}^{\infty} e^{-\rho_J t} \log(1-\varphi_t)\, dt,$$

subject to these constraints.

<u>Remark</u>: previously, we were in effect imposing $\theta_j(t_0-) = \theta_j(t_0)$.

(iii) The Lagrangian form of the problem is to

$$\max \int_{t_0}^{\infty} \left\{ e^{-\rho_J t} \log(1-\varphi_t) + \sum_{j<J} \lambda_j e^{-\rho_j t} \log\left\{ \frac{x_j\, \varphi(t)}{\sum_{i<J} \rho_i x_i\, e^{-\rho_i(t-t_0)}} \right\} - \beta\, \frac{\sum_{j<J} x_j \rho_j e^{-\rho(t-t_0)}}{\varphi_t} \right\} dt$$

where x_j is short for $\theta_j(t_0)$, and $\lambda_j \geq 0$ by complementary slackness.

<u>LEMMA (21/1/2000)</u> At optimality, all the constraints for individual agents $j = 1, \dots, J-1$ hold with equality.

<u>Proof</u> Suppose not; then at optimality we have

$$\int_{t_0}^{\infty} e^{-\rho_j s} \log\left\{ \frac{x_j\, \varphi(s)}{\sum_{i<J} x_i \rho_i e^{-\rho_i(s-t_0)}} \right\} ds \geq \int_{t_0}^{\infty} e^{-\rho_j s} \log\left\{ \frac{\theta_j(t_0-) \sum_{i<J} \theta_i(t_0-)}{\sum_{i<J} \theta_i(t_0-)\, \rho_i\, e^{-\rho_i(s-t_0)}} \right\} ds \equiv A_j$$

for all j, with strict inequality for $j=1$, say. Then the claim is that using λx_1, βx_j $(j = 2, \dots, J-1)$ and $\xi\varphi$ for suitable $\lambda, \xi < 1$, $\beta > 1$, we can actually improve. Clearly the payoff is improved, so we need to consider feasibility. The constraint $\varphi(t_0, t_0) = 1$ will require

(*) $$\xi = \lambda \cdot \int_{t_0}^{\infty} x_1 \rho_1 e^{-\rho_1(s-t_0)}\, \varphi_s^{-1}\, ds + \beta \int_{t_0}^{\infty} \sum_{2}^{J-1} \rho_i e^{-\rho_i(s-t_0)} x_i\, \varphi_s^{-1}\, ds$$

which says that a fixed convex combination of λ and β must equal ξ. For small enough moves feasibility for agent 1 is not affected, so we just need to know that for $2 \leq j < J$

$$\int_{t_0}^{\infty} e^{-\rho_j s} \log\left\{ \frac{\beta\, \xi}{\lambda x_1 \rho_1 e^{-\rho_1(s-t_0)} + \beta \sum_{2}^{J-1} \rho_i x_i e^{-\rho_i(s-t_0)}} \cdot \sum_{1}^{J-1} x_i \rho_i e^{-\rho_i(s-t_0)} \right\} ds \geq 0$$

By moving β up from 1 while holding ξ fixed and the condition (*), this expression gets bigger. It's clear now that this can be done.

Variants of the problem of Thaleia Zariphopoulou (17/12/99)

(i) Suppose we have the same dynamics as on p 9:

$$dS_t = S_t \{ \sigma(t, Y_t)\, dW_t + \mu(t, Y_t)\, dt \}$$

$$dY_t = a(t, Y_t)\, dB_t + b(t, Y_t)\, dt \qquad dB_t\, dW_t = \rho dt$$

and we now consider an agent who consumes, and aims to

$$\max \; E\left[\int_0^T u_0(s, c_s)\, ds + u_1(x_T) \right]$$

If we were to assume that $u_0(s, c_s) = h(s)\, U(c_s)$, $u_1(x) = U(x) \equiv x^{1-R}/(1-R)$, we can simplify the form of the solution

$$V(t,x,y) \equiv \sup E\left[\int_t^T u_0(s, c_s)\, ds + u_1(x_T) \,\Big|\, Y_t = y,\ x_t = x \right]$$

$$= x^{1-R} f(t,y)^\beta.$$

The choice of β is at our disposal. If we do the HJB thing, then with the aid of Maple we get

$$\boxed{\tfrac{1}{2} a^2 f_{yy} - \frac{a^2}{2}\{R - \beta(R + \rho^2(1-R))\} \frac{f_y^2}{Rf} + \left(b + a\rho\theta \frac{1-R}{R}\right) f_y + \dot{f} + \frac{1}{2} \frac{(1-R)}{R\beta}(2rR + \theta^2) f + \frac{(1-R)^{(-1/R)}\, h^{1/R} R}{\beta}\, f^{1-\beta/R} = 0.}$$

When $h \equiv 0$, this reduces to the same as before when we take $\beta = R/\{R + \rho^2(1-R)\}$, otherwise the PDE is non-linear.

(ii) If we use the infinite-horizon version of the above problem, with $h(t) = e^{-\lambda t}$, we should find

$$V(t,x,y) \equiv \sup E\left[\int_t^\infty e^{-\lambda s}\, U(c_s)\, ds \,\Big|\, X_t = x,\ Y_t = y \right]$$

$$= e^{-\lambda t} x^{1-R} f(y)^\beta$$

This removes the time-dependence, but doesn't really make anything better.

(iii) Next thing to try was $U(x) = -\exp(-\kappa x)$; we can then argue that

$$\sup E\left[-\int_0^\infty e^{-\lambda t - \kappa c_t}\, dt \,\Big|\, X_0 = x,\ Y_0 = y \right] = e^{-\kappa r x} f(y)$$

and then HJB gives us

$$\boxed{0 = \tfrac{1}{2} a^2 \rho^2 (f')^2 / f + (b - a\theta\rho) f' + \tfrac{1}{2} a^2 f'' + \left(r - \tfrac{1}{2}\theta^2 - r \log(-rf)\right) f}$$

Once again, getting rid of the non-linearities seems impossible...

Rel: $\tilde{\gamma}(t_0,t) = \sum p_i(t_0)\, e^{-\beta_i t}$, $\zeta(t_0,t) = \tilde{\gamma}(t_0,t) / \tilde{\delta}(t)$

$c_j(t) = e^{-\beta_j t} p_j(t_0) / \zeta(t_0,t)$

$w_j(t) = \dfrac{1}{\zeta(t_0,t)}\, \dfrac{p_j(t_0)\, e^{-\beta_j t}}{\rho_j}$

The large investor and the rule of law (21/1/2000)

(i) Let's return to the problem of the large investor under the assumption that the rule of law prevails, so any deal entered into at t_0 will be honoured fully by all parties. We'll try to carry through the analysis at a slightly more general level, assuming that agent j wishes to maximise

$$E \int_{t_0}^{\infty} e^{-\rho_j s}\, U(c_j(s))\, ds,$$

where $U'(x) = x^{-R}$ for some $R > 0$. Let's also suppose that the consumption streams aggregate to

$$\delta(t) = \exp(\sigma W_t + \mu t).$$

(ii) If the agents in the Pool ($\equiv \{1, 2, \dots, J-1\}$) form an equilibrium together, then for constants $p_j(t_0)$ to be determined, we get

$$(1) \qquad p_j(t_0)\, U'(c_j(s)) e^{-\rho_j s} \equiv p_j(t_0)\, c_j(s)^{-R} e^{-\rho_j s} = \zeta(t_0, s).$$

Thus by market clearing,

$$\tilde{\delta}(t) = \sum_{i<J} c_i(t) = \zeta(t_0,t)^{-1/R} \sum_{i<J} \left(p_i(t_0) e^{-\rho_i t}\right)^{1/R}$$

so

$$(2) \qquad \boxed{\zeta(t_0,t) = \tilde{\gamma}(t_0,t) / \tilde{\delta}(t)^R} \qquad \boxed{\tilde{\gamma}(t_0,t) \equiv \left\{ \sum_{i<J} \left(p_i(t_0) e^{-\rho_i t}\right)^{1/R} \right\}^R}$$

The NPV of agent j's future consumption is

$$w_j(t) = \frac{1}{\zeta(t_0,t)} E\left[\int_t^{\infty} \zeta(t_0,s)\, c_j(s)\, ds \right] = \frac{1}{\zeta(t_0,t)} E\left[\int_t^{\infty} \tilde{\delta}(s)^{1-R} \left(p_j(t_0) e^{-\rho_j s}\, \tilde{\gamma}(t_0,s)^{R-1} \right)^{1/R} ds \right]$$

If we have $\tilde{\delta}_t = \varphi_t \delta_t$ for some <u>deterministic</u> function φ (which, in the case of an equilibrium within in the Pool, would simply be the constant $\sum_{i<J} \theta_i(t_0)$) we can simplify this somewhat to

$$\frac{1}{\zeta(t_0,t)} \int_t^{\infty} \varphi(s)^{1-R} \left(p_j(t_0) e^{-\rho_j s}\, \tilde{\gamma}(t_0,s)^{R-1} \right)^{1/R} \exp\left[\sigma(1-R) W_t + (1-R)\mu s + \tfrac{1}{2}\sigma^2 (1-R)^2 (s-t) \right] ds$$

$$= \frac{\delta(t)^{1-R}}{\zeta(t_0,t)} \int_t^{\infty} \left(p_j(t_0) e^{-\rho_j s}\right)^{1/R} \varphi(s)^{1-R} \cdot \tilde{\gamma}(t_0,s)^{1-1/R} \exp\left[(1-R)(s-t) \left\{ \tfrac{1}{2}\sigma^2(1-R) + \mu \right\} \right] ds.$$

Thus the wealth at time $t \geq t_0$ of agent j is

$$(3) \qquad \boxed{w_j(t) = \frac{\delta(t)^{1-R}}{\zeta(t_0,t)} \int_t^{\infty} \left(p_j(t_0) e^{-\rho_j s}\right)^{1/R} \varphi(s)^{1-R}\, \tilde{\gamma}(t_0,s)^{1-1/R} \exp\left[(1-R)(s-t) \left\{ \tfrac{1}{2}\sigma^2(1-R) + \mu \right\} \right] ds.}$$

For brevity, let's write $\boxed{\alpha \equiv (1-R)\left\{ \tfrac{1}{2}\sigma^2(1-R) + \mu \right\}}$

$$\underline{R=1}: \quad S_t = \frac{1}{\zeta(t_0,t)} \cdot \int_t^\infty \tilde{\delta}(t_0,s)\, \varphi_s^{-1}\, ds = \frac{\psi(t_0,t)}{\zeta(t_0,t)}$$

$$\theta_j(t) = \frac{p_j(t_0)\, e^{-\rho_j t}}{\psi(t_0,t)\, \rho_j}$$

$$w_j(t) = \frac{\varphi(t)^R\, \delta(t)}{\tilde{\delta}(t_0,t)} \int_t^\infty \left(p_j(t_0)\, e^{-\rho_j s}\right)^{1/R} \varphi(s)^{1-R}\, \tilde{\delta}(t_0,s)^{1-1/R}\, e^{\alpha(s-t)}\, ds$$

$$c_j(t) = \left\{ \frac{p_j(t_0)\, e^{-\rho_j t}}{\zeta(t_0,t)} \right\}^{1/R}$$

$$\underline{R=1}: \quad \text{payoff} = E \int_{t_0}^\infty e^{-\rho_j s} \log\left[\frac{p_j(t_0)\, e^{-\rho_j s}}{\tilde{\delta}(t_0,s)}\, \tilde{\delta}_s \right] ds$$

Similarly, the time t price of a share is

(4) $$\boxed{S_t = \frac{\delta_t^{1-R}}{\xi(t_0,t)} \int_t^\infty \tilde{\gamma}(t_0,s)\,\varphi(s)^{-R}\, e^{\alpha(s-t)}\, ds}$$

If we write

(5) $$\boxed{\psi(t_0,t) \equiv \int_t^\infty \tilde{\gamma}(t_0,s)\,\varphi(s)^{-R}\, e^{\alpha(s-t)}\, ds}$$

which is consistent with our earlier usage in the case $R=1$, then we have

(6) $$\boxed{S_t = \frac{\psi(t_0,t)\,\delta_t^{1-R}}{\xi(t_0,t)} = \delta_t \frac{\psi(t_0,t)}{\tilde{\gamma}(t_0,t)}\, \varphi(t)^R.}$$

We deduce that

(7) $$\boxed{\theta_j(t) = \frac{1}{\psi(t_0,t)} \int_t^\infty \left(p_j(t_0)\, e^{-\beta_j s}\right)^{1/R} \varphi_s^{1-R}\, \tilde{\gamma}(t_0,s)^{1-1/R}\, e^{\alpha(s-t)}\, ds}$$

from $w_j(t) = \theta_j(t) S_t$ ... but why is this true? It's because (from (3))

$$dw_j(t) = -\frac{\delta_t \varphi_t \left(p_j(t_0)\, e^{-\beta_j t}\right)^{1/R}}{\tilde{\gamma}(t_0,t)^{1/R}}\, dt + w_j(t)\left\{ \frac{d\delta_t}{\delta_t} + \frac{R\dot{\varphi}(t)}{\varphi(t)}\, dt - \frac{\tilde{\gamma}'(t_0,t)}{\tilde{\gamma}(t_0,t)}\, dt - \alpha\, dt \right\}$$

$$= -c_j(t)\, dt + w_j(t)\left\{ \frac{dS_t}{S_t} + \delta_t\, dt \right\}.$$

The payoff to agent j is

(8) $$\int_{t_0}^\infty e^{-\beta_j s/R}\, p_j(t_0)^{\frac{1}{R}-1}\, \tilde{\gamma}(t_0,s)^{1-\frac{1}{R}}\, \varphi_s^{1-R}\, e^{\alpha(s-t_0)} \frac{ds}{1-R} \cdot \delta_{t_0}^{1-R} = E\int_{t_0}^\infty e^{-\beta_j s}\, U\!\left(\frac{p_j(t_0)^{1/R}\, e^{-\beta_j s/R}}{\tilde{\gamma}(t_0,s)^{1/R}}\, \tilde{\delta}(s) \right) ds,$$

The payoff to agent J will be $\int_{t_0}^\infty U(1-\varphi_s)\, e^{-\beta_J s}\, e^{\alpha(s-t_0)}\, \delta_{t_0}^{1-R}\, ds$

(iii) Let's abbreviate $\boxed{\lambda_j \equiv \lambda_j(t_0) = p_j(t_0)^{1/R}}$ so that the solution to the optimisation problem for the pool on its own would be characterised by letting λ_j solve (from (7))

(9) $$\boxed{\theta_j(t_0-) = \int_{t_0}^\infty \frac{\lambda_j\, e^{-\beta_j s/R}}{\left(\sum_{i<J} \lambda_i\, e^{-\beta_i s/R}\right)^{1-R}}\, e^{\alpha(s-t_0)}\, ds \left\{ \sum_{i<J} \theta_i(t_0-) \right\}^{1-R}} \qquad (j=1,\dots,J-1)$$

This essentially arbitrary choice breaks the freedom in the choice of λ_j. It also results in the condition

(10) $$\boxed{\psi(t_0,t_0) \equiv \int_{t_0}^\infty \tilde{\gamma}(t_0,s)\,\varphi(s)^{-R}\, e^{\alpha(s-t_0)}\, ds = 1}$$

as in the case $R=1$.

$\underline{R=1}$: max $\int_{t_0}^{\infty} e^{-\rho_J s} \log(1-\varphi_s)$ subj to $\int_{t_0}^{\infty} e^{-\rho_j s} \log\left[\frac{p_j(t_0)\, e^{-\rho_j s}}{\sum_{i<J} p_i(t_0)\, e^{-\rho_i s}}\, \varphi_s\right] ds \geq \text{value}_j \quad \forall j$

Maxing over the λ's gives the conditions

$$0 = \int_{t_0}^{\infty} e^{-\alpha s} \sum_i y_i\, e^{-\rho_i s/R} \left(\frac{\lambda_i \varphi_s}{\sum_k \lambda_k e^{-\rho_k s/R}}\right)^{1-R} \left[\frac{\delta_{ip}}{\lambda_i} - \frac{e^{-\rho_p s/R}}{\sum \lambda_k e^{-\rho_k s/R}}\right] ds \qquad , \; p = 1, \ldots, J-1,$$

which corresponds to $J-2$ linearly independent conditions.

The payoff to agent j in the general situation is $(R \neq 1)$

$$(11) \qquad \Pi_j = \int_{t_0}^{\infty} e^{-\rho_j s/R} \lambda_j^{1-R} \delta(t_0)^{1-R} e^{\alpha(s-t_0)} U\left(\frac{\varphi(s)}{\sum \lambda_i e^{-\rho_i s/R}} \right) ds$$

and

$$(12) \qquad \theta_j(t_0) = \int_{t_0}^{\infty} e^{-\rho_j s/R} \lambda_j e^{\alpha(s-t_0)} U\left(\frac{\varphi(s)}{\sum_{i<J} \lambda_i e^{-\rho_i s/R}} \right) (1-R)\, ds = (1-R)\lambda_j^R \delta(t_0)^{R-1} \Pi_j$$

when $R \neq 1$.

<u>What then is J's optimisation problem?</u> We aim to

$$(13) \qquad \max_{\varphi} \int_{t_0}^{\infty} e^{-\rho_J s} e^{\alpha(s-t_0)} \delta(t_0)^{1-R} U(1-\varphi_s)\, ds$$

subject to

$$(14) \qquad \int_{t_0}^{\infty} e^{-\rho_j s/R} \lambda_j^{1-R} \delta(t_0)^{1-R} e^{\alpha(s-t_0)} U\left(\frac{\varphi(s)}{\sum_{i<J} \lambda_i e^{-\rho_i s/R}} \right) ds = \Pi_{0j} \qquad (j=1,\dots,J-1)$$

where the Π_{0j} are computed from (11) using the λ's which solve (9), together with $\varphi_s = \sum \theta_i(t_0-)$.
Now (14) cannot fix the λ_j uniquely, so we need some normalisation condition, such as, for example, (10), or $\sum \lambda_j = 1$, or $\lambda_1 = 1$.

If we take our normalisation condition to be the last of these, $\lambda_1 = 1$, then the Lagrangian form is to

$$(15) \qquad \max_{\varphi, \lambda} \int_{t_0}^{\infty} e^{\alpha s} \left\{ e^{-\rho_J s} U(1-\varphi_s) + \sum_{i<J} y_i e^{-\rho_i s/R} \lambda_i^{1-R} U\left(\frac{\varphi_s}{\sum_{i<J} \lambda_i e^{-\rho_i s/R}} \right) \right\} ds$$

so differentiating gives

$$e^{-\rho_J s} (1-\varphi_s)^{-R} = \sum_{i<J} \frac{y_i e^{-\rho_i s/R} \lambda_i^{1-R}}{\{\sum \lambda_i e^{-\rho_i s/R}\}^{1-R}} \cdot \varphi(s)^{-R}$$

$$(16) \qquad \therefore \quad \boxed{\frac{\varphi(s)}{1-\varphi(s)} = \left\{ \frac{\sum_{i<J} y_i e^{-\rho_i s/R} \lambda_i^{1-R} e^{\rho_J s}}{\{\sum_{i<J} \lambda_i e^{-\rho_i s/R}\}^{1-R}} \right\}^{1/R}}$$

for some multipliers y_i. This looks a lot simpler for the case $R=1$ than what we had before!

(iv) If the rule of law holds, there is nothing to prevent the agents in the market entering at time t_0 an agreement that at time $t \geq t_0$ agent j will be consuming $\varphi_j(t)$ of the total dividend. What happens then?

$$\theta_j(t) = \theta_j(t_0)\, e^{-\rho_j (t-t_0)} / \psi(t_0, t)$$

Here we would find J trying to

$$\max \; E\int_{t_0}^{\infty} e^{-\rho_J s}\, U\Big(\big(1-\sum_{i<J}\varphi_i(s)\big)\delta_s\Big)\,ds$$

subj to

$$E\int_{t_0}^{\infty} e^{-\rho_j s}\, U(\varphi_j(s)\delta_s)\,ds = \Pi_{0j} \qquad (j=1,\dots,J-1)$$

The Lagrangian form of this problem would give for each $j=1,\dots,J-1$

$$e^{-\rho_J s}\Big(1-\sum_{i<J}\varphi_i(s)\Big)^{-R} = y_j\, e^{-\rho_j s}\,\varphi_j(s)^{-R} \equiv g(s)^{-R}$$

so that

$$\boxed{\begin{aligned}&\varphi_j(s) = g(s)\, y_j^{1/R} e^{-\rho_j s/R}, \qquad g(s) = \Big\{1-\sum_{i<J} g(s)\, y_i^{1/R} e^{-\rho_i s/R}\Big\}\, e^{\rho_J s/R}\\ \therefore\quad & g(s) = \Big\{e^{-\rho_J s/R} + \sum_{i<J} y_i^{1/R} e^{-\rho_i s/R}\Big\}^{-1}.\end{aligned}}$$

Notice that this is different from the previous calculation.

(V) <u>If there was no compulsion on agents to stay with their obligations, would the agreed deal break down at some time</u> $\tau > t_0$?

For simplicity, let's first consider the case $\boxed{R=1}$. The deal could break down for one of three possible reasons:

(a) (Some subset of) the Pool might prefer to go on their own

(b) J might prefer to go on his own

(c) J might be able to secure a better deal by renegotiating.

Let's examine (a) firstly, and suppose that some subset $A \subseteq \{1,\dots,J-1\}$ prefers to go on its own, that is,

$$\int_{\tau}^{\infty} e^{-\rho_j s}\log\left\{\frac{\rho_j\,\theta_j(\tau)\, e^{-\rho_j(s-\tau)}}{\sum_{i\in A}\rho_i\,\theta_i(\tau)\, e^{-\rho_i(s-\tau)}}\cdot \sum_{i\in A}\theta_i(\tau)\right\}ds$$

$$\geq \int_{\tau}^{\infty} e^{-\rho_j s}\log\left\{\frac{\rho_j\,\theta_j(t_0)\, e^{-\rho_j(s-t_0)}}{\tilde{\gamma}(t_0,s)}\cdot\varphi(s)\right\}ds \qquad \forall j\in A$$

or again

$$\int_{\tau}^{\infty} e^{-\rho_j s}\log\left\{\frac{\sum_{i\in A}\theta_i(t_0)\, e^{-\rho_i(\tau-t_0)}/\psi(t_0,\tau)}{\sum_{i\in A}\rho_i\,\theta_i(t_0)\, e^{-\rho_i(s-t_0)}}\;\frac{\tilde{\gamma}(t_0,s)}{\varphi(s)}\right\}ds \geq 0 \qquad \forall j\in A$$

If we write $\gamma_A(t_0,s) \equiv \sum_{i\in A}\rho_i\,\theta_i(t_0)\, e^{-\rho_i(s-t_0)}$, $\Gamma_A(t_0,s) \equiv \int_s^{\infty}\gamma_A(t_0,u)\,du$, then when we take

A to be the entire pool we have $\gamma_A(t_0,s) \equiv \tilde{\gamma}(t_0,s)$, and the condition for the subset A to want to break off is

$$\int_\tau^\infty e^{-\rho_j s} \log\left\{ \frac{\Gamma_A(t_0,\tau)}{\gamma_A(t_0,s)} \frac{\tilde{\gamma}(t_0,s)\varphi_s^{-1}}{\psi(t_0,\tau)} \right\} ds \geq 0 \qquad \text{for all } j\in A.$$

Equivalently, for all $j \in A$,

$$\int_\tau^\infty \rho_j e^{-\rho_j(s-\tau)} \log\left(\frac{\tilde{\gamma}(t_0,s)\varphi_s^{-1}}{\gamma_A(t_0,s)} \right) ds \geq \log\left\{ \frac{\psi(t_0,\tau)}{\Gamma_A(t_0,\tau)} \right\}$$

$$= \log\left[\frac{\int_\tau^\infty \gamma_A(t_0,s) \frac{\tilde{\gamma}(t_0,s)\varphi_s^{-1}}{\gamma_A(t_0,s)} ds}{\int_\tau^\infty \gamma_A(t_0,u)\, du} \right]$$

$$\geq \frac{\int_\tau^\infty \gamma_A(t_0,s) \log \frac{\tilde{\gamma}(t_0,s)\varphi_s^{-1}}{\gamma_A(t_0,s)} ds}{\Gamma_A(t_0,\tau)}$$

by Jensen. Now multiply this inequality by $\theta_j(t_0) e^{-\rho_j(\tau-t_0)}$ and sum on j to get the inequality

$$\Gamma_A(t_0,\tau) \geq \sum_{j\in A} \theta_j(t_0) e^{-\rho_j(\tau-t_0)}$$

Since this inequality is in fact an <u>equality</u>, the conclusion is that all the earlier inequalities must be equalities, and so the application of Jensen would force us to have

$$\frac{\tilde{\gamma}(t_0,s)\,\varphi_s^{-1}}{\gamma_A(t_0,s)} = \text{constant}$$

<u>Conclusion</u>: there is no subset of the Pool that would <u>prefer</u> to break off at time τ. At best the members of A would be indifferent, which would happen iff $\tilde{\gamma}(t_0,s)\varphi_s^{-1}/\gamma_A(t_0,s)$ were constant.

Now how about (b)? J prefers at time τ to stay with the deal iff

$$\int_\tau^\infty e^{-\rho_J s} \log(1-\varphi_s)\, ds \geq \int_\tau^\infty e^{-\rho_J s} \log\Big\{1 - \sum_{i<J} \theta_i(\tau)\Big\} ds$$

$$= \int_\tau^\infty e^{-\rho_J s} \log\left[\frac{\int_\tau^\infty \tilde{\gamma}(t_0,u)\varphi_u^{-1}[1-\varphi_u]\, du}{\int_\tau^\infty \tilde{\gamma}(t_0,u)\varphi_u^{-1}\, du} \right]$$

If we write $\theta_p(t) \equiv \sum_{i<J} \theta_i(t)$ for short, we have

$$\theta_p(t) = \int_t^\infty \tilde{\gamma}(t_0,s)\,ds \Big/ \int_t^\infty \tilde{\gamma}(t_0,s)\,\varphi_s^{-1}\,ds$$

so that

$$\dot{\theta}_p(t) = -\frac{\tilde{\gamma}(t_0,t)}{\psi(t_0,t)} + \frac{\theta_p(t)}{\psi(t_0,t)} \cdot \frac{\tilde{\gamma}(t_0,t)}{\varphi(t)} = -\frac{\dot{\psi}(t_0,t)}{\psi(t_0,t)}\left\{\theta_p(t) - \varphi(t)\right\}$$

so that θ_p is being pushed away from φ. Thus

$$\begin{cases} \underline{\text{if } \varphi \text{ is decreasing for}}\ t \geq t^*, \underline{\text{then}}\ \theta_p(t) \leq \varphi(t) \quad \forall t \geq t^*, \underline{\text{and}}\ \theta^p \underline{\text{ is decreasing in}}\ [t^*,\infty) \\ \underline{\text{if } \varphi \text{ is increasing for}}\ t \geq t^*, \underline{\text{then}}\ \theta_p(t) \geq \varphi(t) \quad \forall t \geq t^*, \underline{\text{and}}\ \theta^p \underline{\text{ is increasing in}}\ [t^*,\infty) \end{cases}$$

More systematically, we could attempt to incorporate this constraint by way of a Lagrangian term; the Lagrangian would now be for some increasing f^n A

$$\int_{t_0}^\infty e^{-\rho_J t}\log(1-\varphi_t)\,dt + \sum_{i<J}\int_{t_0}^\infty y_i\,e^{-\rho_i t}\log\varphi_t\,dt$$

$$- \int_{t_0}^\infty \left\{ \int_\tau^\infty e^{-\rho_J s}\log(1-\varphi_s)\,ds - \int_\tau^\infty e^{-\rho_J s}\log(1-\theta_p(s))\,ds \right\} dA_\tau .$$

If we perturb this from optimal φ to $\varphi+\eta$ with η small, the first-order change will be

$$\int_{t_0}^\infty \left\{ -\frac{\eta_t e^{-\rho_J t}}{1-\varphi_t} + \sum_{i<J}\frac{y_i\,\eta_t\,e^{-\rho_i t}}{\varphi_t} + \frac{\eta_t}{1-\varphi_t}\cdot\int_{t_0}^t dA_\tau \cdot e^{-\rho_J t}\right.$$

$$\left. - \int_{t_0}^\infty \eta_t\,\frac{\tilde{\gamma}(t_0,t)}{\varphi_t^2}\left\{\int_{t_0}^t \frac{e^{-\rho_J\tau}}{\rho_J}\,\frac{\theta_p(\tau)}{(1-\theta_p(\tau))\,\psi(t_0,\tau)}\,dA_\tau\right\} \right\}$$

so at optimality we'd need to have

$$\boxed{\frac{e^{-\rho_J t}}{1-\varphi_t}\left\{1-A_t\right\} = \frac{1}{\varphi_t}\sum_{i<J} y_i\,e^{-\rho_i t} - \frac{\tilde{\gamma}(t_0,t)}{\varphi_t^2}\int_{t_0}^t \frac{\theta_p(\tau)\,e^{-\rho_J\tau}}{\rho_J\,(1-\theta_p(\tau))\,\psi(t_0,\tau)}\,dA_\tau}$$

Note that φ is implicitly in θ_p and ψ.

Remarks on an agency problem (21/2/2000)

(i) An interesting preprint 'Portfolio performance and agency' by Phil Dybvig, Heber Farnsworth & Jennifer Carpenter studies the following problem (with slightly different notation). A principal hires a fund manager to manage his assets. The fund manager makes effort $a \in [0,1]$ and receives a signal S about the random outcome X of the market. The random variables S and X are real-valued, with joint density

$$f(s,x|a) = (1-a)\, f_0(s,x) + a\, f_I(s,x)$$

with respect to Leb×Leb. This depends on the effort a expended; if no effort is expended, the joint density f_0 is assumed to be of product form (i.e. signal and market outcome independent). DFC also assume that the marginals of f_0 and f_I agree. It certainly seems reasonable that the X-marginals should be the same, less reasonable that the S-marginals should, but let's proceed, highlighting places where this second assumption is invoked.

The aim is to design a contract where if signal s is reported by the manager, and market outcome x occurs, then the principal receives $V(s,x)$ and the manager $\varphi(s,x)$; the contract design should

Principal(P) $\quad \max \iint U_P(V(s,x))\, f(s,x|a)\, ds\, dx \quad$ subject to

Manager(M) $\quad \iint U_A(\varphi(s,x))\, f(s,x|a) - c(a) = u_0$,

and

Budget (B) $\quad \forall s, \quad \int \{ V(s,x) + \varphi(s,x) \}\, p(x)\, dx = w_0$

where U_P and U_A are respectively the utilities of principal and agent (henceforth, we assume as in DFC that $U_A = U_P = \log$), $c(a)$ is the cost (in utils) to the agent for expending effort a, and $p(x)$ is the time-0 cost of a contingent claim which pays 1 in market outcome x, with w_0 the time-0 wealth of the principal. We assume c is strictly increasing and strictly convex.

This would be the situation if the principal were able to observe the agent's effort and his signal. However, if the principal cannot observe the effort, he will want to design the contract so that in addition to (M) and (B), the constraint

(IC effort) $\quad \iint U_A(\varphi(s,x)) \{ f_I(s,x) - f_0(s,x) \}\, ds\, dx = c'(a)$

holds, so that the contract is <u>incentive-compatible for effort</u>; the effort which the principal plans that the manager should make is in fact the same as the manager would choose to make with the given contract. Such a problem is called <u>second-best</u>, in contrast to the <u>first-best</u> problem where the principal can see both effort and signal.

But the problem really gets interesting if we suppose that the principal cannot see the agent's effort <u>or</u> signal, the so-called <u>third-best</u> problem. Now the principal will want the contract to satisfy additionally

(IC truth) $\forall s, \forall t,$

$$\int U_A(\varphi(t,x))\, f(s,x|a)\, dx \le \int U_A(\varphi(s,x))\, f(s,x|a)\, dx$$

which is to say that the manager would never want to mis-report a signal. DFC replace this condition by

(IC truth') $$\int \frac{\partial}{\partial t} U_A(\varphi(t,x))\Big|_{t=s} f(s,x|a)\, dx = 0$$

but let's not do that for now.

(ii) We can cast the problem into Lagrangian form: the Lagrangian is

$$\iint \left[\log V(s,x) + y\, g(s,x) \right] f(s,x|a)\, ds\, dx - y\, c(a)$$

$$- \int \lambda(s) \int \left(V(s,x) + e^{g(s,x)} \right) p(x)\, dx\, ds$$

$$+ \beta \iint g(s,x) \left(f_I(s,x) - f_0(s,x) \right) ds\, dx - \beta\, c'(a)$$

$$- \int dt \int ds\; \theta(s,t) \int \left(g(s,x) - g(t,x) \right) f(s,x|a)\, dx,$$

where we abbreviate $\log \varphi(s,x) = g(s,x)$. The maximisation should take place over V, g and a, but we'll follow DFC in assuming that a is fixed, work out the optimal contract for that fixed a, with optimisation over a at the end. For the second-best problem, we'd have $\theta \equiv 0$, and for the first-best we'd also have $\beta = 0$. Optimising the Lagrangian gives us

$$\boxed{\begin{aligned} V(s,x) &= \frac{f(s,x|a)}{\lambda(s)\, p(x)} \\ \varphi(s,x) &= \frac{y\, f(s,x|a') + \int \theta(t,s)\, f(t,x|a)\, dt - f(s,x|a) \int \theta(s,t)\, dt}{\lambda(s)\, p(x)} \end{aligned}}$$

$$a' = \frac{ay + \beta}{y}$$

$$\int_0^1 \cos^{2n} 2\pi x \, dx = \frac{1}{2\pi} \binom{2n}{n} 2^{-2n}$$

where $a' = (ay+\beta)/y$.

The nice log utility gives us more tractable forms; we can, for example, use the budget constraint (B) to discover that

$$\boxed{W_0 = \lambda(s)^{-1}\left\{ f(s|a) + y f(s|a') + \int \theta(t,s) f(t|a)\,dt - f(s|a)\int \theta(s,t)\,dt \right\}}$$

where $f(s|a) = \int f(s,x|a)\,dx$ — if the marginal of S is the same for f_0, f_I, then this doesn't depend on a.

The difficult situation remaining is to sort out (IC truth) – conditions (M) and (IC effort) are obtained by adjusting y and β. So we want to consider

$$\int \log\left[\frac{y f(s,x|a') + \int\theta(t,s) f(t,x|a)\,dt - f(s,x|a)\int \theta(s,t)\,dt}{f(s|a) + y f(s|a') + \int \theta(t,s) f(t|a)\,dt - f(s|a)\int\theta(s,t)\,dt}\right] f(s^*, x|a)\,dx$$

and show that for each s^*, the expression is maximised over s at $s = s^*$.

(iii) <u>An example</u> Let's suppose that both S and X take values in $[0,1]$, with the marginal laws both uniform, and let's suppose that

$$f_I(s,x) = 1 + b\cos 2\pi(s-x)$$

where $0 < b \le 1$. This allows for positive correlation between S and X. In fact the correlation is $3b/\pi^2$, which is 0.3040 when $b=1$, so correlation upto 30%.

Can we check out the (IC truth) condition? We need to compute for $0 \le \alpha < 1$

$$h_0(\alpha) \equiv \int_0^1 \log(1 + \alpha\cos 2\pi x)\,dx$$

$$= -\sum_{n\ge 1} \frac{(-\alpha)^n}{n}\int_0^1 \cos^n 2\pi x\,dx$$

$$= -\sum_{n\ge1} \frac{(+\alpha)^{2n}}{2n}\cdot\frac{1}{2\pi}\cdot\binom{2n}{n}2^{-2n}$$

Now $2\pi h_0'(\alpha) = -\sum_{n\ge1} \alpha^{2n-1}\binom{2n}{n}2^{-2n} = -\alpha^{-1}\left\{\frac{1}{\sqrt{1-\alpha^2}} - 1\right\}$, so

$$\boxed{h_0(\alpha) = -\frac{1}{2\pi}\log\left(\frac{2}{1+\sqrt{1-\alpha^2}}\right)}$$

Likewise,

$$h_1(\alpha) \equiv \int_0^1 \log(1+\alpha\cos 2\pi x)\cdot\cos 2\pi x\,dx$$

$$= -\sum_{n\geq 1} \frac{(-\alpha)^n}{n} \int_0^1 \cos^{n+1} 2\pi x \, dx$$

$$= \frac{1}{2\pi} \sum_{n\geq 1} \frac{\alpha^{2n-1}}{2n-1} \binom{2n}{n} 2^{-2n}$$

so that $2\pi h_1'(\alpha) = +\alpha^{-2}\left(\frac{1}{\sqrt{1-\alpha^2}} - 1\right)$, $\boxed{h_1(\alpha) = -\frac{\sqrt{1-\alpha^2}-1}{2\pi\alpha}}$ for $\alpha \in [0,1]$

Suppose we seek multipliers θ such that

$$\theta(s,t) = \bar\theta + \gamma \cos 2\pi(s-t).$$

Then $\int \theta(t,s) f(t,x|a)\,dt = \int (\bar\theta + \gamma \cos 2\pi v)\,(1 + ab\cos 2\pi(v+s-x))\,dv$

$$= \bar\theta + \frac{\gamma ab}{2} \cos 2\pi(s-x)$$

and so our expression for φ would be

$$\varphi(s,x) = \frac{y + ya'b\cos 2\pi(s-x) + \frac{1}{2}\gamma ab \cos 2\pi(s-x) - \bar\theta\, ab \cos 2\pi(s-x)}{\lambda(s)\, p(x)}$$

and $\lambda(s) = (1+y)/w_0$, independent of s. The IC truth condition now requires that

$$\int \log\left(y + (ya'b + \tfrac{1}{2}\gamma ab - \bar\theta ab)\cos 2\pi(s-x)\right) \{1 + ab\cos 2\pi(s^* - x)\}\,dx$$

should be maximised at $s = s^*$. But this expression is

$$\log y + h_0(\alpha) + ab\cos 2\pi(s^*-s)\, h_1(\alpha) \qquad \left[\alpha \equiv y^{-1}(ya'b + \tfrac{1}{2}\gamma ab - \bar\theta ab)\right]$$

which clearly is maximised at $s = s^*$, since $h_1(\alpha) \geq 0$. The only thing we need to watch is that $\alpha \in [0,1]$. The solution involves the two parameters y, α, which we would fix from the scalar constraints (M) and (IC effort).

This particular example is not so interesting, because the second-best solution is already (IC truth) !!?

(iv) Another example The particular form of $f(s, x|a)$ isn't necessarily the only one we might consider – the effort doesn't have to enter linearly. If we took instead

$$f(s,x|\rho) = \exp\left[-\frac{x^2 - 2\rho s x + s^2}{2\sigma^2(1-\rho^2)}\right] / 2\pi\sigma^2\sqrt{1-\rho^2}$$

then the form of the problem remains unchanged except that (IC effort) now reads

$$\iint g(s,x) \frac{\partial f}{\partial \rho}(s,x|\rho)\,ds\,dx = c'(\rho)$$

$X|S \sim N(\rho S, \sigma^2(1-\rho^2))$, $S|X \sim N(\rho X, \sigma^2(1-\rho^2))$

We have

$$\frac{\partial f}{\partial \rho}(s,x|\rho) = f(s,x|\rho)\left[\rho\sigma^2(1-\rho^2) - \rho(x^2+s^2) + (1+\rho^2)xs\right] / \sigma^2(1-\rho^2)^2$$

so that

$$\varphi(s,x) = \frac{f(s,x|\rho)}{\lambda(s)p(x)}\left\{y + \frac{\beta}{\sigma^2(1-\rho^2)^2}\left\{(1+\rho^2)xs - \rho(x^2+s^2) + \rho\sigma^2(1-\rho^2)\right\} - \int\theta(s,t)\,dt + \frac{\int\theta(t,s)\,f(t,x|\rho)\,dt}{f(s,x|\rho)}\right\}$$

If on the other hand we use (IC truth'), the terms in θ in the Lagrangian get replaced by

$$\iint g(s,x)\left\{\theta_s' f(s,x|\rho) + \theta_s f'(s,x|\rho)\right\} ds\,dx$$

and so we have rather nicely

$$\boxed{\varphi(s,x) = \frac{f(s,x|\rho)}{\lambda(s)p(x)}\left[y + \frac{\beta}{\sigma^2(1-\rho^2)^2}\left\{(1+\rho^2)xs - \rho(x^2+s^2) + \rho\sigma^2(1-\rho^2)\right\} + \theta_s' + \theta_s\frac{\rho x - s}{\sigma^2(1-\rho^2)}\right]}$$

The budget constraint becomes $\forall s$

$$w_0 = \lambda_s^{-1} f(s)\left\{y + \theta_s' - \theta_s\,.\,s/\sigma^2\right\}$$

but where now?

Inverting the Wiener-Hopf transforms. (17/3/2000)

(i) This issue arose in the credit-risk work with Bianca. For a spectrally-negative Lévy process, we have the left WH factor

$$\psi_\lambda^-(z) \equiv E \exp z\underline{X}(T_\lambda) = \int_0^\infty \lambda e^{-\lambda t} dt \int_{-\infty}^0 z e^{zx} P(x < \underline{X}(t))\, dx$$

$$= \frac{\lambda}{\lambda - \psi(z)} \cdot \frac{\beta^*(\lambda) - z}{\beta^*(\lambda)}$$

where $\psi \circ \beta^* = id$. Thus this WH factor is in effect a double LT of the quantity of interest, $P(x < \underline{X}(t)) = P(H_x > t)$, and we need to do an inversion.

The algorithms of Abate & Whitt (Ann Appl. Prob. **4** 719–740 (1994), ORSA J. Computing 7, 36–43 (1995)) provide neat ways of inverting LTs; the method involves evaluating the LT at regularly-spaced points, but for our current problem it will be impossible in general to find β^* if given explicitly what ψ is ... so how do we proceed?

Suppose we have a bivariate LT

$$\tilde{f}(s_1, s_2) \equiv \int_0^\infty\int_0^\infty e^{-(s_1 t_1 + s_2 t_2)} f(t_1, t_2)\, dt_1\, dt_2$$

This can be inverted by the recipe

$$f(t_1, t_2) = \int_{\Gamma_1} \frac{dz_1}{2\pi i} \int_{\Gamma_2} \frac{dz_2}{2\pi i}\, e^{z_1 t_1 + z_2 t_2} \tilde{f}(z_1, z_2)$$

where Γ_1 is some line $a_1 + i\mathbb{R}$, Γ_2 is a line $a_2 + i\mathbb{R}$; this is the starting point of the AW method.

(ii) For our application, we have $\lim_{x \uparrow 0} P(x < \underline{X}(t)) = 0$, $\lim_{t \downarrow 0} P(x < \underline{X}(t)) = 1$, so the function we FT is not continuous. If we express

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + bz + \int_{-\infty}^0 (e^{zx} - 1)\, \nu(dx)$$

then for $x < 0$ fixed, we have $P(\underline{X}(t) \leq x) \sim t\,\nu(x)$ as $t \downarrow 0$, and for $t > 0$ fixed, $P(\underline{X}(t) > x) \sim 2|x| / \sqrt{2\pi\sigma^2 t}$ as $x \uparrow 0$. It would be nice to subtract off a known function with known LT to make something that is C^1. It's not clear what choice we could make to achieve this, but at least we can make something continuous by subtracting $|x|/(t + |x|)$; we have

$$\int_0^\infty e^{-\lambda t} \int_{-\infty}^0 e^{zx} \frac{-x}{t - x}\, dt\, dx = \int_0^\infty dt \int_0^\infty dv\; e^{-\lambda t - zv} \frac{v}{t + v}$$

$$= \int_0^\infty dy \int_0^\infty dt \int_0^\infty dv\; e^{-(\lambda + y)t - (y + z)v}\, v$$

which is the derivative wrt z of

$$-\int_0^\infty dy \int_0^\infty dt \int_0^\infty dv\; e^{-(\lambda + y)t - (y + z)v}$$

Even for a general SN Lévy pr, we have $\Psi_0(\zeta(y)) - \Psi(\zeta(y)) = o(|y|)$ as $|y| \to \infty$.

$$= -\int_0^\infty dy \left(\frac{1}{\lambda+y} \cdot \frac{1}{z+y} \right) = \frac{1}{\lambda - z} \log\left(\frac{z}{\lambda}\right).$$

Thus
$$\int_0^\infty e^{-\lambda t} dt \int_0^\infty e^{-zy} \frac{t}{t+y} = \frac{1}{(\lambda-z)^2}\left[-1 + \frac{\lambda}{z} + \log\left(\frac{z}{\lambda}\right)\right]$$

and we could subtract this from the original LT to give better integrability. Alternatively, and probably better, we could subtract off the LT which we get in the case of no jumps.

(iii) Simplifying for the moment to the inversion of a single-dimensional LT
$$\tilde{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\, dt,$$
we have
$$g(x) = \int_\Gamma \frac{dz}{2\pi i} \tilde{g}(z)\, e^{zx}$$

where Γ is a curve of the form $a + i\mathbb{R}$. However, by Cauchy's theorem, we could take any other contour of similar shape provided we can neglect the end errors. It would be most convenient to have
$$g(x) = \int_{\Gamma_0} \frac{d\zeta}{2\pi i} \psi'(\zeta)\, \tilde{g}(\psi(\zeta))\, e^{x\psi(\zeta)}$$

where the contour $\psi(\Gamma_0)$ is 'close to' Γ. How would we choose Γ_0? If $\psi_0(z) \equiv \frac{1}{2}\sigma^2 z^2 + bz$ is the Lévy exponent of the drifting BM, we know that $\psi_0(a+iy) \sim \psi(a+iy)$ as $|y| \to \infty$, so this suggests taking
$$\Gamma_0 = \psi_0^{-1}(a + i\mathbb{R})$$
so that the curve would be
$$y \mapsto \left(\sqrt{b^2 + 2\sigma^2(a+iy)} - b\right)/\sigma^2 \equiv \zeta(y) \equiv \psi_0^{-1}(a+iy)$$

It's not too hard to show that $\psi_0(\zeta(y)) - \psi(\zeta(y)) \to -\gamma(0)$ as $|y| \to \infty$, so at the ends of the curve, $\psi \circ \zeta$ looks like $a+iy$. The branch of $\sqrt{\cdot}$ to be used is with the cut along $(-\infty, 0]$, real positive on $\mathbb{R}^+$. Our representation of $g(x)$ therefore becomes

$$\boxed{g(x) = \int_{y=-\infty}^{\infty} \frac{dy}{2\pi} \cdot \underbrace{\frac{\psi'(\zeta(y))}{\sqrt{b^2 + 2\sigma^2(a+iy)}}}_{i\frac{d\zeta}{dy}\,\cdot\,\psi'(\zeta(y))} \cdot \tilde{g}\big(\psi(\zeta(y))\big) \exp\{x\psi(\zeta(y))\}}$$

This is the starting point for the numerical inversion. We will just try putting the y-values down at equally-spaced points as in the A-W algorithm, and see what we get.

(iv) Our inversion expression has $f(t,x) = P(\bar{X}_t > -x) = P(H_{-x} > t)$ for $x, t > 0$, and

$$\tilde{f}(\lambda, z) = \int_0^\infty e^{-\lambda t}\,dt \int_0^\infty e^{-zx}\,dx\; f(t,x) = \frac{\beta^*(\lambda) - z}{\{\lambda - \psi(z)\}\,\beta^*(\lambda)\, z},$$

and recovers f via

$$f(t,x) = \int_{a_1+i\mathbb{R}} \frac{dz_1}{2\pi i} \int_{a_2+i\mathbb{R}} \frac{dz_2}{2\pi i}\; g'(z_1)\, \tilde{f}(g(z_1), z_2)\, \exp\{t g(z_1) + x z_2\},$$

where $g \equiv \psi \circ \psi_0^{-1}$. Numerically, we approximate the double integral by

$$\frac{h_1 h_2}{4\pi^2} \sum_{n=-N}^{N} \sum_{m=-N}^{N} g'(a_1 + inh_1)\, \tilde{f}(g(a_1+inh_1), a_2 + imh_2)\, \exp\{t g(a_1+inh_1) + x(a_2+imh_2)\}$$

and then take $a_1 = A_1/(2tl_1)$, $a_2 = A_2/(2xl_2)$, $h_1 = \pi/(tl_1)$, $h_2 = \pi/(xl_2)$ to make the terms in the sum (nearly) alternating sign. This makes our numerical approximation

$$\frac{1}{4txl_1l_2} \sum_{n=-N}^{N} \sum_{m=-N}^{N} g'\left(\left(\tfrac{1}{2}A_1 + in\pi\right)/(tl_1)\right) \tilde{f}\left(g\left(\frac{\frac{1}{2}A_1+in\pi}{tl_1}\right), \frac{\frac{1}{2}A_2 + im\pi}{xl_2}\right) \cdot$$

$$\cdot \exp\left\{ t g\left(\frac{\frac{1}{2}A_1 + in\pi}{tl_1}\right) + \frac{A_2}{2l_2} + i\frac{m\pi}{l_2} \right\}$$

NB For f to be concave, need $\alpha+\beta<1$, as well as $\alpha, \beta \in (0,1)$.

A very simple model of tax + production. (12/4/2000)

1) Suppose government taxes at rate τ_t and spends at rate g_t. Government expenditure maintains the level of society θ_t, which evolves as

(1) $$\boxed{\dot\theta_t = g_t - b\,\theta_t}$$

If society is providing labour at rate ℓ_t, level of capitalisation is k_t, production will be at rate $F(\ell_t, k_t, \theta_t)$, which is increasing in all variables. Assume $F(\ell, k, \theta) = \ell\, f(k/\ell, \theta)$, as usual, where f is concave, increasing in both variables. Let D_t be level of government debt, r_t the rate of interest, so that

(2) $$\boxed{\dot D_t = r_t D_t + g_t - \tau_t}$$

Suppose also that k evolves as

(3) $$\boxed{\dot k_t = i_t - a k_t}$$

where i_t is the investment rate. We therefore have the budget constraint

(4) $$\boxed{F(\ell_t, k_t, \theta_t) = c_t + i_t + \tau_t - r_t D_t}$$

The payoff to society is

(5) $$\int_0^\infty U(t, c_t, \ell_t)\, dt$$

where U is increasing in c, decreasing in ℓ, and concave in both.

2) If we look for a <u>steady-state solution</u>, we've got

(6) $$g + rD = \tau, \quad g = b\theta, \quad ak = i, \quad F(\ell, k, \theta) = c + i + g = c + ak + b\theta.$$

For this to make any sense, let's also assume

(7) $$U(t, c, \ell) = e^{-\rho t}\, U(c, \ell)$$

The optimisation problem then is to pick c, ℓ, k and θ to maximise $U(c,\ell)$ subject to $F(\ell, k, \theta) = c + ak + b\theta$.

<u>For example</u>: Take

(8) $$\begin{cases} U(c,\ell) = \dfrac{\left(c(1-\ell)^\lambda\right)^{1-R}}{1-R} = \dfrac{c^{1-R}(1-\ell)^{\lambda-\lambda R}}{1-R} \\ f(k,\theta) = k^\alpha \theta^\beta \end{cases}$$

for some $\alpha, \beta \in (0,1)$, $R > 0$, $R \neq 1$. The Lagrangian to be maximised is therefore

(9) $$L = \frac{c^{1-R}(1-\ell)^{\lambda-\lambda R}}{1-R} + \gamma\left(\ell\left(\frac{k}{\ell}\right)^\alpha \theta^\beta - c - ak - b\theta\right)$$

and the various FOCs are

$$\boxed{\begin{array}{ll} (10) & c^{-R}(1-\ell)^{\lambda-\lambda R} = \gamma \\ (11) & \lambda c^{1-R}(1-\ell)^{\lambda-\lambda R-1} = \gamma(1-\alpha)\,\ell^{-\alpha} k^\alpha \theta^\beta \\ (12) & \alpha\, k^{\alpha-1} \ell^{1-\alpha} \theta^\beta = a \\ (13) & \beta\, k^\alpha \ell^{1-\alpha} \theta^{\beta-1} = b \end{array}}$$

The last two tell us that

$$\frac{\alpha}{\beta} \cdot \frac{\theta}{\kappa} = \frac{a}{b}$$

and that

$$\ell^{-\alpha} \kappa^{\alpha} \theta^{\beta} = \frac{a}{\lambda} \cdot \frac{\kappa}{\ell}$$

Now looking at (11)/(10) gives us that

$$\frac{\lambda c}{1-\ell} = (1-\alpha) \frac{a\kappa}{\alpha \ell} \quad \therefore \quad \boxed{\kappa = \frac{\alpha}{a(1-\alpha)} \frac{\lambda \ell c}{1-\ell} = \frac{\alpha}{a(1-\alpha)} \frac{\lambda \ell}{1-\ell} \frac{(1-\ell)^{(\lambda-\lambda R)/R}}{\gamma^{1/R}}}$$

This virtually determines the solution, because once we've chosen ℓ, we have from (12) that

$$\alpha \, \ell^{1-\alpha} \, \kappa^{\alpha+\beta-1} \left(\frac{\beta a}{b\alpha}\right)^{\beta} = a$$

so that the value of γ is now fixed (in terms of ℓ) from the last two equations. All that remains is to pick ℓ so that the budget constraint holds.

After some calculations, I get that

$$\ell = \frac{1-\alpha}{(1-\alpha)+\lambda(1-\alpha-\beta)}$$

The interest rate r will simply be ρ, by considering MRS of consumption now for consumption later. The level of debt appears to be indeterminate here. Notice that we need not only $\alpha, \beta \in (0,1)$ but also $\alpha + \beta < 1$, which is needed for concavity of f.

Time-lagged investment in a log-Lévy asset (20/4/00)

1) Suppose we have a risky asset $S_t = S_0 \exp(Z_t)$, where Z is a Lévy process, $Z_t = \sigma W_t + at + J_t$, where J is the jump component (assume it's FV for simplicity now).

The wealth of an agent investing in this will be governed by

$$dw_t = r w_t \, dt + \theta_t \left\{ \frac{dS_t}{S_{t-}} - r \, dt \right\}$$

where

$$dS_t = S_{t-} \left\{ dZ_t + \tfrac{1}{2} d\langle Z^c \rangle_t + e^{\Delta Z_t} - 1 - \Delta Z_t \right\}$$

Working with discounted wealth, and discounted asset prices, we can reduce to the case $r=0$ for simplicity, and we get

$$dw_t = \theta_t \left\{ dZ_t + \tfrac{1}{2} d\langle Z^c \rangle_t + e^{\Delta Z_t} - 1 - \Delta Z_t \right\}.$$

If the agent is trying to max $E\, U(w_T)$ for $T>0$ fixed, $U(x) = x^{1-R}/(1-R)$ $(R>0, R \neq 1)$, he will take $\theta_t = \pi w_{t-}$ for some constant π, and then

$$w_t = w_0 \exp\left[\pi Z_t + \tfrac{1}{2}\pi(1-\pi)\langle Z^c \rangle_t \right] \prod_{s \le t} \left(1 + \pi (e^{\Delta Z_s} - 1) \right).$$

The agent therefore aims to pick π to maximise

$$\boxed{\begin{aligned} &\frac{1}{1-R} E \exp\left\{ (1-R)\pi \left(Z_T + \tfrac{1}{2}(1-\pi)\langle Z^c \rangle_T \right) \right\} \prod_{s \le T} \left(1 + \pi(e^{\Delta Z_s} - 1) \right)^{1-R} \\ &= \frac{1}{1-R} \exp\left\{ (1-R)\pi T \left(a + \tfrac{1}{2}\sigma^2 - \frac{\sigma^2 R}{2}\pi \right) \right\} \exp\left[T \int \left\{ (1+\pi(e^x - 1))^{1-R} - 1 \right\} \nu(dx) \right] \end{aligned}}$$

where ν is the Lévy measure of the jumps.

On the other hand, if the agent had to decide at time 0 to put proportion p of wealth into the risky asset, by time T the wealth is $1 - p + p e^{Z_T}$ (if $w_0 = 1$), so he aims to pick p so as to max

$$\frac{1}{1-R} E \left(1 - p + p e^{Z_T} \right)^{1-R}$$

2) For simplicity, let's suppose that J is a compound Poisson process, with jumps $\sim F$ coming at rate λ. If $0<R<1$, let's also assume that $E\, e^{(1-R)V} < \infty$, where V generically denotes a random variable with law F.

We'll now think what happens to an agent with horizon h, thought of as small. The payoff of the Merton investor is easily expanded in powers of h, but for the fixed-policy investor we need the expansion in powers of h of

$$h \mapsto E(1+X)^{1-R}, \quad X \equiv p(e^{Z_h} - 1).$$

Need $E\, e^{2V} < \infty$, otherwise

$$E\left(1+p(e^{V}-1)\right)^{1-R} Y^2 (1+\theta Y)^{-1-R} \leq E\left(q+pe^{V}\right)^{1-R} Y^2 \left(1-\frac{pe^{V}}{q+pe^{V}}\right)^{-1-R}$$

$$= E\left(q+pe^{V}\right)^2 Y^2 \cdot q^{-1-R}$$

The obvious thing to do here is to break the expectation up according to the number of jumps prior to h. We need to deal with 3 terms:

$$E\left[(1+p(e^{Z_h}-1))^{1-R};\ \text{no jumps before } h\right],\quad E\left[(1+p(e^{Z_h}-1))^{1-R};\ \text{one jump before } h\right],$$

$$E\left[(1+p(e^{Z_h}-1))^{1-R};\ \text{at least 2 jumps before } h\right]$$

Let's write N_t for the number of jumps by time t, and take the terms one by one. The first is

$$e^{-\lambda h} E\left[(1+p(e^{\sigma W_h+ah}-1))^{1-R}\right]$$

$$= e^{-\lambda h} E\left[\sum_{k=0}^{N-1} \frac{X^k}{k!}\frac{\Gamma(2-R)}{\Gamma(2-R-k)} + \frac{X^N}{N!}\frac{\Gamma(2-R)}{\Gamma(2-R-N)}(1+\theta X)^{1-R-N}\right]$$

where $\theta\in(0,1)$ is some random variable. As we found in earlier work, $E|X|^k \le C_k\, h^{k/2}$, so we will need to go out to $N=4$ to get a remainder which is $O(h^2)$ – at that value of N, we have $(1+\theta X)^{1-R-N} = (1+\theta X)^{-3-R} \le (1-p)^{-3-R}$. Expanding out, we get (Maple)

$$1+\left\{(1-R)p\left(a+\tfrac12\sigma^2 - Rp\tfrac{\sigma^2}{2}\right)-\lambda\right\}h + O(h^2)$$

As for the second term, we have

$$\lambda h e^{-\lambda h} E\left(1+p(e^{\sigma W_h+ah+V}-1)\right)^{1-R}$$

$$= \lambda h e^{-\lambda h} E\left(1+p(e^V-1)+p(e^{\sigma W_h+ah}-1)e^V\right)^{1-R}$$

$$= \lambda h e^{-\lambda h} E\left[(1+p(e^V-1))^{1-R}\Big\{1+\underbrace{\frac{pe^V(e^{\sigma W_h+ah}-1)}{1+p(e^V-1)}}_{\equiv Y}\Big\}^{1-R}\right]$$

$$= \lambda h e^{-\lambda h} E\left(1+p(e^V-1)\right)^{1-R}\left\{\sum_{k=0}^{N-1}\frac{Y^k}{k!}\frac{\Gamma(2-R)}{\Gamma(2-R-k)} + \frac{Y^N}{N!}\frac{\Gamma(2-R)}{\Gamma(2-R-k)}(1+\theta Y)^{1-R-N}\right\}$$

If we take $N=2$, then the remainder term in the sum will be bounded by const. h, and the term $k=1$ in the sum is $O(h)$, as we see by conditioning on V and evaluating $E(Y|V)$. Thus we <u>do</u> in fact have that the middle term is

$$\lambda h e^{-\lambda h} E\left(1+p(e^V-1)\right)^{1-R} + O(h^2)$$

As for the third and final term, we have $O(h^2)$ immediately if $R>1$, and for $0<R<1$, we have the inequality $(x+y)^{1-R}\le x^{1-R}+y^{1-R}$ for $x,y\ge 0$. The assumed finiteness of $E\exp\{(1-R)V\}$ now shows that the final term is $O(h^2)$. Thus the first-order terms in the two expansions agree, so the loss of efficiency (over a time interval of length h) is $O(h^2)$: therefore

even with a log-Lévy asset, if we divide $[0,T]$ into M equal intervals of length $h\equiv T/M$, the loss of efficiency will still be $O(h)$ over the time horizon to T.

A question of Freddy Delbaen (9/5/2000)

1) Marc Yor asked me whether there is a continuous infinite-variation process H which is not a semimartingale but such that $H_t + B_t$ is a semimartingale in its own filtration (a question due to Delbaen).

Could we decide this by some random Fourier series approach?

Let's take $(\xi_t)_{0\le t\le 1}$ to be a Brownian bridge, which we can extend antisymmetrically to $[-1,1]$ by $\xi_t = -\xi(-t)$, and then we can compute the Fourier coefficients

$$Z_k = \int_{-1}^{1} \xi(t)\, e^{i\pi t k}\, dt = 2i\int_0^1 \xi(t) \sin(\pi t k)\, dt \qquad (k\in\mathbb{Z}),$$

and then $\xi_t = \sum_{k\in\mathbb{Z}} Z_k\, e^{-itk\pi}/2 = \frac{1}{2}\sum_{k\ge1} 2\,\mathrm{Re}\left(Z_k\, e^{-itk\pi}\right) = \sum_{k\ge 1}\left(2\int_0^1 \xi(u)\sin(\pi k u)\,du\right)\sin(k\pi t).$

The Fourier coefficients Z_k are Gaussian, and

$$E(Z_k Z_j) = -4\int_0^1 ds\int_0^1 dt\; \sin(k\pi t)\sin(j\pi s)\,(s\wedge t - st)$$

$$= -4\left\{\int_0^1 ds\int_0^s dt\; t\sin(\pi k t)\sin(\pi j s) + \dots - \int_0^1 s\sin(\pi j s)\,ds\int_0^1 t\sin(\pi k t)\,dt\right\}$$

$$= -4\left\{\frac{(-1)^{j+k}\,j}{k(k^2-j^2)\pi^2} + \frac{(-1)^{j+k}\,k}{j(j^2-k^2)\pi^2} - \frac{(-1)^{j+k}}{\pi^2 jk}\right\}$$

$$= \begin{cases} 0 & \text{if } j\ne k \\ -\dfrac{2}{\pi^2 k^2} & \text{if } j=k \end{cases}$$

Thus we can make a Brownian bridge by the recipe

$$\xi_t = \sum_{k\ge1} \frac{\sqrt{2}}{\pi k} X_k \sin(\pi k t)$$

where the X_k are IID $N(0,1)$. Could we now try to build H as a random Fourier series?

2) If we have $Y_j = d\tilde{P}_j/dP_j$, where P_j is law of $N(0,\sigma_j^2)$, $\tilde{P}_j$ is law of $N(\mu_j,\tilde{\sigma}_j^2)$, we have after some calculations that

$$E\sqrt{Y_j} = \exp\left\{-\frac{\mu_j^2}{2(\sigma_j^2+\tilde{\sigma}_j^2)}\right\}\frac{\sqrt{\sigma_j\tilde{\sigma}_j}}{\sqrt{\frac{1}{2}(\sigma_j^2+\tilde{\sigma}_j^2)}}$$

The two product measures are equivalent iff $\prod E\sqrt{Y_j} > 0$, that is

$$-\infty < \sum_{j\ge1}\left\{\frac{-\mu_j^2}{2(\sigma_j^2+\tilde{\sigma}_j^2)} + \frac{1}{2}\log\frac{\sigma_j\tilde{\sigma}_j}{\frac{1}{2}(\sigma_j^2+\tilde{\sigma}_j^2)}\right\}$$

that is, iff

$$\sum_{j\ge1}\frac{\mu_j^2}{\sigma_j^2+\tilde{\sigma}_j^2} < \infty \quad\text{and}\quad \sum \log\frac{\sigma_j^2+\tilde{\sigma}_j^2}{2\sigma_j\tilde{\sigma}_j} < \infty$$

or again, iff $\sum_j \mu_j^2/\sigma_j^2 < \infty$ and $\sum_j \frac{(\sigma_j - \tilde{\sigma}_j)^2}{\sigma_j \tilde{\sigma}_j} < \infty$

3) How could we pick some equivalent law by choice of μ_j, $\tilde{\sigma}_j^2$? The aim would be to make the resulting process a Brownian bridge + some cts. infinite variation process. Let's actually construct an infinite variation function, so we have $\tilde{\sigma}_j = \sigma_j$ (if possible!). However, this appears not to work. If we took as our function $\sum_{j\geq 1} \mu_j \sin(j\pi t)$ then we would require $\sum \mu_j^2 j^2 < \infty$ for absolute continuity. However, this then means that the FS for the derivative, $\pi \sum j \mu_j \cos(j\pi t)$, is convergent in L^2, so the derivative exists as an L^2 function, therefore we don't have infinite variation. There's really not much room to build infinite variation here...

Some remarks on a search model of Shouyong Shi (9/6/2000)

1) In the paper "Money and prices: a model of search and bargaining" (JET 67 (1995), 467–496), Shi considers a continuum of agents indexed by the circle of circumference 2, each agent being either a producer or a money-holder. Agents meet at random. Each producer has the opportunity to produce good of type x, where x is chosen at random from the circle; an agent desires only goods which are within z of his position measured along the circumference of the circle. Agents are not allowed to consume their own production. When two agents meet, if neither desires the type of the other's production opportunity (which happens with probability $(1-z)^2$) then nothing happens. If i desires j's good but not vice versa, then i may pay j money in return for good produced if i is a money-holder, but if i is a producer as well, then nothing happens. When two producers meet, there is mutual production if there is double coincidence of wants, else nothing happens.

Shi assumes that money comes in indivisible units, and that agents either hold money or a production opportunity, not both. After a trade, either agent who is not holding money immediately gets a new production opportunity. Could we extend his model to allow divisible money?

2) First a few remarks on the interaction that takes place between two agents when they trade. The basic problem of economics is to determine what happens when two agents meet, and a solution approach is given by Shi, in terms of a sequence of bids between the agents. There is also the Nash bargaining approach (not discussed by Shi), where the idea is that if S_i is the surplus to agent i, S_j the surplus to agent j then we choose the allocation to maximise $S_i^{\theta} S_j^{1-\theta}$ for some parameter θ (agent i's bargaining weight). The two approaches are very similar, and also the Nash approach is equivalent to an approach I've long believed in. Here's how.

3) Suppose agents 0 and 1 with utilities U_0, U_1 and initial bundles x_0, x_1 meet to trade, and suppose that we say that the increases in utility must be the same for both agents (or more generally

$$U_0(x_0-z) - U_0(x_0) = \kappa\left(U_1(x_1+z) - U_1(x_1) \right)$$

where z is the transfer from 0 to 1, and $\kappa>0$ is constant). Then the optimisation problem in Lagrangian form is equivalent to

$$\max_z \; U_0(x_0-z) - \lambda\left\{ U_0(x_0-z) - \kappa U_1(x_1+z) \right\}$$

which leads to

$$(1-\lambda)\ \nabla U_0(x_0-z) = \lambda K\ \nabla U_1(x_1+z)$$

On the other hand, the Nash approach would lead us to

$$\max \quad \theta \log\{U_0(x_0-z) - U_0(x_0)\} + (1-\theta)\log\{U_1(x_1+z) - U_1(x_1)\}$$

giving

$$\frac{\theta\ \nabla U_0(x-z)}{U_0(x_0-z) - U_0(x_0)} = \frac{(1-\theta)\ \nabla U_1(x_1+z)}{U_1(x_1+z) - U_1(x_1)}$$

So the two families of solutions, indexed by bargaining power θ or by relative importance K, result in the same allocations. Maybe the Nash story is better, since the answer isn't affected if one of the agents doubles his utility! The Nash criterion (Econometrica $\underset{\sim}{18}$, 155-162, 1950) arises from an axiomatic approach if you want to think of it that way*.

4) Let's see how we might extend the model of Shi to include divisible money. So if agents i, j meet, holding x_i, x_j in money, and each with a production opportunity, then there are two possibilities where something happens:

(i) each wants the other's good, so they produce

$$q_i = q(x_i, x_j), \quad q_j = q(x_j, x_i)$$

(Symmetrically, because all agents are supposed identical)

and i gives j monetary amount

$$g(x_i, x_j) = -g(x_j, x_i)$$

(ii) i wants j's good, but not the other way round. Then

i pays $\tilde{g}(x_i, x_j)$ to j in return for $\tilde{q}(x_j, x_i)$ of good

The exact forms of $g, q, \tilde{g}, \tilde{q}$ will be discussed later. If $V(x)$ is the value to holding cash x, and F is the steady-state dist^n of cash balances held, then

$$\begin{aligned} V(x) = \int_0^\infty \beta e^{-(\beta+r)t} \int F(dy) \Big[& z^2 \{U(q(y,x)) - q(x,y) + V(x - g(x,y))\} \\ & + z(1-z)\ \{U(\tilde{q}(y,x)) + V(x - \tilde{g}(x,y))\} \\ & + z(1-z)\{-\tilde{q}(x,y) + V(x + \tilde{g}(y,x))\} + (1-z)^2\ V(x)\Big]\ dt \end{aligned}$$

This gives us a relation that (V, F) must satisfy:

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* Either approach extends naturally to more than two agents.

$$\boxed{\begin{aligned}\{\beta+r-(1-z)^2\}V(x) = \beta\int\Big[&z^2\{u(q(y,x))-q(x,y)+V(x-g(x,y))\}\\ &+z(1-z)\{u(\tilde q(y,x))+V(x-\tilde g(x,y))\}\\ &+z(1-z)\{-\tilde q(x,y)+V(x+\tilde g(y,x))\}\Big]\,F(dy)\end{aligned}}$$

Furthermore, since F is steady state, we must have for any test function ψ

$$\boxed{\begin{aligned}\int\psi(x)F(dx) = \int F(dx)\int F(dy)\Big[&z^2\psi(x-g(x,y))+z(1-z)\,\psi(x-\tilde g(x,y))\\ &+z(1-z)\,\psi(x+\tilde g(y,x))+(1-z)^2\psi(x)\Big]\end{aligned}}$$

Special case $\psi = V$ gives

$$\boxed{r\int V(x)F(dx) = \beta\iint F(dx)F(dy)\Big[z^2\{u(q(y,x))-q(x,y)\}+z(1-z)\{u(\tilde q(y,x))-\tilde q(x,y)\}\Big]}$$

which is nice, even if we can't find solutions (V,F).

5) When $u(x) = x^{1-R}/(1-R)$, the Nash solution for bargaining between two agents i and j maximizes $\sqrt{S_i}\sqrt{S_j}$ (assuming equal bargaining weight) or $S_i^{\theta}S_j^{1-\theta}$ more generally.

Thus for double coincidence of wants

$$\theta\,\frac{\nabla S_i}{S_i}+(1-\theta)\,\frac{\nabla S_j}{S_j}=0$$

with $S_i = u(q_j)-q_i+V(x_i-g)-V(x_i)$, $S_j = u(q_i)-q_j+V(x_j+g)-V(x_j)$, whence

$$\frac{\theta}{S_i}\begin{pmatrix}1\\-u'(q_j)\\V'(x_i-g)\end{pmatrix} = \frac{1-\theta}{S_j}\begin{pmatrix}u'(q_i)\\-1\\V'(x_j+g)\end{pmatrix}$$

so we have

$$\begin{aligned}1 &= \lambda u'(q_i)\\ u'(q_j) &= \lambda\\ V'(x_i-g) &= \lambda V'(x_j+g)\end{aligned}$$

for some λ, implying

$$\boxed{q_i q_j = 1}$$

and $\boxed{V'(x_i-g)/V'(x_j+g) = u'(q_j)}$.

On the other hand, if i desires j's good but not vice versa, we have

$$S_i = u(\tilde q)+V(x_i-\tilde g)-V(x_i), \quad S_j = -\tilde q+V(x_j+\tilde g)-V(x_j)$$

and then

$$\frac{\theta}{S_i}\begin{pmatrix} U'(\tilde{q}) \\ -V(x_i-\tilde{q}) \end{pmatrix} = \frac{1-\theta}{S_j}\begin{pmatrix} 1 \\ -V'(x_j+\tilde{q}) \end{pmatrix}$$

What Shi does is slightly different, and amounts to maximising $\tilde{S}_i \tilde{S}_j$, where $\tilde{S}_i = V(x_i) + S_i$ is the post-exchange utility. For the double coincidence of wants, it's exactly the same as what is done here.

A public/private investment model of Arrow & Kurz (22/6/2000)

1) In Ch III of the book "Public investment, the rate of return and optimal fiscal policy", Arrow + Kurz take a model which after various transformations can be expressed as follows. The dynamics of capital are

(1) $$\dot{k}_t = \varphi(k_p(t), k_g(t)) - c(t) \qquad , \qquad k_t \equiv k_g(t) + k_p(t)$$

and the objective is to max

(2) $$\int_0^\infty e^{-\lambda t}\, U(c_t, k_g(t))\, dt .$$

If we use the value-function approach, the value function v must satisfy

(3) $$\sup_{c, k_g} \left[-\lambda v(k) + v'(k)\{\varphi(k-k_g, k_g) - c\} + U(c, k_g) \right] = 0$$

so we obtain the system of equations

$$\boxed{\begin{aligned}
&(4a) \quad -\lambda v(k) + v'(k)\{\varphi(k-k_g, k_g) - c\} + U(c, k_g) = 0 \\
&(4b) \quad U_c(c, k_g) = v'(k) \\
&(4c) \quad U_g(c, k_g) + v'(k)\{\varphi_g - \varphi_p\}(k-k_g, k_g) = 0
\end{aligned}}$$

If we know the value of v at some point, then we can solve the three equations (numerically) for the unknowns $c, k_g, v'(k)$, and thus can proceed to solve the system, once we know the value at one point.

2) How do we find the value at one point? Along an optimally-controlled trajectory k_t, the expressions (4) remain always zero, so differentiating with respect to t in (4a) and using (4b), (4c), we find that

$$0 = \{-\lambda v'(k) + v'(k)\varphi_p(k-k_g, k_g) + v''(k)(\varphi(k-k_g, k_g) - c)\}\dot{k}_t$$

so

$$0 = -\lambda v'(k) + v'(k)\varphi_p(k-k_g, k_g) + v''(k)(\varphi(k-k_g, k_g) - c).$$

If we look for a constant solution, we must have

$$(5)\quad \boxed{\begin{aligned}
&(a) \quad \varphi(k-k_g, k_g) = c \\
&(b) \quad \varphi_p(k-k_g, k_g) = \lambda \\
&(c) \quad U_c(c, k_g) = v'(k) \\
&(d) \quad U_g(c, k_g) + v'(k)\{\varphi_g - \varphi_p\}(k-k_g, k_g) = 0
\end{aligned}}$$

Simplest case has $\varphi(k_p, k_g) = A\, k_p^a\, k_g^b - \gamma(k_p + k_g)$, $U(c, k_g) = (c^\alpha k_g^{1-\alpha})^{1-R}/(1-R)$. Now a neat way to proceed is to impose values on k_g^*, k_p^*, c^*, and to fix a, b, α, R, then use 5(e), 5(d) to pick A, 5(a) to pick γ, 5(b) to set λ, and finally deduce $v'(k^*)$ from 5(c).

Clearly, $v(k^*) = U(c^*, k_g^*)/\lambda$

Writing $p \equiv v'(k)$, we shall find that the stationary solution (k_g^*, k_p^*, c^*, p^*) is determined by these four equations.

3) Can we find derivatives wrto k at k^*? We have

$$\begin{cases} -\lambda v + p(\varphi - c) + u \equiv 0 \\ u_c \equiv p \\ u_g + p(\varphi_g - \varphi_p) \equiv 0 \end{cases}$$

at the good arguments $k_g(k)$, $k_p(k)$, $c(k)$, so if we differentiate the first we get

(6) $$\boxed{p(\varphi_p - \lambda) + p'(\varphi - c) \equiv 0}$$

At the stationary point, this tells us nothing extra, but differentiating again gives

(7) $$p\left(\varphi_{pp}(1-k_g') + \varphi_{pg} k_g'\right) + p'\left(\varphi_p(1-k_g') + \varphi_g k_g' - c'\right) = 0$$

and differentiating (4b), (4c) gives

$$\begin{cases} p' = u_{cc} c' + u_{cg} k_g' & \text{(8a)} \\ u_{gc} c' + u_{gg} k_g' - \dfrac{p'}{p} u_g + p\left\{\varphi_{gp}(1-k_g') + \varphi_{gg} k_g' - \varphi_{pp}(1-k_g') - \varphi_{pg} k_g'\right\} = 0 & \text{(8b)} \end{cases}$$

These can be solved for (p', k_g', c') at k^*

4) Since the root-finder may propose negative values, we really need to work with log values, $K = \log k$, $C = \log c$, $K_p = \log k_p$, $K_g = \log k_g$, and think of everything as functions of K, so $v(k) \equiv V(\log k) \equiv V(K)$, and $v'(k) = e^{-K} V'(K)$. The system we then have to solve is

$$\left.\begin{aligned} -\lambda V + p(\varphi - e^C) + u &= 0 \\ u_c &= p \\ u_g + p(\varphi_g - \varphi_p) &= 0 \\ k_p + k_g &= k \end{aligned}\right\} \qquad p \equiv v'(k) = e^{-K} V'(K)$$

which is a non-linear 1st order ODE in the 4 variables (V, C, K_g, K_p). We know how to find the values at $K^* = \log k^*$, and the derivatives c', k_p', k_g' at k^*, so we have (for example)

$$C'(K^*) = \frac{d}{dK} \log c(e^K)\Big|_{K=K^*} = \frac{c'(k^*)}{c(k^*)} \cdot k^*$$

5) Now let's see about a stochastic version of this, where $dk = \sigma(k)dW + (\varphi(k_p, k_g) - c)\,dt$, with the same objective. It appears that we still have HJB

$$\sup_{c, k_g} \left[-\lambda V + \tfrac{1}{2}\sigma^2 V'' + (\varphi - c) V' + U(c, k_g) \right] = 0$$

but in contrast to the deterministic problem, it's hard to see any fixed values.

An alternative approach would be to discretise the diffusion, make the problem a discrete one, and then carry out policy improvement. To deal with the discretisation in a general context, if we have a diffusion $dX_t = \sigma(X_t)dW_t + \mu(X_t)dt$ which we approximate as a Markov chain on the points $\{x_j\}$, let's just assume locally that σ, μ are constant, so the exit time from $[x_{j-1}, x_{j+1}]$ is as for a BM with constant drift. We have easily

$$P(X_\tau = x_{j+1} \mid X_0 = x_j) = \frac{s(x_j) - s(x_{j-1})}{s(x_{j+1}) - s(x_{j-1})} \qquad s(x) \equiv \exp\left[-2\mu(x_j)\,x / \sigma(x_j)^2\right]$$

and $E[\tau \mid X_0 = x_j] = \left\{ (s(x_j) - s(x_{j-1}))(x_{j+1} - x_j) + (s(x_{j+1}) - s(x_j))(x_{j-1} - x_j) \right\} / \mu(x_j)\,(s(x_{j+1}) - s(x_{j-1}))$

We then define the Q-matrix locally by $q_j \equiv E(\tau \mid X_0 = x_j)^{-1}$ and $q_{j,j+1}/q_j = P(X_\tau = x_{j+1} \mid X_0 = x_j)$

Then the value function V satisfies

$$0 = U(c) + (Q - \lambda)V$$

with boundary conditions at x_0, x_N, and we can solve these linear equations very easily and do policy improvement. The boundary conditions could be either to make $V_0 = V_N = 0$, or else $V_0 = \lambda^{-1} U(\varphi(k_p, k_g), k_g)$ where k_p, k_g are chosen to max, subj to $k_p + k_g = k(x_0)$, and similarly for V_N

Or again we could just do a straightforward DP argument.

Lagrangian approach to Cuoco-Liu (30/6/2000)

1) In the preprint 'A martingale characterization of consumption choices and hedging costs with margin requirements', Cuoco & Liu consider the situation where the wealth process w_t evolves as

$$\boxed{(1) \quad dw_t = w_t\left[r_t\,dt + \pi_t\cdot(\sigma_t dW_t + (\mu_t - r_t)dt) + g(t,\pi_t)\,dt\right] - c_t\,dt - dC_t}$$

where C is some non-decreasing process, $C_0 = 0$, and the initial wealth $w_0 = x$ is given. The problem is to minimise x subject to the requirement that $w_T \geq Y$ for some given non-negative Y, with given non-negative consumption stream $(c_t)_{t\geq 0}$.

2) Introduce a Lagrange multiplier process λ_t, $d\lambda_t = \alpha_t\cdot dW_t + \beta_t\,dt$. The function g is assumed concave in π for each t; introduce the notation

$$\tilde{g}(t,\nu) = \sup_{\pi}\{ g(t,\pi) - \pi\cdot\nu \}$$

a convex function. Now if we take the dynamics (1) and integrate λ_t on each side, the LHS would give

$$\int_0^T \lambda_t\,dw_t = \lambda_T w_T - \lambda_0 x - \int_0^T w_s\,d\lambda_s - \int_0^T w_s\,\pi_s^T\sigma_s\alpha_s\,ds$$

and the RHS gives

$$\int_0^T \lambda_t w_t\left(\pi_t\cdot\sigma_t dW_t + \{r_t + g(t,\pi_t) + \pi_t\cdot(\mu_t - r_t)\}dt\right) - \int_0^T \lambda_t c_t\,dt - \int_0^T \lambda_t\,dC_t$$

and these two things have to agree whatever the process λ. We can also introduce the Lagrangian RV Λ to absorb the constraint $w_T \geq Y$, and reformulate the problem as the Lagrangian problem

$$\min_{x,w,\pi,C,Z}\; x + E\Big[\lambda_T w_T - \lambda_0 x - \int_0^T w_s\,d\lambda_s - \int_0^T w_s\,\pi_s^T\sigma_s\alpha_s\,ds$$

$$- \int_0^T \lambda_t w_t\{r_t + g(t,\pi_t) + \pi_t\cdot(\mu_t - r_t)\}dt + \int_0^T \lambda_t c_t\,dt + \int_0^T \lambda_t\,dC_t$$

$$+ \Lambda\,(Y + Z - w_T)\Big]$$

$$= \min_{x,w,\pi,C,Z}\; (1-\lambda_0)x + E\Big[-\int_0^T w_s\left(\beta_s\,ds + \pi_s^T\sigma_s\alpha_s\,ds + \lambda_s(r_s + g(s,\pi_s)) + \pi_s(\mu_s - r_s)\right)ds$$

$$+ \int_0^T \lambda_t c_t\,dt + (\lambda_T - \Lambda)w_T + \int_0^T \lambda_t\,dC_t + \Lambda(Y+Z)\Big]$$

For this minimisation to be well posed, we need various conditions on the multipliers:

$$\boxed{\begin{array}{l} \lambda_0 \leq 1, \quad \lambda_T \geq \Lambda \geq 0, \quad \Lambda Z = 0, \quad \lambda_t \geq 0, \\ \sup_{\pi}\left[\beta_t + \pi^T(\sigma_t\alpha_t + \lambda_t(\mu_t - r_t)) + \lambda_t g(t,\pi) + \lambda_t r_t\right] \leq 0 \end{array}}$$

Other questions : Constraints on cash value of portfolio (constraints on hedge!)

Transactions costs

real options

Proof of super-hedging

Investment/consumption under transactions costs.

Recursive utility?

Zariphopoulou things?

3) Introducing now the notation

$$\nu_t \equiv r_t - \mu_t - \lambda_t^{-1}\sigma_t\alpha_t$$

we shall have

$$\beta_t + \lambda_t r_t + \lambda_t \tilde{g}(t,\nu_t) \le 0$$

so that in terms of ν_t we find that

$$\boxed{\begin{aligned}\alpha_t &= \lambda_t \sigma_t^{-1}\{r_t - \mu_t - \nu_t\} \\ \beta_t &\le -\lambda_t(r_t + \tilde{g}(t,\nu_t))\end{aligned}}$$

Peforming the minimisation, we get that the minimised Lagrangian is

$$E\left[\int_0^T \lambda_t c_t\,dt + \Lambda Y\right]$$

where we have $\lambda_t \ge 0$, $\lambda_0 \le 1$, $\Lambda \le \lambda_T$, $d\lambda_t = \lambda_t\sigma_t^{-1}(r_t - \mu_t - \nu_t)\cdot dW_t + \beta_t\,dt$

Taking the sup over dual-feasible multiplier processes, we have the lower bound for x:

$$\sup_\nu E\left[\int_0^T \xi_\nu(t)\,c_t\,dt + \xi_\nu(T)\,Y\right]$$

where $\quad d\xi_\nu(t) = \xi_\nu(t)\left\{ \sigma_t^{-1}(r_t - \mu_t - \nu_t)\cdot dW_t - (r_t + \tilde{g}(t,\nu_t))\,dt\right\}, \quad \xi_\nu(0) = 1$

This is exactly the expression of Cuoco-Liu... but now we have to close the gap.

4) Let's now look at the problem considered by Shreve + Wystup which was (as I recall) very like the above except that the constraint was on the <u>amount</u> invested rather than proportion:

$$dw_t = r_t w_t\,dt + \theta_t\{\sigma_t dW_t + (\mu_t - r_t)\,dt\} \qquad \text{with } \theta_t \in K \quad \forall t.$$

Once again, with Lagrange multiplier process $d\lambda_t = \alpha_t dW_t + \beta_t\,dt$, if we take the dynamics of w and integrate both sides against λ we get the Lagrangian form

$$\min_{w,x,\theta}\; x + E\left[\lambda_T w_T - \lambda_0 x - \int_0^T w_s\beta_s\,ds - \int_0^T \theta_s^T\sigma_s\alpha_s\,ds - \int_0^T \lambda_s\{r_s w_s + \theta_s^T(\mu_s - r_s)\}\,ds - \Lambda(w_T - Y - Z)\right]$$

For a finite minimum we'll need $\lambda_0 = 1$, $\lambda_T \ge \Lambda$, and if $\tilde{f}_K(y) = \sup_{x\in K}(-x\cdot y)$, we can carry out the minimisation, provided also that $\beta_t + \lambda_t r_t \le 0$ for all t. We get

$$E\left[\Lambda Y - \int_0^T \lambda_t \tilde{f}_K(r_t - \mu_t - \lambda_t^{-1}\sigma_t\alpha_t)\,dt\right]$$

If we now take the sup over the multipliers, we'll achieve

$$\boxed{\sup_{\nu} E\left[Z_T^{\nu} Y - \int_0^T Z_s^{\nu} \tilde{f}_K(\nu_s)\, ds \right]}$$

where we're writing $\nu_t = r_t - \mu_t - \lambda_t^{-1}\sigma_t \alpha_t$, and Z^{ν} for λ, so that

$$dZ_t^{\nu} = Z_t^{\nu}\left[\sigma_t^{-1}(r_t - \mu_t - \nu_t)\, dW_t - r_t\, dt \right]$$

(by pushing β_t up to its bound $-\lambda_t r_t$).

5) We can now understand the 'face-lifting' which happens when $Y = g(X_T)$, where $X_t = \log S_t = \sigma W_t + \tilde{\mu} t$, $\tilde{\mu} = \mu - \frac{1}{2}\sigma^2$, in the case where everything is constant, and we have just one risky asset. We solve the dual problem by setting

$$V(t,x) \equiv \sup_{\nu} E\left[(Z_t^{\nu})^{-1} \left\{ Z_T^{\nu} Y - \int_t^T Z_s^{\nu} \tilde{f}_K(\nu_s)\, ds \right\} \Big| X_t = x \right]$$

as the value of the problem. Then we shall have as usual

$$Z_t^{\nu} V(t, X_t) \quad \underline{\text{is a supermartingale, and a martingale under optimal } \nu}$$

so taking Itô expansion gives

$$\sup_{\nu}\left[-rV + \dot{V} + \tfrac{1}{2}\sigma^2 V'' + \tilde{\mu} V' + (r - \mu - \nu)V' - \tilde{f}_K(\nu) \right] = 0$$

which comes down to

$$-rV + \dot{V} + \tfrac{1}{2}\sigma^2 V'' + (r - \tfrac{1}{2}\sigma^2)V' - f_K(V') = 0$$

where $f_K(x) = 0$ if $\underline{a} \le x \le \bar{a}$; $= -\infty$ else, where $[\underline{a}, \bar{a}] = K$ is the permitted interval for θ. So we see that V' <u>must</u> lie always in K, and V actually solves the BS PDE, with terminal condition

$$V(T, x) = \hat{g}(x) \equiv \inf\left\{ f(x) \,;\; f(y) \ge g(y)\ \forall y,\ \ \underline{a} \le f'(y) \le \bar{a}\ \ \forall y \right\}$$

The effect of the limit probability measure is a bit bizarre; we take $\nu_t = 0$ for $t < T$, and then ν_T is a δ-function, which kicks the process X into a region where $g = \hat{g}$; the loss you get from $-\tilde{f}_K(\nu)$ exactly balances the gain in $\hat{g}(X_\cdot)$ at T.

This behaviour serves as a warning; the limit of the probabilities which we use to approximate the sup needs not be a prob^y on $C[0,T]$!

This duality characterisation is very neat, but hard to make good use of; we are only going to get a tight bound if among the martingales we use for the minimisation is one which is close to the M of the decomposition of the Snell envelope process Y...

Monte Carlo in Monte Carlo (2/7/2000)

This is just a little observation on the Davis-Karatzas paper, whic[h] compute the price of an American option by simulation. The generic Ame[rican ...] calculate

$$\sup_\tau E Z_\tau$$

where τ is a stopping time with values in $[0,T]$. The Snell envelope process

$$Y_t \equiv \sup_{t\le\tau\le T} E[Z_\tau \mid \mathcal{F}_t] = Y_0 + M_t - A_t$$

is a supermartingale with a Meyer decomposition, and $Y_t \ge Z_t \quad \forall t$. In what follows, σ will denote a generic random time with values in $[0,T]$ — not necessarily a stopping time. We have

$$\sup_\sigma E[Z_\sigma - M_\sigma] \le \sup_\sigma E[Y_\sigma - M_\sigma] = \sup_\sigma E[Y_0 - A_\sigma] \le E Y_0 = Y_0$$

which is the value we are seeking. On the other hand, if τ^* is the optimal stopping time, and X is any martingale null at 0, we have

$$\sup_\sigma E[Z_\sigma - X_\sigma] \ge E[Z_{\tau^*} - X_{\tau^*}] = E[Z_{\tau^*}] = E Y_0 = Y_0$$

and so

$$\boxed{Y_0 = \min_{X\in\mathcal{M}_0} \sup_\sigma E[Z_\sigma - X_\sigma] = \min_{X\in\mathcal{M}_0} E\left[\sup_{s\le T}(Z_s - X_s)\right]}$$

We therefore have a dual characterisation of the value! This gives us a way to generate upper bounds.

To try to estimate Y_0, I'd make up a couple of sets $A_1 \subset A_2$ of randomly-chosen martingales (as stochastic integrals wrt to basic mgs of the problem, with piecewise constant integrands of random magnitude), then generate the sample paths repeatedly, and compute the sample mean. Doing this for both sets A_1, A_2 could allow Richardson extrapolation; we could also make use of control variates for problems where we knew the answer. If we had some problems related to this one, where we knew the correct answer, we could use these to guide our choice of martingales.

This is quite similar to the Pontryagin principle, or the Hamiltonian method of Bismut...
It seems a lot clearer like this though

Further examples of the Lagrangian semi-martingale principle (5/7/2000)

1) Let's take a transactions costs problem from Cvitanić & Karatzas, Math. Fin. $\underline{6}$, 133–165, and try out the general Lagrangian semimartingale principle on it. Thus we have cash and share accounts

$$\begin{cases} dX_t = r X_t\,dt + (1-\varepsilon)\,dM_t - (1+\delta)\,dL_t \\ dY_t = Y_t(\sigma dW_t + \mu dt) - dM_t + dL_t , \qquad Y_0 = y \text{ given} \end{cases}$$

and the aim is to super-replicate a contingent claim (C_0, C_1) at time T for minimal initial cash. The vector contingent claim is understood to be C_0 in the cash account, C_1 in the share account, so we have to have equivalently

$$X_T + (1-\varepsilon)Y_T \geq C_0 + (1-\varepsilon)C_1 \qquad \underline{\text{and}} \qquad X_T + (1+\delta)Y_T \geq C_0 + (1+\delta)C_1$$

as explained in C+K. We'll also have to have that at all times (X_t, Y_t) is in the solvency region:

$$X_t + (1-\varepsilon)Y_t \geq 0, \qquad X_t + (1+\delta)Y_t \geq 0$$

Introduce the Lagrangian semimartingales

$$d\xi_t = \alpha_t\,dW_t + \beta_t\,dt, \qquad d\eta_t = a_t\,dW_t + b_t\,dt$$

and then we get the Lagrangian form of the problem:

$$\min_{x, X, Y} x + E\Big[\xi_T X_T - \xi_0 x - \int_0^T X_s\beta_s\,ds - \int_0^T \xi_s\big(r_s X_s\,ds + (1-\varepsilon)dM_s - (1+\delta)dL_s\big)$$
$$+ \eta_T Y_T - \eta_0 y - \int_0^T Y_s(b_s + a_s\sigma_s)\,ds - \int_0^T \eta_s\big(\mu_s Y_s\,ds - dM_s + dL_s\big)$$
$$+ \Lambda_0\big(C_0 + (1-\varepsilon)C_1 - X_T - (1-\varepsilon)Y_T\big) + \Lambda_1\big(C_0 + (1+\delta)C_1 - X_T - (1+\delta)Y_T\big)\Big]$$

and various necessary conditions on the multipliers can now be read off:

$$\xi_0 = 1$$

$$(-\beta_s - r_s\xi_s)X_s + (-b_s - a_s\sigma_s - \eta_s\mu_s)Y_s \geq 0 \qquad (ds \text{ terms})$$

$$\eta_s - (1-\varepsilon)\xi_s \geq 0 \qquad (dM_s \text{ terms})$$

$$(1+\delta)\xi_s - \eta_s \geq 0 \qquad (dL_s \text{ terms})$$

$$(\xi_T - \Lambda_0 - \Lambda_1)X_T + (\eta_T - (1-\varepsilon)\Lambda_0 - (1+\delta)\Lambda_1)Y_T \geq 0 \qquad (\text{time-}T \text{ terms})$$

so we'll need the multiplier processes to satisfy

$$\xi_0 = 1, \quad \beta_s \leq -r_s\xi_s, \quad b_s \leq -\eta_s\mu_s - a_s\sigma_s,$$

$$(1-\varepsilon) \leq \frac{\eta_s}{\xi_s} \leq (1+\delta), \qquad \xi_s, \eta_s \geq 0$$

$$\xi_T \geq \Lambda_0 + \Lambda_1, \quad \eta_T \geq \Lambda_0(1-\varepsilon) + \Lambda_1(1+\delta)$$

And minimising the Lagrangian over x, X and Y when these conditions hold give us a minimised

Value for the Lagrangian

$$E\left[-\eta_0 y + (\Lambda_0+\Lambda_1) C_0 + ((1-\varepsilon)\Lambda_0 + (1+\delta)\Lambda_1) C_1 \right]$$

Now we'll sup this thing, part of which leads us to take $\Lambda_0+\Lambda_1 = \xi_T$, $(1-\varepsilon)\Lambda_0 + (1+\delta)\Lambda_1 = \eta_T$ and then to push β, b as high as they can go:

$$\beta_s = -r_s \xi_s \quad , \quad b_s = -\mu_s \eta_s - a_s \sigma_s$$

If we now define $Z^0_t \equiv e^{rt}\xi_t$, $Z^1_t \equiv S_t \eta_t$, a little routine Itô calculus gives us that Z^0, Z^1 are local martingales, so if $\tilde{S}_t \equiv e^{-rt} S_t$, the condition on η_t/ξ_t becomes

$$1-\varepsilon \leq \frac{Z^1_t}{Z^0_t \tilde{S}_t} \leq 1+\delta$$

and we have $Z^0_0 = 0$, $Z^1_0 = S_0\eta_0 \equiv p\eta_0$ in CK notation. The dual problem is then to find

$$\sup \; E\left[Z^0_T e^{-rT} C_0 + Z^1_T C_1/S_T - \frac{Z^1_0}{p} y \right],$$

just as in their Thm 4.1.

2) With the same dynamics as above, suppose we have consumption out of the cash account and the goal is to

$$\max \; E\left[\int_0^T U(s, c_s)\, ds + u(X_T, Y_T) \right]$$

for given initial $X_0 = x$, $Y_0 = y$. Using the Lagrangian semimartingales ξ, η as before, we shall obtain the problem

$$\max_{X,Y,c} E\Big[\int_0^T U(s,c_s)ds + u(X_T,Y_T) - \Big\{ \xi_T X_T - \xi_0 x - \int_0^T X_s \beta_s ds - \int_0^T \xi_s (r_s X_s ds + (1-\varepsilon)\,dM_s - (1+\delta)\,dL_s - c_s ds) \Big\}$$
$$- \Big\{ \eta_T Y_T - \eta_0 y - \int_0^T Y_s (b_s + \sigma_s a_s)\, ds - \int_0^T \eta_s (\mu_s Y_s ds - dM_s + dL_s) \Big\} \Big]$$

so we derive certain conditions on the Lagrange multipliers:

$$(1-\varepsilon)\xi_s - \eta_s \leq 0, \quad \eta_s - (1+\delta)\xi_s \leq 0, \quad U'(s, c_s) = \xi_s, \quad u_x(X_T, Y_T) = \xi_T, \; u_y(X_T,Y_T) = \eta_T$$

$$X_s(\beta_s + r_s \xi_s) + Y_s(b_s + \sigma_s a_s + \mu_s \eta_s) \leq 0$$

Maximised Lagrangian is therefore

$$E\left[\int_0^T \tilde{U}(s, \xi_s)\, ds + \tilde{u}(\xi_T, \eta_T) + \xi_0 x + \eta_0 y \right]$$

which can be expressed in terms of Z^0, Z^1 as for the first example.

3) Here's a real option example, expressed in slightly more concrete terms than really necessary. There's a 2-d BM W, and we can trade on the first component, so we can make gains-from-trade processes

$$dX_t = r X_t dt + \theta_t \{ \sigma e_1^T dW_t + (\mu - r) dt \} , \quad X_0 = x .$$

We are now going to sell a contingent claim Y measurable on both BMs up to time T, and we aim to $\max E\, U(X_T - Y)$. With the Lagrangian semimartingale $d\lambda_t = \alpha_t dW_t + \beta_t dt$, we get the Lagrangian form

$$\max_{\theta, X} E\left[U(X_T - Y) - \lambda_T X_T + \lambda_0 x + \int_0^T X_s \beta_s ds + \int_0^T \alpha_s^1 \sigma \theta_s ds + \int_0^T \lambda_s (r X_s + \theta_s (\mu - r)) ds \right]$$

Therefore
$$\left. \begin{aligned} U'(X_T - Y) &= \lambda_T \\ \beta_s + \lambda_s r &= 0 \\ \sigma \alpha_s^1 + \lambda_s (\mu - r) &= 0 \end{aligned} \right\}$$

This tells us that $d\lambda_s = \lambda_s \{ -\frac{(\mu - r)}{\sigma} dW_s^1 + \psi_s dW_s^2 - r ds \}$, so λ is a discounted change of measure martingale which fixes the drift of the first component to be r (the minimal martingale measure), and the payoff is

$$E\left[\tilde{U}(\lambda_T) + \lambda_T Y + \lambda_0 x \right]$$

which we now attempt to minimize over all possible λ_T.

A simple search model for money (8/8/2000)

(i) An individual agent experiences random needs, coming as a Poisson process of rate μ. The needs are IID with distⁿ F. Once a need arrives, the agent is immediately matched to another agent who can produce and they reach a bargaining solution. If the need was z and the agent with the need, agent 0, receives q from the producing partner, agent 1, then the loss to agent 0 will be $L(z-q)$, where L is increasing, non-negative and convex. Assume all agents identical, so that it is meaningful to talk of the value function $V(x)$ for cash x.

In the exchange, if agent 0 gives

q_0 of good, ξ of cash for q_1 of good

then his gain will be

$$G_0 = V(x-\xi) - V(x) - \varphi(q_0) + L(z) - L(z-q_1)$$

and agent 1's gain will be

$$G_1 = V(y+\xi) - V(y) + U(q_0) - \varphi(q_1)$$

where increasing convex φ is the cost of producing, increasing concave U is the utility from consumption.

The standard Nash bargaining solution would be to pick q_0, ξ, q_1 to maximise

$$(1-\theta)\log G_0 + \theta \log G_1$$

resulting in FOCs:

$$\frac{1-\theta}{G_0}\begin{bmatrix} \varphi'(q_0) \\ \\ L'(z-q_1) \\ \\ V'(x-\xi) \end{bmatrix} = \frac{\theta}{G_1}\begin{bmatrix} U'(q_0) \\ \\ \varphi'(q_1) \\ \\ V'(y+\xi) \end{bmatrix}$$

Since we don't know V, these won't be easy to solve.

(ii) To make progress, let's do the following. Assume that x_0 will be so small compared to x, y that we can assume $V'(x-\xi) = V'(x)$, $V'(y+\xi) = V'(y)$. Alternatively, think of each agent as an infinitesimal representative of a household, so that the change in the household's value is actually the marginal change. This latter interpretation is very much in the spirit of a lot of the search model literature. In the hope of finding something reasonably explicit, let's now assume

$$\boxed{L(z) = \tfrac{1}{2}\alpha (z^+)^2, \qquad \varphi(z) = \tfrac{1}{2}\gamma z^2}$$

so that if we set $\lambda \equiv V'(x)/V'(y)$ we find the best choice for q_1 will be

$$\boxed{q_1 = \frac{\alpha}{\lambda\gamma + \alpha} z}$$

and the best choice of q_0 will solve

$$\boxed{\gamma q_0 = \lambda u'(q_0)}$$

Keeping this general for the moment, let's write $\psi(\lambda/\gamma)$ for the unique solution q_0 to this equation. We therefore have to have

$$\frac{\theta\lambda}{q_0} = \frac{1-\theta}{q_1}$$

or more expanded

$$\theta\lambda\{\xi V'(y) + u(q_0) - \varphi(q_1)\} = (1-\theta)\{-\xi V'(x) - \varphi(q_0) + L(z) - L(z-q_1)\}$$

so

$$\boxed{\xi V'(x) = (1-\theta)\left\{\frac{\alpha z^2}{2}\left(1 - \left(\frac{\lambda\gamma}{\alpha+\lambda\gamma}\right)^2\right) - \tfrac{1}{2}\gamma q_0^2\right\} + \lambda\theta\left\{\tfrac{1}{2}\gamma z^2\left(\frac{\alpha}{\alpha+\lambda\gamma}\right)^2 - u(q_0)\right\}}$$

Thus what we find is that

$$\boxed{\xi = \frac{1}{V'(x)}\left[z^2 f_1(\lambda) + f_2(\lambda)\right], \quad \lambda \equiv V'(x)/V'(y)}$$

(iii) Now we've understood this, let's suppose we've got M households, with holdings of cash $x_j(t)$ at time t in household j. If we now write

$$g(\lambda) \equiv E z^2 \cdot f_1(\lambda) + f_2(\lambda)$$

we can specify the dynamics of $x(t)$:

$$\boxed{\frac{dx_j}{dt} = -\sum_{i\neq j} \frac{\mu}{V'(x_j)}\, g\!\left(\frac{V'(x_j)}{V'(x_i)}\right)\frac{1}{M} + \sum_{i\neq j} \frac{\mu}{V'(x_i)}\, g\!\left(\frac{V'(x_i)}{V'(x_j)}\right)\cdot\frac{1}{M}}$$

This assumes that agents may go to others in their own household in order for their needs to be satisfied. However, this is a little too simple: the value to any particular household will in general depend on the holdings of all the other households! If $V^i(x)$ denotes the value to household i of a configuration x, we must therefore get instead

$$\boxed{\frac{dx_j}{dt} = \frac{\mu}{M} \sum_{i \neq j} \left\{ -\frac{1}{v_{ji}}\, g\left(\frac{v_{ji}}{v_{ij}}\right) + \frac{1}{v_{ij}}\, g\left(\frac{v_{ij}}{v_{ji}}\right) \right\}}$$

where

$$v_{ji} \equiv \frac{\partial V^j}{\partial x_j} - \frac{\partial V^j}{\partial x_i}$$

is the rate at which household j's value changes as cash is moved from i to j.

(iv) How about the equations for the evolution of V^i? If we imagine that households are made up of N individuals each of mass $1/N$, then if household j is chosen to suffer a need, k to meet it, then amount of money

$$\frac{1}{v_{jk}}\, g\left(v_{jk}/v_{kj}\right) \equiv c_{jk}$$

is transferred from j to k, also agent j produces $p_0(v_{jk}/v_{kj})$ for agent k, and agent k produces $p_1(v_{jk}/v_{kj})$ for agent 0 $[p_0(\lambda) \equiv \psi(\lambda/\gamma)$, $p_1(\lambda) = \alpha E z/(\alpha + \gamma\lambda)]$, so we get the dynamics

$$V^i(x) = \frac{NM\mu}{\rho + NM\mu} \sum_{j=1}^{M} \sum_{k=1}^{M} M^{-2} \left\{ V^i\left(x + \frac{c_{jk}}{N} e_k - \frac{c_{jk}}{N} e_j\right) - \frac{\delta_{ji}}{N}\left(\varphi(p_0) + h(\lambda)\right) + \frac{\delta_{ki}}{N}\left(U(p_0) - \varphi(p_1)\right) \right\}$$

where $h(\lambda) = \frac{1}{2}\alpha E z^2 \left(1 - \left(\frac{\lambda\gamma}{\alpha + \lambda\gamma}\right)^2\right)$ is the expected loss due to unmet need, with $\lambda = v_{jk}/v_{kj}$.

Hence after some calculations

$$\boxed{\begin{aligned}\rho V^i = \frac{\mu}{M} \sum_j \sum_k \Big\{ & c_{jk}\left(v_{ij} - v_{ik}\right) - \delta_{ij}\left(\varphi\left(p_0\left(\frac{v_{jk}}{v_{kj}}\right)\right) + h\left(\frac{v_{jk}}{v_{kj}}\right)\right) \\ & + \delta_{ki}\left\{ U\left(p_0\left(v_{jk}/v_{kj}\right)\right) - \varphi\left(p_1\left(v_{jk}/v_{kj}\right)\right) \right\} \Big\}\end{aligned}}$$

(v) In an abstract formulation, if $V(x) = (V^i(x))_{i=1,\dots,M}$ and $DV = (D_j V^i(x))_{i=1,\dots,M,\, j=1,\dots,M}$ then we know that the rate at which utility flows to agent i is just a function φ_i of $DV(x)$, and $\dot{x} = G(DV(x))$. If we discount at rate ρ, we shall get

$$V(t,x) = \int_t^\infty e^{-\rho(s-t)} \varphi(DV(s,x_s))\,ds$$

and hence

$$-\rho V(t,x) + DV \cdot \dot{x} + \dot{V} = \varphi(DV)$$

so

$$\boxed{-\rho V + DV \cdot G(DV) + \dot{V} = \varphi(DV)}$$

The boxed equation on the previous page is just the time-invariant form of this evolution. It's obvious that in fact $V(t,x)$ doesn't depend on t, resulting in simplification.

(vi) A rather trivial special case would arise when each household has cash $1/M$ in equilibrium; nothing ever changes, and

$$\rho V(\tfrac{1}{M}) = \mu \left\{ U(p_0(1)) - \varphi(p_1(1)) - \varphi(p_0(1)) - h(1) \right\}$$

which we can solve in terms of the parameters of the problem.

Interesting questions/observations.

1) If X is a Brownian motion with drift c, $S_t = \sup_{u \le t} X_u$, then

$$P(S_t \geq a) = \bar{\Phi}\left(\frac{a-ct}{\sqrt{t}}\right) + e^{2ca}\,\bar{\Phi}\left(\frac{a+ct}{\sqrt{t}}\right)$$

2) DGH asks this; can we take a BMB and find stopping times $(\tau_t)_{t\geq 0}$ such that $B(\tau_t)$ is a BM with unit drift?

3) Uwe Küchler asks: if μ is a prob'y measure on $[0,1]$, does the function $u \mapsto \int_0^1 \cos ux \, \mu(dx)$ have infinitely many zeros? No, take $\mu(dx) = c.(1-x)^2 dx$