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# Example of a convex utility not generated by a concave function (6/9/89)

Suppose that one is given a convex continuous utility on $(\mathbb{R}^+)^n$; that is, $u : (\mathbb{R}^+)^n \to \mathbb{R}$ is continuous, and for each $a$,

$$U_a \equiv \{ x : u(x) \geq a \} \quad \text{is convex}$$

Is it always possible to find some concave $\varphi : (\mathbb{R}^+)^n \to \mathbb{R}^+$ such that $\varphi = g \circ u$, where $g$ is ~~cts~~ strictly increasing? If so, one could ignore the "preference structure" from which utility theory starts, and simply work with a concave utility function.

Here is an example, $n=2$, to show that this is impossible. We shall take

$$U_a = \{ (x,y) : xy \geq a \} \qquad \text{for } a \in [0,1] \cup [2,\infty)$$

and otherwise

$$U_a = \{ (x,y) : y \geq f_a(x) \},$$

where the functions $f_a$ are to be specified.

We shall want

(i) $\frac{1}{x} < f_a(x) < \frac{2}{x}$ for $a \in (1,2)$, $f_a(x) \downarrow \frac{1}{x}$ as $a \downarrow 1$, $f_a(x) \uparrow \frac{2}{x}$, $a \uparrow 2$;

(ii) $f_a(\cdot)$ is decreasing and convex for each $a$;

(iii) $x f_a(x) \to 2$ as $x \to \infty$ for each $a \in (1,2)$.

Why will this fix it? If the concave function $\varphi$ took the value $\alpha \in (1,2)$ on the indifference surface $\partial U_a$, where $a \in (1,2)$, then we must have by concavity that for each $x$

$$\alpha = \varphi((x, f_a(x))) \geq \frac{\frac{2}{x} - f_a(x)}{1/x} \varphi((x, \tfrac{1}{x})) + \frac{f_a(x) - 1/x}{1/x} \varphi((x, \tfrac{2}{x}))$$

$$= \{ 2 - x f_a(x) \} . 1 + \{ x f_a(x) - 1 \} . 2$$

and as $x \to \infty$, this lower bound rises to $2$ $\therefore$ $\alpha = 2$ — this is impossible.

The function

$$f_a(x) = \frac{1}{x} + \frac{1}{c_a + x} \equiv \frac{1}{x} + \frac{a-1}{2(2-a) + x(a-1)} \qquad \left[ c_a \equiv \frac{2(2-a)}{a-1} \right]$$

for $x > 0$, $1 < a < 2$, satisfies the conditions demanded.

MLE of drift and variance of drifting BM under redacted observation? (11/9/89)

Let $X_t \equiv \sigma B_t + ct$ be a drifting Brownian motion. Steve Satchell wants to know the joint law of

$$X_t, \quad S_t \equiv \sup_{u\le t} X_u, \quad I_t \equiv \inf_{u\le t} X_u$$

so that he can from this deduce MLE for $\sigma$, $\mu$.

1) I calculate that for $a \le x, y \le b$,

$$\boxed{\begin{aligned} &P^x\left[X_t \in dy,\ S_t \le b,\ I_t \ge a\right] \\ &= \frac{dy}{b-a} \exp\left\{ \frac{c(y-x)}{\sigma^2} - \frac{c^2 t}{2\sigma^2} \right\} q\left( \frac{\sigma^2 t}{(b-a)^2} ; \frac{x-a}{b-a}, \frac{y-a}{b-a} \right) \end{aligned}},$$

where

$$\begin{aligned} q(t; x, y) &\equiv P^x\left[B_t \in dy,\ H_0 \wedge H_1 > t\right] / dy \\ &= \sum_{n\in\mathbb{Z}} \{ p(t; x+2n, y) - p(t; 2-x+2n, y) \}, \end{aligned}$$

and $\dfrac{\partial q}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 q}{\partial x^2} = \dfrac{1}{2}\dfrac{\partial^2 q}{\partial y^2}$, $q(t;\cdot,\cdot)$ vanishing if $x$ or $y \in \{0,1\}$.

One thing one can say immediately from this: $\boxed{\hat{c} = (y-x)/t}$

What's now needed is to calculate the density, by $\partial^2/\partial a\,\partial b$ of the boxed expression, and then to maximise over $\sigma$. The density is an unwieldy mess, and it doesn't look too good. It's probably better to work from

$$\begin{aligned} P^x\left[X_t \in dy,\ S_t \le b,\ I_t \ge a\right] = e^{(c(y-x) - c^2t/2)\sigma^{-2}} dy \cdot \sum_n \{ &p(\sigma^2 t; x-a+2n\delta, y-a) \\ &- p(\sigma^2 t; -x+a+(2n+2)\delta, y-a) \} \end{aligned}$$

where $\delta \equiv b-a$.

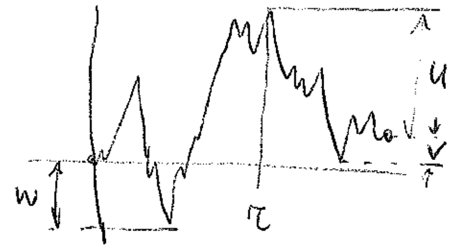
Even so, it's not v. encouraging.

2) By the usual sort of trickery, one can get expressions for $\mathcal{L}(X_T, S_T, I_T)$, where $T$ is indept exp. r.v., but in the end it must come back to the same once the LT has been undone.

Just for the record, we have:

if $T \sim \exp(\beta)$, and $\alpha \equiv \sigma^{-2}\{\sqrt{c^2+2\sigma^2\beta} - c\}$, if $S_T = X_\tau$, then we have

$$\begin{cases} S_T \sim \exp(\alpha) \\ P[W \in d\xi \mid S_T = a] = \dfrac{\mu e^{\mu a/\sigma^2}}{\sigma^2} \dfrac{\sinh \mu a/\sigma^2}{\sinh^2 \mu(a+\xi)/\sigma^2} \, d\xi \end{cases}$$




independent of

$$\begin{cases} U \sim \exp(\alpha + 2c/\sigma^2) \\ P(V \in dv \mid U=b) = \dfrac{\mu e^{\mu b/\sigma^2}}{\sigma^2} \dfrac{\sinh \mu b/\sigma^2}{\sinh^2 \mu(b+v)/\sigma^2} \, dv \end{cases}$$

$$\left(\mu \equiv \sqrt{c^2+2\sigma^2\beta}.\right)$$

[Up until time $\tau$ when we first reach $S_T$, one is observing a BM with drift $\mu$, so

$P^0$( get below $-\xi$ before hit $a$)

$$= \frac{1-e^{-2\mu a}}{e^{2\mu\xi} - e^{-2\mu a}}$$

from which the above formulae can be deduced.]

3) If we just consider the joint law of $S_t, X_t$, it is given by

$$P(S_t \in dx, S_t - X_t \in dy) = \frac{2(x+y)}{\sigma^3\sqrt{2\pi t^3}} e^{-(x+y)^2/2\sigma^2 t + c(x-y)/\sigma^2 - c^2 t/2\sigma^2}$$

which yields MLES

$$\hat{c} = (x-y)/t, \quad \hat{\sigma}^2 = 4xy/3t$$

So in other words,

$$\boxed{\hat{c} = X_t/t, \quad \hat{\sigma}^2 = 4 S_t(S_t - X_t)/3t.}$$

Is $\hat{\sigma}^2$ unbiased? We can compute from the law of $(S_T, S_T - X_T)$ that

$$E\hat{\sigma}^2 = \frac{2\sigma^2}{3}$$

which is very far from unbiased!!

4) There's an unbiased estimator for $\sigma^2$ here. Let $\tilde{\alpha} = \alpha + 2c/\sigma^2$, and notice that

$$\int_0^\infty \beta e^{-\beta t} \, E e^{-\lambda S_t - \rho(S_t - X_t)} \, dt = \frac{\alpha}{\alpha+\lambda} \frac{\tilde{\alpha}}{\tilde{\alpha}+\rho}$$

$$\therefore \int_0^\infty \beta e^{-\beta t} \, E S_t (S_t - X_t) \, dt = \frac{1}{\alpha \tilde{\alpha}} = \frac{\sigma^4}{2\sigma^2 \beta} = \sigma^2/2\beta$$

implying that

$$E\, S_t(S_t - X_t) = t\sigma^2/2$$

and hence

$$\boxed{E\left[ I_t (I_t - X_t) + S_t(S_t - X_t) \right] = t\sigma^2}$$

Can one compute the variance of this estimator?

5) I've computed the density of the range $S_T - I_T$, using the information in 2) above. It turns out that the range has density (when $\sigma \equiv 1$, $\theta = \sqrt{c^2 + 2\beta}$)

$$\frac{2\theta}{2\beta \sinh^3 \theta a} \Big[ c^2 \cosh ca \sinh^2 \theta a - c\theta \sinh ca \sinh 2\theta a + \theta^2 \cosh ca \,(1 + \cosh^2 \theta a) - 2\theta^2 \cosh \theta a \Big] \quad (a > 0).$$

So this really does tell us that it's futile to look for the law of $(S_T, I_T, X_T)$.

6) By differentiating again in 4), we conclude that

$$E\left(S_t^2 (S_t - X_t)^2\right) = t^2\sigma^4/2 = E\left(I_t^2 (I_t - X_t)^2\right).$$

If $Y_1 \equiv I_t(I_t - X_t)$, $Y_2 \equiv S_t(S_t - X_t)$, $Y \equiv Y_1 + Y_2$, we have trivially

$$\|Y\|_2 \leq \|Y_1\|_2 + \|Y_2\|_2 = \sqrt{2}\, t\sigma^2$$

and so

$$\boxed{\mathrm{var}(Y) = E(Y^2) - (EY)^2 \leq t^2\sigma^4.}$$

If $Y_1, Y_2$ were independent, we'd have $\mathrm{var}(Y) = t^2\sigma^4/2$, so it's all pretty similar.

7) Computing the variance of the estimator $Y$ explicitly looks hard. If we were

to stop at $T \sim \exp(\beta)$ instead of $T$, we'd be looking for $E\left[S_T(S_T-X_T)(X_T+I_T)I_T\right]$, which also seems hard to compute — and then the LT must be inverted.

Let's just record here the results of some calculations, in case they may be useful later. Wlog, $\sigma \equiv 1$; we can always scale it in later.

If $\gamma_1 = c+\sqrt{c^2+2\beta}$, $\gamma_2 = \sqrt{c^2+2\beta}-c$, we have

$$E e^{i\theta X_T} = \frac{\gamma_1}{\gamma_1+i\theta}\cdot\frac{\gamma_2}{\gamma_2-i\theta} = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}\left(\frac{1}{\gamma_1+i\theta}+\frac{1}{\gamma_2-i\theta}\right)$$

So $P(X_T>0) = \gamma_1/(\gamma_1+\gamma_2)$.

Rate of $\beta$-marked excursions into $\mathbb{R}^+$ (resp $\mathbb{R}^-$) is $\frac{1}{2}\gamma_1$ (resp. $\frac{1}{2}\gamma_2$) so that $L_T \sim \exp(\sqrt{c^2+2\beta}) = \exp((\gamma_1+\gamma_2)/2)$

Rate of excursions which get out to $x>0$ but contain no mark

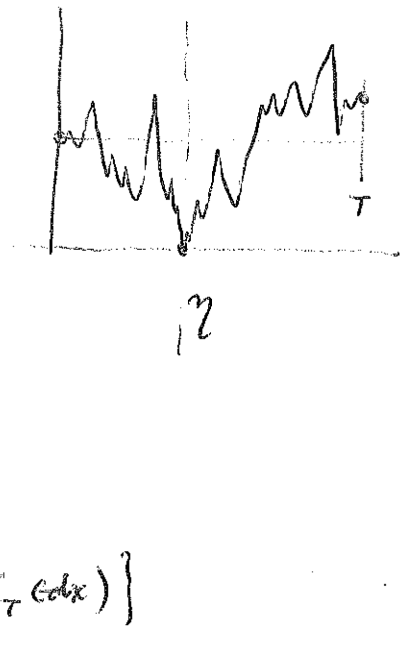
$$= \frac{\mu}{e^{2\mu x}-1} \qquad (\mu \equiv \sqrt{c^2+2\beta})$$

One way to proceed is to condition on $X_T = x$, $L_T = l$, $L(T,x) = l'$, and calculate the laws of $S_T - X_T$, $-I_T$ given this info (assume $x>0$ to begin with). This seems to be unapproachable (even low-order moments of these things are going to be hard), but it's worth just noting

$$E\left[e^{-\alpha L(T,x)-\gamma L(T,0)} \,\middle|\, X_T = x\right]$$

$$= \mu^2\left[\mu^2+\alpha\mu+\gamma\mu+\alpha\gamma(1-e^{-2\mu x})\right]^{-1}.$$

Another way in is to split at the minimum; to the right of $\eta$ we see a marked excursion of $B_t+ct$ and to the left, once we reverse time, a marked excursion of $B_t-ct$, so it should be possible to analyse these.




To the right of $\eta$, we get for $0<x<y$

$$P^\varepsilon(H_y<T<H_0,\ X_T\in dx)$$

$$= P^\varepsilon(H_y<T<H_0)\ P^y(T<H_0,\ X_T\in dx)$$

$$= \frac{e^{\gamma_2\varepsilon}-e^{-\gamma_1\varepsilon}}{e^{\gamma_2 y}-e^{-\gamma_1 y}}\left\{P^y(X_T\in dx) - P^y(H_0<T)\,P^0(X_T\in dx)\right\}$$

$$\sim \frac{(\gamma_1+\gamma_2)\varepsilon}{e^{\gamma_2 y} - e^{-\gamma_1 y}} \cdot \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} e^{-\gamma_1 y}\left\{ e^{\gamma_1 x} - e^{-\gamma_2 x} \right\}$$

and $P^{\varepsilon}(T<H_0) = 1 - e^{-\varepsilon\gamma_1} \sim \varepsilon\gamma_1$, so looking at this gives

$$n^c( X_T \in dx, S_T > y \,/\, T<\zeta) = \frac{\gamma_2 \{ e^{\gamma_1 x} - e^{-\gamma_2 x}\}}{e^{(\gamma_1+\gamma_2)y} - 1}$$

with a similar expression for the piece before $\eta$, but calculating anything in closed form here looks hopeless. Numerical things could be done, but there's still the LT to invert...

8) If $\mu$ is enormous, we have that roughly $S_t = \mu t$, $S_t - X_t \approx \exp(2\mu/\sigma^2)$ indept of $I_t$, $I_t - X_t$. This says that approximately

$$Y_1 \equiv S_t(S_t - X_t) \sim \exp\left(\frac{2}{\sigma^2 t}\right)$$

and approximately

$$\operatorname{var}(Y_1+Y_2) = 2\operatorname{var}(Y_2) = \sigma^4 t^2/2 .$$

The joint law of $(S_t, S_t - X_t, I_t, I_t - X_t)$ changes with $c$, and even of $S_t(S_t - X_t)$, cos the tail of this can't be exponential.

9). For $a<0<b$, it's easy (by solving the ODE with bcs, for example) to show that

$$E\left[ e^{i\omega X(T_\lambda)} : S_{T_\lambda} < b, I_{T_\lambda} > a \right] = \varphi \, \frac{e^{2\mu(b-a)} - 1 - e^{i\omega b - \alpha b}(e^{-2\mu a} - 1) - e^{i\omega a - \beta a}(e^{2\mu b} - 1)}{e^{2\mu(b-a)} - 1}$$

where $\mu = \sqrt{c^2+2\lambda}$, $\varphi = \dfrac{\lambda}{\lambda + \frac{1}{2}\omega^2 - i\omega c}$, $\alpha = -c-\mu$ $\beta = -c+\mu$.

[note: changed terminology!]

We can get the LT of this; for $\zeta, \delta, \gamma$ close enough to $0$,

~~Trying to LT this in a, b looks a dead loss.~~

$$E \exp\{\zeta X(T_\lambda) + \gamma S(T_\lambda) + \delta I(T_\lambda)\} = \frac{\lambda}{\lambda - \frac{1}{2}\zeta^2 - \zeta c}\left[ 1 + \sum_{n\ge 0} \frac{\gamma\delta}{(2n+1)\mu - \zeta - \gamma - c}\left\{ \frac{1}{\delta + 2n\mu} - \frac{1}{\delta + 2(n+1)\mu} \right\} \right.$$

$$\left. + \sum_{n\ge 0} \frac{\gamma\delta}{(2n+1)\mu + \zeta + \delta + c}\left\{ \frac{1}{2n\mu - \gamma} - \frac{1}{2(n+1)\mu - \gamma} \right\} \right]$$

## Integrability of Pick f$^{ns}$ in terms of the boundary values. (12/10/89)

1) Suppose given a Pick f$^{n}$

$$f(z) = \int \mu(dx)\left[\frac{1}{x-z} - \frac{x}{1+x^2}\right] + cz + a$$

and we want to consider when

$$\mathrm{Im}\int_1^\infty f(it)\,\frac{i\,dt}{it} \quad \text{is cgt}$$

$$= \int_1^\infty \frac{dt}{t}\int \mu(dx)\,\frac{t}{x^2+t^2}$$

$$= \int \mu(dx)\left[\frac{1}{x}\tan^{-1}\left(\frac{t}{x}\right)\right]_1^\infty = \int \frac{\mu(dx)}{x}\tan^{-1}(x)$$

As the condition for integrability of $\mathrm{Im}\; i\int_1^\infty \frac{f(it)}{it}\,dt$ is

$$\boxed{\int \mu(dx)\,(1+|x|)^{-1} < \infty.}$$

(together with $c=0$, of course.)

2) Now let's consider the function $F(z) \equiv \int_i^z f(w)\,dw/w$, and see how the imaginary part behaves, assuming the boxed condition holds.

For any $C>0$, we consider the behaviour as $R\to\infty$ of

$$\mathrm{Im}\{-F(iR) + F(CR+iR)\}$$

$$= \mathrm{Im}\int_0^{CR} dx\,\frac{f(x+iR)}{x+iR}$$

$$= \mathrm{Im}\int_0^C \frac{dt}{t+i}\int \mu(dx)\,\frac{1}{x-R(t+i)}$$

[ASSUMING: $a=0=c$]

$$= \int_0^C dt\int \frac{\mu(dx)}{x}\,\mathrm{Im}\left\{\frac{1}{t+i} + \frac{R}{x-R(t+i)}\right\}$$

$$= \int_0^C dt\int \frac{\mu(dx)}{x}\left\{\frac{-1}{1+t^2} + \frac{R^2}{(x-Rt)^2+R^2}\right\}$$

$$= \int_0^C dt\int \frac{\mu(dx)}{x}\,\frac{\frac{2tx}{R} - x^2/R^2}{(1+t^2)\left(1+(t-x/R)^2\right)}$$

$$= \int \frac{\mu(dx)}{x} \Big[ -\tan^{-1}(t) + \tan^{-1}(t - x/R) \Big]_0^c$$

$$= \int \frac{\mu(dx)}{x} \Big\{ -\tan^{-1}(c) + \tan^{-1}(c - x/R) - \tan^{-1}(-x/R) \Big\}$$

$$= \int \frac{\mu(dx)}{x} \tan^{-1}(x/R) + \int \frac{\mu(dx)}{x} \Big\{ \tan^{-1}(c - \tfrac{x}{R}) - \tan^{-1}(c) \Big\}$$

which is bounded in modulus by

$$\int \frac{\mu(dx)}{|x|} \tan^{-1}\left(\frac{|x|}{R}\right) + \int \frac{\mu(dx)}{|x|} \left( \left|\frac{x}{R}\right| \wedge \pi \right)$$

$$\to 0 \quad \text{as } R \to \infty \qquad \text{UNIFORMLY IN } C \text{ !!}$$

3) Let's now suppose that $\rho$ is real-valued $f^n$ on $\mathbb{R}$, integrable, non-negative and let $\tilde{\rho}(z) = \int \rho(x)(x-z)^{-1} dx$ be its extension into $\mathbb{H}$. Assuming the boxed condition, with $f(z) \equiv \int \mu(dx)(x-z)^{-1}$, I want to prove that

$$\lim_{N \to \infty} \operatorname{Im} \int_1^N i\,dt \; f(it)\tilde{\rho}(it) \qquad \underline{\text{exists and is finite}}$$

[Precisely this situation arises in the case of trying to build the Green's $f^n$. Note that if $\rho$ were a delta $f^n$ at $0$, the situation is more simple.]

$$\operatorname{Im} \int_1^\infty i\,dt \; f(it)\tilde{\rho}(it) = \operatorname{Im} \int_1^\infty i\,dt \int \frac{\rho(v)\,dv}{v - it} \cdot \int \frac{\mu(dx)}{x - it}$$

$$= \operatorname{Im} \int \rho(v)\,dv \int \mu(dx) \int_1^\infty \frac{i\,dt}{x-v} \left( \frac{1}{v - it} - \frac{1}{x - it} \right)$$

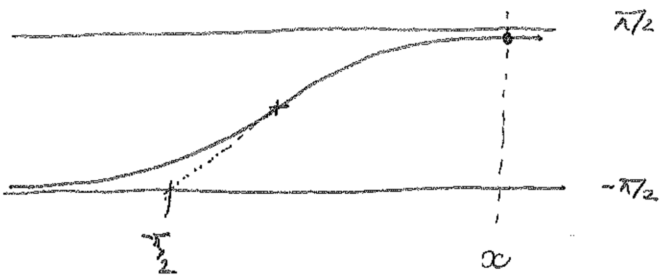
$$= \int \rho(v)\,dv \int \mu(dx) \int_1^\infty \frac{dt}{x - v} \left( \frac{v}{v^2+t^2} - \frac{x}{x^2+t^2} \right)$$

$$= - \int \rho(v)\,dv \int \mu(dx) \; \frac{\tan^{-1}(v) - \tan^{-1}(x)}{v - x}$$

(Putting in an infinite upper limit immediately is harmless) Now the problem is one of estimating $(\tan^{-1}(x+h) - \tan^{-1}(x))/h$. Wlog, we take $x > 0$ to begin with, and consider the graph of $\tan^{-1}(x)$.

We even have $0 \le \dfrac{\tan^{-1}(x) - \tan^{-1}(v)}{x - v} \le \dfrac{c}{1 + |x| \vee |v|}$

Cos for $x > 0 > v$, the difference quotient is $\le 1 \wedge \frac{\pi}{|x|} \wedge \frac{\pi}{|v|}$
and for $x \ge v \ge 0$, we have $\dfrac{\tan^{-1}(x) - \tan^{-1}(v)}{x - v}$ is maximised when $v = 0$, to value $\dfrac{\tan^{-1}(x)}{x} \le \dfrac{c}{1+|x|}$.




For $x>0$, it is clear from concavity of $\tan^{-1}$ in $\mathbb{R}^+$, and from the picture, that the difference quotient satisfies the bound

$$\frac{\tan^{-1}(x+h) - \tan^{-1}(x)}{h} \leq \frac{\tan^{-1}(x) + \pi/2}{x + \pi/2} \qquad \forall h \in \mathbb{R}$$

whence for all $x$, by symmetry,

$$\frac{\tan^{-1}(x+h) - \tan^{-1}(x)}{h} \leq \frac{\tan^{-1}(|x|) + \pi/2}{|x| + \pi/2} .$$

Thus $-\operatorname{Im} \int_1^\infty i\,dt\, f(it)\tilde{\rho}(it) \leq \int \rho(v)\,dv \int \mu(dx)\, \frac{\pi}{|x|+\pi/2} < \infty$.

4) <u>The above argument generalises to any non-tangential approach to $\infty$.</u>

Define

$$F(z) = \int_i^z dw\, f(w)\, \tilde{\rho}(w)$$

[Again, take care: in general, $f(z) = \int \frac{\mu(dx)}{x-z}$ + real const.]

$$= \int \rho(v)\,dv \int \mu(dx) \int_i^z \frac{dw}{v-x}\left(\frac{1}{w-v} - \frac{1}{w-x}\right)$$

so

$$\operatorname{Im} F(z) = \int \rho(v)\,dv \int \frac{\mu(dx)}{v-x} \operatorname{Im}\left(\Big[\log(w-v) - \log(w-x)\Big]_i^z\right)$$

$$= \iint \frac{\mu(dx)\rho(v)\,dv}{v-x} \left\{ \tan^{-1}\left(\frac{-a+v}{b}\right) - \tan^{-1}\left(\frac{-a+x}{b}\right) - \tan^{-1}(+v) + \tan^{-1}(+x) \right\}$$

where $z = a+ib$. Now let's suppose we let $b \to \infty$, while keeping $|a| \leq Kb$.

We have

$$\left\{\tan^{-1}\left(\frac{v-a}{b}\right) - \tan^{-1}\left(\frac{x-a}{b}\right)\right\}(v-x)^{-1} \leq \text{const}\left\{1 + \left|\frac{x-a}{b}\right|\right\}^{-1} b^{-1}$$

$$= \frac{c}{b + |x-a|}$$

When $b > 1$, we have $b + |x-a| > 1$

and also

$$b + |x-a| \geq \text{const}\,(|a| + |x-a|)$$

$$\geq \text{const}\,|x|$$

so $b + |x-a| \geq \text{const}\,.\, 1 \vee |x| \geq \text{const}\,(1 + |x|)$.

Hence $0 \le \iint \frac{\mu(dx)\,p(v)\,dv}{v-x} \left\{ \tan^{-1}\left(\frac{v-a}{b}\right) - \tan^{-1}\left(\frac{x-a}{b}\right) \right\} \le c \iint \frac{\mu(dx)\,p(v)\,dv}{b+|x-a|} \to 0$

as $(a,b)\to\infty$ non-tangentially, by dominated cgce. Thus we may (still assuming the boxed condition) define the non-negative function

$$\boxed{h(a+ib) = \iint \frac{\mu(dx)\,p(v)\,dv}{v-x} \left\{ \tan^{-1}\left(\frac{v-a}{b}\right) - \tan^{-1}\left(\frac{x-a}{b}\right) \right\}}$$

which is harmonic in $\mathbb{H}$, and, when added to $\log\left|\frac{z+i}{z-i}\right|$, gives something satisfying the boundary conditions.

Thus the boxed condition is sufficient for transience.

Deducing moving barrier from first-passage distribution (10/11/89)

If $f : \mathbb{R}^+ \to \mathbb{R}$ is smooth, and the initial law of BM is $\mu$, let $\tau \equiv \inf\{t : B_t > f(t)\}$, and let $q_t(x) \equiv P(B_t \in dx, \tau > t)$.

Suppose that you know the function

$$\rho(t) = P(\tau \in dt)/dt$$

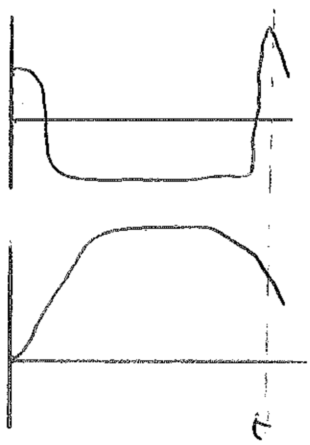
$$= -\frac{\partial}{\partial t} \int q_t(x)\,dx$$

– can one work out what the barrier $f$ was?

One has the boundary condition $q_t(f(t)) = 0$, and can show quite easily that

$$\frac{\partial q_t}{\partial t} = \frac{1}{2}\frac{\partial^2 q_t}{\partial x^2} \quad \text{in } C = \{(t,x) : x < f(t)\}.$$

hence $\boxed{\rho(t) = -\frac{1}{2}\frac{\partial q_t}{\partial x}(f(t))} = \dot{q}_t(f(t))/2\dot{f}(t)$



Can we (by repeat differentiation?) get the unknown $q$ out of the equations, and find a d.e. for $f$ in terms of $\rho$? This cannot be expected, because the values + derivatives of $q$ near $f$ do not depend only on the behaviour of $f$ near the point — as these two pictures show

## Limit laws in extreme values (27/11/89)

One takes i.i.d. r.v.s. $X_n$, letting $X_n^* \equiv \sup\{X_k : k \le n\}$, and considers a situation where for some constants $a_n, b_n$,

$$(X_n^* - b_n)/a_n \Rightarrow G$$

where $G$ is a non-degenerate distribution. Assume that the sequence $(b_n)$ is monotone increasing, and that $(a_n)$ is monotone.

1) For any $t>0$,

$$P(X^*_{[tn]} \le a_n x + b_n) = F(a_n x + b_n)^{[tn]} \to G(x)^t$$

$$= P(X^*_{[tn]} \le a_{[tn]} y_n + b_{[tn]})$$

where $y_n = (a_n x)/a_{[tn]} + (b_n - b_{[tn]})/a_{[tn]} \equiv \frac{a_n x}{a_{[tn]}} + c_{n,t}$

Now we may wlog assume that $G(0) \in (0,1)$, so we know that if $G$ were strictly increasing at $G^{-1}(G(0)^t)$, then the $y_n$ must be (when $x=0$) converging. But this is true for all but countably many $t>0$, so we have

$$\lim_n c_{n,t} \equiv \lim_n \frac{b_n - b_{[tn]}}{a_{[tn]}} \equiv \mu(t)$$

exists for all but countably many $t$. Let $N$ be the countable set of bad $t$. By choosing $x$ at will, we can ensure that

$$\lim_n \frac{a_{[tn]}}{a_n} \equiv \lambda(t) \qquad \text{exists } \forall t \notin N$$

and easily for $s,t \notin N$, $\lambda(st) = \lambda(s)\lambda(t)$. Together with monotonicity of $\lambda$, this implies that for some $\gamma$,

$$\boxed{\lambda(t) = t^\gamma \qquad \text{for all } t.}$$

2) This forces the convergence of $c_{n,t}$ for every $t$. Thus

$$\frac{b_n - b_{[tn]} + b_{[tn]} - b_{[stn]}}{a_{[tn]}} = \frac{b_n - b_{[stn]}}{a_{[stn]}} \cdot \frac{a_{[stn]}}{a_{[tn]}} \to \mu(st)\, s^\gamma$$

But the LHS goes to $\mu(t) - \mu(s^{-1})$, implying

$$\boxed{\mu(t) - \mu(s^{-1}) = s^{\gamma}\mu(st)} \qquad \text{for } s, t > 0,$$

so in particular, $\mu(1) = 0$.

When $\gamma \neq 0$, picking $c = \mu(\tfrac{1}{2})(1-2^{\gamma})^{-1}$, we get

$$2^{\gamma}(\mu(2t) - c) = \mu(t) - c$$

whence

$$\mu(t) = c + A t^{-\gamma}$$

and the condition $\mu(1) = 0$ implies that for $t > 0$

$$\boxed{\mu(t) = A(t^{-\gamma} - 1)}$$

When $\gamma = 0$, we get $\boxed{\mu(t) = -A \log t}$. The condition $\mu(t) = -\mu(t^{-1})t^{-\gamma}$ implies $A = 1$.

3) Going back to the first identity in section 1), we can now say that for $\gamma \neq 0$

$$G(x)^t = G(t^{-\gamma}x + t^{-\gamma} - 1)$$

Writing $G(x) \equiv \exp(-h(x+1))$ yields

$$t\,h(x+1) = h(t^{-\gamma}(x+1))$$

and so $h(x) = C\, x^{-1/\gamma}$, and

$$\boxed{G(x) = \exp\{-C(1+x)^{-1/\gamma}\}}.$$

The case $\gamma = 0$ goes a bit differently, but ends up with

$$\boxed{G(x) = \exp(-Ce^{-x})}$$

4) For the Gaussian distn, we get $b_n \approx \sqrt{2\log n - \log\log n}$, $a_n = 1/b_n$ will do – so the $a_n \to 0$ !! We have $\gamma = 0$ for this one.

5) Charles Goldie says that Resnick's book has a lot to say on this...

## Skew-reflecting BMs again 28/11/89.

In the Greens function approach to this problem, one builds the Greens $f^n$ as the sum of $c.\log(|z+i|/|z-i|)$ and the imaginary part of $f$, where

$$\mathrm{Im}\left(\frac{f'}{i\psi}\right) = \mathrm{Im}\,\rho \qquad \text{on } \mathbb{R},$$

where $\rho$ is a nice Pick $f^n$,

$$\mathrm{Im}\,\rho(x) = \frac{1}{1+x^2}\,\mathrm{Im}\left(\frac{-1}{i\psi(x)}\right) \qquad x \in \mathbb{R}.$$

To obtain a proof of the implication $\int \mathrm{Re}\,\psi(x)\,(1+|x|)^{-1}dx = +\infty \Rightarrow$ recurrence, one natural thing to try is to build a candidate for the Greens $f^n$ and show that it has impossible properties.

Thus we would consider

$$f(z) - f(i) = \int_i^z dw \;\; i\psi(w)\,\rho(w)$$

$$= \int_i^z dw \int \mu(dx)\left[\frac{1}{x-w} - \frac{x}{1+x^2}\right]\int \frac{\rho(y)\,dy}{y-w} \qquad \frac{\mu(dx)}{dx} = \mathrm{Im}\,i\psi$$

(since $\rho \in L^1$, $-1/i\psi$ being a Pick $f^n$)

$$= \int \mu(dx)\int \rho(y)dy \int_i^z dw\left\{\frac{1}{x-y}\left(\frac{1}{w-x} - \frac{1}{w-y}\right) + \frac{x}{1+x^2}\,\frac{1}{w-y}\right\}$$

$$= \int \mu(dx)\int \rho(y)\,dy\left[\frac{1}{x-y}\left(\log(w-x) - \log(w-y)\right) + \frac{x}{1+x^2}\log(w-y)\right]_i^z$$

The Imaginary part is

$$= \int \mu(dx)\int \rho(y)\,dy\left[\frac{1}{x-y}\left\{\tan^{-1}\left(\frac{x-a}{b}\right) - \tan^{-1}\left(\frac{y-a}{b}\right)\right\} + \frac{x}{1+x^2}\tan^{-1}\left(\frac{y-a}{b}\right)\right.$$
$$\left. - \frac{1}{x-y}\left(\tan^{-1}(x) - \tan^{-1}(y)\right) - \frac{x}{1+x^2}\tan^{-1}(y)\right]$$

(*) $$= \int \frac{\mu(dx)}{1+x^2}\int \rho(y)\,dy\left[\frac{1+xy}{x-y}\left\{\tan^{-1}\left(\frac{x-a}{b}\right) - \tan^{-1}\left(\frac{y-a}{b}\right)\right\} + x\tan^{-1}\left(\frac{x-a}{b}\right)\right.$$
$$\left. - \frac{1+xy}{x-y}\left\{\tan^{-1}(x) - \tan^{-1}(y)\right\} - x\tan^{-1}(x)\right]$$

Now we are going to add to this $c \log |z+i|/|z-i| \geq 0$, and if we could show that (*) were bounded above, this would force the process to be recurrent.

One obvious way to do that is to prove that the integrand is bounded above. Unfortunately, that is not true.

Indeed, if we fix $b=1$, and consider the integrand

$$\frac{1+xy}{x-y}\left(\tan^{-1}(x-a) - \tan^{-1}(y-a)\right) + x\tan^{-1}(x-a)$$
$$- \frac{1+xy}{x-y}\left(\tan^{-1}x - \tan^{-1}y\right) - x\tan^{-1}x \leq C, \text{ we suppose.}$$

Let $a \to -\infty$; we have

$$x\left(\frac{\pi}{2} - \tan^{-1}(x)\right) - \frac{1+xy}{x-y}\left[\tan^{-1}(x) - \tan^{-1}(y)\right] \leq C$$

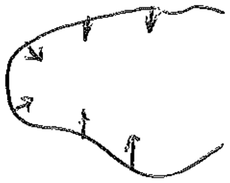
But if we now consider the case where $x > 0 > y \equiv -\eta$, we have

$$x\left(\frac{\pi}{2} - \tan^{-1}(x)\right) + \frac{x\eta}{x+\eta}\left(\tan^{-1}(x) + \tan^{-1}(\eta)\right) + O(1) \leq C$$

– but if $\eta = x$, say, this gets arbitrarily large as $x \to \infty$.

Recurrence/transience of skew-reflecting BM by mappings (5/12/89)

1) The idea is to multiply $\varphi$ by some analytic $f^n$ and then integrate so as to get a domain looking like:



or some such.

For technical convenience, let's take $\Phi$ defined by

$$i\,\Phi(z) \equiv \int_i^z dw\ \frac{i\psi(w)}{w} \qquad (z \in \mathbb{H})$$

The purpose of this is that the arg of $\frac{1}{w}$ decreases from $0$ on $\mathbb{R}^+$ to $-\pi$ on $\mathbb{R}^-$, and so successfully twists around the tangent.

Thus

$$i\,\Phi(z) = \int_i^z \frac{dw}{w}\left\{\int \mu(dt)\left(\frac{1}{t-w}-\frac{t}{1+t^2}\right)+c_1 w + c_2\right\}$$

$$\left[\frac{\mu(dt)}{dt} = \frac{\operatorname{Im} i\psi(t)}{\pi}\right. \qquad c_1 \geq 0,\ c_2 \text{ real.}$$

$$= \int \mu(dt) \int_i^z dw \left\{\left(\frac{1}{w}-\frac{1}{w-t}\right)\frac{1}{t} - \frac{t}{(1+t^2)w}\right\} + c_1(z-i) + c_2(\log z - \log i)$$

$$= \int \mu(dt)\left\{\frac{1}{t(1+t^2)}(\log z - \log i) - \frac{1}{t}\big(\log(z-t) - \log(z-i)\big)\right\} + c_1(z-i) + c_2(\log z - \log i)$$

The imaginary part of the integral is

$$\int \frac{\mu(dt)}{t}\left\{\frac{1}{1+t^2}\tan^{-1}\left(\frac{-a}{b}\right) - \tan^{-1}\left(\frac{t-a}{b}\right) + \tan^{-1}(t)\right\}$$

$$= \int \frac{\mu(dt)}{t}\left\{\frac{1}{1+t^2}\left(\tan^{-1}\left(\frac{-a}{b}\right) - \tan^{-1}\left(\frac{t-a}{b}\right)\right) - \frac{t^2}{1+t^2}\left(\tan^{-1}\left(\frac{t-a}{b}\right) - \tan^{-1}(t)\right) + \frac{\tan^{-1}(t)}{1+t^2}\right\}.$$

The final term contributes simply a finite constant. As for the others, the first integrand satisfies

$$0 \geq \frac{\tan^{-1}(-a/b) - \tan^{-1}\left(\frac{t-a}{b}\right)}{t} \geq -\frac{1}{b}\left(1 + \left|\frac{a}{b}\right| \vee \left|\frac{t-a}{b}\right|\right)^{-1}$$

$$\geq -\left(b + |a| \vee |t-a|\right)^{-1}$$

$$\geq -|a|^{-1}$$

Thus the integral of the first piece is $\leq 0$, is well defined in $\bar{\mathbb{H}} \setminus \{0\}$, but explodes to $-\infty$ as $z = ib \downarrow 0$.

For the middle piece, we have

$$\frac{t}{1+t^2}\left\{\tan^{-1}(t) - \tan^{-1}\left(\frac{t-a}{b}\right)\right\}$$

$$= \frac{t}{1+t^2}\left\{\tan^{-1}(t) - \frac{\pi}{2}\operatorname{sgn}(t)\right\} + \frac{t}{1+t^2}\left\{\frac{\pi}{2}\operatorname{sgn}(t) - \tan^{-1}\left(\frac{t-a}{b}\right)\right\},$$

the first part of which gives only a harmless constant, and the second part of which contains the real interest. The integral will be globally bounded if $\int \mu(dt)(1+|t|)^{-1} < \infty$, and otherwise will explode as $z = ib \to \infty$.

Thus: $\boxed{\operatorname{Im} i\Phi \text{ is bounded above iff } c_1 = 0 \text{ and } \int \frac{\mu(dt)}{1+|t|} < \infty}$

If $\operatorname{Im} i\Phi$ is bdd above, since $i\Phi$ satisfies the b.c. $\operatorname{Im}\left(\frac{f'}{i\varphi}\right) = 0$ on $\mathbb{R}$ it follows that the local mg $\operatorname{Im} i\Phi(Z_t)$ is a.s. convergent, so that $Z$ has to be transient. What about the converse implication?

## Grossissement + Girsanov? 22/12/89.

The formulae which keep appearing in grossissement are very reminiscent of Girsanov-type results (a fact which Jacod SLN 1118, 15–35 has also noticed).

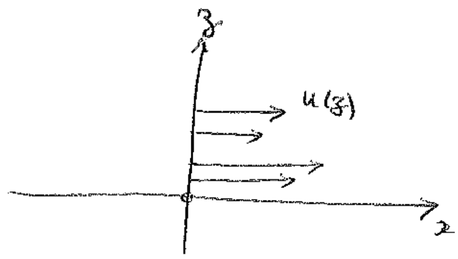
The form of the results suggests that if $\mathcal{F}$ is the original filtration, $L$ an honest time, and $\mathcal{G}$ the smallest enlargement making $L$ a stopping time, then there is some measure $\tilde{P}$ equivalent to $P$ such that

$$M \text{ is an } (\mathcal{F},P)\text{-martingale} \Rightarrow M \text{ is a } (\mathcal{G},\tilde{P})\text{-martingale}.$$

To see that this is impossible in general, take $L = \sup\{t < H_1 : B_t = 0\}$. Then if $\tilde{P} \sim P$, we have to have that $B$ is a $(\mathcal{G},\tilde{P})$-Brownian motion – and yet after the stopping time $L$ it goes up.

## Stochastic methods in dispersion theory. (13/1/90)

1) There seems to be a lot of interest in diffusions in a horizontal flow (in a river?) where the $Z$-component behaves as an autonomous positive-recurrent diffusion, and $X$ is given by




$$X_t = X_0 + B_t + \int_0^t u(Z_s)\,ds.$$

Without loss of generality, we assume that $Z$ is in natural scale on interval $I$, with instantaneous reflection at the end-points of $I$ (if any). Thus

$$dZ_t = \sigma(Z_t)\,dW_t \quad (+\, dL_t^- - dL_t^+)$$

and $\int_I m(dx) \equiv \int_I \sigma(x)^{-2}\,dx < \infty$.

Interest centres on the asymptotic drift and variance of $X$ as $t \to \infty$; this is trivially the same problem as deciding the asymptotic behaviour of the additive functional

$$A_t = \int_0^t u(Z_s)\,ds.$$

Now by the ergodic theorem (R+W p 300) we can conclude that if

$\int_I |u(x)|\, m(dx) < \infty$, then

$$\frac{1}{t}\int_0^t u(Z_s)\,ds \xrightarrow{a.s.} \int u(x)\, m(dx) / m(I), \quad t \to \infty.$$

Thus we may subtract off from $u$ its mean w.r.t. $m$, and we are left with only the interesting bit of deciding the "central limit" type of asymptotics.

Let's assume therefore that

$$\boxed{\int_I u(x)\, m(dx) = 0 \quad ; \quad \int_I \left(\int_a^x 2u(y)\, m(dy)\right)^2 dx \equiv v < \infty .}$$

(Here, $a$ denotes the lower end of $I$.) If we define $F : I \to \mathbb{R}$ by the requirements

$$\tfrac{1}{2}\sigma^2 F'' = u, \qquad F' = 0 \text{ at both ends of } I \ \left(\text{possible because of } \int u\, dm = 0\right)$$

and the additive constant fixed in some arbitrary way, then applying Itô's formula to $F(Z_t)$ yields

$$F(Z_t) - F(Z_0) = \int_0^t F'(Z_s)\,\sigma(Z_s)\, dW_s + \int_0^t u(Z_s)\, ds$$

(the local time terms dropping out because $F' = 0$ at each end)

$$\equiv M_t + A_t .$$

Now we know that $M_t = \tilde{W}\left(\int_0^t F'(Z_s)^2 \sigma(Z_s)^2 ds\right)$ (Thm IV.34.1, R+W)

Hence, using the ergodic theorem once more,

$$\boxed{\frac{1}{\sqrt{t}} M_t \xrightarrow{\mathcal{D}} N\left(0,\ v/m(I)\right)}$$

We know that $t^{-\frac{1}{2}} F(Z_t) \xrightarrow{P} 0$, since $Z_t \xrightarrow{\mathcal{D}} m(\cdot)/m(I)$, by, for example, the coupling argument on p 301 R+W. Put all this together and you have the CLT

$$\boxed{\frac{1}{\sqrt{t}}\left(A_t - t \int u(x)\, m(dx) \big/ m(I)\right) \xrightarrow{\mathcal{D}} N\left(0,\ v/m(I)\right).}$$

Remark. The 'flow in a pipe' example is also covered by this, cos the diffusion factors into a radial part and a longitudinal part.

2) Particles hitting a side of the region. The set-up is like in the first situation except that now the diffusion $Z$ in natural scale hits one of the endpoints of $I$, it sticks there. One wants to know the $X$ displacement at that time.

But the principle is obvious. If $\tau$ is the hitting time,

$$E^z e^{i\theta X_\tau} = E^z e^{i\theta A_\tau - \frac{1}{2}\theta^2\tau} \qquad \text{(assuming } X_0 = 0\text{, wlog)}$$

$$\equiv f(z),$$

say, and now all one needs to do is solve

$$\boxed{\tfrac{1}{2}\sigma^2 f'' = (\tfrac{1}{2}\theta^2 - i\theta u) f}$$

with the boundary conditions $f = 1$ at absorbing bdy, $f' = 0$ at reflecting bdy, $f$ bdd near infinite bdy.

Of course, it's not necessary to have $Z$ in natural scale — more generally, solve

$$\tfrac{1}{2}\sigma^2 f'' + b f' = (\tfrac{1}{2}\theta^2 - i\theta u) f.$$

One trivial example is where $\sigma$ is const, $b = -\mu < 0$ is const, and $u$ is also constant. Then for $z > 0$ we get

$$E^z e^{i\theta X_\tau} = \exp\left[ -z \, \frac{(\mu^2 + \sigma^2\theta^2 - 2i\theta\sigma^2 u)^{\frac{1}{2}} - \mu}{\sigma^2} \right]$$

One can also do, say, a two-layer flow, but the calculations lead to very little useful.

3) Flow in tidal estuary!

This time, we take the flow to be dependent on time as well as on $z$, so that

$$dX_t = dB_t + \cos\omega t \; u(Z_t)\, dt.$$

The methodology needed here is somewhat different. Let's suppose once again that $Z$ is a positive-recurrent diffusion in natural scale and without any real loss of generality that $Z_0$ is distributed according to the equilibrium distribution $\pi$ (we'll explain what happens more generally later). Now define for $n \geq 1$

$$\xi_n \equiv \int_{(n-1)\lambda}^{n\lambda} \cos \omega t \; u(Z_t)\, dt \qquad (\lambda \equiv 2\pi/\omega).$$

Then if $\mathcal{F}_n = \sigma(\{Z_s : s \leq n\})$, we have that the sequence $(\xi_n)_{n\geq 1}$ is stationary ergodic (because the underlying diffusion $Z$ is ergodic). It's clear that $E\xi_1 = 0$ (assuming $\int |u(x)| \pi(dx) < \infty$ for the integrals to make sense), hence by the Birkhoff ergodic theorem,

$$n^{-1} S_n \equiv n^{-1}(\xi_1 + \cdots + \xi_n) \xrightarrow{a.s.} 0.$$

Once again, we want to know about the CLT behaviour. Let's quote a result from P. Hall + C. C. Heyde, Martingale Limit Theory and its Application Academic Press, New York 1980. Their Theorem 5.2 says thus.

"If $(\xi_n)$ is stationary ergodic, $E\xi_0 = 0$, $E\xi_0^2 < \infty$, $X_0 \in \mathcal{F}_0$ and if

(i) $\sum_{k\geq 1} E(\xi_k\, E(\xi_N \mid \mathcal{F}_0))$ is convergent for each $N$

(ii) $\lim_{N\to\infty} \sum_{k\geq K} E(\xi_k\, E(\xi_N \mid \mathcal{F}_0)) = 0$ uniformly in $K \geq 1$,

then $\lim n^{-1} E S_n^2 \equiv \sigma^2$ exists, and if $\sigma > 0$ then $S_n/\sigma\sqrt{n} \xrightarrow{\mathcal{D}} N(0,1)$".

To fix the integrability, let's assume that

$$\int \pi(dx) \left( \int_0^\infty |P_s \tilde{u}(x)|\, ds \right)^2 < \infty$$

where $\tilde{u}(x) \equiv u(x) - \int \pi(dy)\, u(y)$ is $u$ centred. Note that for each $x$, $P_s \tilde{u}(x) \to 0$ as $s \to \infty$, by a coupling argument, so this isn't quite such a wild thing to assume! The result we seek may be true under weaker assumptions, but that's rather incidental!

Now we check the hypotheses of the theorem.

$$E(\xi_N \mid \mathcal{F}_0) = \int_{(N-1)\lambda}^{N\lambda} \cos \omega t \; P_t u(X_0)\, dt = \int_{(N-1)\lambda}^{N\lambda} \cos \omega t \; P_t \tilde{u}(X_0)\, dt,$$

so

$$\sum_{k\geq K} E(\xi_k\, E(\xi_N / \mathcal{F}_0)) = E\left[ \int_{(K-1)\lambda}^{\infty} \cos \omega s \; \tilde{u}(X_s)\, ds \int_{(N-1)\lambda}^{N\lambda} \cos \omega t \; P_t \tilde{u}(X_0) \right]$$

$$= E \int_{(k-1)\lambda}^{\infty} ds \cos\omega s \; P_s \tilde{u}(X_0) \int_{(N-1)\lambda}^{N\lambda} \cos\omega t \; P_t \tilde{u}(X_0)\, dt$$

which is integrable because of the obvious domination

$$\leq \int \pi(dx) \int_{(k-1)\lambda}^{\infty} ds \; |P_s \tilde{u}(x)| \cdot \int_{(N-1)\lambda}^{N\lambda} |P_t \tilde{u}(x)|\, dt$$

$$\leq \int \pi(dx) \left( \int_0^\infty |P_s \tilde{u}(x)|\, ds \right)^2 < \infty$$

by assumption. Thus both conditions of the quoted result hold, and we have the central limit result. Identifying the limit variance in general does not appear to be very useful.

Special case If $Z$ is Brownian motion, and $u(z) = -\alpha z$, then Young, Rhines + Garrett studied the problem J. Phys. Oceanography. 12 515–527, 1982.

Thus
$$X_t = X_0 + B_t - \alpha \int_0^t Z_s \cos\omega s \; ds \equiv X_0 + B_t + A_t .$$

It's immediate that $A_t$ is a zero-mean Gaussian process, and it's not hard to compute

$$\boxed{E A_t^2 = \frac{\alpha^2 t}{2\omega^2}(1 + 2\sin^2\omega t) - \frac{2\alpha^2 \sin\omega t}{\omega^3}\left\{1 - \frac{3}{4}\cos\omega t\right\}},$$

either directly, or by firstly integrating by parts. Thus the variance of $X_t$ is

$$\mathrm{var}(X_t) = t + t\,\frac{\alpha^2}{2\omega^2}(1 + 2\sin^2\omega t) + O(1)$$

which does not appear to agree with YRG formula (7), where they say that the "effective horizontal diffusivity" is (making allowance for different terminology)

$$1 + \frac{\alpha^2}{2\omega^2}$$

So what has happened? What has happened is explained in their § 2.b, where they start the diffusion according to the "density"

$$\theta(x, z, 0) = \frac{\exp(-z^2/2b)}{\sqrt{2\pi b}}$$

To make sense of this, we interpret it as a limiting case of the starting law where

$X_0, Z_0$ are indept Normals, zero-mean variances $b, v$ respectively, where $v$ is very large, and we scale up the density by $\sqrt{2\pi v}$. One can now work out the joint law of $X_t, Z_t$, (which is, of course, Gaussian), and let $v \to \infty$ to get the formula for $\theta(x, z, t)$ which they obtain.

Notice that by using the wrong methodology, they have missed an important feature;

$$\frac{\mathrm{var}(X_t)}{t} \sim 1 + \frac{\alpha^2}{2\omega^2}(1 + 2\sin^2\omega t)$$

and <u>not</u> $\dfrac{\mathrm{var}(X_t)}{t} \sim 1 + \dfrac{\alpha^2}{2\omega^2}$, as YRG seem to think!

## Polymer carpet in a potential (2/2/90)

The idea is that a piece of polymer sticking to a flat surface is modelled as a BES(3) in the direction normal to the surface, with 2 indept BMs in the other two directions, <u>but</u> with the paths weighted by $\exp\left(-\int_0^\infty V(X_s)\,ds\right)$, where $X$ is the BES(3) component. This will convert $X$ into a new diffusion. But the best way to deal with this is in reversed time, so that $X$ becomes a BM coming in from $+\infty$, conditioned not to be killed before it hits 0. We have that if

$$\psi(x) \equiv E^x \exp\left\{-\int_0^{H_0} V(B_s)\,ds\right\} \qquad H_0 \equiv \inf\{t : B_t = 0\}$$

then $\frac{1}{2}\psi'' - V\psi = 0$, and the BM conditioned to get in before killing has generator

$$\mathcal{G} = \tfrac{1}{2} D^2 + \frac{\psi'}{\psi} D .$$

[Strictly speaking, $\int_0^\infty V(X_s)\,ds < \infty$ iff $\int x V(x)\,dx < \infty$, so weighting by the factor $\exp -\int_0^\infty V(X_s)\,ds$ produces triviality if $\int x V(x)\,dx = \infty$. All the same, one can interpret this in some way (as above) by conditioning to reach a high level (or come in from a high level) without killing.]

What's the local time process of this diffusion? By NRW Thm 2, it satisfies

$$dL_x = 2\sqrt{L_x}\,dW_x + 2\left\{1 + \frac{\psi'}{\psi}(x) L_x\right\} dx$$

We have $\frac{1}{2}\psi'' = V\psi$, and $\frac{1}{2}\ell' = 1 + \psi'/\psi \, \ell$, and can eliminate $\psi$, by

$$\frac{1}{2}\ell'' = \frac{\psi''}{\psi}\ell - \frac{\psi'^2}{\psi^2}\ell + \frac{\psi'}{\psi}\ell'$$

$$= 2V\ell - \ell^{-1}\left(\tfrac{1}{2}\ell' - 1\right)^2 + \ell'\left(\tfrac{1}{2}\ell' - 1\right)/\ell$$

$$\therefore \quad 2\ell\ell'' = 8V\ell^2 - (\ell' - 2)^2 + 2\ell'(\ell' - 2)$$

$$\boxed{\therefore \quad 2\ell\ell'' - \ell'^2 - 8V\ell^2 + 4 = 0}$$ (Kalvis saw this)

$\ell(0) = 0$, $\ell \geq 0$, (bounded?)

If we set $\ell = \varphi^2$, we get

$$\boxed{\varphi^3\varphi'' - 2V\varphi^4 + 1 = 0}$$ if it's any easier.

Example: $\ell(x) = A(1 - e^{-\alpha x})$. This gives

$$V(x) = \frac{4 - A^2\alpha^2 \, e^{-\alpha x}\{3 - 2e^{-\alpha x}\}}{8A^2(1 - e^{-\alpha x})}$$

In order that $V$ is bdd near 0, $A\alpha = 2$, and then

$$V(x) = \alpha^2 \frac{1 - e^{-2\alpha x}}{8} \geq c\,\ell(x) \text{ iff } c \leq \alpha^3/8.$$

[Kalvis says that writing $\ell = y^2$ simplifies the ODE – to

$$y'' - 2yV + y^{-3} = 0$$ --- any easier? ]

so that $l_x = E L_x$ solves

$$d l_x = 2\left\{1 + \frac{\psi'}{\psi}(x)\, l_x\right\} dx$$

which is solved by

$$\boxed{l_x = \psi(x)^{+2} \int_0^x 2\psi(y)^{-2}\, dy}$$

The interesting thing here is that one sometimes in polymer physics writes $V = V_0 + U$, where $V_0$ is the underlying potential, $U$ is the 'mean-field' interaction; and one must have then that

$$U(x) \propto l_x .$$

This consistency condition restricts the forms of $U$ possible; we must have ($V_0 \equiv 0$)

$$\begin{cases} (\psi^2 U)' = 2\psi^{-2} \\ \tfrac{1}{2}\psi'' - U\psi = 0 \end{cases} \Rightarrow \quad \psi\psi''' - 3\psi'\psi'' = 4\psi^2 ,$$

for example. We also have $\psi(0) = 1$, $\psi \geq 0$ decreasing, $U_0 = 0$, $U_0' = 2$, whence $\psi''(0) = 0$.

## Polymers between plates (2/2/90)

Take a diffusion in $[0,b]$ with generator $\mathcal{G}$, and kill at rate $\lambda$, to get a new Mkv process on $[0,b] \cup \{\partial\}$. Let's enlarge the statespace to $[0,b] \cup \{\partial\} \cup \{\Delta\}$, where $\Delta$ is another graveyard state, to which the process goes if killed in another way.

Physicists have to think in terms of modifying the laws $\mathbb{P}^x$ of this killed process by forming

$$\tilde{\mathbb{E}}^x[Y] \equiv \mathbb{E}^x\left[ Y \exp\left\{-\int_0^\infty V(X_s)\, ds\right\}\right] / \psi(x)$$

where

$$\psi(x) \equiv \mathbb{E}^x \exp\left\{-\int_0^\infty V(X_s)\, ds\right\}$$

and where $V : [0,b] \cup \{\partial\} \cup \{\Delta\} \longrightarrow \mathbb{R}$, $V(\partial) = 0 = V(\Delta)$. This is like killing $X$ at rate $V$, and then conditioning on $\lambda$-killing before $V$-killing. Thus if the process goes to $\Delta$ when V-killed, it's like conditioning on being in $\partial$ ultimately.

One has that

$$f(X_t) - \int_0^t \{ \mathcal{G} f(X_s) + \lambda (f(\partial) - f(X_s)) \} ds \text{ is a mg}$$

and so $\psi(x) \equiv E^x \exp(-\int_0^\infty V(X_s)\, ds)$ must solve

$$\boxed{\mathcal{G}\psi + \lambda(1-\psi) - V\psi = 0.} \qquad (\psi(\partial) = 1, \ \psi(\Delta) = 0)$$

When one h-~~trans~~forms, the generator $\mathcal{L}$ goes to $\frac{1}{h}\mathcal{L}(h\,\cdot)$, and in this case we get a new generator

$$\boxed{\begin{aligned} \tilde{\mathcal{L}} f &\equiv \frac{1}{\psi}\mathcal{L}(\psi f) \\ &= \mathcal{G} f + \frac{\psi'}{\rho\psi} f' - \frac{\lambda f}{\psi} \end{aligned}}$$

where $\mathcal{G} = \frac{1}{2\rho} D^2 + bD$. So the new diffusion has an additional drift, and killing at rate $\lambda/\psi$.

It would be nice to get an invariant measure for this, plus information about probabilities of crossing the gap $[0, b]$.

<u>Particles being carried by waves and diffusing</u> (2/2/90).

Consider the SDE

$$dX_t = \sigma\, dB_t + f(X_t - ct)\, dt$$

where $f$ is periodic, period 1. Let $Y_t \equiv X_t - ct$, so that

$$dY_t = \sigma\, dB_t + \{ f(Y_t) - c \}\, dt$$

is an autonomous diffusion. Let $\bar{f} \equiv \int_0^1 f(t)\, dt$. Wlog, suppose $\bar{f} < c$.

The aim is to find out about the asymptotic distribution of $X_t$, which we do via $Y_t$. The scale function $s$ of $Y$ is given by

$$s'(y) = \exp\left[ -\int_0^y 2\sigma^{-2} (f(v) - c)\, dv \right]$$

$$= \exp\left(-\int_0^y g(v)\,dv\right) \qquad , \quad g(v) \equiv 2(f(v)-c)/\sigma^2$$

Notice that

$$s'(y+1) = \exp\left(-\int_0^{y+1} g(v)\,dv\right)$$

$$= s'(y) \exp\left(-\int_0^1 g(v)\,dv\right)$$

since $g$ is periodic, period 1. Write $\rho \equiv \exp\left(\int_0^1 g(v)\,dv\right) \in (0,1)$; we have

$$\boxed{s'(y) = \rho\, s'(y+1)}$$

and so $s'$ grows exponentially: in particular, $\int_{-\infty} s'(y)\,dy < \infty$, and so the diffusion $Y \to -\infty$ (surprise, surprise!)

Let $T_0 \equiv 0$, $T_{n+1} \equiv \inf\{t > T_n : |Y_t - Y_{T_n}| = 1\}$. Then the process $S_n \equiv Y_{T_n}$ is a random walk on the integers.

We have that $s(0) - s(-1) = \rho\,(s(1) - s(0))$, and so

$$p \equiv P^0(Y \text{ reaches } 1 \text{ before } -1) = \frac{\rho}{1+\rho} < \tfrac{1}{2}.$$

Thus by the CLT

$$\boxed{\frac{S_n - n(p-q)}{(4npq)^{\frac{1}{2}}} \xrightarrow{\mathcal{D}} N(0,1)} \qquad (q \equiv 1-p).$$

If $\mu \equiv ET_1$, then $T_n/n \xrightarrow{a.s.} \mu$ and so we conclude that

$$\frac{Y_t - t(p-q)/\mu}{(4tpq/\mu)^{\frac{1}{2}}} \xrightarrow{\mathcal{D}} N(0,1)$$

and from this we get that

$$\boxed{X_t \sim \left(c + \frac{p-q}{\mu}\right)t + \sqrt{\frac{4tpq}{\mu}}\; N(0,1) \quad \text{for large } t}$$

How to calculate $\mu$? Putting $Y$ into natural scale gives diffusion with speed measure $m(dx) = (\sigma\, s' \circ s^{-1}(x))^{-2}\, dx$, so for $a<0<b$

$$\mu = E^0(\tau_{ab}) = 2\int_{s(a)}^{s(0)} \frac{(s(b)-s(0))(y-s(a))}{s(b)-s(a)} \frac{dy}{\sigma^2 s'(s^{-1}(y))^2}$$

$$+2\int_{s(0)}^{s(b)} \frac{(s(0)-s(a))(s(b)-y)}{s(b)-s(a)} \frac{dy}{\sigma^2 s'(s^{-1}(y))^2}$$

So if we take $a=-1$, $b=1$, assume $s$ normalised so that $s(1)-s(0)=1$, $s(0)-s(-1)=\rho$, we've got

$$\mu = \frac{2}{\sigma^2(1+\rho)}\left\{\int_{-1}^{0} \frac{s(x)-s(-1)}{s'(x)}\,dx + \rho\int_0^1 \frac{s(1)-s(x)}{s'(x)}\,dx\right\}$$

$\therefore$

$$\boxed{\mu = \frac{2}{\sigma^2(1+\rho)} \int_0^1 \frac{dx}{s'(x)} \left\{ s(x)-s(0) + \rho\,(s(1)-s(x))\right\}}$$

## Some observations on an example of Hinch + Leal JFM52 683. (15/2/90)

Let $P(X) \equiv I - XX^T$ be projection onto $T_X S^2$, and let $K$ be a fixed matrix. The problem considered by Hinch + Leal is a special case of the SDE

$$dX = \varepsilon P(X)\,dB - \varepsilon^2 X\,dt + P(X)\,KX\,dt$$

for a diffusion on $S^2$. In the H-L example, $K = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \alpha_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

1). K symmetric. This case is particularly easy, because the Brownian motion on $S^2$ has been perturbed by the gradient of a function $X^TKX/2\varepsilon^2 \equiv h(X)$, say. To check this, let $V$ be the v.f. $P(X)KX$, and let $U$ be some other Vf on $S^2$. Then, with $\langle\cdot,\cdot\rangle$ denoting the inner product on $S^2$ with the metric tensor given by $\varepsilon^{-2}\times$ Euclidean one, and $(\cdot,\cdot)$ the usual inner product on $\mathbb{R}^3$, we have

$$\langle V, U\rangle_X = \varepsilon^{-2}\big(P(X)KX, U\big)_X = \varepsilon^{-2}\,(KX, U)_X = Uh(X)$$
$$\equiv \langle \operatorname{grad} h, U\rangle_X .$$

Thus the generator of the diffusion in local coordinates is

$$\frac{1}{2}\,\frac{1}{\sqrt{\det g}\;e^{2h}}\,D_i\left(\sqrt{\det g}\;g^{ij}\,e^{2h}D_j\right) = \frac{1}{2}\varepsilon^2\Delta + g^{ij}D_ih\,D_j ,$$

so that $e^{2h}$ <u>is an invariant density for the diffusion</u>.

2) An observation which might be helpful is that if $Y$ solves the SDE in $\mathbb{R}^3$

$$dY = |Y|\,dB + KY\,dt$$

then $Y/|Y|$ has the same law as $X$.

3) In general, if one has a diffusion with generator $\frac{1}{2}\Delta + V$ on a compact manifold, then an invariant density $\pi$ must satisfy

$$\mathcal{G}^*\pi \equiv \tfrac{1}{2}\Delta\pi - V\pi - \pi\operatorname{div}V = 0$$

4) Going back to the $(\theta,\varphi)$ parametrisation of $S^2$, the generator of BM is

$$\frac{1}{2}\left(\operatorname{cosec}^2\varphi\,\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\varphi^2} + \cot\varphi\,\frac{\partial}{\partial\varphi}\right)$$

Now the most general $K$ can be split into $A = \frac{1}{2}(K-K^T)$, $S = \frac{1}{2}(K+K^T)$, and $A$ represents a rotation about some axis, wlog the N-S pole. This then just adds on the v.f. $\alpha\frac{\partial}{\partial\theta}$, corresponding to $A = \begin{pmatrix}0 & -\alpha & 0\\ \alpha & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$. The expression for the v.f. $P(X).SX$ in $(\theta,\varphi)$ coordinates is a mess, but even if $S = \begin{pmatrix}0 & b & 0\\ b & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ one gets the ugly coupled SDEs

$$\boxed{\begin{cases} d\theta = \varepsilon\operatorname{cosec}\varphi\,dB + (\alpha + b\cos 2\theta)\,dt \\ d\varphi = \varepsilon\,d\tilde{B} + \left\{\frac{1}{2}\varepsilon^2\cot\varphi + \frac{b}{2}\sin 2\theta\sin 2\varphi\right\}dt \end{cases}}$$

Any offers?!

## Estimating $\sigma^2$ from $X_t, S_t, I_t$ again (21/2/90)

1) We seek some function $g$ such that

$$E^{c,\sigma^2}[\,g(X_t, S_t, I_t)\,] = t\sigma^2$$

for all $t \geq 0$, $\sigma^2$, $c \in \mathbb{R}$. Here $P^{c,\sigma^2}$ is law of $\sigma B_t + ct \overset{\mathcal{D}}{=} B_{\sigma^2 t} + ct$. This expectation is

$$E^{c/\sigma^2, 1}\left[g(X_{t\sigma^2}, S_{t\sigma^2}, I_{t\sigma^2})\right]$$

so abbreviating $P^{c,1}$ to $P^c$, we require

$$E^c\, g(X_t, S_t, I_t) = t \qquad \forall t \geq 0 \;\; \forall c \in \mathbb{R}$$

implying

$$E^0\left[g(X_t, S_t, I_t)\, e^{cX_t}\right] = t\, e^{c^2 t/2} \qquad \forall c \in \mathbb{R} \;\; \forall t \geq 0.$$

Hence in particular,

$$E^0\left[g(X_t, S_t, I_t) \,\middle|\, X_t = x\right] = t \qquad \forall x \in \mathbb{R},\; t \geq 0.$$

Reworking this into a statement about $X_{T_\lambda}, S_{T_\lambda}, I_{T_\lambda}$, we get $(\theta \equiv \sqrt{2\lambda})$

$$\boxed{E^0\left[\,g(X_{T_\lambda}, S_{T_\lambda}, I_{T_\lambda}) \,\middle|\, X_{T_\lambda} = x\right] = \frac{\theta|x| + 1}{\theta^2}, \quad x \in \mathbb{R},\; \lambda > 0.}$$

We already know that there exist such $g$ (for example, $g(x, s, j) = 2s(s-x)$), but since the joint law of $S_{T_\lambda}, I_{T_\lambda}$ given $X_{T_\lambda} = x$ is so complicated, this looks still impossible.

2) Just for the record, we get the following. Conditional on $X_{T_\lambda} = x \geq 0$, the value of $-I_T \equiv -I_{T_\lambda} \sim \exp(2\theta)$. If we split at the time of the minimum, then we see a BM with drift $-\theta$ running down from 0 to the minimum, and, backwards from $T$, a BM with drift $-\theta$ running down from $x$ to the minimum. Here

$$P\left[S_T \leq a \,\middle|\, X_T = x,\; I_T = -v\right] = \frac{s(a) - s(0)}{s(a) - s(-v)} \cdot \frac{s(a) - s(x)}{s(a) - s(-v)} \qquad \forall\; a \geq x,$$

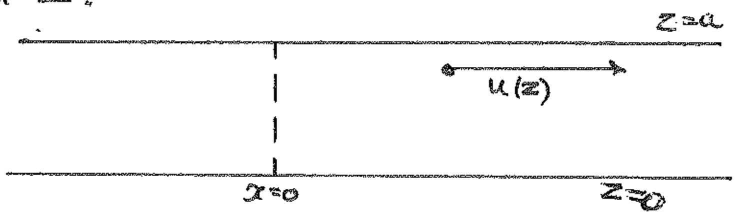
where $s(x) = e^{2\theta x}$ is the scale function. From this, various manipulations give

$$\boxed{p(u, v \,|\, x) = \frac{2\theta^2 e^{-\theta x}}{\sinh^3 \theta(x+u+v)} \left\{ \sinh\theta(x+v)\, \sinh\theta(x+u) + \sinh\theta u\, \sinh\theta v \right\}}$$

for the proof that $S_T = x+u$, $I_T = -v$ given $X_T = x$. The LT could be inverted in terms of BES(3) first-passage densities, but those are not known explicitly either.

## Notes on some fluid flow papers (2/3/90)

(a) N. G. Barton JFM 136. The idea is that one has a flow in the $x$ direction with velocity $u(z)$ at height $z$.




One assumes that the $z$-component of the motion is a 1-dimensional diffusion in $[0,a]$, and that the $x$-component is simply

$$X_t = \int_0^t u(Z_s)\,ds.$$

The idea is to solve the diffusion equation

$$\frac{\partial c}{\partial t} + u(z)\frac{\partial c}{\partial x} = \mathcal{G}_z c, \quad c(t;0,z) = e^{i\omega t} g(z).$$

If we write $\varphi(t;x,z) \equiv c(-t;x,z)$, then this is rephrased as

$$\boxed{\mathcal{G}_z\varphi - u(z)\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial t} = 0, \quad \varphi(t;0,z) = e^{-i\omega t} g(z).}$$

Letting $(\xi_t, \zeta_t)$ be diffusion with generator $\mathcal{G}_z - u(z)\frac{\partial}{\partial x}$, we have the probabilistic solution

$$\varphi(t;x,z) = E^{(x,z)}\left[e^{-i\omega\tau} g(\zeta_\tau)\right] e^{-i\omega t},$$

where $\tau = \inf\{u : \xi_u = 0\}$.

We can recast this in various ways; if $\tau_t \equiv \inf\{s : \int_0^s u(\zeta_v)\,dv > t\}$, and we time change by $\tau$, then we ~~transform~~ $(\xi,\zeta)$ into a new diffusion whose $\xi$-component moves leftward at constant speed 1, and we find ourselves looking for

$$E^{(0,z)}\left[e^{-i\omega A_t} g(\tilde{\zeta}_t)\right], \qquad A_t \equiv \int_0^t u(\tilde{\zeta}_s)^{-1}\,ds.$$

This is an essentially intractable problem; if one could solve it, one would know the law at time $t$ of $\tilde{S}$, and there are no diffusions in compact intervals that I know of where any useful closed-form expression exists for this.

(b) J. E. Houseworth JFM 142. The situation is like that of the previous paper, but now the problem is to compute

$$E \int_0^\infty I_{[0,b]}(X_s)\,ds,$$

or more generally $E\int_0^\infty f(X_s)\,ds = E\int_0^\infty f(t)\,u(J_t)^{-1}\,dt$, where $A_t \equiv X_t = \int_0^t u(Z_s)\,ds$, $\tau$ is inverse to $A$, $J \equiv Z\circ\tau$. In special cases, we might have a chance of $R_\lambda(\frac{1}{u})$, but that's about all.

(c) H. Yasuda JFM 148. The situation is like the two preceding ones, but now the velocity field depends on time, specifically

$$u(t,z) = \sin \omega t\; u(z).$$

The aim is to compute the law of $\int_0^t u(s, Z_s)\,ds$. This is a gruesome task for fixed $t$, but the asymptotics as $t\to\infty$ can be handled as on pp 20–21.

(d) R. Smith JFM 152 443–454. Like the others, except that now there is a depth-dependent killing. This is a steady-state model,

$$(\mathcal{G}_z + u(z)\tfrac{\partial}{\partial x} - \alpha(z))\,c = 0, \qquad c(0,z) = q(z)$$

where $q$ is known, so $c(x,z) \equiv E^{(x,z)}[q(Z_\tau)]$ exactly as before.

(e) G. I. Taylor, Proc Roy Soc A 219 186–203, 1953.

In the classical 'flow in a pipe' problem, the $Z$-component moves in $[0,a]$ like a BES(2) process, with generator

$$\tfrac{1}{2}\sigma^2\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz}\right)$$

and $u(z) = u_0\,(1 - z^2/a^2)$. The diffusion in the axial direction is ignored, and the mean velocity is easily computed to be $\frac{1}{2}u_0$, asymptotic variance of position $X_t$ is

$$t \cdot \frac{u_0^2\,a^2}{48\,\sigma^2} \qquad \text{by result on p. 19.}$$

Ron Smith (JFM $\underset{\sim}{130}$, 299–314) performs a diffeomorphism of a region to make it nicer. If $(M,g)$ is a Riem. mfd, we define grad $f$ by $(U, \operatorname{grad} f) = Uf$ $\forall\, U \in \mathfrak{X}(M)$ and div by $\int_M \varphi \operatorname{div} U \, dm = -\int_M (U, \operatorname{grad}\varphi)_g \, dm$

Heat equation says

$$\boxed{\frac{\partial}{\partial t}\left(\int v\, c\, dm\right) = \int v \operatorname{div}(K \operatorname{grad} c)\, dm} \qquad \forall\, v \in C_c^\infty(M)$$

Now if $\varphi : (M,g) \to (N,h)$ is a diffeomorphism with inverse $\psi$, I make it that

$$\frac{\partial}{\partial t}\left(\int_M v\, c\, dm\right) = \frac{\partial}{\partial t}\left(\int_N (vc)\circ\psi \, dn\right) = \int_M v \operatorname{div}(K \operatorname{grad} c)\, dm$$

$$= \int_N v\circ\psi \operatorname{div}(K^* \operatorname{grad}(c\circ\psi))_h \, dn$$

where $K^* = D\varphi\, K\, (D\varphi)^T$, equivalently,

$$(U^*, K^* V^*)_h = (U, V)_h \qquad \forall\, U, V \in \mathfrak{X}(M).$$

[In coordinates, $\operatorname{grad} f = (g^{ik} D_k f) D_i$ , $\operatorname{div} U = (\det g)^{-\frac{1}{2}} D_i \left( (\det g)^{\frac{1}{2}} U^i \right)$ ]

(f) R. Smith JFM (1990) Takes flow in a narrow gap to be essentially given by a 2-dimensional manifold $S \times [0, \varepsilon]$, where the width $\varepsilon$ can vary with position in $S$, and with time. The velocity field $u$ decomposes into $(u_S, w)$ in $S$, and across the gap respectively. The generator is assumed to be of the form $\nabla_S \cdot (K_S \nabla_S) + \partial_n (K \partial_n) + u_S \cdot \nabla_S + w \partial_n$ (where we assume constant density, for simplicity. The idea is terribly simple; at each $x \in S$, one has a diffusion with generator $\partial_n (K(x, \cdot) \partial_n) + w(x, \cdot) \partial_n$ across the gap, and one simply replaces this term with the asymptotic mean drift + diffusion according to the recipe computed on p. 19.

(g) Total dosage problem

Suppose one releases stuff across the stream at $x=0$ (and this is not just restricted to parallel-sided regions, as the method of R. Smith JFM 130 299-314 shows that one can apply a diffeomorphism to a varying channel and reduce to this) in a time-dependent way so that the concentration $c(t; x, y)$ is known at $x=0$ for all $t, y$. Suppose we define

$\int c(t; x, y)\, dt \equiv C(x, y)$ the total dosage, and suppose that

$c(t; 0, y) = 0 \quad \forall |t| > N, \forall y$. Then from the diffusion equation,

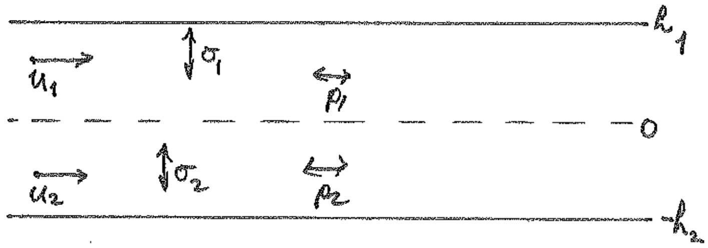
$$\mathcal{G}^* C = 0, \qquad C(0, y) \text{ is known,}$$

so to compute $C$, we just need to calculate the solution to the Dirichlet problem for the reversed diffusion, which is not hard. We certainly get a bound for total dosage, and a spatially-even input of stuff keeps the highest dosage smallest.

(h) One can similarly characterise the solution of $\dot{c} = \mathcal{G}^* c$ as a solution to the Dirichlet problem for the reversed diffusion:

$$c(t; x, y) = E^{*(x,y)} \left[ c(0; X_t, Y_t) \right].$$

(i) Chickwendu + Ojiakor JFM $\underset{\sim}{152}$. In this example, a two-zone model for the flow field is considered, with velocity $u_j$, transverse diffusion $\sigma_j$, longitudinal diffusion $\rho_j$ in zone $j$. By rescaling the zones, we could take $\sigma_1 = \sigma_2 = 1$, wlog.



If we now let $A_t \equiv \int_0^t I_{(0,h_1)}(Z_s)\,ds$ be the time spent in the upper zone, it's really just a one-dimensional problem; we can compute (for example)

$$E^0\left[e^{-\alpha A_T - \beta(T-A_T)} ; Z_T \in dx\right] / dx$$

$T \sim \exp(\lambda)$, $\frac{1}{2}\theta^2 = \lambda+\alpha$, $\frac{1}{2}\tilde{\theta}^2 = \lambda+\beta$

$$= \frac{\theta^2 \cosh\theta(h_1-x)}{\cosh\theta h_1 \left\{ \theta\tanh\theta h_1 + \underbrace{\tilde{\theta}}_{} \tanh\tilde{\theta} h_2 \right\}} \qquad (x>0)$$

const other than 1 is possible here for skewed case.

from which in principle one can derive the law of $(X_T, Z_T)$.

(j) Polymers as Brownian paths of exponential length. If we think of a polymer suspension as a load of non-interacting Brownian path fragments of lengths $\sim \exp(\lambda)$, starting points distributed as a Poisson process rate $\alpha$ through $\mathbb{R}^3$, then near an obstacle, there will be thinning of the Poisson process corresponding to the polymers hitting the obstacle. For example, at distance $x$ from a flat wall, prob of thinning is $e^{-x\sqrt{2\lambda}}$, distance $x$ from sphere radius $r$ it's $r(r+x)^{-1}e^{-\theta x}$, between two plates $2a$ apart, distance $x$ from one, it's $\operatorname{sech}\theta a \cosh\theta(x-a)$.

(k) Polymers at a sticky wall. If one takes an infinite flat wall, with stickiness $c>0$, and considers a BM $B$ in $\mathbb{R}^+$ reflected at the wall, then

$$f(|B_t|)\exp(-\lambda t + cL_t)) \quad \text{is a martingale}$$

if $\frac{1}{2}f'' = \lambda f$, $f'(0) = \lambda c f(0)$ (here $B$ is ordinary non-sticky BM). By solving this de and time-changing by inverse to $t+cL_t$ in order to make the wall sticky, we get, for example,

$$P^x\left(\sup_{[0,T]} X_s > \xi\right) = P^x(H_\xi < T) = \frac{f(x)}{f(\xi)} = \frac{\theta\cosh\theta x + \lambda c\sinh\theta x}{\theta\cosh\theta\xi + \lambda c\sinh\theta\xi},$$

the law of the sup of the path. Or if one wants to know the dist$^n$ of the no of times it crosses over to $\delta>0$ and gets back (ie there is a "loop"), one has

$$P(\text{no of loops is} \geq n) = p^n,$$

where

$$p \equiv \frac{f(0)}{f(\delta)}\,e^{-\theta\delta}.$$

This also answers the question of the dist$^n$ of the number of crossings of a gap.

(l) Rephrasing Boltzmann weighting in terms of change of drift. Suppose we tries to transform Wiener measure using a Boltzmann weighting
$$\propto \exp\{-\int_0^1 u(X_s)ds\}.$$

Then $E[\exp(-\int_t^1 u(X_s)ds)\,|\,\mathcal{F}_t] = \varphi(1-t, X_t)$, where $\varphi(0,x) \equiv 1$, and
$$\boxed{\tfrac{1}{2}\varphi'' - \dot{\varphi} - u\varphi = 0.}$$

Thus the Boltzmann weighting is equivalent to introducing a drift $\varphi'(t,x)/\varphi(t,x)$. There seem to be few examples one can compute exactly, but, of course, one can very easily convert a CM drift into the equivalent Boltzmann weighting (usually with a time-dependent $u$).

Here are two explicit examples.

(i) $u(x) = \lambda x$. In this case,
$$M_t \equiv E\left(\exp - \lambda \int_0^1 B_s ds \,|\, \mathcal{F}_t\right) = e^{-\lambda\int_0^t B_s ds}\,\varphi(t, B_t),$$
where $\varphi(t,x) \equiv \exp\{-\lambda x(1-t) + \lambda^2(1-t)^3/6\}$, yielding a new measure with drift
$$-\lambda(1-t).$$

(ii) $u(x) = \lambda x^2$. The problem is to compute
$$E^{\xi} \exp(-\lambda \int_0^T B_s^2 ds)$$

- the case $\xi = 0$ being well known (it's $(\operatorname{sech}\theta T)^{\frac{1}{2}}$, $\theta \equiv \sqrt{2\lambda}$, by Ray-Knight.) Squaring what we want, the essential part of the problem is to compute the law of time spent in $[0,T]$ by BM before $L(\cdot,0)$ reaches the value $2\xi^2$. We do this by marking BM at rate $\alpha \equiv \frac{1}{2}\gamma^2$ while in $[0,T]$; the rate of marked excursions turns out to be $\frac{\gamma}{2}\tanh\gamma T$. Assembling this gives
$$E^{\xi} \exp - \lambda \int_0^T B_s^2 ds \equiv \varphi(T,x) = (\operatorname{sech}\theta T)^{\frac{1}{2}} \exp\left(-\frac{x^2\theta}{2}\tanh\theta T\right).$$
The drift $\varphi'/\varphi$ is simply
$$-\theta x \tanh\theta(1-t).$$

(m) Boltzmann weighting of polymers of exponential length. We dealt with this question on pp 24–25. Again, explicit solutions are scarce, but here's one, where we take 1-dimensional BM, killed at rate $\lambda$, with potential $V = \alpha I_{[-a,a]}$. The PDE for $\psi$ is

$$\tfrac{1}{2}\psi'' = \begin{cases} \lambda(\psi - 1) & \text{off } [-a,a] \\ (\lambda+\alpha)\left(\psi - \frac{\lambda}{\lambda+\alpha}\right) & \text{on } [-a,a] \end{cases}$$

with $\psi \in C^1$. This gets solved by

$$\psi(x) = \begin{cases} 1 - A e^{-\theta(|x|-a)}, & |x| \geq a \\ \frac{\lambda}{\lambda+\alpha} + B\,\gamma \cosh \gamma x, & |x| \leq a, \end{cases}$$

where $\theta \equiv \sqrt{2\lambda}$, $\gamma \equiv \sqrt{2\lambda + 2\alpha}$, and

$$(\lambda+\alpha)\left(\theta \cosh \gamma a + \gamma \sinh \gamma a\right)\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha\gamma \sinh \gamma a \\ \alpha\theta \end{pmatrix}.$$

The additional drift $\psi'/\psi$ and modified killing $\lambda/\psi$ are immediate.

(ii) <u>Self-consistent potential for BM in a gap, of exp($\lambda$) length.</u> We modify BM of exp($\lambda$) duration in an interval $[0,a]$ by a Boltzmann weighting $\exp\{-\int_0^\tau V(X_s)ds\}$. Want this $V$ to be the "interaction correction", so need that $V$ is fundamental e-function of $\tilde{\mathcal{L}}^*$, where $\tilde{\mathcal{L}}$ is the transformed generator; see p 25. This looks not particularly tractable.

<u>More on estimation of variance from high, low, closing price</u> 12/3/90.

Assuming zero drift, on a unit time interval, let $\hat{\sigma}_0^2 \equiv X_1^2$, the crudest estimate of variance, let $\hat{\sigma}_1^2 \equiv S_1(S_1 - X_1) + I_1(I_1 - X_1)$ be the good one, and let $\hat{\sigma}_2^2 \equiv 0.511(S_1 - I_1)^2 - 0.019(X_1(S_1+I_1) - 2I_1S_1) - 0.383 X_1^2$ be the Garman-Klass estimator. Then all are unbiased, and

$$\boxed{\operatorname{var}(\hat{\sigma}_0^2) = 2\sigma^4, \quad \operatorname{var}(\hat{\sigma}_1^2) = 0.331\,\sigma^4, \quad \operatorname{var}(\hat{\sigma}_2^2) = 0.27\,\sigma^4.}$$

So the good estimator $\hat{\sigma}_1^2$ is not too much worse than the 'optimal' G-K estimator, even in the ideal case of zero drift. (81.5% as good, in fact.)

If we were allowed to use $n$ observations, and took the natural unbiased estimator based on $X_1, \ldots, X_n - X_{n-1}$, we'd have $(n-1)^{-1}\sum_1^n (y_j - \bar{y})^2$ ($y_j \equiv X_j - X_{j-1}$) which is unbiased, but has variance $\sigma^4/(n-1)$ --- so asymptotically half as bad as $\hat{\sigma}_0^2$!!

## Self-consistency for polymers, both chain, and branching (11/4/90)

1) Let's consider first a BM killed at rate $\lambda$, started in $(0,\infty)$ and conditioned not to hit $0$ before killing. If we introduce a potential $V$, let $A_t \equiv \int_0^t V(X_s)ds$ be the corresponding additive functional, and reweight the original law by $\exp(-A_\zeta)$, where $\zeta$ is the $\lambda$-killing time, then we computed this on p. 25; the diffusion has generator

$$\check{\mathcal{L}} f = \tfrac{1}{2} f'' + \frac{\psi' f'}{\psi} - \lambda f/\psi$$

where $\tfrac{1}{2}\psi'' + \lambda(1-\psi) - V\psi = 0, \quad \psi(0) = 0, \quad 0 \le \psi \le 1.$

If we were to take $\varphi(x) \equiv \tilde{E}^x\left[\int_0^\zeta g(X_t)\,dt\right]$ - expectation for the reweighted law - then $\varphi$ solves

$$\tfrac{1}{2}\varphi'' + \frac{\psi'\varphi'}{\psi} - \frac{\lambda\varphi}{\psi} + g = 0,$$

and thus with $g = \delta_a$ we have to solve

$$\begin{cases} \tfrac{1}{2}\varphi'' + \dfrac{\psi'\varphi'}{\psi} - \dfrac{\lambda\varphi}{\psi} = 0 & \text{in } (0,a)\cup(a,\infty) \\ \varphi'(a+) - \varphi'(a-) = -2, & \varphi(0) = 0, \quad \varphi \ge 0 \end{cases}$$

This can be re-expressed as

$$\begin{cases} \tfrac{1}{2}(\varphi\psi)'' = \varphi\psi(\lambda + V) & \text{in } (0,a)\cup(a,\infty) \\ \varphi'(a+) - \varphi'(a-) = -2, & \varphi(0) = 0, \quad \varphi \ge 0 \end{cases}$$

Now let $u_\pm$ be the increasing/decreasing solutions of $\tfrac{1}{2}u'' = (\lambda+V)u$. The function $U \equiv \varphi\psi$ can be expressed as

$$U(x) = \begin{cases} c\,\{u_+(x)\,u_-(0) - u_-(x)\,u_+(0)\} & 0 \le x \le a \\ c\,\{u_+(a)\,u_-(0) - u_-(a)\,u_+(0)\}\,\dfrac{u_-(x)}{u_-(a)} & a \le x \end{cases}$$

where $c$ is chosen to make the jump of derivative correct; we have

$$c = \frac{2\,u_-(a)\,\psi(a)}{u_-(0)}\left\{u_-(a)\,Du_+(a) - u_+(a)\,Du_-(a)\right\}^{-1}$$

and

$$\boxed{\lim_{x \downarrow 0} \varphi(x) = \check{E}^0\left[L(\zeta, a)\right] = \frac{2\,\psi(a)\,u_-(a)}{\psi'(0)\,u_-(0)}.}$$

2) Let's now consider a BBM which gets killed at rate $\lambda$, splits into two at rate $\alpha \le \lambda$.

We want again to stick this thing in a potential $V$, so want to compute for $x>0$

$$\psi(x) \equiv E^x\Big[\exp\Big\{-\sum_{\text{particles}}\int_{\beta_i}^{\zeta_i} V(X_s^i)\,ds\Big\} : \text{no particle hits } 0\Big]$$

where $\beta_i$ is birth time of $i^{\text{th}}$ particle, $\zeta_i$ the death time, $X^i_\cdot$ the path of the $i^{\text{th}}$ particle. Keeping track of the killing at zero is notationally cumbersome, so let's absorb this into the V-killing by adding $\delta_0$ to $V$. If we let $(\tilde{R}_\lambda)_{\lambda>0}$ be the resolvent of Brownian motion with V-killing, then if $T$ is $\exp(\gamma)$, $\gamma \equiv \lambda+\alpha$, we have

$$E^x\Big[f(X_T)\exp\Big(-\int_0^T V(X_s)ds\Big)\Big] \equiv g(x), \quad \text{say},$$

will solve

$$\gamma \tilde{R}_\gamma f = g.$$

Hence by considering what happens to the BBM at time of first split/death, we have that $\psi$ must satisfy

$$\boxed{\psi = \alpha \tilde{R}_\gamma \psi^2 + \lambda \tilde{R}_\gamma 1 .}$$

Next, let's characterise the transformed BBM. The statespace is $\{\partial,\partial'\}\cup\big(\bigcup_{n\geq 1}(0,\infty)^n\big)\equiv E$, where $\partial$ is graveyard for death by natural causes ($\lambda$-killing), and $\partial'$ the graveyard for death by V-killing. If $f: E\to[0,1]$ is smooth enough, we can specify the generator explicitly:

$$\mathcal{G}f(\partial) = \mathcal{G}f(\partial') = 0,$$

$$\begin{aligned}\mathcal{G}f(x_1,\dots,x_k) = {} & \frac{1}{2}\sum_{i=1}^k \frac{\partial^2 f}{\partial x_i^2} + \sum_1^k V(x_i)\{f(\partial')-f(x_i)\} \\ & + \lambda\sum_1^k\{f(x_1,\dots,\hat{x}_i,\dots,x_k)-f(x)\} \\ & + \alpha\sum_1^k\{f(x_1,\dots,x_k,x_i)-f(x)\}\end{aligned}$$

where the particles are ordered oldest first. The harmonic $f^n$ $h$ by which we h-transform is $h(\partial)=1$, $h(\partial')=0$, $h(x_1,\dots,x_k) = \psi(x_1)\cdots\psi(x_k)$. We compute the new generator $\tilde{\mathcal{G}}$ in the usual way: $\tilde{\mathcal{G}}f = \frac{1}{h}\mathcal{G}hf$.

Using the equivalent form

$$(\gamma - \tfrac{1}{2}D^2 + V)\psi = \alpha\psi^2 + \lambda$$

of the boxed equation above, one can work out the transformed BBM;

$$\boxed{\text{the particles diffuse according to } \mathcal{A}f = \tfrac{1}{2}f'' + \frac{\psi' f'}{\psi}, \text{ killed at rate } \frac{\lambda}{\psi}, \text{ split rate } \alpha\psi.}$$

What about "monomer density"? Fixing some $f_0 \in C_b(\mathbb{R}^+)$, we consider

$$u(x) \equiv \tilde{E}^x [\text{total integral of } f_0 \text{ along all parts of the polymer}].$$

We shall have

$$u(x) = \left(\frac{\lambda}{\psi} + \alpha\psi - \mathcal{G}\right)^{-1} f_0 + \left(\frac{\lambda}{\psi} + \alpha\psi - \mathcal{G}\right)^{-1} 2\alpha\psi u.$$

whence

$$\boxed{\left(\frac{\lambda}{\psi} + \alpha\psi - \mathcal{G}\right) u = f_0 + 2\alpha\psi u.}$$

<u>No stochastic converse to Kronecker's Lemma (12/4/90)</u>

If $Y_n$ are indep<sup></sup>t, mean zero, finite variance, and $a_n \uparrow \infty$, then

$$\sum \frac{EY_n^2}{a_n^2} < \infty \Rightarrow \sum Y_n/a_n \text{ cgt} \Rightarrow \frac{S_n}{a_n} \equiv \frac{Y_1 + \dots + Y_n}{a_n} \xrightarrow{a.s.} 0.$$

To see that the converse doesn't hold in general, take $Y_i$ i.i.d. $N(0,1)$, so

$$\frac{S_n}{a_n} \equiv \frac{B_n}{a_n} \xrightarrow{a.s.} 0 \quad \text{iff} \quad (2n\log\log n)^{\frac{1}{2}}/a_n \to 0$$

and $\displaystyle\sum_1^T Y_n/a_n = \int_0^T \frac{dB_s}{a(s)}$ is a.s. cgt iff $\displaystyle\int_0^\infty \frac{ds}{a(s)^2} = \sum_1^\infty \frac{1}{a_n^2} < \infty$

But in general $a_n^{-2} = \varepsilon_n/(2n\log\log n)$, where $\varepsilon_n \to 0$, and that isn't always cgt.

<u>SDE for the local time process of BES(n), $n \geq 3$. (12/4/90)</u>

We know that if we take a BES(3) process, the local time process $Z_x = L(\infty, x)$, $x \geq 0$, is a BESQ(2) process. Since a BES(n) can be obtained from a BES(3) by change of speed + scale, each $n \geq 3$, we can do some straightforward calculations and conclude that the loc. time process $Z_x$ for BES(n) satisfies

$$\boxed{dZ_x = 2\sqrt{Z_x}\, dW_x + \left\{2 - \frac{n-3}{x} Z_x\right\} dx.}$$

Thus $Z$ is expressible as the sum of two iid processes with common prescription

$$dY_x = 2\sqrt{Y_x}\, dW_x + \left\{1 - \frac{n-3}{x} \cancel{Y}_x\right\} dx, \quad Y_0 = 0$$

and $Y^{1/2}$ will solve

$$\boxed{dV_x = dW_x - \frac{n-3}{2x} V_x\, dx, \quad V_0 = 0}$$

Writing $\alpha \equiv (n-3)/2$, we can express the solution of this thing explicitly:

$$x^\alpha V_x = \int_0^x t^\alpha dW_t = x^\alpha W_x - \int_0^x \alpha t^{\alpha-1} W_t dt$$

so that
$$V_x = W_x - x^{-\alpha} \int_0^x \alpha t^{\alpha-1} W_t dt.$$

More on limit theorems for 1-dimensional transient diffusions (17/4/90)

Let $X$ be a transient diffusion in nat scale on $(0,1]$, 1 reflecting, and suppose that the tail $\sigma$-field is nontrivial, $M_t \equiv c(X_t) - t$, with $c(x) \equiv E^1(H_x)$. Suppose that

$$\frac{f(X_t)}{g(t)} \xrightarrow{a.s.} Z \qquad (t \to \infty)$$

Then we must have $\dfrac{g(t)}{g(t+a)} \to e^{\beta a}$ $(t \to \infty)$ and $Z = \text{const}.\ e^{\beta M_\infty}$.

We use the fact that $H_{1/2}$ has a positive density to deduce that for a.e. $a \in \mathbb{R}^+$, $P^{1/2}$-a.s. $f(X_t)/g(t+a) \xrightarrow{a.s.}$ . Thus for a.e. $a$, $g(t+a)/g(t) \to$ as $t \to \infty$. Let $N$ be the Lebesgue null set where the convergence of $g(t+a)/g(t)$ fails. Then for $a \in G \equiv N^c$,

$$\frac{g(t+a)}{g(t)} \longrightarrow \lambda(a)$$

and evidently $G$ is a semigp, and if we consider $G$ as a subset of $\mathbb{R}$, then $G$ is a group. Now if $A_n = \{x \in [0,1] : 2^{-n}x \in N\}$, then $A_n$ is a null set, so $\exists\, x_0 \in [0,1]$ s.t. $2^{-n}x_0 \in G$ for all $n \in \mathbb{Z}$, and so $\lambda(a) = e^{\beta a}$ for some $\beta \in \mathbb{R}$, for $a \in G$. Monotonicity of $g$, and continuity of $\lambda$, together force $G = \mathbb{R}$.

Now let $\tilde{f} = f \circ c^{-1}$. We have $\tilde{f}(M_t + t)/g(t) \to Z$ a.s., so

$$\frac{\tilde{f}(M_t+t)}{g(t)} = \frac{\tilde{f}(M_\infty + s)}{g(M_\infty + s)} \cdot \frac{g(M_\infty + s)}{g(M_\infty + s - M_t)} \xrightarrow{a.s.} Z \qquad [s \equiv M_t + t - M_\infty]$$

But $g(M_\infty + s)/g(M_\infty + s - M_t) \to e^{\beta M_\infty}$, and so must have $\tilde{f}(u)/g(u)$ cgt, whence stated result.

Asymptotics of perturbed ODEs (19/4/90)

Phil Griffin asks about what happens to the solution of

$$dX_t = \varepsilon\, dB_t + b(X_t)\, dt \qquad X_0 = 0,$$

as $\varepsilon \downarrow 0$. Let's suppose without any great loss of generality that $b$ is odd, $x\, b(x) > 0 \quad \forall x \neq 0$.

One thing one could do is to compute the mean exit time from $[-1,1]$ and see how that behaves as $\varepsilon \downarrow 0$. We get

$$\tfrac{1}{2} E\, T = \varepsilon^{-2} \int_0^1 s'(y)^{-1} dy \,\{s(1) - s(y)\}, \quad \text{where } s'(x) \equiv \exp\Big\{-2\int_0^x b(u)\,du/\varepsilon^2\Big\}$$

is the derivative of the scale function, as usual. So we want to know about the asymptotics as $\varepsilon \downarrow 0$ of

$$\varepsilon^{-2} \int_0^1 dy \int_y^1 dt \;\exp\Big\{-\frac{2}{\varepsilon^2}\int_y^t b(u)\,du\Big\},$$

in particular, under what conditions on $b$ does this thing remain bounded?

Let $g(t) \equiv \int_0^t b(u)\,du$, an increasing $f^c$, with inverse $\gamma$, say. We want to know about the behaviour as $\lambda \to \infty$ of

$$\lambda \int_0^1 dy \int_y^1 dt \;\exp\{-\lambda(g(t) - g(y))\}$$

$$= \lambda \int_0^{g(1)} dv \int_v^{g(1)} ds \;e^{-\lambda(s-v)} / b(\gamma_s)\, b(\gamma_v)$$

$$= \lambda \int_0^\infty \frac{dy}{\lambda} \int_y^\infty \frac{dt}{\lambda} \;e^{-(t-y)} / b(\gamma_{t/\lambda})\, b(\gamma_{y/\lambda}) \qquad \text{if } b \equiv +\infty \text{ on } (g(1),\infty)$$

Thus if $f \equiv b \circ \gamma$, we want to know about

$$\frac{1}{\lambda}\int_0^\infty dy \int_y^\infty dt \;\frac{e^{-(t-y)}}{f(t/\lambda) f(y/\lambda)} = \int_0^\infty \frac{dy}{\lambda}\int_0^\infty ds \;\frac{e^{-s}}{f(\frac{y}{\lambda}) f(\frac{y+s}{\lambda})}$$

$$= \int_0^\infty \frac{dv}{f(v)} \int_0^\infty \frac{e^{-s}\,ds}{f(v + s/\lambda)} \uparrow \int_0^\infty \frac{dv}{f(v)^2} \quad \text{as } \lambda \uparrow \infty$$

$$= \int_0^\infty \frac{dt}{b(t)}$$

Thus we have the neat criterion:

$$\boxed{\text{the mean exit time from } [-1,1] \text{ remains bdd iff } \int_{0+} b(t)^{-1}\,dt < \infty.}$$

[This is too clean not to have been done before ... the condition is the condition for there to exist a non-zero solution to $\dot{x}_t = b(x_t)$.]

See R. Bafico + R. Baldi Rend. Sem. Math. Univ. Politec. Torino (1982) 23–27 [MR 84e: 60111]
P. Baldi Ann Sci Clermont Ferrand II Math 20 (41–52) (1982) [MR 84k: 60101]

On some approaches to self-repelling BM (22/4/90)

For now, let's just look at the one-dimensional case.

Model 1 Let $X$ solve

$$(*) \qquad X_t = B_t + \int_0^t ds \int_0^s du\, f(X_s - X_u)$$

where $f$ is Lipschitz (which implies $\exists !$ solution). We earlier did some heuristics, so think we know the asymptotic form of the solution (simulations and heuristic arguments by Rick Durrett suggest that for the case $f \in L^1$, the ultimately linear growth we expected probably does not happen; it will go linearly for a long time perhaps, but will then flip over and go back the other way). However, if $|f(x)| \sim |x|^{-\beta}$, $0 < \beta < 1$, we still believe the scaling, that $X_t \sim t^\alpha$ where $\alpha = 2/(1+\beta)$.

(i) If we assume that $f \geq 0$, this should make things easier - certainly the pr. $\to \infty$. If now we also assume that $\boxed{f(x) \sim x^{-\beta}}$ $(x \to \infty)$, we would consider the rescaled process $x_t \equiv X_{tT}/T^\alpha$ $(0 \leq t \leq 1)$ which satisfies

$$x_t = T^{-\alpha} B_{tT} + T^{2-\alpha} \int_0^t ds \int_0^s du\, f\left(T^\alpha (x_s - x_u)\right).$$

We suspect that $x$ looks like the exact solution $\xi_t = \text{const}\ t^{-\alpha}$ which we get by setting $B \equiv 0$ and $f(x) = x^{-\beta} I_{(x>0)}$ in $(*)$, and so could write $x_t = \xi_t + \eta_t$, where $\eta$ should be small. Then

$$\eta_t = T^{-\alpha} B_{tT} + T^{2-\alpha} \int_0^t ds \int_0^s du \left\{ f(T^\alpha(x_s - x_u)) - f(T^\alpha(\xi_s - \xi_u)) \right\}$$
$$+ T^{2-\alpha} \int_0^t ds \int_0^s du \left\{ f(T^\alpha(\xi_s - \xi_u)) - c\left(T^\alpha(\xi_s - \xi_u)\right)^{-\beta} \right\}$$

and for $T$ large, the first and third terms combine to give something small;

$$\eta_t = \delta_t + T^{2-\alpha} \int_0^t ds \int_0^s du \left\{ f(T^\alpha(\xi_s - \xi_u + \eta_s - \eta_u)) - f(T^\alpha(\xi_s - \xi_u)) \right\}.$$

It is tempting to seek some "real-variable" proof that if $\delta_t$ is uniformly small, then so must $\eta$ be. However, this seems to be impractical; if we had $f(x) = x^{-\beta} I_{(x>1)}$, then $X \equiv 0$ is the solution, even for small $\delta$... !!

This may not be so wild, though; when $x \leq 0$, the third term in the penultimate displayed expression could be too big to neglect.

One thing may be worth noticing: approximating $\delta$ uniformly by smooth $\delta'$ gives solutions converging to the true sol$^n$ $\eta$, so could assume $\delta$ smooth if it helps.

(ii) Another way we could proceed would be to approximate the process by splitting at the times when we first hit level $-n$ (so take drift $-f$, with $f \geq 0$, so as to get SDE for local time going in forward direction), and between $H_{-n}$ and $H_{-n-1}$ take the drift to be $\varphi_n(x) = -\int_0^{H_{-n}} f(x - X_u)\,du \equiv -\int f(x-y) Z^n_y\,dy$, where $Z^n_y \equiv L(H_{-n}, y)$. Now if $\mathcal{J}^n \equiv Z^n - Z^{n-1}$, we have that

$$d\mathcal{J}^{n+1}_x = 2\sqrt{\mathcal{J}^{n+1}_x}\,dW^n_x + 2\{ I_{(x \leq -n)} - \mathcal{J}^{n+1}_x\,\varphi_n(x)\}\,dx, \quad \mathcal{J}^{n+1}_{-n-1} = 0.$$

This way, the process $(Z^n)_{n\in\mathbb{N}}$ is Markovian. Can compute $E[H_{-n-1} - H_{-n} \mid \mathcal{F}_{H_{-n}}]$ from this (or otherwise):

$$E\left[H_{-n-1} - H_{-n} / \mathcal{F}_{H_{-n}}\right] = 2\int_{-n-1}^{-n} da\ \exp\left(2\int^a \varphi_n(y)\,dy\right)\int_a^\infty \exp\left(-2\int^x \varphi_n(t)\,dt\right).$$

One snag: this approximation suffers the drawback that even if one looked at the crossing times to $-n\delta$ for some arbitrarily small $\delta$, one cannot expect that the thing we're considering should converge to the process of interest.

(iii) If we assume that $f$ is a symmetric function, then

$$2\int_0^t ds \int_0^s du\ f(X_s - X_u) = \iint \mu_t(dx)\,\mu_t(dy)\,f(x-y),$$

where $\mu_t$ is occupation measure of the path til time $t$. Can we get lower bound for this, and deduce a liminf for growth rate? Perhaps, but note that

$$\min \iint \mu(dx)\,\mu(dy)\,f(x-y) = 0$$

in the case where $f$ is supported in $[\frac14, \frac34] \cup [-\frac34, -\frac14]$, and $\mu$ is conc on $\mathbb{Z}$, so nothing this crude can work.

However, if $f(x) \geq \alpha I_{[-a,a]}$ for some $\alpha, a > 0$, then one can get a lower bound.

(iv) A large-deviations viewpoint would shrink $[0,T]$ by Brownian scaling, setting $x_t \equiv T^{-1/2} X_{tT}$, $\beta_t \equiv T^{-1/2} B_{tT}$ so that

$$x_t = \beta_t + T^{3/2}\int_0^t ds \int_0^s du\ f\left(T^{1/2}(x_s - x_u)\right).$$

If $\xi_t$ is the expected limit of $T^{-\alpha} X_{tT}$, then we could take

$$C_T = \{ x : |x_t - T^{\alpha - 1/2}\xi_t| > \varepsilon T^{\alpha - 1/2} \text{ for some } 0 \leq t \leq 1\}$$

as the set where a large deviation occurs, and look for

$$\inf_{x \in C} \int_0^1 \left( \dot{x}_s - T^{3/2} \int_0^s du \, f(\sqrt{T}(x_s - x_u)) \right)^2 .$$

The variational problem is ugly, but perhaps one could get this directly ...? Eg. some positive lower bound would help.

<u>Model II</u> This is the one which physicists would prefer. Reweight the Wiener measure with

$$\exp\left\{ - \int_0^T dt \int_0^T ds \, \psi(X_t - X_s) \right\}.$$

To find most likely paths, try to

$$\min \quad \tfrac{1}{2} \int_0^T \dot{x}_s^2 \, ds \; + \; \int_0^T dt \int_0^T ds \, \psi(x_t - x_s).$$

Variational problem gives us that, if $f(x) = \psi'(x) - \psi'(-x)$, then

$$\ddot{x}_u = \int_0^T f(x_u - x_s) \, ds .$$

If we fix $x_T = \xi$, then $x$ solves the integrated form of this equation:

$$x_t = \frac{t\xi}{T} - \frac{t}{T} \int_t^T dv \int_v^T du \int_0^T ds \, f(x_u - x_s) + \frac{T-t}{T} \int_0^t dv \int_0^v du \int_0^T ds \, f(x_u - x_s).$$

Could try iterative solution by successive approximations; if $\delta_t^n \equiv |x_t^n - x_t^{n-1}|$, we can show (assuming $f$ is Lipschitz, constant $K$) that

$$\delta_t^{n+1} \le \frac{5K}{2} \, t(T-t) \int_0^T \delta_u^n \, du$$

whence

$$\int_0^T \delta_u^{n+1} \, du \le \frac{5KT^2}{4} \int_0^T \delta_u^n \, du .$$

So, provided $T$ is small enough, this works. In general, there is no unique solution, as the case $f(x) = -x/2\pi$, with $x_u = \lambda \sin u$, shows; for any $\lambda$ this gives a solution!

Pair of independent BMs rubbing each other out (25/4/90)

Take $B, \tilde{B}$ two indept BMs on $\mathbb{R}$, $W = (B+\tilde{B})/\sqrt{2}$, $\tilde{W} = (B-\tilde{B})/\sqrt{2}$, and assume wlog that $W_0 = 0$, $\tilde{W}_0 = \xi > 0$. We shall kill at rate $\gamma$ in local time at 0 of $\tilde{W}$. What's the distribution at time $t$ of the killed process $(W, \tilde{W})$?

1) If $T \sim \exp(\lambda)$, $\lambda \equiv \frac{1}{2}\theta^2$, then

$$E^\xi\left[e^{ia\tilde{W}_T - \gamma L_T}\right] = E^\xi\left[e^{ia\tilde{W}_T} : T < H_0\right] + E^\xi\left[e^{ia\tilde{W}_T - \gamma L_T} : T > H_0\right]$$

$$= E^\xi\left[e^{ia\tilde{W}_T}\right] - e^{-\theta\xi} E^0\left[e^{ia\tilde{W}_T}\right] + e^{-\theta\xi} E^0\left[e^{ia\tilde{W}_T - \gamma L_T}\right]$$

$$= \frac{\lambda}{\lambda + \frac{1}{2}a^2}\left(e^{ia\xi} - e^{-\theta\xi}\right) + e^{-\theta\xi}\frac{\theta}{\theta+\gamma}\frac{\lambda}{\lambda+\frac{1}{2}a^2}$$

$$= \frac{\lambda}{\lambda+\frac{1}{2}a^2}\left(e^{ia\xi} - \frac{\gamma}{\gamma+\theta}e^{-\theta\xi}\right).$$

Hence easily

$$E\left[e^{ia\tilde{W}_T + ibW_T - \gamma L_T}\right] = \frac{\lambda}{\lambda + \frac{1}{2}(a^2+b^2)}\left[e^{ia\xi} - \frac{\gamma}{\gamma+\rho}e^{-\xi\rho}\right], \quad \rho \equiv \sqrt{2\lambda + b^2}$$

and so with $B_0 = 1 = -\tilde{B}_0$,

$$E \exp(i\alpha B_T + i\beta\tilde{B}_T - \gamma L_T) = \frac{\lambda}{\lambda + (\alpha^2+\beta^2)/2}\left[e^{i(\alpha-\beta)} - \frac{\gamma}{\gamma + (2\lambda + (\alpha+\beta)^2/2)^{\frac{1}{2}}}\exp - \sqrt{4\lambda + (\alpha+\beta)^2}\right].$$

Hence

$$\boxed{E\exp(i\alpha B_T - \gamma L_T) = \frac{\lambda}{\lambda+\alpha^2/2}\left\{e^{i\alpha} - \frac{\gamma}{\gamma+\sqrt{2\lambda+\alpha^2/2}}e^{-\sqrt{4\lambda+\alpha^2}}\right\}}$$

Some inversion of ~~transforms~~ is possible:

$$\exp - \sqrt{4\lambda + \alpha^2} = \int_0^\infty e^{-\lambda t - \alpha^2 t/4}\sqrt{2}\, e^{-1/t}\frac{dt}{\sqrt{2\pi t^3}},$$

$$\frac{\gamma}{\gamma + \sqrt{2\lambda + \alpha^2/2}} = \int_0^\infty e^{-\lambda t}\,dt\left(\int_0^\infty \gamma e^{-\gamma y}\frac{y e^{-y^2/2t}}{\sqrt{2\pi t^3}}\,dy\right)e^{-\alpha^2 t/4}.$$

Also, if $\hat{a}(s,\lambda) \equiv \int_0^\infty \lambda e^{-\lambda t}dt \int e^{isx}a(t,x)\,dx$, where $a$ is density of $B$ under killing, we get that the double ~~transform~~ of $\dot{a} - \frac{1}{2}a''$ is

$$-\lambda e^{is} + (\lambda + \tfrac{1}{2}s^2)\hat{a}(s,\lambda) = \frac{-\lambda\gamma\, e^{-\sqrt{4\lambda+s^2}}}{\gamma + \sqrt{2\lambda + s^2/2}}.$$

The double transform on the RHS inverts to

$$-\int_0^t \frac{e^{-x^2/t}}{\sqrt{\pi t}}\,du\,\frac{\sqrt{2}}{2\pi(u(t-u))^{3/2}}\int_0^\infty \gamma e^{-\gamma y}\, y e^{-y^2/2u}\,dy\; e^{-1/(t-u)}$$

$$= -\frac{e^{-x^2/t}}{\sqrt{\pi t}}\int_0^\infty \gamma e^{-\gamma y}\,(y+\sqrt{2})\,e^{-(y+\sqrt{2})^2/2t}\,\frac{dy}{\sqrt{2\pi t^3}} \quad \ldots \text{ this is a mess.}$$

2) Can also identify the transition mechanism directly.

If we take a BM and kill at rate $\gamma$ in local time at 0, then the trans density is for $a>0$

$$\tilde{p}_t(a,x) = \{p_t(a,x) - p_t(a,-x)\}\, I_{(x>0)} + \varphi_t(a,|x|),$$

where $\boxed{\varphi_t(a,x) \equiv \int_0^\infty e^{-\gamma l}\,(l+a+|x|)\,e^{-(l+a+|x|)^2/2t}\,\frac{dl}{\sqrt{2\pi t^3}}}\,,$

with $\tilde{p}_t(-a,-x) = \tilde{p}_t(a,x)$. When $B_0 = 1 = -\tilde{B}_0$, we have $\tilde{W}_0 = \sqrt{2}$, $W_0 = 0$, so that

$$P(W_t \in dy,\ \tilde{W}_t \in dx) = p_t(y)\,\tilde{p}_t(\sqrt{2},x)\,dx\,dy,$$

from which we can compute the density of $B_t = \frac{1}{\sqrt{2}}(W_t + \tilde{W}_t)$; it's

$$\boxed{P(B_t \in dx)/dx \equiv q_t(x) = \sqrt{2}\int_{-\infty}^\infty p_t(v)\,\tilde{p}_t(\sqrt{2},\, x\sqrt{2}-v)\,dv.}$$

Note that $\tilde{p}_t(a,x) = \tilde{p}_t(x,a)$, and

$$\tilde{p}_t(a,x) = p_t(a,x) - \int_0^\infty (1-e^{-\gamma l})\,(l+a+|x|)\,e^{-(l+a+|x|)^2/2t}\,\frac{dl}{\sqrt{2\pi t^3}}$$

$$= p_t(a,x) - \delta_t(a,x), \text{ say}$$

The fact that this is not $C^1$ at 0 seems to rule it out as giving $\dot{a} - \frac{1}{2}a'' = -q_t(x)\,q_t(-x)$.

[I'm sure I decided this conclusively, but can't lay hands on the notes just now.]

## More on the RBM problem (28/4/90)

The various expressions in the paper (BMNVSR II) are extremely clumsy; we can do better, as follows. We started from

$$G^\partial(z, z_0) = \operatorname{Im} \frac{i}{\pi}\left\{\log(z-\bar{z}_0) - \log(z-z_0)\right\}$$

and wanted to express $G$ as

$$G(z,z_0) = G^\partial(z,z_0) + \operatorname{Im} f(z,z_0).$$

Write $\varphi(z) = i\psi(z)$, $\mu(dx) \equiv \operatorname{Im}\varphi(x)\, dx/\pi$. The boundary condition is

$$\operatorname{Re}\left[\frac{i}{\varphi(z)}\left(f'(z) + \frac{i}{\pi}\left(\frac{1}{z-\bar{z}_0} - \frac{1}{z-z_0}\right)\right)\right] = 0 \qquad \text{for } z \in \mathbb{R},$$

so $$\operatorname{Im}\frac{\pi f'(z)}{\varphi(z)} = \operatorname{Im}\left[-\frac{i}{\varphi(z)}\left\{\frac{1}{z-\bar{z}_0} - \frac{1}{z-z_0}\right\}\right] \qquad \text{for } z \in \mathbb{R},$$

which suggests we take

$$\frac{\pi f'(z)}{\varphi(z)} = -\frac{1}{\varphi(z)}\, i\left(\frac{1}{z-\bar{z}_0} - \frac{1}{z-z_0}\right) - \left\{\underbrace{\frac{i}{\varphi(z_0)}\,\frac{1}{z-z_0}}_{\text{to remove pole at } z_0} - \underbrace{\frac{i}{\overline{\varphi(z_0)}}\,\frac{1}{z-\bar{z}_0}}_{\text{to correct for the imaginary part on } \mathbb{R} \text{ introduced by 1st correction.}}\right\}$$

If now we abbreviate $\sigma \equiv -1/\varphi(z_0) \equiv \alpha + i\beta$, we get

$$\boxed{-i\pi f'(z) = \frac{\sigma}{z-z_0}\left\{\varphi(z) - \varphi(z_0)\right\} - \frac{\bar{\sigma}}{z-\bar{z}_0}\left\{\varphi(z) - \overline{\varphi(z_0)}\right\}.}$$

$$\boxed{\pi f(z) - \pi f(0) = \int \frac{\mu(dx)}{|x-z_0|^2}\left\{i\sigma(\bar{z}_0 - x) - i\bar{\sigma}(z_0-x)\right\}\log(z-x) \; - 2\beta c_1 z}$$

And so in the transient case, assuming $c_2 = 0$, we can get the neat statement

$$\boxed{\pi \operatorname{Im} f(z) = \int \mu(dx)\left\{\frac{i\sigma}{z_0-x} - \frac{i\bar{\sigma}}{\bar{z}_0-x}\right\}\tan^{-1}\left(\frac{x-a}{b}\right)}$$

Since $i\sigma(\bar{z}_0 - x) - i\bar{\sigma}(z_0-x) = 2\beta x - 2\beta x_0 + \alpha y_0$, we can rework the last into

$$\frac{\pi}{2}\operatorname{Im} f(z) = \int \frac{\mu(dx)}{|x-z_0|^2}(\beta x - \beta x_0 + \alpha y_0)\tan^{-1}\left(\frac{x-a}{b}\right)$$

$$= \int_{-\infty}^{\infty} \frac{b\, du}{b^2+(u-a)^2}\int_u^\infty \frac{\mu(dx)}{|x-z_0|^2}(\beta x - \beta x_0 + \alpha y_0).$$

The assumption $c_2 = 0$ is easily shown equivalent to $\int \frac{\mu(dx)}{|x-z_0|^2}(\beta x - \beta x_0 + \alpha y_0) = 0$, as this last exhibits a non-negative bdd Im f.

<u>Some more pieces of the self-repelling BM story</u> (1/5/90).

Again, we consider $dX_t = dB_t + \int_0^t f(X_t - X_s)\,ds$, with $f \geq 0$, $f(0) > 0$. If $f$ is symmetric, the total push until time $t$ is

$$\int_0^t ds \int_0^s du\, f(X_s - X_u) = \frac{1}{2} \int_0^t ds \int_0^t du\, f(X_s - X_u),$$

so it makes sense to consider the problem of

$$\min \int_0^T G(dx) \int_0^T G(dy)\, f(x-y)$$

where $G$ is a probability measure.

(i) If $G$ were actually optimal, variational considerations show that there must be $c > 0$ such that

(*) $$\int G(dy)\, f(x-y) \geq c \qquad \forall x \in [0,T], \text{ with equality on } \mathrm{supp}(G).$$

Moreover, these conditions are also sufficient for optimality

<u>Case 1</u>: $f(x) =$ 

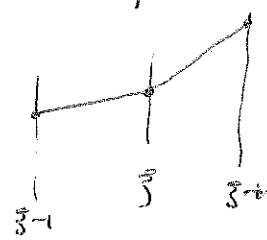
.

The condition (*) gets rephrased in this case as

$$\gamma(x) \equiv -\int_{x-1}^x G(y)\,dy + \int_x^{x+1} G(y)\,dy \geq c \quad \text{, with equality on } \mathrm{supp}(G) \equiv S$$

Suppose $\sup S < T$; then consider some $x \in (\sup S, T)$. As $x$ rises, $\int_x^{x+1} G(y)dy - \int_{x-1}^x G(y)dy$ must decrease (the first term is constant), and it starts at value $c$ — this is a contradiction of $\gamma(x) \geq c$, so $\sup S = T$, $\inf S = 0$ by similar reasoning.

Now let $(a,b)$ be a maximal interval in $S^c$. If $\gamma(x) > c$ for some $x \in (a,b)$ then there are $a < \xi < \eta < b$ where $\gamma'(\xi) > 0 > \gamma'(\eta)$. But $\gamma'(\xi) > 0$ is geometrically:

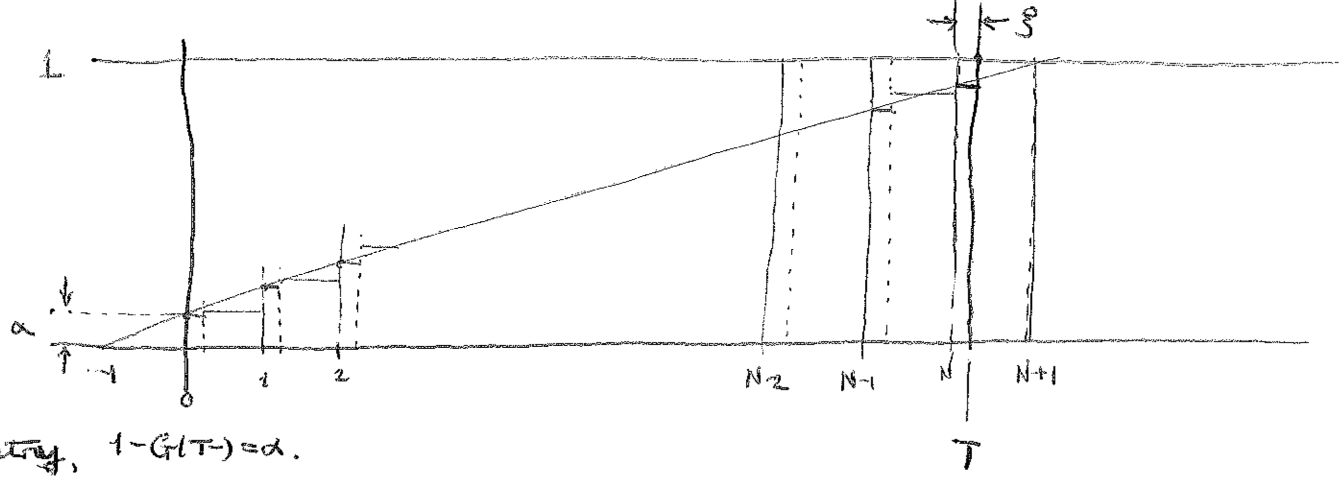



and as $\xi$ moves right, the central point doesn't rise, but the two outside ones do — so $\gamma'(\eta) < 0$ is impossible in $(a,b)$.

Conclusion: if $\gamma(x) \geq c$, then actually $\gamma(x) = c$ for all $x$. Hence

$$\gamma'(x) = G(x+1) - 2G(x) + G(x-1) = 0 \quad \forall\, x \in [0,T).$$

Let's assume now that $T = N + \xi$, where $0 < \xi < 1$, and write $\alpha \equiv G(0)$. We have for $0 \leq x < 1$ that $G(1+x) = 2G(x)$, so in particular $G(1) = 2G(0)$ and from $\gamma'$ get $G(k) = (k+1)\alpha$ for $k = 0, 1, \dots, N+1$. But $G(N+1) = 1$, so $\boxed{\alpha = 1/(N+2)}$ More generally, for $0 \leq x < 1$, $G(k+x) = (k+1)G(x)$.



By symmetry, $1 - G(T-) = \alpha$.

If some value $\beta$ is taken on $(0,1)$, then $G(N-1+v) = N\beta \leq 1-\alpha$ (where $v$ is the point where the value $\beta$ is taken). In particular, $\beta \leq \frac{N+1}{N}\alpha < 2\alpha$, so there must be a jump at 1.

If $\delta_k$ is the size of the jump at $k$, then $\delta_k = \alpha - k\varepsilon$ for some $\varepsilon \geq 0$; $\varepsilon$ is the jump at $\xi$, so by symmetry this is the jump at $T-\xi$, which is $\delta_N = \alpha - N\varepsilon$. Hence $\varepsilon = \alpha/(N+1)$. Can check that the $G$ thus constructed has property $(*)$, with

$$c = \int_0^1 G(y)\,dy = \frac{\xi}{N+2} + \frac{1-\xi}{N+1}.$$

So optimal solution just linearly interpolates the points $(N, \frac{1}{N})$.

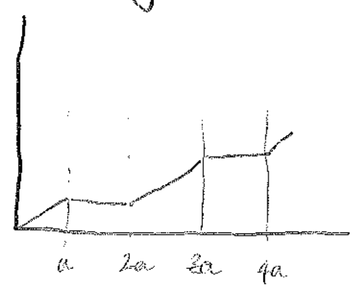
Using uniform dist$^{n}$ gives $(3T-1)/3T^2$.

Case 2: Take $f(x) = I_{[-a,a]}$. We shall further specialise by assuming $T = (2n+1)a$.

In this case, $(*)$ becomes the condition that $\exists\, c$ such that

$$G(x+a) - G(x-a) \geq c \quad \text{for all } x \in [0,T] \text{ with equality on } S \equiv \operatorname{supp}(G).$$

In this instance we can build $G$ explicitly: with mass $\frac{1}{n+1}$ in each of the intervals $(2ka, (2k+1)a)$ uniformly spread through it. The minimised value is $1/(n+1)$.



If $T$ is not of the form $(2n+1)a$, we define $N \equiv \left[\frac{T}{2a} - \frac{1}{2}\right]$ and get the obvious bounds
$$\frac{1}{N+1} \geq \min \iint_0^T G(dx)\, G(dy)\, f(x-y) \geq \frac{1}{N+2}$$
which describes the behaviour asymptotically very well.
[If we use a uniform dist$^{n}$, we get $a(2T-a)/T^2$ ]

(ii) Suppose $a>0$ is fixed, and that $X$ is a cts function on $[0,t]$, with $X_0=0$, and max and min values $\beta > \alpha$ resp.. <u>What's the least that</u>
$$J \equiv \int_0^t ds \int_0^s du \;\; I_{\{X_s - X_u \leq a\}}$$
<u>can be?</u>

Suppose firstly we fix the local time process $\ell_x \equiv L(t,x)$, and ask to minimise $J$ subject to the constraint that its local time should be $\ell$. The obvious intuitive notion is that should visit highest levels latest. So we suspect that if $\varphi(x) = \int_\alpha^x L(t,y)dy$ then $\xi_t \equiv \inf\{x : \varphi(x) > t\}$ will be optimum, and
$$\int_0^t ds \int_0^s du \;\; I_{\{\xi_s - \xi_u \leq a\}} = \int_\alpha^\beta L(t,x)\,dx \int_\alpha^x L(t,y)\, I_{\{x-y\leq a\}}\, dy$$
$$= \int_\alpha^\beta L(t,x)\,dx \int_{x-a}^x L(t,y)\,dy .$$

Thus the claim is that
$$J = \int_\alpha^\beta dx \int_0^t L(ds,x) \int_{x-a}^\infty L(s,v)\,dv \geq \int_\alpha^\beta dx\; L(t,x) \int_{x-a}^x L(t,y)\,dy .$$

The difference is
$$\int_\alpha^\beta dx \int_0^t L(ds,x) \left[ \int_x^\infty L(s,v)\,dv - \int_{x-a}^x (L(t,y) - L(s,y))\,dy \right]$$
$$\geq \int_\alpha^\beta dx \int_0^t L(ds,x) \left[ \int_x^\infty L(s,v)\,dv - \int_{-\infty}^x (L(t,y) - L(s,y))\,dy \right]$$
$$= \int_\alpha^\beta dx \int_0^t L(ds,x) \left\{ s - \int_{-\infty}^x L(t,y)\,dy \right\} = 0 .$$

Thus if we now hold $\alpha, \beta$ fixed, and minimise over $\ell$, we are trying to minimise over $\ell$
$\frac{1}{2} \int_\alpha^\beta \ell_x\, dx \int_\alpha^\beta \ell_y\, dy \;\; I_{(|x-y|\leq a)}$ which we know about; it's at least $at^2/(\beta-\alpha+3a)$.

The decreasing function $I_{(-\infty,a]}$ generalises to any decreasing $f^x$ — the functional $J$ is minimised by an increasing path.

SO if $f \geq 0$ is <u>decreasing</u>, or <u>symmetric and decreasing in $\mathbb{R}^+$</u>, then

$$Z_t \equiv \int_0^t ds \int_0^s du\, f(X_s - X_u) \geq f(a) \cdot \tfrac{1}{2} \int \ell_x dx \int \ell_y dy\, I_{\{|x-y| \leq a\}}$$

$$\geq \frac{a\, f(a)\, t^2}{\beta - \alpha + 3a}$$

If we now write $\rho$ for the range $\beta - \alpha$ of the path, and set $f(a) = a^{-\beta}$ for large $a$, say (with now $\beta$ some parameter in $(0,1)$) this lower bound is maximised by the choice $a = \theta\rho$, $\theta \equiv (1-\beta)/3\beta$. So if $t$ is large, and approximating $\rho$ by $Z$, we have

$$\rho \geq \frac{(\theta\rho)^{1-\beta} t^2}{\rho + 3\theta\rho} = \rho^{-\beta} \frac{\theta^{1-\beta} t^2}{1+3\theta}$$

$$\boxed{\therefore\ \rho \geq \left(\frac{\theta^{1-\beta}}{1+3\theta}\right)^{\frac{1}{1+\beta}} t^{\alpha}} \qquad \left(\alpha = \frac{2}{1+\beta}\right)$$

So this argument shows that for this case, the growth must be at least const. $t^\alpha$.

(iii) Consider the physicists' preferred model, where we weight Wiener measure by

$$\exp\left\{- \int_0^1 ds \int_0^1 dt\, \psi(X_s - X_t)\right\}$$

where $\psi$ is symmetric, $\geq 0$, Lipschitz, decr in $\mathbb{R}^+$, say. To find the "most likely path" we want to

$$\min \int_0^1 ds \int_0^1 dt\, \psi(x_s - x_t) + \tfrac{1}{2} \int_0^1 \dot{x}_s^2 ds \equiv \min J(x)$$

over paths $x$. One way is to fix $x_0 = \xi$, minimise over $x$, then minimise over $\xi$. If we consider the example where $\psi(x) = K I_{\{|x| \leq 1\}}$, and we fix $x_0 = 0$, then we can see that <u>a monotone path from $x_0$ to $x_1$ need not be optimal</u>

Indeed, taking the path

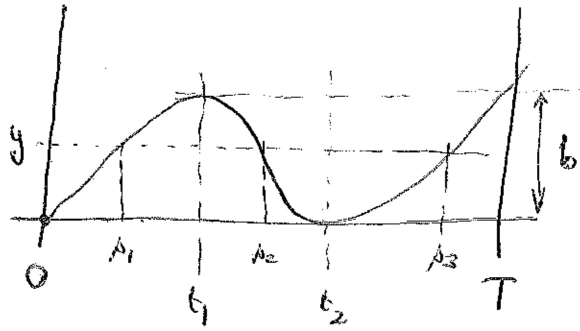
$$x_t = \begin{cases} ct & 0 \leq t \leq \tfrac{1}{2} \\ c(1-t) & \tfrac{1}{2} \leq t \leq 1 \end{cases}$$

can compute

$$J(x) = \frac{4K}{c^2}(c-1) + \tfrac{1}{2}c^2 \leq \frac{4K}{c} + \tfrac{1}{2}c^2;$$

by choosing $c = (4K)^{1/3}$, we get an upper bound of $\frac{3}{2} \cdot 4^{2/3} K^{2/3}$. The straight path from $x_0 = 0$ to $x_1 = 0$ has $J = K$, so for large enough $K$, it's better to go up then down.

Nonetheless, it's always going to be better to go up, rather than up-down-up. The smooth piece of path shown here can always be replaced by a monotone piece of path from $(0,0)$ to $(T,b)$ which leaves the local time unaltered (and so $\iint \psi(x_s - x_t)$ ) but which reduces $\int \dot{x}_s^2\, ds$.




How? Well if $\lambda_y$ is loc time at level $y \in (0,b)$, define

$$\varphi_x = \int_0^x \lambda_y\, dy$$

and now define $v_t = \varphi^{-1}(t)$. Then $v$ is monotone, goes from $(0,0)$ to $(T,b)$, has the same local time $\lambda$ as $x$, and

$$\int_0^T \dot{v}_t^2\, dt = \int_0^T \frac{dt}{\lambda(v_t)^2} = \int_0^b \frac{dy}{\lambda_y}$$

But we have $\lambda_y = \dfrac{1}{\dot{x}_{s_1}} + \dfrac{1}{\dot{x}_{s_2}} + \dfrac{1}{\dot{x}_{s_3}} \equiv \lambda_1(y) + \lambda_2(y) + \lambda_3(y)$, say,

and
$$\int_0^T \dot{x}_t^2\, dt = \int_0^b \left[ \frac{1}{\lambda_1(y)} + \frac{1}{\lambda_2(y)} + \frac{1}{\lambda_3(y)} \right] dy$$
$$\geq 3 \int_0^b dy/\lambda_y$$

so the integral of the squared derivative is a lot less

## Bound on growth rate for compactly supported $f \geq 0$ (7/5/90)

Let's suppose $f \geq 0$, with support $\subseteq (-1,1)$. Consider a reflecting drifting BM

$$dY_t = dB_t + 7\gamma^4 b\, dt + dL_t, \quad Y_0 = 0$$

where $b \equiv \sup f(x)$. If $\tilde{H}$ is the first time $Y$ reaches 1, then we pick $\gamma > 0$ so small that $P(\tilde{H} > 5\gamma) \geq \frac{1}{2}$.

Now back to $X_t = B_t + \int_0^t (\int_0^s f(X_s - X_u)\, du)\, ds$. Let $H_n = \inf\{t : X_t = n\}$. Suppose that $H_N \leq \gamma N$. Then $H_n - H_{n-1} \leq 2\gamma$ for at least half of the $n \leq N$. If we had that $H_n - H_{n-1} \leq 2\gamma$, then the upward push on $X$ at time $H_n + t$ cannot be more than $(2\gamma + t)\, b$ (if $X_{H_n+t} \geq n$) or $+\infty$ if $X_{H_n+t} < n$. Thus $X_{H_n+t}$ will be dominated by

$$dY_t = dB_t + b(2\gamma + t)\, dt + dL_t$$

and so on $[0, 5\gamma]$ is dominated $dY_t = dB_t + 7\gamma b\, dt + dL_t$. Hence, by choice of $\gamma$, $P(H_{n+1} - H_n > 5\gamma) \geq \frac{1}{2}$. Thus at least half of the level crossings $n \to n+1$ immediately following a quick level crossing (one which took $\leq 2\gamma$) will take $\geq 5\gamma$ on average. So, on average, time to reach $N$ will be $\geq \frac{N}{2} \cdot \frac{1}{2} \cdot 5\gamma = 5\gamma N/4$. This tells us that $P(H_N \leq \gamma N) \to 0$.

## Skew-reflecting BM - possible methodology (8/5/90).

We know that the Greens functions for conformally equivalent domains map across in the obvious way (ie. $\tilde{G}(z, z') = G(\varphi(z), \varphi(z'))$). Thus if we took $\theta$ as given in $\mathbb{R} \setminus [-1, 1]$ then extended it by $\theta(x) = -\theta(1/x)$ $\forall x \in \mathbb{R}$, since the Green's $f^n$ in $\log(\mathbb{H})$ is now obviously symmetric about the imaginary axis, we can obtain the Green's $f^n$ for RBM killed when it hits the unit circle, just as $G(z, z_0) - G(z, \bar{z}_0^{-1})$ $(|z|, |z_0| > 1)$. If we know $P^z$(hit unit circle before $\infty$), which we can obtain in same way by Burdzy-Marshall conformal map, then if we keep jumping out from $0$ to $z$ (large), we get

$$E^z(\text{time taken to reach unit circle}) = \int_{|z_0|>1} \{G(z, z_0) - G(z, \bar{z}_0^{-1})\}\, |z_0|^{-2(1+\eta)}\, dz_0 / P^z(\text{hit } 0 \text{ before } \infty)$$

In principle, these things are all known, and we can use this to see whether an extension beyond $H_0$ is possible. Moreover, when we know $G_1(z, z_0) \equiv G(z, z_0) - G(z, \bar{z}_0^{-1})$, we can work out the exit dist$^n$ given that we escape at unit circle before $\infty$, just by considering

$$\lim_{\varepsilon \downarrow 0} G(z, (1+\varepsilon)e^{i\theta}) \Big/ \left\{ \int_0^{\pi} G(z, (1+\varepsilon)e^{i\omega})\, d\omega \right\}.$$

Getting any of these things in a manageably compact form may not be easy.

Note that $\psi(z) = \psi(1/z)$ if $\theta(x) = -\theta(1/x)$, $x \in \mathbb{R}$.

## Limit laws for 1-dimensional diffusions (11/5/90)

To complete the result with David Hobson, we can argue that $c(X_t)/t \xrightarrow{P} 1$ is equivalent to $c$ slowly varying. The only implication left to prove is $c(X_t)/t \xrightarrow{P} \Rightarrow \frac{H_x}{c(x)} \xrightarrow{P} 1$.

Now for any $\varepsilon > 0$,

$$P[c(X_t) < (1-\varepsilon)t] \to 0$$

$$= P[X_t > c^{-1}((1-\varepsilon)t)] \geq P[H_{c^{-1}((1-\varepsilon)t)} > t]$$

so that $P(H_x > c(x)/(1-\varepsilon)) \to 0$. Any limit law of $\frac{H(x)}{c(x)}$ must therefore be concentrated on $[0,1]$; but since $(H_x/c(x))_{x>0}$ is bdd in $L^2$ and UI, must have that the mean of any limit law is 1. Hence $H_x/c(x) \xrightarrow{\mathcal{D}} 1$.

## Ciesielski-Taylor, & Marc Yor's approach (11/5/90).

We computed the law of BES(n) local time, now what's the law of local time til first hit 1 of Bes(n)? Can use the NRW stuff to show that it solves

$$dZ_x = 2\sqrt{Z_x}\, dW_x + \left\{2 - 2Z_x \frac{n-1}{2|x|}\right\} dx, \qquad Z_{-1} = 0$$

so that $Z$ can be expressed as the sum of the squares of two i.i.d. processes

$$dU_x = dW_x - \frac{n-1}{2|x|} U_x\, dx, \qquad U_{-1} = 0,$$

exactly as before (p.38). A little manipulation and we get

$$U_x = |x|^\beta \int_{-1}^x |y|^{-\beta}\, dW_y \qquad (\beta = (n-1)/2)$$

and now we use the neat result of Yor: for $\varphi \in L^2([0,1]^2)$,

$$\int_0^1 ds \left(\int_0^1 \varphi(s,u)\, dB_u\right)^2 \stackrel{\mathcal{D}}{=} \int_0^1 du \left(\int_0^1 \varphi(s,u)\, dB_s\right)^2$$

with the function $\varphi(s,u) = \left(\frac{u}{s}\right)^\beta I\{u<s\}$ !

## Estimates for the counter-example on RBM (12/5/90)

In the construction of the counterexample, I need to consider the asymptotics of

$$(*) \qquad \frac{I_0'(b)K_0(b) - I_0(b)K_0'(b)}{I_0'(b)K_0(a) - K_0'(b)I_0(a)}$$

as $b \downarrow 0$, and $a \downarrow 0$, where $\alpha \equiv \log a < \beta \equiv \log b$. The numerator is exactly $b^{-1}$. We want to bound the denominator below, so that we can bound this

5) Just for the record, let's have the Perkins' decomposition of $L(1,x)$, from p 261, SLN 1118.
If we set $I_t \equiv \inf\{X_s : s \le t\}$, $A_1(t,x) \equiv \int_0^{t\wedge 1} I_{\{X_s \le x\}}\, ds$, $\mathcal{E}_x(1)$ the $\sigma$-field generated by $X(\tau_1(\cdot,x))$, then

$$\tfrac{1}{2}L(1,x) - I_{\{x \ge I_1\}}\Big[(x\wedge X_0) + (x\wedge X_1) - 2I_1 + \int_{I_1}^{x\wedge S_1}\Big\{-\frac{\frac{1}{2}L_1^y + (X_0-y)_+}{1-\int^y L_1^z\,dz} + I_{\{X_1\le x\}}\frac{1}{\frac{1}{2}L_1^y + (X_0-y)_+}\Big\}L_1^y\,dy,$$

is an $\mathcal{E}_x(1)$-martingale.

---

expression above. We use $I_0'(b) \ge b/2$, $K_0(a) \equiv I_0(a)\int_a^\infty \frac{dy}{y\, I_0(y)^2} \ge \frac{-\log a}{I_0(1)^2}$,
and, from the bound $I_0(x) \le 1 + x(I_0(1)-1)$, we get

$$-K_0'(x) = \frac{1}{x I_0(x)}\{1 - x I_0'(x) K_0(x)\}$$

$$\ge \frac{1}{x(1+cx)}\{1-cx\} \qquad \text{for some const } c$$

$$\ge \frac{1}{x}\{1-cx\} \qquad \text{for all } x\in(0,1), \text{ some } c>0.$$

Thus $(*)$ is $\le \left\{-\log a \cdot \frac{b^2}{2I_0(1)^2} + 1 - cb\right\}^{-1}$

and in the situation we consider, where we take $\log a = -\frac{1}{b^2}$,
this gives a bound

$$(*) \le \frac{1}{1+\delta} \quad , \text{where } \delta > 0.$$

Thanks to Hsu Pei for noticing that I'd cut a corner there.

# Problems, Questions.

1) De Haan says that if you take bivariate Normal r.v.s, with non-degenerate law, and do the one-dimensional scaling of $X_n^* \equiv \sup\{X_k : k \leq n\}$, $Y_n^*$ which leads to a non degenerate limit law (namely, consider $b_n(X_n^* - b_n)$, where $b_n = \sqrt{2\log n - \log\log n}$) then in the limit the two components are independent.

This says that if you took indep<sup></sup>t BMs $B^{(n)}$ on $[0,t]$, and considered

$$B_t^{(n*)} \equiv \sup\{B_t^{(k)} : k \leq n\}$$

then the limiting law of the processes $(B^{(n*)} - b_n)\, b_n$ would be like the derivative of BM - indep<sup></sup>t r.v.'s at each different time point.

So what can one say about

$$\lim_{n\to\infty} \int_0^t (B_s^{(n*)} - \sqrt{s}\, b_n)\, b_n \sqrt{s}\, ds \quad \ldots \; ?$$

2) Take a positive-recurrent Mkv chain, and suppose $\pi u = 0$.
Can one obtain the asymptotics of

$$E\Big(\sum_{r=1}^{n} u(X_r)\Big)^2 \; ?$$

Pei Hsu: Chung's book (Mkv Chains) Iosifescu
Ron Pyke: Moral Neuts?
Peter March: Baxter + Brosamler, early 70's Math. Scand.

3) Michael Aizenmann says that if $\omega_n$ = no of prime factors of $n$, and if $S_k \equiv \sum_{j=1}^{k} (-1)^{\omega_j}$, and if for each $\varepsilon > 0$ $\exists\, c(\varepsilon)$ s.t.

$$|S_k| \leq c(\varepsilon)\, k^{\frac{1}{2}+\varepsilon} \qquad \forall k$$

then the Riemann hypothesis follows.

4) Chuck Newman asks: take edge-avoiding r.w. on $\mathbb{Z}^2$ started at 0 - never retrace a used edge, pick with equal prob<sup></sup>y from the available choices. You can only get stuck at 0, but are you <u>certain</u> to?