Convertible Bonds | Case where $K=0$

Formulae for S and Y:

$$S = \frac{V - m\rho/r}{n-m} + \frac{\beta m\rho/r - (\beta-1)\xi}{(\alpha+\beta)(n-m)}\left(\frac{V}{\xi}\right)^{-\alpha} + \frac{\alpha m\rho/r - (\alpha+1)\xi}{(\alpha+\beta)(n-m)}\left(\frac{V}{\xi}\right)^{\beta}$$

$$Y = \frac{V - n\rho/r}{n-m} + \frac{\beta n\rho/r - (\beta-1)\eta}{(\alpha+\beta)(n-m)}\left(\frac{V}{\eta}\right)^{-\alpha} + \frac{\alpha n\rho/r - (\alpha+1)\eta}{(\alpha+\beta)(n-m)}\left(\frac{V}{\eta}\right)^{\beta}$$

$$Y(m,\xi) = -\frac{\rho\xi}{m} \Rightarrow \eta = \frac{1}{\Theta}\,\frac{n\rho/r\,(\alpha\Theta^{\beta} + \beta\Theta^{-\alpha} - \alpha - \beta)}{(\alpha+1)\Theta^{\beta-1} + (\beta-1)\Theta^{-(\alpha+1)} - (\alpha+1) - (\beta-1) - \rho\frac{n-m}{m}(\alpha+\beta)}$$

$$\frac{\partial S}{\partial m}(m,\eta) = 0 \Rightarrow \xi' = \frac{n\rho/r\,(\beta\Theta^{\alpha} + \alpha\Theta^{-\beta} - \alpha - \beta) - \eta\left((\beta-1)\Theta^{\alpha+1} + (\alpha+1)\Theta^{-(\beta-1)} - (\alpha+1) - (\beta-1)\right)}{\frac{2}{\sigma^2}\frac{n-m}{\xi}(m\rho - \delta\xi)(\Theta^{-\beta} - \Theta^{\alpha})}$$

Good!!

Note that the denominator of ξ' is positive. [$m\rho - \delta\xi > 0$ is equivalent to $\frac{\partial^2 S}{\partial V^2}(m, \xi(m)) > 0$].

Let $\Theta = e^{-x}$. We have

$$\int_0^x \alpha\beta(e^{\beta t} - e^{-\alpha t})\,dt = \alpha e^{\beta t} + \beta e^{-\alpha t}\Big|_0^x = -\alpha - \beta + \alpha e^{\beta x} + \beta e^{\alpha x} = -\alpha - \beta + \alpha\Theta^{-\beta} + \beta\Theta^{\alpha}$$

$$\int_0^x (\alpha+1)(\beta-1)(e^{(\beta-1)t} - e^{-(\alpha+1)t})\,dt = -(\alpha+1) - (\beta-1) + (\alpha+1)\Theta^{-(\beta-1)} + (\beta-1)\Theta^{\alpha+1}$$

$$\int_0^x \alpha\beta(e^{\alpha t} - e^{-\beta t})\,dt = -\alpha - \beta + \alpha\Theta^{\beta} + \beta\Theta^{-\alpha}$$

$$\int_0^x (\alpha+1)(\beta-1)(e^{(\alpha+1)t} - e^{-(\beta-1)t})\,dt = -(\alpha+1) - (\beta-1) + (\alpha+1)\Theta^{\beta-1} + (\beta-1)\Theta^{-(\alpha+1)}$$

So
$$\eta = \frac{1}{\Theta}\,\frac{n\rho/r\int_0^x \alpha\beta(e^{\alpha t} - e^{-\beta t})\,dt}{\int_0^x (\alpha+1)(\beta-1)(e^{(\alpha+1)t} - e^{-(\beta-1)t})\,dt - \rho\frac{n-m}{m}(\alpha+\beta)} = \frac{1}{\Theta}\,\frac{n\rho\int_0^x (e^{\alpha t} - e^{-\beta t})\,dt}{\delta\int_0^x (e^{(\alpha+1)t} - e^{-(\beta-1)t})\,dt - \frac{1}{2}\sigma^2\rho\frac{n-m}{m}(\alpha+\beta)}$$ ✓

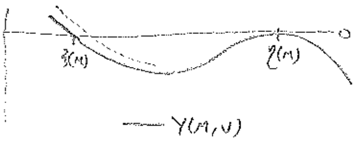
and numerator of ξ' is
$$n\rho/r\int_0^x \alpha\beta(e^{\beta t} - e^{-\alpha t})\,dt - \eta\int_0^x (\alpha+1)(\beta-1)(e^{(\beta-1)t} - e^{-(\alpha+1)t})\,dt.$$

Therefore ξ' has the same sign as
$$\frac{\int_0^x (e^{\beta t} - e^{-\alpha t})\,dt}{\int_0^x (e^{(\beta-1)t} - e^{-(\alpha+1)t})\,dt} - \frac{\delta\eta}{n\rho}.$$

$$= \frac{\int_0^x (e^{\beta t} - e^{-\alpha t})\,dt}{\int_0^x (e^{(\beta-1)t} - e^{-(\alpha+1)t})\,dt} - \frac{1}{e^{-x}}\,\frac{\int_0^x (e^{\alpha t} - e^{-\beta t})\,dt}{\int_0^x (e^{(\alpha+1)t} - e^{-(\beta-1)t})\,dt - \frac{1}{2}\sigma^2\rho\frac{n-m}{m}\frac{\alpha+\beta}{\delta}}$$

If we take $\rho = 0$ then this is positive, according to LCGR's Proposition 1.

Restrict to case where $p=0$ We have $Y(m,\xi(m))=0$, $Y(m,\eta(m)) = \frac{\partial Y}{\partial V}(m,\eta(m)) = 0$, $\xi'(m) \geq 0$.



— $Y(m,V)$

---- $Y(m+\varepsilon,V)$ $(\varepsilon>0)$.

$$\frac{d}{dm} Y(m,\xi(m)) = 0 \quad \therefore \quad \frac{\partial Y}{\partial m}(m,\xi(m)) + \xi'(m)\frac{\partial Y}{\partial V}(m,\xi(m)) = 0$$

$$\Rightarrow \frac{\partial Y}{\partial m}(m,\xi(m)) > 0$$

Now
$$Y(m,V) = \frac{V - n\rho/r}{\lambda - m} + \frac{\beta n\rho/r - (\beta-1)\eta}{(\alpha+\beta)(\lambda-m)}\left(\frac{V}{\eta}\right)^{-\alpha} + \frac{\alpha n\rho/r - (\alpha+1)\eta}{(\alpha+\beta)(\lambda-m)}\left(\frac{V}{\eta}\right)^{\beta}$$

$$\frac{\partial Y}{\partial m}(m,V) = \frac{Y(m,V)}{\lambda-m} + \frac{\alpha\beta n\rho/r - (\alpha+1)(\beta-1)\eta}{(\alpha+\beta)(\lambda-m)}\left(\frac{\eta'}{\eta}\right)\left\{\left(\frac{V}{\eta}\right)^{-\alpha} - \left(\frac{V}{\eta}\right)^{\beta}\right\}$$

$$\therefore 0 < \frac{\partial Y}{\partial m}(m,\xi(m)) = \frac{n\rho - \delta\eta}{\frac{1}{2}\sigma^2(\alpha+\beta)(\lambda-m)}\left(\frac{\eta'}{\eta}\right)\left(\Theta^{-\alpha} - \Theta^{\beta}\right)$$

So provided $\eta > \frac{n\rho}{\delta}$, we have that $\eta' < 0$.

[$\eta > \frac{n\rho}{\delta}$ is equivalent to $\frac{\partial^2 Y}{\partial V^2}(m,\eta(m)) < 0$].

CM Lévy processes have CM overshoots of exponential levels (17/7/2001)

If we have $\psi_\lambda^+(z) \equiv E \exp z\bar{X}(T_\lambda)$, $\psi_\lambda^-(z) \equiv E \exp z\underline{X}(T_\lambda)$ for some Lévy process X, then there is a well-known expression for the overshoot of an exponential level. If we assume $H_x \equiv \inf\{t : (X_t - x)(X_0 - x) < 0\}$ then for $\alpha > 0$

$$(1a) \quad \int_0^\infty \alpha e^{-\alpha x} E \exp\{-\lambda H_x + z(X(H_x) - x)\} dx = \frac{\alpha}{\alpha + z}\left[1 - \frac{\psi_\lambda^+(-\alpha)}{\psi_\lambda^+(z)}\right],$$

$$(1b) \quad \int_0^\infty \alpha e^{-\alpha x} E \exp\{-\lambda H_{-x} + z(X(H_{-x}) + x)\} dx = \frac{\alpha}{\alpha - z}\left[1 - \frac{\psi_\lambda^-(\alpha)}{\psi_\lambda^-(z)}\right].$$

See, for example, the survey article of Bingham.

In the case of a CM Lévy process, the Wiener-Hopf factors have a Pick function representation

$$(2) \quad \psi_\lambda^+(z) = \int_{[0,\infty]} \frac{v F_\lambda^+(dv)}{v - z}, \qquad \psi_\lambda^-(z) = \int_{[-\infty,0]} \frac{v F_\lambda^-(dv)}{v - z}$$

so the RHS of (1a) can be expressed as

$$(3) \quad \frac{1}{\psi_\lambda^+(z)} \int_{[0,\infty]} \frac{\alpha v F_\lambda^+(dv)}{(v+\alpha)(v-z)} = \int_{[0,\infty]} \frac{\alpha v F_\lambda^+(dv)}{(\alpha+v)(v-z)} \Big/ \int_{[0,\infty]} \frac{v F_\lambda^+(dv)}{v - z}.$$

Is this a Pick function? Taking $z = x + iy$, $y > 0$, and writing the numerator in (3) as $\tilde{A} + i\tilde{B}$, the denominator as $A + iB$, what we have to prove is that

$$A\tilde{B} \geq \tilde{A}B,$$

that is,

$$\int_{[0,\infty]} \frac{v(v-x) F_\lambda^+(dv)}{(v-x)^2 + y^2} \cdot \int_{[0,\infty]} \frac{\alpha v\, y F_\lambda^+(dv)}{((v-x)^2+y^2)(v+\alpha)} \geq \int_{[0,\infty]} \frac{\alpha v(v-x) F_\lambda^+(dv)}{(\alpha+v)((v-x)^2+y^2)} \int_{[0,\infty]} \frac{v y F_\lambda^+(dv)}{(v-x)^2+y^2}$$

The terms involving x in the numerator cancel, transforming the inequality to

$$\frac{\displaystyle\int_{[0,\infty]} v \cdot \frac{v F_\lambda^+(dv)}{(v-x)^2+y^2}}{\displaystyle\int_{[0,\infty]} \frac{v F_\lambda^+(dv)}{(v-x)^2+y^2}} \geq \frac{\displaystyle\int_{[0,\infty]} v \cdot \frac{\alpha}{\alpha+v} \cdot \frac{v F_\lambda^+(dv)}{(v-x)^2+y^2}}{\displaystyle\int_{[0,\infty]} \frac{\alpha}{\alpha+v} \cdot \frac{v F_\lambda^+(dv)}{(v-x)^2+y^2}}$$

But this inequality is evident, since the reweighting $\alpha/(\alpha+v)$ of the measure $v F_\lambda^+(dv)/((v-x)^2+y^2)$ shifts mass toward lower values.

An observation from linear algebra (31/7/01)

1) Suppose that $\{t_1, \dots, t_N\}$ are N distinct reals, and $\{z_1, \dots, z_N\}$ are N distinct reals. Then the $N \times N$ matrix

$$A = (a_{ij}) \equiv (\exp(z_i t_j))$$

is non-singular. Why? This can be proved by induction on N. For $N = 2$,

$$\det(A) = e^{z_1 t_1 + z_2 t_2} - e^{z_1 t_2 + z_2 t_1} = 0 \text{ iff } (z_1 - z_2)(t_1 - t_2) = 0$$

For the inductive step, we may wlog suppose that $z_1 > z_j \ \forall j > 1$, $t_1 > t_j \ \forall j > 1$, and A will be non-singular iff the matrix

$$\tilde{A} = (\tilde{a}_{ij}) \equiv \left(\exp(z_i t_j - z_i t_1 - t_j z_1 + z_1 t_1)\right)$$

is non-singular. We may therefore replace z_i by $\tilde{z}_i = z_i - z_1$, t_j by $\tilde{t}_j \equiv t_j - t_1$ and assume $z_1 = t_1 = 0$, $z_i < 0$, $t_i < 0$ for $i > 1$. Then

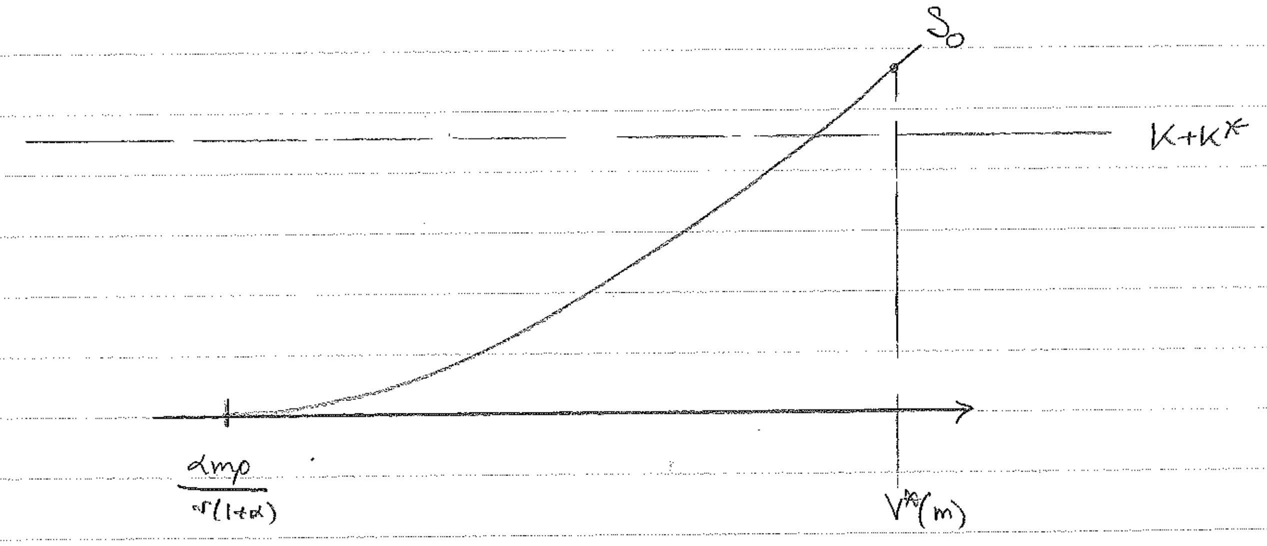
$$A = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 1 \cdots & 1 \\ \hline 1 & & & & \\ 1 & & B & & \\ \vdots & & & & \\ 1 & & & & \end{array}\right)$$

where B is nonsingular by inductive hypothesis. Subtracting the first row from all others, A is non-singular iff $B - 11^T$ is non-singular iff $I - (B^{-1}1)1^T$ is nonsingular. But it is easy to see that $I - uv^T$ is nonsingular iff $v^T u \neq 1$, so the condition for non-singularity becomes that

$$1^T B^{-1} 1 \neq 1$$

But $1^T B^{-1} 1 - 1 = 1^T (B^{-1} - n^{-1} I) 1$, so if we can show that $nI - B$ is non-singular we are done. But this is clear, since the entries of B are all in $(0,1)$.

2) There's an application; if we try to fit a yield curve at m points as a linear combination $\sum_{j=1}^{m} c_j \exp(-\lambda_j t_i)$, this can always be done!



S_0
$K+K^*$
$\frac{\alpha m\rho}{r(1+\alpha)}$
$V^A(m)$

Lemmas for the convertible bond question 8/8/01

Back to the problem of the callable convertible bond, assuming

$$\rho/r \le K^* < (K + \beta\rho/r)/(\beta - 1)$$

which ensures that calling will occur, and also that the calling takes place at $V^*(m) \equiv n(K+K^*) - mK$, at which time the share is worth $K+K^*$. In the analysis to follow, we assume unless stated otherwise that m is small enough that calling happens before conversion. We already know that there is a unique $\xi(m)$ for each m with the properties that

$$S = \frac{\partial S}{\partial V} \text{ at } V = \xi(m), \quad S = K + K^* \text{ at } V = V^*(m),$$

and S has the form

$$S(m,V) = \frac{V - m\rho/r}{n-m} + a(m)\, V^{-\alpha} + b(m)\, V^{\beta}$$

$$a(m) = \frac{\beta m\rho - (\beta-1) r \xi(m)}{(\alpha+\beta) r (n-m)} \xi(m)^{\alpha}, \quad b(m) = \frac{\alpha m\rho - (\alpha+1) r \xi(m)}{(\alpha+\beta) r (n-m)} \xi(m)^{-\beta}$$

Lemma 1 We have

$$\boxed{\frac{\alpha m\rho}{r(\alpha+1)} < \xi(m) < \frac{\beta m\rho}{r(\beta-1)}.}$$

Proof. If we were to select $\xi(m) = \alpha m\rho / r(\alpha+1)$ as the point for smooth pasting, then the form of S would be

$$S_0(m,V) = \frac{V - m\rho/r}{n-m} + a(m)\, V^{-\alpha}$$

$$= \frac{V - m\rho/r}{n-m} + \left(\frac{V}{\xi}\right)^{-\alpha} \frac{m\rho}{1+\alpha} \cdot \frac{1}{r(n-m)}$$

so if we evaluate at $V = V^*(m)$, we find

$$S_0(m, V^*) = K + K^* + \frac{m(K^* - \rho/r)}{n-m} + \left(\frac{V}{\xi}\right)^{-\alpha} \frac{m\rho}{(1+\alpha) r (n-m)} > K + K^*$$

so that S_0 is too high at V^*. [If the actual value of ξ was $< \alpha m\rho / r(1+\alpha)$, there would be some $x \in (\alpha m\rho/r(1+\alpha), V^*(m))$ where S_0 crosses the true S, so the difference between the two would be

$$S = S_0 + c\left(\left(\frac{V}{x}\right)^{-\alpha} - \left(\frac{V}{x}\right)^{\beta}\right)$$

for some positive c. Since S_0 is convex, the derivative of S to the left of $\alpha m\rho/r(1+\alpha)$ would be negative, a contradiction; so $\xi(m) > \alpha m\rho / r(1+\alpha)$.] ... (not needed)

On the other hand, if we used $\beta m\rho / r(\beta-1)$ as a guess at ξ we would be looking at

Don't need $\frac{\beta m\rho}{r(\beta-1)} < V^*(m)$ for this $\longrightarrow$

Notice that $\frac{\beta\rho}{(\beta-1)r} > \frac{\rho}{\delta}$, so if K^* satisfies the inequality $(\beta-1)K^* > \beta\rho/r$ we shall automatically have the condition $K^* > \rho/\delta$ needed to ensure $V^*(m) > n\rho/\delta$ at $m=n$.

$$S_0(m,V) = \frac{V - mp/r}{n-m} - \left(\frac{V}{\xi}\right)^{\beta} \frac{mp}{\beta-1} \frac{1}{r(n-m)}$$

which is concave in V, therefore $S_0(m,V) \le 0$. By continuity, there must be a value in $(\alpha mp/r(\alpha+1), \ \beta mp/r(\beta-1))$ where $S(m,V^*) = K+K^*$, and by uniqueness of ξ, this value is $\xi(m)$. $\square$

Corollary $\boxed{b(m) < 0 < a(m)}$

We can now prove the monotonicity of ξ.

Lemma 2. The function $\xi(\cdot)$ is strictly increasing

Suppose we increase m to $m+\varepsilon$, but we try taking the exact same point $\xi(m)$ for smooth fit to zero; so we're considering the function

$$S(m+\varepsilon, V) = \frac{V-(m+\varepsilon)p/r}{n-m-\varepsilon} + \frac{\beta(m+\varepsilon)p - r(\beta-1)\xi}{r(n-m-\varepsilon)(\alpha+\beta)}\left(\frac{V}{\xi}\right)^{-\alpha} + \frac{\alpha(m+\varepsilon)p - r(\alpha+1)\xi}{r(n-m-\varepsilon)(\alpha+\beta)}\left(\frac{V}{\xi}\right)^{\beta}$$

whence

$$(n-m-\varepsilon)S_0(m+\varepsilon,V) = (n-m)S(m,V) + \varepsilon\left[\frac{p}{(\alpha+\beta)r}\underbrace{\left\{\beta\left(\frac{V}{\xi}\right)^{-\alpha} + \alpha\left(\frac{V}{\xi}\right)^{\beta}\right\}}_{\text{increasing, convex in } V \ge \xi} - p/r\right]$$

We then have

$$S_0(m+\varepsilon, V^*(m+\varepsilon)) > \frac{n-m}{n-m-\varepsilon} S(m, V^*(m+\varepsilon))$$

$$> \frac{n-m}{n-m-\varepsilon}\left[K+K^* - \frac{\varepsilon K}{n-m}\right]$$

$$> K+K^*$$

By the previous argument, we know that if we shift ξ all the way up to $\beta mp/r(\beta-1)$ we'll have $S(m+\varepsilon, V^*(m+\varepsilon)) < 0$, so somewhere in between we get exactly $K+K^*$. Thus $\xi(m+\varepsilon) > \xi(m)$. $\square$

Lemma 3. (14/8/01). There exists $\varepsilon > 0$ such that

$$\frac{\partial S}{\partial V}(m, V^*(m)-) > 1/n \qquad \text{for } 0 < m \le \varepsilon$$

and there exists $m^* > 0$ such that

$$\frac{\partial S}{\partial V}(m, V^*(m)-) = 1/n \quad \text{at} \quad m = m^* \quad \text{provided} \quad \boxed{(\beta-1)K^* > \beta p/r}$$

Proof. If we consider the function

$$S(m,V) = \frac{V - mp/r}{n-m} - \frac{m(K^* - p/r)}{n-m}\left(\frac{V}{V^*}\right)^{\beta}$$

it is clear that $S_0(m, V^*(m)) = K + K^*$, and $S_0(m, 0) < 0$, so there exists (for $m > 0$) some $c(m) > 0$ such that

$$S(m,V) = S_0(m,V) + c(m)\left\{\left(\frac{V}{V^*}\right)^{-\alpha} - \left(\frac{V}{V^*}\right)^{\beta}\right\}$$

$$= \frac{\theta V^* - m\rho/r - m(K^*-\rho/r)\theta^\beta}{n-m} + c(m)\left\{\theta^{-\alpha} - \theta^\beta\right\} \qquad \left(\theta = \frac{V}{V^*(m)}\right)$$

$$= \theta(K+K^*) - \frac{m}{n-m}\left\{(K^*-\rho/r)\theta^\beta - \theta K^* + \rho/r\right\} + c(m)\left\{\theta^{-\alpha} - \theta^\beta\right\}$$

and for a given value of m, $c(m)$ is chosen so that the inf over $0 < \theta \le 1$ of this expression is zero. Equivalently, with $m > 0$, and writing $\lambda = (n-m)/m$, $\tilde{c}(m) = (n-m)c(m)/m$, we require

$$\inf_{0<\theta\le 1} \lambda\theta(K+K^*) + f_0(\theta) + \tilde{c}(m)\left\{\theta^{-\alpha} - \theta^\beta\right\} = 0 \qquad (*)$$

where we define

$$f_0(\theta) = \theta K^* - \rho/r - (K^*-\rho/r)\theta^\beta$$

a concave function, negative at 0, vanishing at 1.

(i) What happens when m is small? Considering the slope of S at V^*-, we have

$$\frac{\partial S}{\partial V}(m, V^*-) = \frac{1}{n-m} - \frac{\beta m(K^*-\rho/r)}{(n-m)V^*} - \frac{c(m)}{V^*}(\alpha+\beta)$$

$$\le \frac{1}{n}$$

iff $$\frac{m}{(n-m)V^*}\left\{\beta(K^*-\rho/r) + \tilde{c}(m)(\alpha+\beta)\right\} \ge \frac{m}{n(n-m)}$$

iff $$\tilde{c}(m) \ge \left\{\frac{V^*}{n} - \beta(K^*-\rho/r)\right\}/(\alpha+\beta)$$

As $m \downarrow 0$, the RHS here tends to $(K+K^* - \beta(K^*-\rho/r))/(\alpha+\beta)$ which is positive, in view of the assumption that $K^* \in (\rho/r, (K+\beta\rho/r)/(\beta-1))$. Thus there exists some positive constant δ such that for $0 < m < \delta$

$$\frac{V^*}{n} - \beta(K^*-\rho/r) \ge \tfrac{1}{2}\left\{K+K^* - \beta(K^*-\rho/r)\right\} > 0$$

The condition $(*)$ cannot now be satisfied for small m if $\frac{\partial S}{\partial V}(m,V^*) \le \frac{1}{n}$, and the first conclusion of the lemma follows.

... about the behaviour as $m \uparrow n$? As $m \uparrow n$, $\lambda = (n-m)/m \downarrow 0$, and $\tilde{c}(m) \uparrow$ to ... for which

$$f_0(\theta) + \tilde{c}(n)(\theta^{-\alpha} - \theta^{\beta}) = 0.$$

... of ... $0 < \theta < 1$. The derivative of this function is

$$K^* - \beta(K^* - p/r)\theta^{\beta-1} - \tilde{c}(n)(\alpha\theta^{-\alpha-1} + \beta\theta^{\beta-1}),$$

... $f_1(\theta) = f_0(\theta) + \tilde{c}(n)(\theta^{-\alpha} - \theta^{\beta})$) since we know $f_1(\theta) \geq 0$, $f_1(1) = 0$, it must be ... derivative $f_1'(1) \leq 0$. Could it actually be zero? If we work out $f_1'(1)$, we obtain

$$f_1'(1) = -(\beta-1)K^* + \beta p/r - \tilde{c}(n)(\alpha+\beta)$$

... be < 0 in view of the inequality assumed for K^*.

$$\ldots(m, V^*(m)) = \frac{V^*(m) - \beta m(K^* - p/r) - (\alpha+\beta)\tilde{c}(m)\, m}{(n-m)V^*(m)}$$

$$\sim \frac{n\{K^* - \beta(K^* - p/r) - (\alpha+\beta)\tilde{c}(n)\}}{(n-m)\; V^*(m)} \qquad (m \uparrow n)$$

$$\to -\infty$$

... $f_1'(1) < 0$.

$\square$

... the form of Y; if we aim to solve $\mathcal{L}Y + \frac{\delta V - np}{n-m} = 0$, with the ... condition that $Y(m, V^*(m)) = K$, $\frac{\partial Y}{\partial V}(m, V^*(m)) = 0$, we find the solution

$$Y(m, V) = \frac{V - np/r}{n-m} + \tilde{A}(m)\left(\frac{V}{V^*(m)}\right)^{-\alpha} + \tilde{B}(m)\left(\frac{V}{V^*(m)}\right)^{\beta}$$

$$\tilde{A}(m) = \frac{\beta np + \beta r K(n-m) - (\beta-1) r V^*(m)}{r(n-m)(\alpha+\beta)}$$

$$\tilde{B}(m) = \frac{\alpha np + \alpha r K(n-m) - (\alpha+1) r V^*(m)}{r(n-m)(\alpha+\beta)}$$

... solution Y from earlier work, but with $\eta(m)$ replaced by $V^*(m)$). ... the following result about the behaviour of Y_0.

$$f_1(1) = 0 = f_1'(1), \quad f_1''(1) = \theta^{\beta-2}(\alpha+1)\beta(\beta-1) + \theta^{-\alpha-2}(\beta-1)\alpha(\alpha+1)\Big|_{\theta=1} = (\alpha+\beta)(\alpha+1)(\beta-1) > 0$$

<u>Lemma 4.</u> (i) $Y_0(m, \xi(m)) > 0$ for m in some neighbourhood of 0

(ii) $\lim_{m\uparrow n} Y_0(m,\xi(m)) = -\infty$ <u>provided</u> $K^* > \beta\rho/r(\beta-1)$ <u>and</u> $K^* > \rho/\delta$.

[In fact, $\rho_\delta < \beta\rho/r(\beta-1)$ in any case.]

<u>Proof</u> (i) Because $K^* < (K+\beta\rho/r)(\beta-1)$, we have for some $\varepsilon>0$

$$(\beta-1)K^* = K + \beta\rho/_r - \varepsilon$$

and thus

$$r(\alpha+\beta)(n-m)\tilde{A}(m) = \beta n\rho + \beta r K(n-m) - (\beta-1)r\big((n-m)K + nK^*\big)$$

$$= r(n-m)K - nr(K+\beta\rho/_r - \varepsilon) + \beta n\rho$$

$$= -mK + nr\varepsilon$$

$$> nr\varepsilon/_2$$

provided $m < nr\varepsilon/K$. Now evidently $\dfrac{\xi(m)-n\rho/r}{n-m}$ and $\tilde{B}(m)\left(\dfrac{\xi(m)}{V^*(m)}\right)^\beta$ remain bounded in a neighbourhood of zero, but $Y_0(m,\xi(m))$ is the sum of those two terms, plus

$$\tilde{A}(m)\left(\frac{\xi(m)}{V^*(m)}\right)^{-\alpha} > \tilde{A}(m)\left(\frac{\beta m\rho}{r(\beta-1)V^*(m)}\right)^{-\alpha}$$

which behaves like $m^{-\alpha}$ for small α.

(ii) Writing $\theta \equiv V/V^*(m)$, we can re-express Y_0:

$$Y_0 = \frac{\theta V^* - n\rho/r}{n-m} + \frac{\beta n(\rho/_r - K^*) + V^*}{(n-m)(\alpha+\beta)}\theta^{-\alpha} + \frac{\alpha n(\rho/_r - K^*) - V^*}{(n-m)(\alpha+\beta)}\theta^\beta$$

so that

$$\frac{(n-m)Y_0}{n} = \frac{V}{n} - \frac{\rho}{r} - (K^* - \rho/_r)\frac{\alpha\theta^\beta + \beta\theta^{-\alpha}}{\alpha+\beta} + \frac{V^*}{n}\frac{\theta^{-\alpha}-\theta^\beta}{\alpha+\beta}$$

$$= \theta K^* - \rho/_r - (K^*-\rho/_r)\frac{\alpha\theta^\beta+\beta\theta^{-\alpha}}{\alpha+\beta} + K^*\frac{\theta^{-\alpha}-\theta^\beta}{\alpha+\beta} + \frac{(n-m)K}{n}\left\{\frac{\theta^{-\alpha}-\theta^\beta}{\alpha+\beta} + \theta\right\}$$

$$= K^*\left(\theta - \frac{\alpha\theta^\beta+\beta\theta^{-\alpha}}{\alpha+\beta} + \frac{\theta^{-\alpha}-\theta^\beta}{\alpha+\beta}\right) + \rho/_r\left(\frac{\alpha\theta^\beta+\beta\theta^{-\alpha}}{\alpha+\beta} - 1\right) + O(n-m)$$

$$\equiv -K^* f_1(\theta) + \rho/_r f_2(\theta) + O(n-m)$$

So we would be finished if we could prove that $\boxed{\lim_{m\uparrow n} -K^* f_1\left(\frac{\xi(m)}{V^*(m)}\right) + \rho/_r f_2\left(\frac{\xi(m)}{V^*(m)}\right) < 0.}$

$$\begin{aligned} r(\alpha+1)(\beta-1) - \delta\alpha\beta &= (r-\delta)\,\alpha\beta - r + r(\beta-\alpha) \\ &= (r-\delta)\,\frac{2r}{\sigma^2} - r + r\left(\frac{-(r-\delta-\sigma^2/2)}{\sigma^2/2} \right) \\ &= 0 \end{aligned}$$

Also we have $\quad \alpha - \beta + 1 = \dfrac{2(r-\delta)}{\sigma^2}$

However, the function of θ that interests us, $-K^* f_1(\theta) + \frac{\rho}{r} f_2(\theta)$, vanishes with its first derivative at $\theta = 1$, and can be written

$$\theta K^* + \left(\frac{\beta\rho}{r} - (\beta-1)K^*\right)\frac{\theta^{-\alpha}}{\alpha+\beta} + \left(\frac{\alpha\rho}{r} - (\alpha+1)K^*\right)\frac{\theta^{\beta}}{\alpha+\beta}$$

If we assume $K^* > \beta\rho/(r(\beta-1))$, then the function is concave; since we know that $\xi(m) \le m\rho/\delta$ (see Lemma 5 below) and $K^* > \rho/\delta$, we have

$$\frac{\xi(m)}{V^*(m)} \le \frac{m\rho/\delta}{nK^* + (n-m)K} \uparrow \frac{\rho}{\delta K^*} < 1$$

and this suffices. □

<u>Lemma 5</u> <u>The bound on ξ from Lemma 1 can be improved</u>: <u>for</u> $m < n$,

$$\xi(m) < m\rho/\delta.$$

<u>As</u> $m \uparrow n$, $\theta(m) \equiv V^*(m)/\xi(m)$ <u>converges to the unique solution</u> in $(1,\infty)$ <u>to</u> $\varphi(\theta) = 0$, <u>where</u>

$$\varphi(\theta) \equiv rK^*\left[\alpha+\beta - (\beta-1)\theta^{-\alpha-1} - (\alpha+1)\theta^{\beta-1}\right] + \rho\left[\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta\right], \quad \text{if } K^* > \rho/\delta.$$

<u>Proof</u> By considering the form of $S(m, V^*(m)) = K + K^*$, we see that

$$(n-m)r(\alpha+\beta)(K+K^*) = r\xi\left((\alpha+\beta)\theta - (\beta-1)\theta^{-\alpha} - (\alpha+1)\theta^{\beta}\right) + m\rho\left(\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta\right) \qquad (\dagger)$$

We also have from the fact that $a(m) > 0 > b(m)$ that $S(m,\cdot)$ is initially convex then concave in V. Considering therefore the second derivative of S at $\theta = 1$, we get

$$\begin{aligned} V^2 \frac{\partial^2 S}{\partial V^2} &= \alpha(\alpha+1)\,a(m) + \beta(\beta-1)\,b(m) \\ &= \{\alpha\beta m\rho - (\alpha+1)(\beta-1) r\,\xi(m)\} / (n-m)r \\ &= \frac{\alpha\beta}{(n-m)r}\{m\rho - \delta\xi(m)\} \end{aligned}$$

Using the useful identity

$$\boxed{r(\alpha+1)(\beta-1) = \delta\alpha\beta}$$

If the second derivative of S at $V = \xi(m)$ were ≤ 0, then S would be concave in $[\xi(m), \infty)$, and so could not get up to $K + K^* > 0$ at $V^*(m)$. Hence we must have $\partial^2 S/\partial V^2 > 0$ at $\xi(m)$, implying that $\xi(m) < m\rho/\delta$ as stated.

Now consider the form of S at $V^*(m)$, and write $\xi = V^*/\theta$ to obtain from (†) that as $m \uparrow n$, any limit of $\theta(m)$ must satisfy $\varphi(\theta) = 0$. Now $\theta = 1$ will be a solution, but it is inadmissible since $V^*(m) \geq nK^* > np/\delta > \xi(m) \ \forall m$, by hypothesis on K^*.

Differentiating φ gives

$$\begin{cases} \varphi'(\theta) = \alpha\beta p\,(\theta^{\beta-1} - \theta^{-\alpha-1}) + (\alpha+1)(\beta-1)rK^*(\theta^{-\alpha-2} - \theta^{\beta-2}) \\ \varphi''(\theta) = \alpha\beta p\,((\beta-1)\theta^{\beta-2} + (\alpha+1)\theta^{-\alpha-2}) - (\alpha+1)(\beta-1)rK^*((\alpha+2)\theta^{-\alpha-3} + (\beta-2)\theta^{\beta-3}) \end{cases}$$

Now we observe that $\varphi'(1) = 0$, $\varphi''(1) = (\alpha+\beta)(\alpha\beta p - (\alpha+1)(\beta-1)rK^*) < 0$, and $\varphi(\theta) \to \infty$ as $\theta \to \infty$. So φ and its first derivative vanish at 1, φ is concave at 1; there will therefore be at least one root in $(1,\infty)$. If there were more than one, there would have to be at least 4 distinct zeros of the derivative in $[1,\infty)$; but since φ' is a linear combination of 4 distinct powers of θ, this is impossible (see p 2 !!) $\square$

The result of Lemma 4(ii) can also be strengthened, for which we need a simple result from real analysis.

<u>Proposition 1.</u> <u>Suppose that</u> $t > 0$, <u>and</u> f_1, f_2 <u>are two probability densities on</u> $[0,t]$, <u>such that</u> f_1 <u>is strictly increasing</u>, f_2 <u>is decreasing</u>. <u>Then for any strictly increasing</u> C^1 <u>function</u> h

$$\int_0^t h(s) f_1(s)\,ds > \int_0^t h(s) f_2(s)\,ds.$$

<u>Proof.</u> If $g = f_1 - f_2$, an increasing function integrating to zero, $G(t) = \int_0^t g(s)ds$, we have

$$\int_0^t h(s) g(s)\,ds = \left[h(s)G(s)\right]_0^t - \int_0^t G(s) h'(s)\,ds$$
$$= -\int_0^t G(s) h'(s)\,ds$$

But as $G' = g$ is strictly increasing, G is strictly convex, and so $G(x) < 0 \ \forall 0 < x < t$, since $G(0) = 0 = G(t)$. The result follows. $\square$

Applying this proposition, we have immediately the inequality

$$\frac{\int_0^t e^s\,(e^{(\beta-1)s} - e^{-(\alpha+1)s})\,ds}{\int_0^t (e^{(\beta-1)s} - e^{-(\alpha+1)s})\,ds} > \frac{\int_0^t e^u\,(e^{(\alpha+1)(t-u)} - e^{-(\beta-1)(t-u)})\,du}{\int_0^t (e^{(\alpha+1)(t-u)} - e^{-(\beta-1)(t-u)})\,du}$$

$$= e^t\,\frac{\int_0^t e^{-s}\,(e^{(\alpha+1)s} - e^{-(\beta-1)s})\,ds}{\int_0^t (e^{(\alpha+1)s} - e^{-(\beta-1)s})\,ds}$$

Rearranging and evaluating the integrals converts the inequality to

$$\frac{\alpha e^{\beta t} + \beta e^{-\alpha t} - \alpha - \beta}{\alpha e^{-\beta t} + \beta e^{\alpha t} - \alpha - \beta} > e^{t} \cdot \frac{(\alpha+1) e^{(\beta-1)t} + (\beta-1) e^{-(\alpha+1)t} - \alpha - \beta}{(\alpha+1) e^{-(\beta-1)t} + (\beta-1) e^{(\alpha+1)t} - \alpha - \beta}.$$

Now suppose that we take e^t to be the limit of $V^*(m)/\xi(m)$ as $m \uparrow n$, guaranteed by Lemma 5, and satisfying

$$\frac{\alpha e^{\beta t} + \beta e^{-\alpha t} - \alpha - \beta}{(\alpha+1) e^{(\beta-1)t} + (\beta-1) e^{-(\alpha+1)t} - \alpha - \beta} = \frac{rK^*}{\rho}.$$

The above inequality then gives

$$\frac{rK^*}{\rho} > \frac{\alpha e^{-\beta t} + \beta e^{\alpha t} - \alpha - \beta}{(\alpha+1) e^{-\beta t} + (\beta-1) e^{\alpha t} - (\alpha+\beta) e^{-t}}.$$

If now we write $\theta = e^{-t}$ for the limit of $\xi(m)/V^*(m)$, we have the inequality

$$\frac{rK^*}{\rho} > \frac{\alpha \theta^{\beta} + \beta \theta^{-\alpha} - \alpha - \beta}{(\alpha+1)\theta^{\beta} + (\beta-1)\theta^{-\alpha} - (\alpha+\beta)\theta},$$

and hence

$$\boxed{0 > \frac{\rho}{r}\left(\alpha\theta^{\beta} + \beta\theta^{-\alpha} - \alpha - \beta\right) + K^*\left((\alpha+\beta)\theta - (\alpha+1)\theta^{\beta} - (\beta-1)\theta^{-\alpha}\right).}$$

But this amounts to the boxed inequality at the foot of p7, so we can strengthen Lemma 4(ii) to the following

<u>Lemma 4 (ii)′</u> <u>If</u> $K^* > \rho/\delta$, <u>then</u>

$$\lim_{m \uparrow n} Y_0(m, \xi(m)) = -\infty$$

<u>Corollary</u>. <u>If</u> $K^* > \rho/\delta$, <u>there exists</u> $m^* \in (0, n)$ <u>such that</u> $\frac{\partial Y_0}{\partial V}(m, V^*(m)) = 0$ <u>at</u> $m = m^*$.

Interlude: Some questions from Monique Jeanblanc (10/9/01)

1) Consider the standard dynamics

$$dx = rx\,dt + \theta(\sigma dW + (\mu - r)dt)$$

with the objective

$$\max E \int_0^T U(t, x_t)\,dt$$

This is motivated by considering someone who wishes to pass on the maximal expected utility of bequest to their descendants at the random time of demise.

Introducing the Lagrangian semimartingale $d\xi = \xi(a\,dW - b\,dt)$, we find the dual problem is got from

$$\sup E\left[\int_0^T U(t,x_t)\,dt - \left(x_T\xi_T - x_0\xi_0 + \int_0^T x\xi b\,dt - \int_0^T a\xi\theta\sigma\,dt\right) + \int_0^T \xi(rx + \theta(\mu - r))\,dt\right]$$

$$= \sup E\left[\int_0^T \tilde{U}(t, \xi_t(b_t - r))\,dt + x_0\xi_0\right]$$

with the dual-feasibility conditions

$$a_t\sigma_t + \mu_t - r_t = 0$$

$$b_t - r_t \equiv \gamma_t > 0$$

and complementary slackness $\xi_T x_T = 0$. So $d\xi_t = \xi_t\left(-\frac{\mu - r}{\sigma}dW - (r + \gamma)\,dt\right)$, and therefore $\xi_t = \xi_0 \zeta_t \exp(-\Gamma_t)$, where ζ is the standard state-price density, $\Gamma_t = \int_0^t \gamma_s\,ds$. The dual is therefore

$$\boxed{\min_{\xi_0, \gamma} E\left[\int_0^T \tilde{U}(t, \xi_0\zeta_t e^{-\Gamma_t}\gamma_t)\,dt + \xi_0 x_0\right]}$$

2) Special case: Let's suppose that $U(t,x) = f(t)\,x^{1-R}/(1-R)$, so that $\tilde{U}(t,y) = -f(t)\,(y/f(t))^{1-R'}/(1-R')$, $R' \equiv 1/R$. The dual problem is therefore to

$$\min_{\xi_0, \gamma} E\left[-\int_0^T f(t)^{R'}\left(\xi_0\zeta_t e^{-\Gamma_t}\gamma_t\right)^{1-R'}\frac{dt}{1-R'} + \xi_0 x_0\right]$$

which we can do if we can compute

$$\max_\gamma E\int_0^T f(t)^{R'}\left(\zeta_t e^{-\Gamma_t}\gamma_t\right)^{1-R'}\frac{dt}{1-R'}$$

But if we define

$$V_t \equiv \sup_\gamma E_t\left\{\int_t^T f(s)^{R'}\left(\frac{\zeta_s}{\zeta_t}\,e^{-(\Gamma_s - \Gamma_t)}\gamma_s\right)^{1-R'}\frac{ds}{1-R'}\right\}$$

this is in fact non-random, as a little thought shows. Moreover, the standard HJB story is

$$V_t\,\xi_t^{1-R'} + \int_0^t f(s)^{R'}(\xi_s\gamma_s)^{1-R'}\frac{ds}{1-R'} \quad \text{is a supermartingale etc}$$

Taking the differential,

$$\sup_\gamma \; \dot V \xi^{1-R'} + V\Big\{(1-R')\frac{d\xi}{\xi} - \frac{(1-R')R'}{2}\frac{d\langle\xi\rangle}{\xi^2}\Big\}\xi^{1-R'} + f(t)^{R'}(\xi\gamma)^{1-R'}/_{1-R'} = 0$$

$$\therefore \sup_\gamma \; \dot V + V\Big\{(1-R')(-r-\gamma) - \frac{(1-R')R'}{2}\Big(\frac{\mu-r}{\sigma}\Big)^2\Big\} + f_t^{R'}\gamma^{1-R'}/_{1-R'} = 0$$

Optimal γ solves $\gamma^{-R'} \cdot f^{R'} = (1-R')V$, so we end up with

$$\boxed{\dot V - \frac{(1-R')R'}{2}\Big(\frac{\mu-r}{\sigma}\Big)^2 V - r(1-R')V - f\frac{\big((1-R')V\big)^{1-R}}{1-R} = 0}$$

with be $V_T = 0$.

3) Another question (Monique studies this with Peter Lakner).

We have

$$dx_t = rx_t\,dt + \theta_t(\sigma\,dW_t + (\mu-r)\,dt) - \varepsilon\,dt, \qquad x \geq 0$$

where $\varepsilon, r, \sigma, \mu$ are constant, and the aim is to choose stopping time τ to achieve

$$\max_{0\leq\tau\leq T} E\,U(\tau, x_\tau)$$

[In their formulation, $E\,U(\tau, x_\tau - K)$ for some constant K, but the two are really the same.]

This one appears harder, and can probably only be done numerically. We have the HJB equations for the value function V:

$$\min\Big\{V-U,\; \frac{(\mu-r)^2 V_x^2}{2\sigma^2 V_{xx}} - \dot V\Big\} = 0, \qquad V(T,x) = U(T,x)$$

We expect there to be an exercise boundary, above which $U = V$, with smooth pasting across the boundary. Solving this numerically by backward recursion should be fairly straightforward. There is a dual formulation of the problem, but it's no easier.

Another interlude: a question studied by Stephen Walker + Igor Prünster (15/9/01)

1) The idea is to try to generate a random CDF on $[0,\infty)$ (or $[0,T]$) by the recipe

$$F(t) = Z(t)/Z(\infty)$$

where Z is an increasing process with independent (but not necessarily stationary) increments. The questions of interest then are things like 'What is the law of the mean?'.

To set up notation, if

$$E\exp(-\lambda Z_t) = \exp\left\{-\int_0^t \int (1-e^{-\lambda x})\,\mu_s(dx)\,ds\right\}$$

then we shall have

$$E\exp(-\alpha\xi - \beta\eta) = \exp\left\{-\int_0^t \int \left[1-\exp\{-x(\alpha\varphi(s)+\beta\psi(s))\}\right]\mu_s(dx)\,ds\right\}$$

where

$$\xi \equiv \int_0^t \varphi(s)\,dZ_s, \qquad \eta \equiv \int_0^t \psi(s)\,dZ_s.$$

The interest is in the law of ξ/η, equivalently, the Mellin transform of this thing

$$E\left(\frac{1}{z+\xi/\eta}\right) = E\left(\frac{\eta}{\xi+z\eta}\right).$$

But if we now set

$$\Psi(\alpha,\beta) \equiv E\exp(-\alpha(\xi+z\eta)-\beta\eta) = E\exp\{-\alpha\xi-(\alpha z+\beta)\eta\}$$

we shall have that

$$-\frac{\partial}{\partial\beta}\int_0^\infty \Psi(\alpha,\beta)\,d\alpha\Big|_{\beta=0} = E\left(\frac{1}{z+\xi/\eta}\right) = \int_0^\infty \frac{G(dx)}{x+z}$$

where G is the distribution of ξ/η. If we can extend this function into $\mathbb{C}\setminus\mathbb{R}^-$, we can recover the density of G, assuming it has one;

$$g(x_0) = -\lim_{b\downarrow 0} \mathrm{Im}\int_0^\infty \frac{g(x)\,dx}{x-x_0+ib}\,\frac{1}{\pi}$$

2) Example. Let's take $\int(1-e^{-\lambda x})\,\mu_s(dx) = \lambda^\theta$ for some fixed $\theta\in(0,1)$, and keep everything on $[0,T]$. Use $\varphi(s)=s$, $\psi(s)=1$, and find

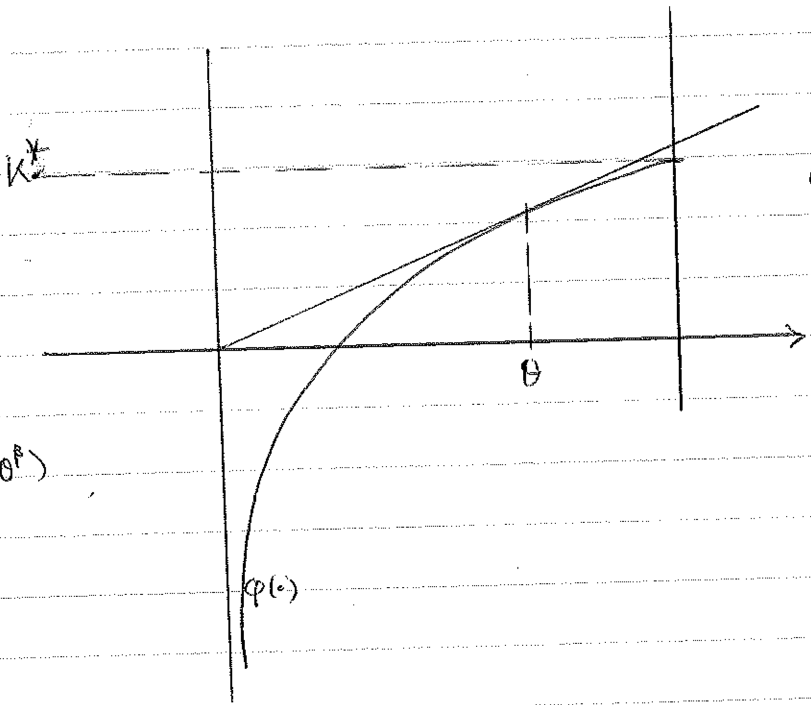
$$\Psi(\alpha,\beta) = \exp\left[-\{(\alpha T+\alpha z+\beta)^{\theta+1} - (\alpha z+\beta)^{\theta+1}\}/(\theta+1)\alpha\right] = \exp\left\{-\int_0^T (\alpha s+\alpha z+\beta)^\theta\,ds\right\}$$

whence

$$\boxed{E\,\frac{1}{z+\xi/\eta} = \frac{(T+z)^\theta - z^\theta}{(T+z)^{1+\theta}-z^{1+\theta}}\cdot\frac{1+\theta}{\theta}}$$

From this, the density at $x\in(0,T)$ of ξ/η is

$$\boxed{-\frac{1+\theta}{\theta}\,\frac{b_1a_2-b_2a_1}{\pi(a_2^2+b_2^2)} \quad \text{where } a_1+ib_1 = (T-x)^\theta - x^\theta e^{i\pi\theta},\quad a_2+ib_2 = (T-x)^{1+\theta}-x^{1+\theta}e^{i\pi(1+\theta)}}$$

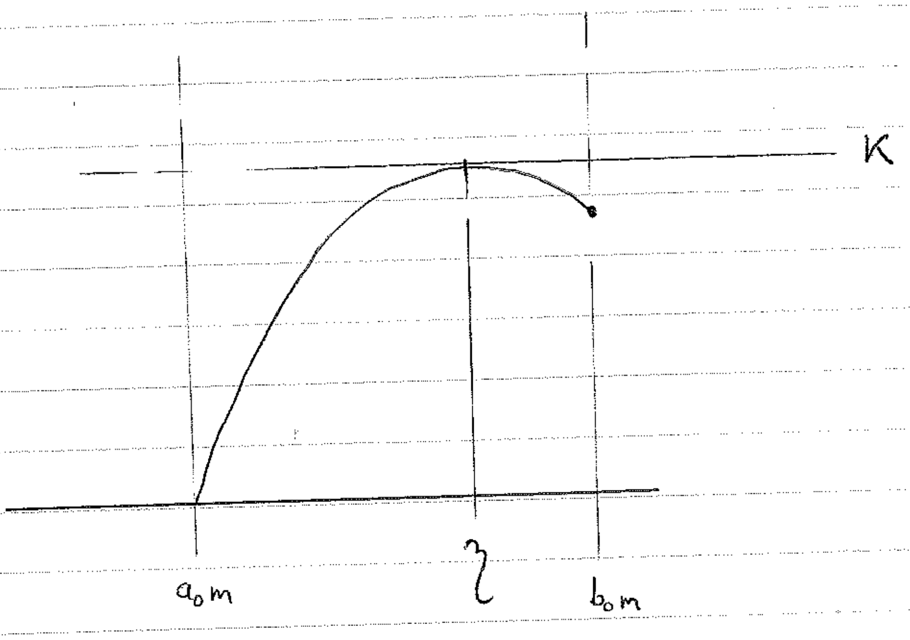


Evidently, $\varphi'(\theta) = b_0 > K^*$.

$$a_0 = \frac{\frac{p}{r} - K^*}{\alpha+\beta}\,\alpha\beta\left(\theta^{-\alpha} - \theta^{\beta}\right),$$

$$m_0 = n\,\frac{K+K^*}{K+b_0}$$

$$\frac{p}{r} = a_0 + \frac{\varepsilon}{\alpha+\beta}\left(\beta\theta^{-\alpha} + \alpha\theta^{\beta}\right) = \frac{\varepsilon}{\alpha+\beta}\left\{(\alpha+1)\beta\theta^{-\alpha} - \alpha(\beta-1)\theta^{\beta}\right\}, \quad \varepsilon = p/r - K^*$$



Lemmas for the CCCR study again (4/10/01)

1) In the situation where

$$K^* < \rho/r$$

we know that for $m \leq m_0$, there is bankruptcy declared at a lower boundary $a_0 m$, with calling (with surrender) at upper boundary $b_0 m$, where $a_0 = \theta\, b_0$, and a_0, θ must solve

$$\begin{cases} a_0 = \varphi(\theta) \equiv f_r - \dfrac{\rho/r - K^*}{\alpha+\beta}\left(\beta\theta^{-\alpha} + \alpha\theta^{\beta}\right) & \text{(a concave incr } f^n,\ \varphi'(1)=0) \\ a_0 = \theta\varphi'(\theta). \end{cases}$$

The value m_0 is where $b_0 m = V^*(m) \equiv n(K+K^*) - mK$. To the left of m_0, we have

$$Y(m,V) \equiv S(m,V) - B(m,V) = \frac{V - n\rho/r}{n-m} + A(m)\,V^{-\alpha} + B(m)\,V^{\beta}$$

and $Y(m, a_0 m) \leq 0$, $Y(m, b_0 m) = (b_0 m - nK^*)/(n-m) \uparrow K$ as $m \uparrow m_0$.

2) Do we know that $Y \leq K$ everywhere to the left of m_0? Wlog, may suppose $Y(m, a_0 m) = 0$. If there was some $m \leq m_0$ where Y rose above K for some $V \in (a_0 m, b_0 m)$, then by pulling Y down at $b_0 m$ we could reduce the max until it were equal to K, with smooth fit to K at some $\eta \in (a_0 m, b_0 m)$. The expression for Y has now been modified to

$$Y_0(m,V) = \frac{V - n\rho/r}{n-m} + \frac{\beta n\rho/r + \beta K(n-m) - (\beta-1)\eta}{(\alpha+\beta)(n-m)}\left(\frac{V}{\eta}\right)^{-\alpha} + \frac{\alpha n\rho/r + \alpha K(n-m) - (\alpha+1)\eta}{(\alpha+\beta)(n-m)}\left(\frac{V}{\eta}\right)^{\beta}$$

and η must be such that $Y_0(m, a_0 m) = 0$, that is,

$$0 = (\alpha+\beta)(a_0 m - n\rho/r) + \left(\frac{n\rho}{r} + K(n-m) - \eta\right)\left\{\beta\left(\frac{a_0 m}{\eta}\right)^{-\alpha} + \alpha\left(\frac{a_0 m}{\eta}\right)^{\beta}\right\} + \eta\left\{\left(\frac{a_0 m}{\eta}\right)^{-\alpha} - \left(\frac{a_0 m}{\eta}\right)^{\beta}\right\}$$

So if we were to write $\eta = \lambda^{-1} a_0 m$, with $\lambda \in (0,1]$, the condition which λ has to satisfy is

$$0 = (\alpha+\beta)(a_0 m - n\rho/r) + \left(\frac{n\rho}{r} + K(n-m) - \lambda^{-1} a_0 m\right)\left(\beta\lambda^{-\alpha} + \alpha\lambda^{\beta}\right) + \lambda^{-1} a_0 m\left\{\lambda^{-\alpha} - \lambda^{\beta}\right\}$$

or equivalently,

$$m\left[-a_0(\alpha+\beta) + (K + \lambda^{-1} a_0)(\beta\lambda^{-\alpha} + \alpha\lambda^{\beta}) - a_0\lambda^{-1}\{\lambda^{-\alpha} - \lambda^{\beta}\}\right] = -(\alpha+\beta)\frac{n\rho}{r} + \left(\frac{n\rho}{r} + nK\right)\left(\beta\lambda^{-\alpha} + \alpha\lambda^{\beta}\right)$$

or again

$$\boxed{m\left[K\left(\beta\lambda^{-\alpha} + \alpha\lambda^{\beta}\right) + a_0\left\{(\alpha+1)\lambda^{\beta-1} + (\beta-1)\lambda^{-\alpha-1} - \alpha - \beta\right\}\right] = \left(\frac{n\rho}{r} + nK\right)\left(\beta\lambda^{-\alpha} + \alpha\lambda^{\beta}\right) - n\rho(\alpha+\beta)/r}$$

3) Now if we define

$$\begin{cases} \varphi_1(\lambda) = K\left(\beta\lambda^{-\alpha}+\alpha\lambda^{\beta}\right) + a_0\left[(\alpha+1)\lambda^{\beta-1}+(\beta-1)\lambda^{-\alpha-1}-\alpha-\beta\right] \\ \varphi_2(\lambda) = \left(\frac{\rho}{r}+K\right)\left(\beta\lambda^{-\alpha}+\alpha\lambda^{\beta}\right) - \frac{\rho}{r}(\alpha+\beta) \end{cases}$$

the condition we require is that λ should solve

$$m\varphi_1(\lambda) = n\varphi_2(\lambda)$$

Clearly, as $\lambda\to 0$, $\varphi_1(\lambda)/\varphi_2(\lambda)\to\infty$, and as $\lambda\to 1$, $\varphi_1(\lambda)/\varphi_2(\lambda)\to 1$, so there will be at least one root $\lambda\in(0,1)$.

4) <u>More insight into the form of Y.</u> We have

$$Y(m,V) = E^V\left[\int_0^\tau e^{-rs}\,\frac{\delta V_s - n\rho}{n-m}\,ds + e^{-r\tau} I_{\{V_\tau = b_0 m\}}\,\frac{b_0 m - nK^*}{n-m}\right]$$

so that

$$\begin{aligned}(n-m)Y(m,\lambda m) &= E^{\lambda m}\left[\int_0^\tau e^{-rs}\delta V_s\,ds + e^{-r\tau}I_{\{V_\tau=b_0 m\}}\,b_0 m\right] \\ &\quad - nE^{\lambda m}\left[\int_0^\tau \rho e^{-rs}\,ds + e^{-r\tau}I_{\{V_\tau=b_0 m\}}\,K^*\right] \\ &\equiv m\,f_1(\lambda) - n\,f_2(\lambda)\end{aligned}$$

exploiting the scaling properties of V, where

$$f_1(\lambda)\equiv E^\lambda\left[\int_0^\tau e^{-rs}\delta V_s\,ds + e^{-r\tau}I_{\{V_\tau=b_0\}}\,b_0\right]$$

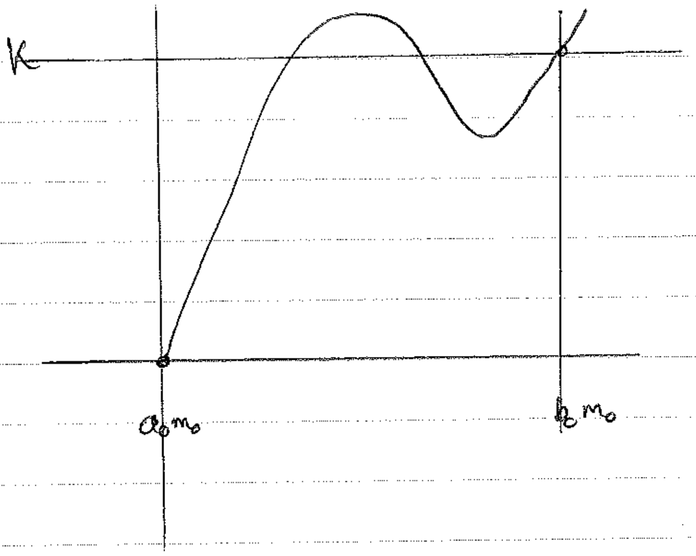
$$f_2(\lambda)\equiv E^\lambda\left[\int_0^\tau \rho e^{-rs}\,ds + e^{-r\tau}I_{\{V_\tau=b_0\}}\,K^*\right]$$

We have
$$\left.\begin{aligned} f_1(V) &= V + A_1V^{-\alpha}+B_1V^{\beta}, & f_1(a_0)=0,\ f_1(b_0)=b_0 \\ f_2(V) &= \rho/r + A_2V^{-\alpha}+B_2V^{\beta}, & f_2(a_0)=0,\ f_2(b_0)=K^* \end{aligned}\right\}$$

<u>Hence</u>

$$\boxed{\begin{aligned} f_1(V) &\equiv V - \frac{\theta b_0}{\theta^{-\alpha}-\theta^{\beta}}\left\{\left(\frac{V}{b_0}\right)^{-\alpha}-\left(\frac{V}{b_0}\right)^{\beta}\right\} \\ f_2(V) &= \frac{\rho}{r} - \frac{\rho}{r}\,\frac{(1-\theta^{\beta})\left(\frac{V}{b_0}\right)^{-\alpha}+(\theta^{\alpha}-1)\left(\frac{V}{b_0}\right)^{\beta}}{\theta^{-\alpha}-\theta^{\beta}} + K^*\,\frac{\left(\frac{V}{a_0}\right)^{-\alpha}-\left(\frac{V}{a_0}\right)^{\beta}}{\theta^{+\alpha}-\theta^{-\beta}} \end{aligned}}$$

Clearly, $\max\limits_{a_0\le\lambda\le b_0}\{m f_1(\lambda) - n f_2(\lambda)\}$ rises with m, and is non-negative. So in order to prove that Y remains bounded by K for $m\le m_0$, $\forall V\in[a_0 m, b_0 m]$, it's enough to get



$$\left(m_0 = n \frac{K+K^*}{K+b_0} \right)$$

the bound on Y at $m = m_0$.

For this it is sufficient to prove that $\frac{\partial Y}{\partial V}(m_0, b_0 m_0) > 0$, for if we knew this and if Y exceeded K somewhere, there would have to be two zeros of the derivative of Y. This can only happen if A, B are of opposite signs; and if this is the case, Y can only be increasing to the right of the larger zero if $B > 0$. But if $B > 0 > A$ we have in fact that Y is increasing!

<u>So the essential is to prove that</u> $\frac{\partial Y}{\partial V}(m_0, b_0 m_0) > 0$.

Doing some Maple on this, we get a great pile of stuff. Better is to go back to the expression for $Y_0(m_0, a_0 m)$, which we must prove is positive:

$$Y_0(m_0, a_0 m) = \frac{a_0 m - np/r}{n - m_0} + \frac{\beta np/r + \beta K(n-m_0) - (\beta-1) b_0 m_0}{(n-m_0)(\alpha+\beta)} \theta^{-\alpha} + \frac{\alpha np/r + \alpha K(n-m_0) - (\alpha+1) b_0 m_0}{(n-m_0)(\alpha+\beta)} \theta^{\beta}$$

which is positive iff

$$0 < (\alpha+\beta)(a_0 m_0 - np/r) + (\beta\theta^{-\alpha} + \alpha\theta^{\beta})\left\{\frac{np}{r} + K(n-m_0) - b_0 m_0\right\} + (\theta^{-\alpha} - \theta^{\beta}) b_0 m_0$$

$$= (\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta) np/r + (\alpha+\beta) a_0 m_0 - nK^*(\beta\theta^{-\alpha} + \alpha\theta^{\beta}) + (\theta^{-\alpha} - \theta^{\beta}) b_0 m_0$$

$$= (\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta)\frac{np}{r} + \left\{\theta^{-\alpha} - \theta^{\beta} + \theta(\alpha+\beta)\right\} b_0 n \frac{K+K^*}{K+b_0} - nK^*(\beta\theta^{-\alpha} + \alpha\theta^{\beta})$$

Now the values of θ, b_0 are not affected by K; the only K-dependence is in the increasing ratio $(K+K^*)/(K+b_0)$. The worst case is therefore when $K = 0$, so it's n+s to prove

$$0 < \frac{p}{r}(\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta) + K^*\left\{\theta^{-\alpha} - \theta^{\beta} + \theta(\alpha+\beta)\right\} - K^*(\beta\theta^{-\alpha} + \alpha\theta^{\beta})$$

However, if we write $p/r - K^* \equiv \varepsilon$, we have a link between p/r and θ:

$$a_0 = p/r - \frac{\varepsilon}{\alpha+\beta}(\beta\theta^{-\alpha} + \alpha\theta^{\beta}) = \frac{\varepsilon}{\alpha+\beta}\alpha\beta(\theta^{-\alpha} - \theta^{\beta}) \Rightarrow \frac{p}{r} = \frac{\varepsilon}{\alpha+\beta}\left(\beta(\alpha+1)\theta^{-\alpha} - \alpha(\beta-1)\theta^{\beta}\right)$$

We can likewise express K^* in terms of θ, so putting it all together, what we have to prove is that for all $\theta \in (0,1]$,

$$\boxed{\left(\beta(\alpha+1)\theta^{-\alpha} - \alpha(\beta-1)\theta^{\beta}\right)\left(\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta\right) > \left(\beta(\alpha+1)\theta^{-\alpha} - \alpha(\beta-1)\theta^{\beta} - \alpha - \beta\right).\left((\beta-1)\theta^{-\alpha} + (\alpha+1)\theta^{\beta} - \theta(\alpha+\beta)\right)}$$

Various Maple plots show this is always true; but Jon points out that if $\alpha = 0.05$, $\beta = 5$, $\theta = 0.2$, we get that LHS − RHS = − 0.02832, so the conjectured inequality does not hold universally.

5) What must in fact be happening is the following. If $\frac{\partial Y}{\partial V}(m_0, b_0 m_0) \geq 0$, then it's the story above, but if not then there is a smaller value m^* at which

$$\sup_{a_0 m \leq V \leq b_0 m} Y(m, V) = K$$

with the sup attained at some point η^*. What happens is that at this particular value m^* the DE solution starts up, and there is a discontinuity in the upper boundary.

6) Where do things begin to go wrong? Suppose we solve for Y with the conditions $\tilde{Y}(m, a_0 m) = -p a_0$, $\tilde{Y}(m, b_0 m) = K$; then assuming $m < m_0$

$$\sup_{a_0 m \le V \le b_0 m} Y(m, V) > K \quad \Rightarrow \quad \frac{\partial \tilde{Y}}{\partial V}(m, b_0 m) < 0$$

However, the converse is not true.

Proof of a conjecture of Stephen Walker (13/10/01)

Lemma Let $(X_n)_{n\geq 0}$ be a non-negative process adapted to the filtration $(\mathcal{F}_n)$ and suppose that (X_n) satisfies the conditions $\sup_n E X_n < \infty$ and

$$(*) \quad P[\Delta A_n > 0 \text{ i.o. }] = 0$$

where $\Delta A_n = E[X_n - X_{n-1} | \mathcal{F}_{n-1}]$, $A_0 = 0$. Then (X_n) is a.s. convergent.

Proof (i) Let's firstly prove the result on the additional assumption that X is bounded. Then if we write the Doob decomposition

$$X_n = M_n + A_n$$

we have that M is a martingale with bounded increments. For a martingale with bounded increments, we know that

$$\{\sup_n M_n < \infty\} = \{M_n \text{ converges a.s.}\} = \{\sup_n M_n < \infty, \inf_n M_n > -\infty\}$$

to within null sets. We also know that A_n is ultimately decreasing, by $(*)$, so if (M_n) did not converge, then the inf of $X_n \equiv M_n + A_n$ would be $-\infty$ ※. Therefore M_n is convergent a.s., A_n is convergent a.s., and X_n is convergent a.s.

(ii) To conclude, replace the original X_n by $X_n \wedge K$, where K is a (large) positive real. Now

$$E(X_n \wedge K | \mathcal{F}_{n-1}) \leq K \wedge E(X_n | \mathcal{F}_{n-1}) \leq K \wedge X_{n-1} \quad \text{if} \quad E(X_n | \mathcal{F}_{n-1}) \leq X_{n-1}$$

which will be true for all but finitely many n. Thus the truncated process $K \wedge X_n$ satisfies $(*)$, so is convergent a.s. Since $\liminf X_n \in L^1$ (Fatou) it must be that X_n is a.s. cgt.

Remarks This seems to be quite a delicate result. Convergence of the A_n is not enough, as we can see from the (deterministic) example where $x_n = \sum_1^n \xi_j / j$, with $\xi \in \{+1, -1\}$ chosen to make x_n oscillate between 1 and 2.

Solving the stochastic Ramsey problem (16/10/01)

(i) Let's consider the public/private sector version of the stochastic Ramsey problem with a consumption tax only. The dynamics are

$$\begin{cases} dk_t = (f(k_t) - \gamma k_t - c_t)\,dt - k_t\, dZ_t \\ dk_p(t) = (f(k_t) - \gamma k_p(t) - \beta_c^{-1} c_t)\,dt - k_p(t)\, dZ_t \end{cases}$$

where $Z_t = \sigma W_t$. These are the equations we get if we assume that the govt cannot charge for the returns to public capital. There is an optimal solution to the govt's optimal consumption/investment problem, given by using consumption $c_t = c^*(k_t^*)$ for the right f^n c^*. The question is whether the govt by choice of $h \equiv \beta_c^{-1}$ as a f^n of k can induce the private sector to follow that trajectory.

(ii) If we use Lagrangian pos. $e^{-\lambda_p t}\varphi, e^{-\lambda_p t}\psi$ for the two constraints, $d\varphi = \varphi(\alpha dZ + \beta dt)$, $d\psi = \psi(a dZ + b dt)$ then the Lagrangian is

$$\max E \int_0^T e^{-\lambda_p t}\left\{ u(c) + \varphi(f - \gamma k - c) + k\varphi(\beta - \lambda_p - \alpha\sigma^2) + \psi(f - \gamma k_p - ch) + k_p\psi(b - \lambda_p - a\sigma^2)\right\} dt$$
$$+ \varphi_0 k_0 + \psi_0 k_p(0)$$

with $\psi_T \geq 0$, $\varphi_T \geq 0$

Maximising over c, k, k_p, we get

$$\begin{cases} u' = \varphi + h\psi \\ \varphi(f' - \gamma + \beta - \lambda_p - \alpha\sigma^2) + \psi(f' - ch') = 0 \\ b - \omega_p - a\sigma^2 = 0 \end{cases} \qquad (\omega_p \equiv \lambda_p + \gamma)$$

Now if we had k^* was govt. optimal path, $\psi = \psi(k^*)$, we get

$$d\psi = \psi'(k^*)\left\{ -k^* dZ + (f(k^*) - \gamma k^* - c^*(k^*))\,dt \right\} + \tfrac{1}{2}\psi''(k^*)(\sigma k^*)^2 dt$$

from which

$$\begin{cases} a\psi = -k^*\psi'(k^*) \\ b\psi = \tfrac{1}{2}(\sigma k^*)^2\psi''(k^*) + \psi'(k^*)\left\{ f(k^*) - \gamma k^* - c^*(k^*)\right\} \end{cases}$$

The final equation is thus

$$\boxed{\tfrac{1}{2}(\sigma k)^2\psi''(k) + \psi'(k)\left\{ f(k) - \gamma k - c^*(k) + \sigma^2 k\right\} - \omega_p\psi(k) = 0.}$$

We can use the first equation to eliminate h from the second, and get an equation for φ alone:

$$\boxed{\tfrac{1}{2}(\sigma k)^2\varphi''(k) + \varphi'(k)\left\{\sigma^2 k + f(k) - \gamma k\right\} + \varphi(k)\left(f'(k) - \omega_p - \frac{c^*\psi'}{\psi}(k)\right) + \psi(k) f'(k) + c^*(k)\left(\frac{u'\psi'}{\psi} - u''\right)(k) = 0}$$

We expect that $U'(c) \to 0$ as $k\uparrow$, so this means we require dec pos sol^{ns} for φ, ψ

The Yeoman problem (23/10/01)

1) The standard Ramsey problem with dynamics $\dot{k} = f(k) - c^*(k)$ and objective

$$\max \int_0^\infty e^{-\rho t}\, u(c_t)\, dt$$

is difficult to solve in closed form for any meaningful problem. But suppose we approach from the other end; suppose we want dynamics (under optimal control)

$$\dot{k} = F(k)$$

and value f^n $V(k)$, solving

$$u'(c^*(k)) = V'(k) \quad , \quad u(c^*(k)) - \rho V(k) + V'(k) F(k) = 0$$

Writing J for the inverse to V', I for the inverse to u', we get $(z \equiv V'(k))$

$$(u \circ I)(z) - \rho (V \circ J)(z) + z (F \circ J)(z) = 0$$

Differentiating this gives

$$\boxed{I'(z) = \{\rho - F'(J(z))\} J'(z) - \frac{F(J(z))}{z}}$$

This must satisfy the two conditions $I' \le 0$ everywhere in $(0,\infty)$, and $I'(\cdot)$ is integrable at infinity. This is easily seen to be equivalent to

$$\int^\infty \frac{F(J(z))}{z}\, dz < \infty.$$

Since $J(z) \to 0$ $(z \to \infty)$ we clearly need $F(0) = 0$ for this to hold.

2) Example If we take $b \in (0,1)$, $a > 0$, $R > 0$, $R \neq 1$ and demand for some $\mu > 0$

$$\begin{cases} F(k) = a k^b - \mu k \\ V(k) = k^{1-R}/(1-R) \end{cases}$$

then $$\boxed{I'(z) = -z^{-1-1/R} \left\{ \frac{\rho + \mu}{R} - \mu \right\} - a\left(1 - \frac{b}{R}\right) z^{-1-b/R}}$$

Hence we require $\boxed{\rho + \mu \ge \mu R \quad \text{and} \quad b \le R,}$ and find that

$$\boxed{I(z) = z^{-1/R} (\rho + \mu - \mu R) + z^{-b/R}\, a\left(\frac{R}{b} - 1\right)}$$

[In fact, $\rho \ge \mu R$ is needed if f is to be inc-;

The dynamics can be solved explicitly:

$$\boxed{k_t = \left\{ \frac{a}{\mu}\left(1 - e^{-\mu(1-b)t}\right) + \mu k_0^{1-b} e^{-\mu(1-b)t} \right\}^{1/1-b}}$$

Clearly $c^*(k) = I(V'(k)) = k(\rho + \mu - \mu R) + k^b a\left(\frac{R}{b} - 1\right)$, $f(k) = k(\rho - \mu R) + a\frac{R}{b} k^b$.

3) Can we do anything for the 2-sector problem? (4/11/01)

Dynamics $\dot{k} = f(k_p^*, k_g^*) - c^* = F(k)$, we would like (1)

Bellman
$$\begin{cases} u(c^*, k^*) - \rho V(k) + V'(k)F(k) = 0 & (2) \\ u_c = V'(k) & (3) \\ u_g = V'(k)(f_p - f_g) & (4) \end{cases}$$

where V is the form we want for the value function

Differentiate (1) wrto k:
$$F'(k) = k_g^{*\prime}(f_g - f_p) + f_p - c^{*\prime} \qquad (5)$$

Differentiate (2) wrto k:
$$0 = u_c c^{*\prime} + u_g k_g^{*\prime} - \rho V' + V'F' + V''F$$
$$= -\rho V'(k) + V'(k) f_p(k_p^*, k_g^*) + V''(k)F(k) \qquad \text{[using (3),(4)]} \qquad (6)$$

From this, with the required form of F and V, we can deduce f_p. This suggests the following

Strategy for finding f, u:

(a) Impose the functions F, V, f_g, and the path $k_p^*(k)$, $k_g^*(k)$;

(b) Deduce u, u_c, u_g, f_p from (2),(3),(4),(6);

(c) Deduce $(c^*)'$ from (5), and hence c^*;

(d) Deduce f from (1);

(e) Check tangent inequality for f and u.

When steps (a)-(d) are completed, we know f, f_p, f_g along the path (k_p^*, k_g^*) and u, u_c, u_g along the path (c^*, k_g^*). The question is whether there is an extension of f and u off the path where they're known which is monotone and concave. Certainly necessary for this would be that $f_p \geq 0$, $f_g \geq 0$ and for all z

$$f(k_p^*(z), k_g^*(z)) \leq f(k_p^*(k), k_g^*(k)) + (k_p^*(z) - k_p^*(k)) f_p(k_p^*(k), k_g^*(k))$$
$$+ (k_g^*(z) - k_g^*(k)) f_g(k_p^*(k), k_g^*(k)) \qquad (7)$$

which is just the tangent inequality referred to in (e). However, this is also <u>sufficient</u>; if for all k the tangent hypeplane to f at $(k_p^*(k)\ k_g^*(k))$ bounds f above along the rest of the path, the function $\bar{f}$ which is just the infimum of all these tangent hypeplanes will be concave, increasing, and will agree with f along the optimal path !!

4) <u>Example.</u> Let's require

$$\boxed{\begin{array}{l} V(k) = k^{1-R}/(1-R), \quad F(k) = ak^b - \mu k, \quad k_p^*(k) = \theta k \\ f_g(k_p^*(k), k_g^*(k)) = \beta + \alpha R k^{b-1} \end{array}}$$

where $R, b, \theta \in (0,1)$, $a, \mu, \alpha, \beta > 0$. For this example, we know we can obtain the path k_t^* in closed form, so it seems a good place to begin. Other conditions will emerge. We have

$$f_p = \rho - \frac{V''}{V'} F = \rho - \mu R + aRk^{b-1}$$

so require $\boxed{\rho > \mu R}$ The DE for c^* is

$$c^{*\prime} = f_p + (k_g^*)'(f_g - f_p) - F'(k)$$

$$= \rho - \mu R + aRk^{b-1} + (1-\theta)\{\beta - \rho + \mu R + (\alpha - a)Rk^{b-1}\} - abk^{b-1} + \mu$$

$$= A + bBk^{b-1}$$

where $A \equiv \mu + (1-\theta)\beta + \theta(\rho - \mu R)$, $B \equiv -a + R(a + (1-\theta)(\alpha - a))/b$. This gives us

$$c^*(k) = Ak + Bk^b$$

so we shall have to have the conditions

$$A \equiv \mu + (1-\theta)\beta + \theta(\rho - \mu R) > 0$$

$$\boxed{B \equiv -a + R(a + (1-\theta)(\alpha - a))/b > 0}$$

The first is implied by earlier conditions on the parameters, the second is an additional requirement. We now deduce the form of f along the optimal path:

$$f = f(k_p^*(k), k_g^*(k)) = F(k) + c^*(k)$$

$$= \{(1-\theta)\beta + \theta(\rho - \mu R)\}k + \frac{R}{b}\{a + (1-\theta)(\alpha - a)\}k^b.$$

To check the tangent inequality for f, we need for any $z, k \geq 0$ that

$$f(k_p^*(z), k_g^*(z)) - f(k_p^*(k), k_g^*(k)) = (z-k)\{\theta(\rho - \mu R) + (1-\theta)\beta\}$$

$$+ \frac{R}{b}\{\theta a + (1-\theta)\alpha\}(z^b - k^b)$$

$$\leq \theta(z-k)f_p + (1-\theta)(z-k)f_g$$

The linear terms on each side cancel, leaving us to prove that

$$\frac{R}{b}\{\theta a + (1-\theta)\alpha\}(z^b - k^b) \leq \theta(z-k)aRk^{b-1} + (1-\theta)(z-k)\alpha R k^{b-1}$$

which is immediate from concavity of x^b.

$$u = \rho V - FV' = \frac{\rho + \mu(1-R)}{1-R} k^{1-R} - a k^{b-R}$$

For $u_g > 0$, we need the conditions $a > \alpha$, $\rho \geq \mu R + f$

Slope of LHS at $k=0$ is $\rho + \mu - \mu R$; slope of RHS at $k=0$ is $A + (1-\theta)(\rho - \mu R - f) = \mu + (1-\theta)(\rho - \mu R) + \theta(\rho - \mu R) = \mu + \rho - \mu R$

Lastly, must check tangent inequality for u; so for any $k, z > 0$, we need to have

$$(z^{1-R} - k^{1-R})\ \frac{\rho + \mu - \mu R}{1-R} - a(z^{b-R} - k^{b-R})$$

$$\leq (c^*(z) - c^*(k))\, k^{-R} + (1-\theta)(z-k)\, k^{-R}\, (f_0 - f_g)$$

$$= \{A(z-k) + B(z^b - k^b)\}\, k^{-R} + (1-\theta)(z-k)\, k^{-R}\left(\rho - \mu R - \beta + (a-\alpha) R\, k^{b-1}\right)$$

If we set $z \equiv k(1+t)$, this inequality becomes on dividing by k^{1-R}

$$\left((1+t)^{1-R} - 1\right) \frac{\rho + \mu - \mu R}{1-R} - a\left\{(1+t)^{b-R} - 1\right\} k^{b-1}$$

$$\leq At + B\left((1+t)^b - 1\right) k^{b-1} + (1-\theta)\, t\, \left(\rho - \mu R - \beta + (a-\alpha) R\, k^{b-1}\right)$$

which is equivalent to the two inequalities

$$\boxed{\begin{cases} \left((1+t)^{1-R} - 1\right) \dfrac{\rho + \mu - \mu R}{1-R} \leq At + (1-\theta)t\,(\rho - \mu R - \beta) \\[2ex] -a\left\{(1+t)^{b-R} - 1\right\} \leq B\left((1+t)^b - 1\right) + (1-\theta)t\,(a-\alpha) R \end{cases}}$$

The first is easy ($x^{1-R}/(1-R)$ is concave, and this is just the tangent inequality at $t=1$). The second is more interesting and requires the condition

$$\boxed{a > \alpha} \qquad \text{(consider } t \to \infty)$$

If we write $\varphi(t) \equiv B\{(1+t)^b - 1\} + (1-\theta)t\,(a-\alpha) R + a\{(1+t)^{b-R} - 1\}$, then easily $\varphi(0) = 0 = \varphi'(0)$ and

$$\varphi''(t) = (1+t)^{b-2}\left[-b(1-b)B + a(R-b)(R+1-b)(1+t)^{-R}\right]$$

We require $\varphi''(0) > 0$, so if we have $\boxed{R > b}$ then that's guaranteed by $a > \alpha$. Easily, φ'' is then positive in $(-1, 0]$, so $\varphi \geq 0$ in $(-1, 0]$. What we see in $(0, \infty)$ is that φ'' is initially > 0, then goes < 0, so the slope φ' increases for a while, then decreases. We will have $\varphi' > 0$ to the right of 0 provided the limit of φ' is > 0; but it is equal to $(1-\theta)(a-\alpha)R > 0$, so we do have $\varphi > 0$ if all of these conditions on the parameters are valid.

5) <u>Stochastic versions?</u> If we consider the dynamics which arise from the stochastic version of the Ramsey problem,

$$dk = (f(k) - c)\,dt - \sigma k\, dW \equiv F(k)dt - \sigma k\, dW \qquad \text{under optimal control}$$

and the related Bellman equation

$$\begin{cases} u(c) - \rho V(k) + \tfrac{1}{2}\sigma^2 k^2\, V''(k) + F(k)\, V'(k) = 0 & \text{on optimal path} \\ u'(c) = V'(k) & \text{at optimality} \end{cases}$$

by differentiating F and the Bellman equation we learn

$$\tfrac{1}{2}\sigma^2 k^2 V''' + \sigma^2 k V'' + F V'' - \rho V' + V' f' = 0$$

whence we discover f', hence f, and then c^*; since $c^* = I(V'(k))$, we now know I.

If we try $F(k) = a k^b - \mu k$, $V(k) = k^{1-R}/(1-R)$ we get very similar answers to the non stochastic problem:

$$\boxed{\begin{aligned} f(k) &= \frac{aR}{b} k^b + \left(\rho - \mu R + \tfrac{1}{2}\sigma^2 R(1-R)\right) k \\ c^*(k) &= a\left(\frac{R}{b} - 1\right) k^b + \left(\rho - \mu R + \mu + \tfrac{1}{2}\sigma^2 R(1-R)\right) k \end{aligned}}$$

This agrees for $\sigma = 0$.

6) The origins of the problem involve scaling out population size, and for this to work we need CRRA felicities. With the formulation above, we could achieve this in one of two ways:

(a) $R = b$, so that

$$f(k) = a k^R + \left(\rho - \mu R + \tfrac{1}{2}\sigma^2 R(1-R)\right) k$$

$$c^*(k) = \left(\rho - \mu R + \tfrac{1}{2}\sigma^2 R(1-R)\right) k + \mu k$$

(b) $\rho - \mu R + \mu + \tfrac{1}{2}\sigma^2 R(1-R) = 0$, so

$$\begin{cases} f(k) = \dfrac{aR}{b} k^b - \mu k \\ c^*(k) = a\left(\dfrac{R}{b} - 1\right) k^b \end{cases}$$

This is OK provided we've got some depreciation rate which is $\geq \mu$.

$$F(\lambda K_p, \lambda K_g, \lambda L) = \lambda F(K_p, K_g, L) \quad \therefore \; F = K_p F_p + K_g F_g + L F_L$$

$$\therefore \; f = k_p f_p + k_g f_g + F_L$$

$$\gamma = \delta + \mu_L + \mu_T - v_{LL} - v_{LT} - v_{TT}$$

NB no tax relief on capital investment - that adds further difficulty to the analysis.

The private sector as continuum of infinitesimal agents (5/11/01)

1) It seems that the way Arrow & Kurz and others handle the private sector is to consider it as a continuum of infinitesimal agents, whose actions therefore do not impact the economy individually. This makes the analysis a bit simpler. We use the notation of WN XIX pp 1–6, or of Peter's draft.

Consider a household of weight ε; its effective labour at time t is $\varepsilon\eta_t \equiv \varepsilon A_t T_t$. If it enters the production process at time t with capital εX, the output of the entire economy at that instant is increased by

$$\varepsilon X F_p + \varepsilon \eta_t F_L \quad + o(\varepsilon)$$

so the dynamics of the wealth of that household will satisfy

$$\varepsilon \dot{X} = \varepsilon X F_p + \varepsilon \eta F_L - \delta \varepsilon X - \varepsilon C_t .$$

If we set $x \equiv X/\eta$, then we obtain the following dynamics for x

$$dx = \{ x f_p + (f - k_p^* f_p - k_g^* f_g) - \gamma x - c \} dt - x\, d(Z^L + Z^T).$$

What happens if govt doesn't appropriate returns from its capital? If the government charges a proportion θ_g of the return to its capital, then the remainder $(1-\theta_g) K_g F_g$ gets distributed pro rata as wages, so the dynamics of x would change to

$$dx = \{ x f_p + (f - k_p^* f_p - \theta_g k_g^* f_g) - \gamma x - c \} dt - x\, d(Z^L + Z^T)$$

at least if there are no taxes.

2) What happens if there are taxes? The total income stream to the small household (after tax) is therefore

$$\varepsilon \{ \beta_k K_p f_p + \beta_w (f - k_p^* f_p - \theta_g k_g^* f_g)\eta + \beta_r r D \}$$

where K_p is the amount of private capital held, D the amount of govt debt. These change according to

$$\begin{cases} \dot{D} = \beta_k K_p f_p + \beta_w (f - k_p^* f_p - \theta_g k_g^* f_g)\eta + r\beta_r D - \beta_c^{-1} C - I_p \\ \dot{K}_p = I_p - \delta K_p \end{cases}$$

So if $X \equiv D + K_p$ is the total wealth of this agent, we shall have

$$\boxed{\begin{aligned} \dot{X} &= r\beta_r X + \beta_w (f - k_p^* f_p - \theta_g k_g^* f_g)\eta - \beta_c^{-1} C \\ r\beta_r &= \beta_k f_p - \delta \end{aligned}}$$

the latter being required for coexistence of private capital + govt debt. Letting $x \equiv X/\eta$, we get

$$\boxed{dx = \{ \beta_k f_p x + \beta_w (f - k_p^* f_p - \theta_g k_g^* f_g) - \beta_c^{-1} c - \gamma x \} dt - x\, d(Z^L + Z^T).}$$

What is the objective? The private agent wants to maximise

$$dx = -x(dZ^L + dZ^T) + \{\varepsilon - \beta_c^T c + r x\} dt$$

$$\begin{cases} \varepsilon \equiv \beta_w (f - k_p^* f_p - \theta_g k_g^* f_g) \\ r = \beta_R f_p - \gamma \end{cases}$$

$$\gamma = \delta + \mu_L + \mu_T - v_{LL} - v_{LT} - v_{TT}$$

$$\tilde{\gamma} = \gamma + v_{LL} + (2-R) v_{LT} + (1-R) v_{TT}$$

$$\hat{\gamma} = \gamma + v_{LL} + (2-S) v_{LT} + (1-S) v_{TT}$$

$$E\left[\int_0^\infty e^{-\rho_p t} L_t\, u\left(\frac{c_t}{L_t}, \frac{k_g^*(t)}{L_t}\right) dt\right] = E\left[\int_0^\infty e^{-\rho_p t} L_t T_t^{1-S} u(c_t, k_g^*(t))\, dt\right]$$

$$= \hat{E}\left[\int_0^\infty e^{-\hat{\lambda}_p t} u(c_t, k_g^*(t))\, dt\right]$$

assuming u is homogeneous of degree $(1-S)$, where $d\hat{P}/dP|_{\mathcal{F}_t} = \exp(M_t - \frac{1}{2}\langle M\rangle_t)$, $M_t = Z_t^L + (1-S) Z_t^T$, and the dynamics of x now taking the form

$$dx = x\, d\xi + (\varepsilon - \beta_c^{-1} c)\, dt$$

where

$$\begin{cases} d\xi = -d(\hat{Z}^L + \hat{Z}^T) + \{\beta_k f_p - \tilde{\gamma} - v_{LL} - (2-S) v_{LT} - (1-S) v_{TT}\}\, dt \equiv -d(\hat{Z}^L + \hat{Z}^T) - \hat{\gamma}\, dt + \beta_k f_p\, dt \\ \varepsilon = \beta_w (f - k_p^* f_p - \theta_g k_g^* f_g) \\ dZ^L = d\hat{Z}^L + (v_{LL} + (1-S) v_{LT})\, dt, \quad dZ^T = d\hat{Z}^T + (v_{LT} + (1-S) v_{TT})\, dt \end{cases}$$

Further, the dynamics of the optimally-controlled k^* can be expressed as

$$dk^* = \{f(k_p^*, k_g^*) - \hat{\gamma} k^* - c^*\}\, dt - k^* d(\hat{Z}^L + \hat{Z}^T), \qquad \hat{\gamma} \equiv \tilde{\gamma} + (R-S)(v_{LT} + v_{TT})$$

So the dynamics faced by the private sector contains lots of terms involving k^*; we want to find the value function $\hat{V} = \hat{V}(x, x/k^*) \equiv \hat{V}(x, z)$

But there's a snag: if we imagine that the optimal x will turn out to be $k_p^* = k_p^*(k^*)$, then $dx = k_p^{*\prime}\, dk^* + \dots = -k^* k_p^{*\prime}\, d(\hat{Z}^L + \hat{Z}^T) + \dots$, so matching the martingale parts we see that

$$x = k_p^* = k^* k_p^{*\prime}$$, that is, we can only have this when $k_p^* = \text{const.}\, k^*$!!

As we know, this is not impossible, but it is restrictive!

A little Itô calculus gives us

$$dz = d\left(\frac{x}{k^*}\right) = z\left\{\beta_k f_p - \frac{f - c^*}{k^*}\right\} dt + \frac{\varepsilon - \beta_c^{-1} c}{k^*}\, dt.$$

The CCR question again: a special case (7/11/01)

In the case where $p = 1$, there are no losses on default so we find the accounting identity

$$mB + (n-m)S = V$$

Hence at ξ we must have $B = \xi/m$, $m\frac{\partial B}{\partial V} = 1$, so (see WN XIX, p14)

$$B = \frac{p}{r}\left\{1 - \frac{\beta}{\alpha+\beta}\left(\frac{V}{\xi}\right)^{-\alpha} - \frac{\alpha}{\alpha+\beta}\left(\frac{V}{\xi}\right)^{\beta}\right\} + \frac{\xi(\beta-1)}{m(\alpha+\beta)}\left(\frac{V}{\xi}\right)^{-\alpha} + \frac{\xi(\alpha+1)}{m(\alpha+\beta)}\left(\frac{V}{\xi}\right)^{\beta}$$

On the other hand, we have the expressions we know for S and Y.

$$S = \frac{V - mp/r}{n-m} + \frac{\beta mp/r - (\beta-1)\xi}{(\alpha+\beta)(n-m)}\left(\frac{V}{\xi}\right)^{-\alpha} + \frac{\alpha mp/r - (\alpha+1)\xi}{(\alpha+\beta)(n-m)}\left(\frac{V}{\xi}\right)^{\beta}$$

$$Y = \frac{V - np/r}{n-m} + \frac{\beta np/r + \beta K(n-m) - (\beta-1)\eta}{(\alpha+\beta)(n-m)}\left(\frac{V}{\eta}\right)^{-\alpha} + \frac{\alpha np/r + \alpha K(n-m) - (\alpha+1)\eta}{(\alpha+\beta)(n-m)}\left(\frac{V}{\eta}\right)^{\beta}$$

$$= S - B$$

By taking the two expressions for Y and matching the coefficients of $V^{-\alpha}$, V^{β} we find a relationship $(\theta \equiv \xi/\eta)$ linking ξ and η; indeed, there are two expressions for η,

$$\begin{cases} \eta = \dfrac{m\beta}{\beta-1} \; \dfrac{np/r\,(\theta^{\alpha}-1) - K(n-m)}{n\theta^{\alpha+1} - m} \\[2ex] \eta = \dfrac{m\alpha}{\alpha+1} \; \dfrac{np/r\,(\theta^{-\beta}-1) - K(n-m)}{n\theta^{-\beta+1} - m} \end{cases}$$

The equation this implies for θ is some linear combination of different irrational powers of θ equal to zero; clearly not soluble in closed form. This shows that it is futile to look for a closed-form solution in general, since there is none in this special case.

NB If y^* were the optimising martingale, and τ^* the optimal stopping time, then we must have $y^*_T = y^*_{\tau^*}$ by convexity of $\tilde{U}$, & we may in fact rewrite this as

$$\min \left\{ x_0 y_0 + A \sup_{\tau} E\left(y_\tau \varphi_\tau + \frac{\tilde{U}(y_\tau)}{A} \right) \right\}$$

A question of José Scheinkman & Thaleia Zariphopoulou (7/11/01)

1) Here's a nice (and difficult!) question which might arise naturally in the context of an executive stock option, where the executive has A (infinitely divisible) options, and initial wealth x_0. If he has exercised m_t by time t, his wealth at time t is

$$x_t = x_0 + \int_0^t \varphi_s \, dm_s$$

where $\varphi_s \geq 0$ is the value for exercising at time s. The goal is to max $E\, U(x_T)$ over all exercise rules.

2) Dual formulation Absorb the dynamics with $dy = y(\alpha dW - \beta dt)$ and do the old Lagrangian trick:

$$\max \; E\Big[U(x_T) + \int_0^T y_t \varphi_t \, dm_t - x_T y_T + x_0 y_0 - \int_0^T \beta_t x_t y_t \, dt + \eta\big(A - \int_0^T dm_s\big) \Big]$$

$$= \max \; E\Big[\tilde{U}(y_T) + x_0 y_0 + A\eta_T + \int_0^T (y_t \varphi_t - \eta_t)\, dm_t \Big] \qquad [\beta_t \geq 0]$$

where (η_t) is the martingale closed on the right by η

$$= E\Big[\tilde{U}(y_T) + x_0 y_0 + A\eta_T \Big]$$

where we have the dual feasibility conditions $\beta \geq 0$, $y_t \varphi_t \leq \eta_t$. We now have the dual problem of minimising this over choice of the dual variables. Notice

$$E\eta_T = E\eta_\tau \geq E(y_\tau \varphi_\tau)$$

for all stopping times τ, so we optimise over η by taking $E\eta_T = \sup_\tau E(y_\tau \varphi_\tau)$, the value of an American pricing problem.

Notice that if we fix y_T, if y is any supermartingale with that given terminal value then you minimise by making y a <u>martingale</u>; so the dual problem is to minimise over all positive martingales y the expression

$$E\, \tilde{U}(y_T) + x_0 y_0 + A \sup_\tau E[y_\tau \varphi_\tau]$$

3) More quickly... If y is a positive martingale, $\tau_z \equiv \inf\{t : m_t > z\}$ then we have

$$E\, U(x_T) \leq E\big[\tilde{u}(y_T) + x_T y_T \big]$$

$$= E\Big[\tilde{u}(y_T) + x_0 y_0 + \big(\int_0^T \varphi_s \, dm_s\big) y_T \Big]$$

$$= E\Big[\tilde{u}(y_T) + x_0 y_0 + \int_0^A \varphi(\tau_z)\, dz \; y_T \Big]$$

$$= E\Big[\tilde{u}(y_T) + x_0 y_0 + \int_0^A \varphi(\tau_z)\, y(\tau_z)\, dz \Big]$$

$$\leq \quad E\tilde{U}(y_T) + x_0 y_0 + A \sup_\tau E(y_\tau \varphi_\tau).$$

4) <u>An example.</u> Suppose we take

$$U(x) = -\gamma^{-1} e^{-\gamma x} \quad , \quad \text{so} \quad \tilde{U}(y) = \frac{y}{\gamma}(\log y - 1)$$

and $\varphi_t = (W_t + \mu t)^+$, where μ is a constant, $\mu \geq \gamma/2$. Let's write the dual martingale y as

$$y_t = y_0 Z_t^b \equiv y_0 \exp\left(\int_0^t b_s \, dW_s - \tfrac{1}{2}\int_0^t b_s^2 \, ds \right) \equiv y_0 \left.\frac{dP^b}{dP}\right|_{\mathcal{F}_t}$$

so that the expression to be minimised will be

$$x_0 y_0 + A \sup_\tau E\left[y_0 Z_\tau^b \varphi_\tau + A^{-1}\tilde{U}(y_0 Z_\tau^b) \right]$$

$$= x_0 y_0 + A y_0 \sup_\tau E\left[Z_\tau^b \varphi_\tau + A^{-1}\frac{Z_\tau^b}{\gamma}\left\{ \log Z_\tau^b + \log y_0 - 1 \right\} \right]$$

$$= x_0 y_0 + \tilde{U}(y_0) + A y_0 \sup_\tau E^b\left[\varphi_\tau + \frac{1}{A\gamma} \log Z_\tau^b \right]$$

$$= x_0 y_0 + \tilde{U}(y_0) + A y_0 \sup_\tau E^b\left[\int_0^\tau I_{\{\varphi_t > 0\}}(b_t + \mu)\, dt + \tfrac{1}{2} \ell_\tau + \frac{1}{2\gamma A}\int_0^\tau b_s^2 \, ds \right]$$

Now because of the assumption $\mu \geq \gamma/2$, the integrand is always non-negative, so the best is to choose $\tau = T$;

$$= x_0 y_0 + \tilde{U}(y_0) + A y_0 \, E^b\left[\int_0^T I_{\{\varphi_t > 0\}}(b_t + \mu)\, dt + \tfrac{1}{2} \ell_T + \frac{1}{2\gamma A}\int_0^T b_s^2 \, ds \right]$$

The problem now is to minimise this over choice of y...? One thing at least can be said; if

$$\boxed{\xi = \inf_b \sup_\tau E^b\left[\varphi_\tau + \frac{1}{2\gamma A}\int_0^\tau b_s^2 \, ds \right]}$$

then the dual value is

$$\inf_{y_0}\left[x_0 y_0 + \tilde{U}(y_0) + A\xi y_0 \right] = -\gamma^{-1} \exp -\gamma(x_0 + A\xi) = U(x_0 + A\xi)$$

Conditions assumed on the parameters:

$a, \mu > 0$, $R, b \in (0,1)$, $R \geq b$, $\theta \in (0,1)$

$$\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R \geq 0 \qquad (f_p \geq 0)$$

$$\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R \geq \beta \qquad (u_g \geq 0)$$

$$a \geq \alpha \qquad (u_g \geq 0)$$

$$\theta a + (1-\theta)\alpha \geq ab/R \qquad (c^* \text{ incr})$$

Two-sector stochastic example (8/11/01)

1) We need this for the taxation questions earlier. We shall look for the following:

$$\boxed{\begin{array}{l} dk = -\sigma k\, d\hat{W} + F(k)\, dt \quad \text{at optimality}, \quad F(k) = a k^b - \mu k \\ V(k) = k^{1-R}/(1-R) \quad , \quad k_p^* = \theta k^* \\ f_g(k) = \beta + \alpha R k^{b-1} \end{array}}$$

We have
$$\begin{cases} F(k) = f(k_p^*(k), k_g^*(k)) - c^*(k) \quad \text{together with the HJB eq}^n \\ U(c^*, k^*) - \rho V + \tfrac{1}{2}\sigma^2 k^2 V'' + F V' = 0 \\ U_c = V' \\ U_g = V' \cdot (f_p - f_g) \end{cases}$$

As before, by differentiating the first two we shall find that

$$f_p = (\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R) + a R k^{b-1}$$

and then

$$c^*(k) = A k + B k^b \quad , \quad \begin{cases} A = \mu + \theta(\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R) + (1-\theta)\beta \\ B = -a + (\theta a + (1-\theta)\alpha) R/b \end{cases}$$

$$f(k_p^*(k), k_g^*(k)) = F(k) + c^*(k)$$
$$= (\theta a + (1-\theta)\alpha)\frac{R}{b} k^b + \{(1-\theta)\beta + \theta(\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R)\} k$$

and
$$U(c^*(k), k_g^*(k)) = \left(\frac{\rho}{1-R} + \tfrac{1}{2}\sigma^2 R + \mu\right) k^{1-R} - a k^{b-R}$$

$$U_g = k^{-R}\left(\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R - \beta + (a-\alpha) R k^{b-1}\right)$$

The tangent inequalities check out fine for this, just by using the earlier result + considering what has changed by using $\sigma \neq 0$.

2) The dynamics of the wealth of the private agent are given by (see p 26)

$$dx = -x\sigma\, d\hat{W} + x(\beta_k f_p - \hat{\gamma})\, dt + (\varepsilon - \beta_t^{\top} c)\, dt$$

So if we are to have $x = k_p^*$ it has to be that

$$\boxed{\theta F(k^*) = \theta k^*(\beta_k f_p - \hat{\gamma}) + \varepsilon - \beta_t^{\top} c}$$

How do we see whether this is optimal for the private agent? If we attempt to absorb this dynamic using the multiplier process $e^{-\lambda t} y_t$, where

$$dy_t / y_t = H_t\, d\hat{W}_t + \Psi_t\, dt,$$

Two-sector stochastic example (8/11/01)

1) We need this for the taxation questions earlier. We shall look for the following:

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$$\begin{aligned} f(k_p^*(k), k_g^*(k)) &= F(k) + c^*(k) \\ &= (\theta a + (1-\theta)\alpha)\frac{R}{b} k^b + \{(1-\theta)\beta + \theta(\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R)\} k \end{aligned}$$

and
$$U(c^*(k), k_g^*(k)) = \left(\frac{\rho}{1-R} + \tfrac{1}{2}\sigma^2 R + \mu\right) k^{1-R} - a k^{b-R}$$

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$$dx = -x\sigma\, d\hat{W} + x(\beta_k f_p - \hat{\gamma})\, dt + (\epsilon - \beta_t^{\top} c)\, dt$$

So if we are to have $x = k_p^*$ it has to be that

$$\boxed{\theta F(k^*) = \theta k^*(\beta_t f_p - \hat{\gamma}) + \epsilon - \beta_t^{\top} c}$$

How do we see whether this is optimal for the private agent? If we attempt to absorb this dynamic using the multiplier process $e^{-\lambda_p t} y_t$, where

$$dy_t / y_t = H_t\, d\hat{W}_t + \Psi_t\, dt,$$

NB: different u from previous page!

$$f = \theta k f_p + (1-\theta) k f_q = k_p^* f_p + k_q^* f_q$$

we get the Lagrangian form

$$\max \hat{E} \int_0^\infty e^{-\lambda_p t} \{ v(c, k_g^*) + xy(\beta_k f_p - \hat{\gamma}) + y(\varepsilon - \beta_c^{-1} c) - \lambda_p xy + \Psi xy - \sigma H xy \} dt$$

where v is the felicity of the private agent, and from this we derive the conditions

$$\boxed{\begin{aligned} v_c &= \beta_c^{-1} y \\ \beta_k f_p - \hat{\gamma} - \lambda_p &+ \Psi - \sigma H \leq 0 \end{aligned}}$$

3) A special case. The CRRA situation where we can scale in U would be good to start with, so let's suppose that $\boxed{\frac{\rho}{1-R} + \frac{1}{2}\sigma^2 R + \mu = 0 = \mu + \beta}$ so that we get simplifications:

$$\begin{cases} f_p = \beta + aRk^{b-1}, \quad f_g = \beta + \alpha R k^{b-1}, \quad f = \beta k + (\theta a + (1-\theta)\alpha) \frac{R}{b} k^b \\ u_c = k^{-R}, \quad u_g = (a-\alpha) R k^{b-R-1}, \quad u = -a k^{b-R}, \quad c^* = Bk^b \end{cases}$$

The tangent inequalities for u, f are both OK. Note that since $\mu < 0$, the net drift F is always > 0! However, provided $\boxed{\beta \leq \frac{1}{2}\sigma^2}$ the diffusion will be recurrent!

Suppose we use Cobb-Douglas v:

$$v = (c^\pi k_g^{1-\pi})^{1-S}$$

and try for a constant β_c. Then $y = \beta_c \pi(1-S) B^{\pi(1-S)-1} (1-\theta)^{(1-\pi)(1-S)} k^{b\pi(1-S) - b + (1-\pi)(1-S)}$

$\equiv \Gamma k^\nu$, and hence

$$dy/y = -\nu\sigma d\hat{W} + \{ \nu\beta + \tfrac{1}{2}\nu(\nu-1)\sigma^2 + a\nu k^{b-1} \} dt$$

If we take the dual complementary slackness condition with equality, this gives us a form of β_k

$$\boxed{\beta_k = \frac{\hat{\gamma} + \lambda_p + (-\nu)\{\frac{1}{2}\sigma^2(\nu+1) + \beta + ak^{b-1}\}}{\beta + aRk^{b-1}}}$$

For this to make sense, we want $\nu \leq 0$, that is, $b(1-\pi(1-S)) < (1-\pi)(1-S)$, which can be achieved. It's not hard to show that $\nu \geq -1$ always, so we get some pretty sensible looking behaviour here.

Lastly, we can compute the tax rate on wages:

$$\boxed{\beta_W = \frac{\theta k\{(\nu+1)(\frac{1}{2}\sigma^2\nu + \beta) - \lambda_p\} + k^b\{\beta_c^{-1} B + \theta a(\nu+1)\}}{\beta R(1-\theta)(1-\theta_g) + Rk^b\{\theta a(1-b) + (1-\theta)\alpha(1-\theta_g b)\}/b}}$$

All looks very reasonable; this solution has no debt.

Writing $\theta = e^{-x}$, we have alternatively

$$(n-m) \quad \xi' \cdot \int_0^x \left(\beta e^{\beta t} + \alpha e^{-\alpha t}\right) dt \cdot \left(\frac{m\rho}{\xi} - \delta\right)$$

$$= -\eta \, \delta \int_0^x \left(e^{\beta s} - e^{-\alpha s}\right) e^{-s} \, ds + n\rho \int_0^x \left(e^{\beta s} - e^{-\alpha s}\right) ds$$

$$= \int_0^x \left(e^{\beta s} - e^{-\alpha s}\right)\left(n\rho - \delta e^{-s} \eta\right) ds$$

Also
$$\eta = \frac{n\rho \int_0^x \left(e^{\alpha s} - e^{-\beta s}\right) ds}{e^{-x}\left\{ \int_0^x \delta \left(e^{\alpha t} - e^{-\beta t}\right) e^{t} \, dt - \frac{\sigma^2}{2} (\alpha+\beta) \rho (n-m)/m \right\}}$$

Simplifying CCCR: take $K=0$ (11/11/01)

1) Talking to Peter Carr, it seems that really only the case $K=0$ matters in practice, so let's now work with that assumption and see whether things simplify very much.

Firstly, for define

$$\boxed{\begin{aligned}\Phi(\zeta,\theta,x) \equiv &\left\{(\alpha+\beta)\theta - (\beta-1)\theta^{-\alpha} - (\alpha+1)\theta^{\beta}\right\}x \\ &+ \left\{\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta\right\}\zeta\rho/r\end{aligned}}$$

then we have expressions for S, and Y, in terms of Φ:

$$\boxed{\begin{aligned}(\alpha+\beta)(n-m)S(m,V) &= \Phi(m, V/\xi(m), \xi(m)) \\ (\alpha+\beta)(n-m)Y(m,V) &= \Phi(n, V/\eta(m), \eta(m))\end{aligned}}$$

We have

$$\Phi_\theta(\zeta,\theta,x) = (\theta^{-\alpha-1}-1)\alpha(\beta-1)\left(x - \frac{\zeta\rho\beta}{r(\beta-1)}\right) + (1-\theta^{\beta-1})\beta(\alpha+1)\left(x - \frac{\alpha\zeta\rho}{r(\alpha+1)}\right)$$

$$\Phi_{\theta\theta}(\zeta,\theta,x) = \theta^{-2}(\alpha+1)(\beta-1)\left[\alpha\theta^{-\alpha}\left(\frac{\zeta\rho\beta}{r(\beta-1)} - x\right) + \beta\theta^{\beta}\left(\frac{\alpha\zeta\rho}{r(\alpha+1)} - x\right)\right]$$

The qualitative behaviour of Φ as a f^n of θ therefore depends on the signs of $\zeta\rho\beta - rx(\beta-1)$, $\zeta\rho\alpha - rx(\alpha+1)$:

Case 1: $x \leq \alpha\zeta\rho/r(\alpha+1)$. Here Φ is convex

Case 2: $\alpha\zeta\rho/r(\alpha+1) < x < \beta\zeta\rho/r(\beta-1)$. This time, $\Phi_{\theta\theta}$ is >0 for small enough θ, but changes sign at some critical value then remains <0; Φ is convex then concave

Case 3: $x \geq \beta\zeta\rho/r(\beta-1)$. This time Φ is concave

The ODE for $\xi(m) \equiv \theta(m)\eta(m)$ now simplifies a little

$$\boxed{\begin{aligned}&(n-m)\frac{2}{\sigma^2}\xi'(\theta^{\alpha}-\theta^{-\beta})(\delta - m\rho/\xi) \\ &= (1-\theta^{\alpha})\{(\beta-1)\theta\eta - n\beta\rho/r\} + (1-\theta^{-\beta})\{(\alpha+1)\theta\eta - n\alpha\rho/r\} + \eta(\alpha+\beta)(1-\theta), \\ &\eta = \frac{n\rho/r\,(\alpha+\beta-\beta\theta^{-\alpha} - \alpha\theta^{\beta})}{(\alpha+\beta)\theta - (\beta-1)\theta^{-\alpha} - (\alpha+1)\theta^{\beta} + (\alpha+\beta)\rho(n-m)\theta/m}\end{aligned}}$$

Writing $\theta = e^{-x}$, we get in these terms

$$\boxed{(n-m)\xi' \int_0^x (\beta e^{\beta t} + \alpha e^{-\alpha t})dt\left(\frac{m\rho}{\xi} - \delta\right) = \int_0^x (e^{+\beta s} - e^{-\alpha s})(n\rho - \delta e^{-s}\eta)\,ds}$$

$$V = \frac{np}{\alpha\beta}\left(\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta\right) = \frac{1}{2}\sigma^2 \frac{np}{r}\left(\beta\theta^{-\alpha} + \alpha\theta^{\beta} - \alpha - \beta\right),$$

$$\Delta = \theta\left\{ \frac{\delta}{(\alpha+1)(\beta-1)}\left((\beta-1)\theta^{-\alpha-1} + (\alpha+1)\theta^{\beta-1} - \alpha - \beta\right) - \frac{1}{2}\sigma^2 (\alpha+\beta)p(n-m)/m \right\}$$

$$= \frac{1}{2}\sigma^2\theta\left\{ (\beta-1)\theta^{-\alpha-1} + (\alpha+1)\theta^{\beta-1} - \alpha - \beta - (\alpha+\beta)p(n-m)/m \right\}$$

$$\frac{\partial\Delta}{\partial m} = \frac{1}{2}\sigma^2\theta \frac{(\alpha+\beta)p}{m^2}$$

with

$$\eta \equiv \frac{V}{\Delta} = \frac{n\rho \int_0^x (e^{\alpha s} - e^{-\beta s})\,ds}{e^{-x}\left\{\int_0^x \delta(e^{\alpha t} - e^{-\beta t})e^{t}dt - \frac{1}{2}\sigma^2(\alpha+\beta)p(n-m)/m\right\}}$$

Jon remarks that $m\rho - \delta\xi > 0 \Leftrightarrow \frac{\partial^2 \xi}{\partial V^2}(m, \xi(m)) > 0$
$\eta > n\rho/\delta \Leftrightarrow \frac{\partial^2 Y}{\partial V^2}(m, \eta(m)) < 0$
both of which should hold for our solution.

Since $Y(m,\xi) + p\xi/m \equiv 0$, it follows that at $(m, \xi(m))$

$$\boxed{\frac{\partial Y}{\partial m} + \xi' \frac{\partial Y}{\partial V} + \frac{p}{m}\left(\xi' - \frac{\xi}{m}\right) = 0}$$

Now $\frac{\partial Y}{\partial m}(m, \xi(m)) = \frac{-p\xi}{m(n-m)} + \frac{2(n\rho - \delta\eta)}{\sigma^2(\alpha+\beta)(n-m)}\left(\frac{\eta'}{\eta}\right)\left\{\theta^{-\alpha} - \theta^{\beta}\right\}$

So if we want $\eta' < 0$ this is equivalent to

$$\boxed{\xi'\left(\frac{\partial Y}{\partial V} + \frac{p}{m}\right) - \frac{p\xi}{m}\,\frac{n}{m(n-m)} < 0}$$

For the case $p=0$, Jon proves this by noting from using my earlier Proposition 1 that $\xi' > 0$, and by geometric insight into shape of Y must have $\frac{\partial Y}{\partial V} < 0$. This leaves case $0 < p \leq 1$.

2) For general p, we can find

$$\frac{\partial Y}{\partial V} + \frac{p}{m} = \left\{\frac{\alpha\beta n\rho}{r\eta}(\theta^{\beta} - \theta^{-\alpha}) + \alpha(\beta-1)\theta^{-\alpha} - \beta(\alpha+1)\theta^{\beta} + (\alpha+\beta)\theta + \theta p(\alpha+\beta)(n-m)/m\right\} \Big/ \theta(\alpha+\beta)(n-m)$$

$$= \frac{2}{\sigma^2}\cdot\frac{\Delta}{(\alpha+\beta)(n-m)}\cdot\frac{1}{\eta}\frac{\partial \eta}{\partial \theta} \qquad \text{(Jon observed this)}$$

$$\frac{\partial \Psi}{\partial \mu} = e^{\mu+\sigma^2/2}\,\bar{\Phi}(\alpha-\sigma) \quad , \quad \frac{\partial^2 \Psi}{\partial \mu^2} = e^{\mu+\sigma^2/2}\,\bar{\Phi}(\alpha-\sigma) + \frac{K e^{-\alpha^2/2}}{\sigma\sqrt{2\pi}}$$

$$\frac{\partial \Psi}{\partial \sigma} = \frac{K e^{-\alpha^2/2}}{\sqrt{2\pi}} + e^{\mu+\sigma^2/2}\,\bar{\Phi}(\alpha-\sigma)$$

Approximating the max call (20/12/01)

1) One of the more difficult problems for the dual approach to pricing American options appears to be the max call. There are d assets, with price processes

$$S_t^i = S_0^i \exp[\sigma_i W_t^i + \mu_i t] \qquad (i=1,\dots,d)$$

where $\mu_i = r - \delta_i - \sigma_i^2/2$. For simplicity, we will restrict attention to the case where the W^i are independent, $\sigma_i = \sigma \ \forall i$, $\delta_i = \delta \ \forall i$. Write $X_t^i \equiv \log S_t^i$, $\bar{X}_t \equiv \max\{X_t^1, \dots, X_t^d\}$.

Introduce the function

$$\Psi(\mu, \sigma, K) \equiv E\,(e^{\mu+\sigma Z} - K)^+$$

$$= e^{\mu+\sigma^2/2}\, \bar{\Phi}(\alpha - \sigma) - K\bar{\Phi}(\alpha), \qquad (\sigma\alpha \equiv \log K - \mu)$$

The max call pays $(\max\{X_t^1, \dots, X_t^d\} - K)^+$ when exercised. It seems a reasonable guess that the value function might be approximately of the form

$$V(t,x) \simeq V_0(t,x) \equiv \Psi\left(m(t,x), \sigma\sqrt{T-t}, K\right) e^{-r(T-t)}$$

where T is the expiry, and we can propose different forms for the mean function m. For example, we could take

$$m(t,x) = \bar{x} + (r-\delta-\sigma^2/2)\tau + \sum_{i=1}^{d} \frac{\sigma\sqrt{\tau}}{a + b\,(\bar{x} - x_i)/\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{a}$$

for parameters $a, b > 0$ to be determined. This behaves correctly if $\bar{x}$ is substantially higher than the other values, and as values get close to $\bar{x}$, this 'effective mean' rises. Or again, we might take

$$m(t,x) = \bar{x} + (r-\delta-\sigma^2/2)\tau + \sum_{i=1}^{d} A\sigma\sqrt{\tau}\exp\{-b(\bar{x}-x_i)/\sigma\sqrt{\tau}\} - A\sigma\sqrt{\tau}$$

2) Let's try the second of these to begin with. If we let j be shorthand for the index of the leading share, then

$$dm(t,X) = \sigma dW^j + dL + \sum_{i\neq j} A\sigma\sqrt{\tau}\, e^{-b(\bar{X}-X_i)/\sigma\sqrt{\tau}} \left\{ \frac{b(\sigma(dW^i - dW^j) - dL)}{\sigma\sqrt{\tau}} + \frac{b^2}{\tau}dt - \frac{dt}{2\tau} - \frac{b(\bar{X}-X_i)}{2\sigma\tau^{3/2}}dt \right\}$$

$$= \left(\sigma - A\sigma\sum_{i\neq j} b e^{-b(\bar{X}-X_i)/\sigma\sqrt{\tau}}\right) dW^j + dL\left(1 - Ab\sum_{i\neq j} e^{-b(\bar{X}-X_i)/\sigma\sqrt{\tau}}\right)$$

$$+ \sum_{i\neq j} Ab\sigma\, e^{-b(\bar{X}-X_i)/\sigma\sqrt{\tau}}\, dW^i + A\sigma\sqrt{\tau}\sum_{i\neq j} e^{-b(\bar{X}-X_i)/\sigma\sqrt{\tau}} \left\{ \frac{b^2 - \frac{1}{2}}{\tau} - \frac{b(\bar{X}-X_i)}{2\sigma\tau^{3/2}} \right\} dt$$

Hence

$$dV_0 = e^{-r\tau}\left\{ r\Psi\,dt + \Psi_\mu\,dm + \tfrac{1}{2}\Psi_{\mu\mu}\,d\langle m\rangle - \frac{\sigma}{2\sqrt{\tau}}\Psi_\sigma\,dt \right\}$$

$$\doteq e^{-r\tau}\Big\{ r\Psi + \Psi_\mu \sum_{i\neq j} e^{-b(\bar{X}-X_i)/\sigma\sqrt{\tau}} A\left(\frac{\sigma}{\sqrt{\tau}}(b^2-\tfrac{1}{2}) - \frac{b(\bar{X}-X_i)}{2\tau}\right) - \frac{\sigma}{2\sqrt{\tau}}\Psi_\sigma$$

$$+\tfrac{1}{2}\Psi_{\mu\mu}\Big(\sum_{i\neq j} e^{-2b(\bar{X}-X_i)/\sigma\sqrt{\tau}}\, b^2A^2 + \big(1 - Ab\sum_{i\neq j} e^{-b(\bar{X}-X_i)/\sigma\sqrt{\tau}}\big)^2\Big)\sigma^2\Big\}\cdot dt$$

$$+\Psi_\mu\Big(1 - bA\sum_{i\neq j} e^{-b(\bar{X}-X_i)/\sigma\sqrt{\tau}}\Big)dL$$

While true, it won't be possible to use this as is, because of the presence of the local time term.

measure in use in the
stochastic setting

$$U(c, k_g) = \lambda c^{\vartheta} k_g^{1-s-\vartheta}$$

Return to the Yesmar problem in Arrow-Kurz 2-sector situation (28/1/02)

(i) One important feature of the AK 2-sector problem <u>not</u> incorporated explicitly earlier is the scaling behaviour of U: we have to have $U(\lambda c, \lambda k_g) = \lambda^{1-S} U(c, k_g) \quad \forall \lambda > 0$, for some $S > 0$.

The basic dynamics of the deterministic situation are given by

$$\begin{cases} \dot{k} = \bar{f}(k_p, k_g) - c = \bar{f}(k_p^*, k_g^*) - c^* \equiv \Phi(k) & \text{under optimal control} \\ U(c, k_g) - \rho V(k) + \Phi(k) V'(k) = 0 \\ U_c = V'(k) \\ U_g = V'(k)(\bar{f}_p - \bar{f}_g) \end{cases}$$

where we think of $k_g^* = k_g^*(k)$, $c^* = c^*(k)$, and are looking for examples where F and V are first selected, then the f and U are deduced.

Scaling of U gives us

$$U(c, k_g) = k_g^{1-S} h(\xi), \quad \xi \equiv c/k_g$$

so

$$\begin{cases} U_c = k_g^{-S} h'(\xi) & = V'(k) \\ U_g = k_g^{-S}\left[(1-S)h(\xi) - \xi h'(\xi)\right] & = V'(k)(\bar{f}_p - \bar{f}_g) \end{cases}$$

Thus the condition that U is increasing in both variables translates to

$$h' > 0, \quad (1-S)h - \xi h' > 0,$$

and concavity is

$$\boxed{\begin{aligned} &h'' < 0, \qquad \xi^2 h'' + 2S\xi h' - S(1-S)h < 0 \\ &(1-S)h h'' + S(h')^2 < 0 \end{aligned}}$$

in terms of h.

As before, differentiating the dynamics, and the BE, together lead to

$$0 = -\rho V' + \bar{f}_p V' + \Phi V''$$

so that f_p is fixed once Φ, V are chosen.

(ii) <u>Special case: $h(\xi) = \lambda \xi^{\nu}$</u> for some $\nu \in (0,1)$. The conditions for concavity are now

$$\boxed{1 - S - \nu > 0, \quad (1-S)(1-\nu) - \nu S > 0, \quad S(1-S) - 2\nu S + \nu(1-\nu) > 0}$$

which for given ν could be satisfied by taking $S \downarrow 0$, for given S could be satisfied by taking $\nu \downarrow 0$.

More simply, observe that

$$\frac{cV'}{U} = c\,\frac{U_t}{U} = \frac{c}{k_g}\,\frac{h'(\xi)}{h(\xi)} = \nu \quad !! \longrightarrow$$

Now if we assumed the form of c^* (and for simplicity abbreviate now to c), we would have
$$\bar{f} = \Phi + c$$
so that f is determined, and hence
$$\bar{f}' = \Phi' + c' = \bar{f}_p k_p' + \bar{f}_g k_g' = \bar{f}_p - k_g'(\bar{f}_p - \bar{f}_g) \qquad (*)$$
However, the condition on U_c gives us
$$\frac{\lambda\nu c^{\nu-1}}{k_g^{\nu-1+S}} = V'(k)$$
determining k_g:
$$k_g = \left(\frac{\lambda\nu c^{\nu-1}}{V'}\right)^{1/\nu-1+S}$$
Again, from U_g/U_c we find that $\bar{f}_p - \bar{f}_g = \frac{(1-S-\nu)}{\nu}\, c/k_g$, so the DE for c above $(*)$ becomes more simply
$$\Phi' + c' = \bar{f}_p - c\cdot\frac{k_g'}{k_g}\frac{(1-S-\nu)}{\nu}$$
$$= \bar{f}_p - \frac{(1-S-\nu)c}{\nu}\left\{\frac{\nu-1}{\nu-1+S}\frac{c'}{c} - \frac{V''}{V'}\frac{1}{\nu-1+S}\right\}$$
$$= \bar{f}_p + \frac{\nu-1}{\nu}\cdot c' - \frac{V''}{V'}\cdot\frac{c}{\nu}$$
Hence
$$\boxed{c' + \frac{V''}{V'}c = \nu(\bar{f}_p - \Phi')}$$
Using the expression for f_p in terms of V, Φ, we can rework the DE and actually solve explicitly!!
$$\boxed{c = \frac{\rho\nu V}{V'} - \nu\Phi + \frac{\text{const}}{V'}}$$
In any given situation, it remains to check $\bar{f}_p > 0$, $\bar{f}_g > 0$, and the tangent inequality for f. If we had fixed depreciation at rate δ, then we'd want $\bar{f}_p > -\delta$, $\bar{f}_g > -\delta$.

(iii) Introducing randomness gives us $dk = (\bar{f} - c)\,dt - \sigma k\, d\tilde{W} = \Phi\, dt - \sigma k\, d\tilde{W}$. The equation for $\bar{f}_p$ changes to
$$0 = \sigma^2 k V'' + \tfrac{1}{2}\sigma^2 k^2 V''' + \Phi V'' - \rho V' + \bar{f}_p V'$$
giving
$$\boxed{c = \frac{\rho\nu V}{V'} - \nu\Phi - \frac{\frac{1}{2}\nu\sigma^2 k^2 V''}{V'} + \frac{\text{const}}{V'}}$$
— check the constant by returning to the Bellman equation...

One-sector Yeoman problem again (14/2/02)

1) The dynamical system

$$dk = (f(k)-c)\,dt - \sigma k\,dW = \Phi(k)\,dt - \sigma k\,dW$$

which arises in the (stochastic version of) the one-sector Ramsey problem has value function V solving the Bellman equation:

$$\sup_c \left\{ U(c) - \rho V + \tfrac{1}{2}\sigma^2 V'' k^2 + (f(k)-c)\,V'(k) \right\} = 0$$

If we want the drift under optimal control to be $\Phi(k)$, we conclude that

$$\begin{cases} U(c) - \rho V + \tfrac{1}{2}\sigma^2 k^2 V''(k) + \Phi(k)\,V'(k) = 0 \\ U'(c) = V'(k) \end{cases}$$

Differentiating the Bellman equation gives

$$\begin{aligned} 0 &= U'(c).c' - \rho V' + \sigma^2 k V'' + \tfrac{1}{2}\sigma^2 k^2 V''' + \Phi V'' + \Phi' V' \\ &= U'(c)\{f' - \Phi'\} - \rho V' + \sigma^2 k V'' + \tfrac{1}{2}\sigma^2 k^2 V''' + \Phi V'' + \Phi' V' \\ &= V' f' - \rho V' + \tfrac{1}{2}\sigma^2 k^2 V''' + \sigma^2 k V'' + \Phi V'' \end{aligned}$$

This gives f in terms of the assumed primitives Φ and V of the problem. We still need to check that f is increasing concave, and $c, f \geq 0$, in any given example, but this is in effect the template for all Yeoman problems. Need c increasing with k also, as $U'(c) = V'(k)$

2) $\Phi(k) = ak^b - \mu k$, $V(k) = k^{1-R}/(1-R)$ gives

$$\left.\begin{aligned} f(k) &= Rak^b/b + k\left(\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R\right) \\ c(k) &= (R-b)ak^b/b + k\left(\rho + \tfrac{1}{2}\sigma^2 R(1-R) - \mu R + \mu\right) \end{aligned}\right\}$$

OK if $b \in (0,1)$, $R > b$, $\rho + (1-R)(\tfrac{1}{2}\sigma^2 R + \mu) > 0$

3) If we assume that $U'(c) = c^{-R}$, some calculations lead to a relation between V and Φ:

$$V'\Phi' + V''\Phi = \rho V' - \tfrac{1}{2}\sigma^2 k^2 V''' - \sigma^2 k V'' + \tfrac{1}{R}(V')^{-1/R} V''$$

whence for $R \neq 1$ we get

$$f = \frac{\text{const}}{V'} + \rho \frac{V}{V'} - \frac{\tfrac{1}{2}\sigma^2 k^2 V''}{V'} - \frac{R}{1-R}(V')^{-1/R}$$

We may now ask for a given concave increasing V whether the f so defined is concave increasing; if so, we have an economically meaningful stochastic Ramsey problem (so long as the diffusion k remains non-negative)

For example, if $V(k) = k^{1-S}/(1-S)$, and $0 < S < 1 < R$, we have a valid solution with production function

$$\boxed{f(k) = k\left(\tfrac{1}{2}\sigma^2 S + \frac{\rho}{1-S}\right) + \frac{R}{R-1}\,k^{(S/R)}}$$

NB in the context of growth models, only CRRA utilities U have any meaning

$$d\eta^{-1} = \eta^{-1}\left\{ -dZ^L + (v_{LL} - \mu_L - r)\,dt \right\}$$

To avoid awful notational clashes, let's make the following conventions:

F is production function

Φ is the drift in the optimally controlled process:

$$dk^* = k^*\left(dz^0 - dz^L\right) + \Phi(k^*)\,dt$$

so $\Phi(k) = f(k_p^*(k), k_g^*(k)) - \gamma_g k - c^*(k)$

The 2-sector Arrow-Kurz model with different endowments (14/2/02)

(i) Let's consider firstly the government problem, where we take for the dynamics

$$\begin{cases} dK = K\,dZ^0 + \big(f(K_p, K_g, L) - C - \delta K\big)\,dt \\ dK_p = dI_p - \delta K_p\,dt \\ dK_g = dI_g - \delta K_g\,dt \end{cases}$$

and the assumptions concerning the population process L_t and the technology process T_t are

$$dL_t = L_t\,(dZ_t^L + \mu_L\,dt)\,, \qquad dT_t = \tau T_t\,dt$$

both starting at 1. As before, Z^0 and Z^L are two (multiples of) BMs, $dZ^i dZ^j = v_{ij}\,dt$, and we set $\eta = LT$. Government's objective is to max

$$E\int_0^\infty e^{-\rho_g t} L_t\, U\Big(\frac{C_t}{L_t}, \frac{K_g(t)}{L_t}\Big)\,dt$$

$$= E\int_0^\infty e^{-\rho_g t} L_t T_t^{1-R}\, U(c_t, k_g(t))\,dt \qquad \begin{pmatrix} c_t \equiv C_t/\eta_t\,,\ k_g(t) \equiv K_g(t)/\eta_t \\ U \text{ homogeneous of degree } 1-R \end{pmatrix}$$

$$= E\int_0^\infty e^{-\lambda_g t}\, e^{Z_t^L - \frac{1}{2}v_{LL}t}\, U(c_t, k_g(t))\,dt \qquad [\lambda_g \equiv \rho_g - \mu_L - (1-R)\tau]$$

$$= \tilde{E}\int_0^\infty e^{-\lambda_g t}\, U(c_t, k_g(t))\,dt$$

where under $\tilde{P}$, $dZ^L = dz^L + v_{LL}\,dt$, $dZ^0 = dz^0 + v_{0L}\,dt$. The dynamics of $k \equiv K/\eta$ are

$$dk = k\,(dZ^0 - dZ^L) + \big[f(k_p, k_g) - \gamma_p k - c\big]\,dt \qquad [\gamma_p \equiv \delta + \mu_L + \tau + v_{0L} - v_{LL}]$$
$$= k\,(dz^0 - dz^L) + \big[f(k_p, k_g) - \gamma_g k - c\big]\,dt \qquad [\gamma_g = \delta + \mu_L + \tau]$$

and the equations which arise for the value function are

$$\boxed{\begin{gathered} \max_{c,\,k_g}\Big\{U(c, k_g) - \lambda_g V(k) + \tfrac{1}{2}\sigma^2 k^2 V''(k) + \big(f(k_p, k_g) - \gamma_g k - c\big)V'(k)\Big\} = 0 \\ U_c = V' \\ U_g = V'\,(f_p - f_g) \end{gathered}}$$

with $\sigma^2 = v_{00} + v_{LL} - 2v_{0L}$. Apart from the interpretations of the constants, <u>these equations are exactly what we got before</u>.

(ii) Now bring in the private sector's optimisation. Let's suppose that output of the economy gets split

$$\begin{cases} K_p\,dZ^0 + K_p F_p\,dt & \text{due to private capital} \\ K_g\,dZ^0 + K_g F_g\,dt & \text{due to govt capital} \\ L F_L\,dt & \text{due to labour.} \end{cases}$$

Let's also assume that the government appropriates <u>none</u> of the returns on government capital, but instead levies taxes on consumption, income, returns on capital, and returns on govt. debt D. Thus the

$$\tilde{f}(k) = f(k_p^*(k), k_g^*(k)) - \gamma_g k$$

budget equation for the private sector will be

$$\beta_k\,(K_p\,dZ^0 + K_p \bar{F}_p\,dt) + \beta_W\left(F dt - K_p\,dZ^0 - K_p \bar{F}_p\,dt + K\,dZ^0\right) + r\beta_r D\,dt \qquad (*)$$
$$= dI_p + \beta_c^{-1} C\,dt + dD,$$

or more simply

$$\boxed{(\beta_k - \beta_W)\,(K_p\,dZ^0 + K_p \bar{F}_p\,dt) + \beta_W\,F\,dt + r\beta_r D\,dt = dI_p + \beta_c^{-1} C\,dt + dD}$$

Rearranging, we get

$$\boxed{d(K_p + D) = K_p\left[(\beta_k - \beta_W)\,dZ^0 + \left((\beta_k - \beta_W)\bar{F}_p - \delta\right) dt\right] + r\beta_r D\,dt + (\beta_W F - \beta_c^{-1} C)\,dt}$$

This has a tempting (but misleading) interpretation as an equation for the evolution of the wealth $K_p + D$ of the private sector. It's not quite as simple as it might appear, since K_p is an argument of F and $\bar{F}_p$, also indirectly in the β's, if they happen to be functions of k.

(iii) <u>Special case</u>: If we return to the situation at the bottom of p.37, we can look for a solution of the form

$$\boxed{V(k) = k^{1-S}/(1-S), \quad \Phi(k) = \bar{f}(k) - c(k) = ak^b - \mu k, \quad U(c,k_g) = \lambda\, c^{\nu}\, k_g^{1-R-\nu}}$$

for chosen $b \in (0,1)$, $a, \mu > 0$, and $\nu < 0$. Can we find a solution of the simple form

$$\boxed{k_g^*(k) = \theta k, \quad c^*(k) = \Gamma k^b, \quad \bar{f}(k_p, k_g) = \Lambda\, k_p^{\alpha}\, k_g^{\beta} + \psi k}$$

for positive $\theta, \Gamma, \alpha, \beta, \Lambda$? It turns out that we can provided various conditions hold:

(1) $\dfrac{\rho}{1-S} + \mu + \sigma^2 S/2 = 0$ (for ρ, read λ_g)

(2) $R - S = (1-\nu)(1-b) > 0$

(3) $\lambda = \theta^{\nu+R-1}/\nu\,\Gamma^{\nu-1}$

(4) $\Gamma = -a\nu$

(5) $a(1-\nu) = \Lambda\,(1-\theta)^{\alpha}\,\theta^{\beta}$

(6) $\alpha + \beta = b$

(7) $\psi = -\mu$

(8) $-\theta\alpha + (1-\theta)\beta = \dfrac{1-\theta}{1-\nu}(1-R-\nu)$

(9) $\alpha, \beta \in (0,1)$

So $\boxed{\alpha = \dfrac{S(1-\theta)}{1-\nu} > 0, \quad \beta = b - \alpha}$

Now (1) uniquely determines $S>1$ from $\rho \equiv \partial_g$, μ and σ; next, (2) determines what $R>0$ should be. Checking the two sides of the equation for c at the foot of p 37 gives agreement, using (1) and (4). The expression for k_g on p37 reduces to θk, using (3). The Bellman equation holds and $U_c = V'$. We can work out

$$\bar{f} = a(1-\nu)k^b - \mu k \quad (= \Phi + c) \quad = \Lambda k_p^{\alpha} k_g^{\beta} + \Psi k \qquad \text{if (5), (7) hold}$$

so

$$\bar{f}_p = aSk^{b-1} - \mu \qquad \text{(from the expression for } f_p \text{ in terms of } F, V)$$

$$= \alpha \Lambda k_p^{\alpha-1} k_g^{\beta} + \Psi \qquad \text{(after some simplifications)}$$

We also need to check the optimality condition

$$U_g = V'(\bar{f}_p - \bar{f}_g).$$

But we already have $U_c = V'$, so

$$\frac{U_g}{U_c} = \frac{1-R-\nu}{\nu} \frac{c}{k_g} = -a \cdot \frac{1-R-\nu}{\theta} \cdot k^{b-1} = \bar{f}_p - \bar{f}_g,$$

as required. Note that this form of the solution works just as well for the original randomness structure.

(iv) <u>The private sector's optimisation problem.</u> We are now going to think of the private sector as consisting of a very large number $L_0 \sim 1/\varepsilon$ of identical households, each of whom is acting individually to maximise his optimality criterion

$$E \int_0^\infty e^{-\rho_p t} u\left(\frac{\Delta C_t}{L_t/L_0}, K_g(t)/L_t\right) dt \tag{10}$$

where ΔC_t is the small amount of consumption of the individual household. What is the budget equation to be satisfied? If we consider the aggregate budget equation (*) of the private sector and think what happens if there is a small perturbation to the government's optimal solution, we see the budget equation for the single household to be

$$\boxed{\begin{aligned} &p_k \Delta K_p \left(dZ^0 + f_p dt\right) + p_W \frac{T_t L_t}{L_0}\left\{(1-\theta)k_g^* dZ^0 + \left(f - k_p^* f_p - \theta k_g^* f_g\right)dt\right\} + r p_r \Delta D\, dt \\ &= d\Delta K_p + \delta \Delta K_p\, dt + d\Delta D + p_c^{-1} \Delta C\, dt \end{aligned}} \tag{11}$$

where the derivatives of f, and the tax rates, <u>are evaluated along the optimal trajectory</u> k^*. How can this be justified, because if we perturb the budget equation (*) we get not only terms like $p_k \Delta K_p$, but also terms $p_k' \cdot \Delta k^* \cdot K_p$, which will be of the same order? The rationalisation is that the total output of the economy will indeed be perturbed by such terms, but that the changes will be absorbed by the entire population, so the effects on an individual can be neglected. The individual household receives the market return on its $O(1)$ private capital and the market wage for its $O(1)$ labour.

The aggregate output of the economy is

$$L_t T_t \left\{ k_t\, dZ^0_t + \left((f - k_p^* f_p - k_g^* f_g) + k_p^* f_p + k_g^* f_g \right) dt \right\}$$

so the return per unit of labour will be

$$T_t \left[(1-\theta) k_g^* (dZ^0 + f_g\, dt) + (f - k_p^* f_p - k_g^* f_g)\, dt \right],$$

which explains the perturbed budget equation.

We suppose that u is homogeneous of degree $1-R_p$, so that the household's objective is to

$$\max E \int_0^\infty e^{-\rho_p t}\, u\left(\frac{\Delta c_t . L_0}{L_t}, \frac{K_g(t)}{L_t} \right) dt$$

$$= \max E \int_0^\infty e^{-\rho_p t}\, T_t^{1-R_p}\, u\left(\frac{\Delta c_t . L_0}{\eta_t}, k_g^*(t) \right) dt$$

$$= \max E \int_0^\infty e^{-\lambda_p t}\, u\left(\frac{\Delta c_t . L_0}{\eta_t}, k_g^*(t) \right) dt \qquad (\lambda_p \equiv \rho_p - (1-R_p)\tau)$$

This shows that we need to work with intensive variables, $k_p \equiv L_0 \Delta K_p/\eta$, $\Delta_p \equiv L_0 \Delta D/\eta$, $c \equiv L_0 \Delta c/\eta$, (which we believe should turn out to be the starred quantities). After some calculations we have the budget equation (with $x \equiv k_p + \Delta_p$)

$$\boxed{\begin{aligned} dx &= k_p\left[\beta_p\, dZ^0 - dZ^L + (v_{LL} - \mu_L - \tau - \beta_p v_{0L} - \delta + \beta_p f_p)\, dt \right] \\ &\quad + \Delta_p\left[-dZ^L + (v_{LL} - \mu_L - \tau + r\beta_r)\, dt \right] - \beta_c^{-1} c\, dt \\ &\quad + \beta_w \left\{ (1-\theta) k_g^*\, dZ^0 + (f - k_p^* f_p - \theta k_g^* f_g)\, dt \right\} - v_{0L}\beta_w (1-\theta) k_g^*\, dt \\ &= k_p\left[\beta_p\, dZ^0 - dZ^L + (\beta_p f_p - \delta - \beta_p v_{0L} + \mu_0)\, dt \right] + \Delta_p\left[-dZ^L + (r\beta_r + \mu_0)\, dt \right] - \beta_c^{-1} c\, dt \\ &\quad + \left[A\, dZ^0 + B\, dt \right] \end{aligned}} \qquad (12)$$

where $\mu_0 \equiv v_{LL} - \mu_L - \tau$, $A \equiv \beta_w (1-\theta) k_g^*$, $B \equiv \beta_w \left(f - k_p^* f_p - \theta k_g^* f_g - v_{0L}(1-\theta) k_g^* \right)$. This budget equation has the familiar form: return on wealth invested in private capital + return on wealth invested in govt debt − consumption + return due to wage income.

Observe: if (as we plan) both k_p^* and Δ_p^* are to be functions of k^*, then $x^* = k_p^* + \Delta_p^* = h(k^*)$ and we have

$$dx^* = h'(k^*)\, k^* (dZ^0 - dZ^L) + \text{FV terms}$$

On comparing coefficients of dZ^L we conclude that $h(k) = k h'(k)$, that is, for some constant Γ

Check transversality condition! →

we must have

$$\boxed{k_p^* + \Delta_p^* = \Gamma k^*} \tag{13}$$

Next comparing the coefficients of dZ^0 we learn that we would have to have

$$\boxed{\beta_k k_p^* + A \equiv \beta_k k_p^* + \beta_W (1-\theta) k_g^* = \Gamma k^*} \tag{14}$$

But can this be done?

(V) Solving the private sector's optimisation problem. Introduce a Lagrangian process $e^{-\lambda_p t}\psi$, where $d\psi = \psi[a(dZ^0 - dZ^L) + b_0 dt]$ (the point is that we shall be wanting the multiplier process also to be a function of k^*, which requires a driving stochastic term $dZ^0 - dZ^L$). The Lagrangian form of the problem therefore is

$$\max E\int_0^\infty e^{-\lambda_p t}\Big[u(c, k_g^*) + \psi\{x(r\beta_r + \mu_0) + k_p(\beta_k f_p - \delta - \beta_k v_{0L} - r\beta_r) - \beta_c^{-1} c + B\}$$

$$+ x\psi(b_0 - \lambda_p) + a\psi\{(k_p\beta_k + A)v_{00} - (x + k_p\beta_k + A)v_{0L} + x v_{LL}\}\Big]dt \quad + x_0\psi_0$$

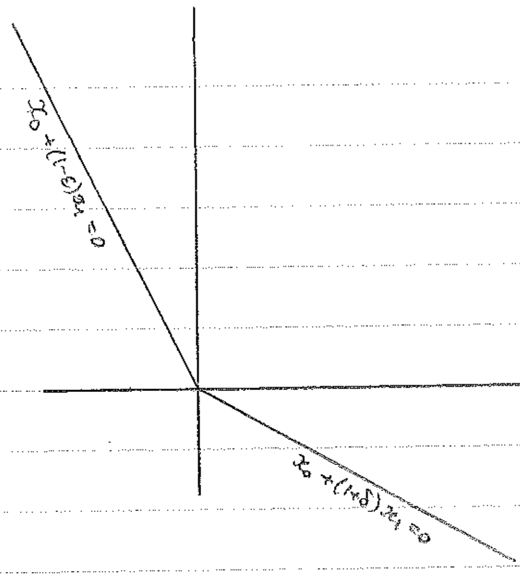
$$= \max E\int_0^\infty e^{-\lambda_p t}\Big[\tilde{u}(\beta_c^{-1}\psi, k_g^*) + \psi k_p\{\beta_k f_p - \delta - \beta_k v_{0L} - r\beta_r + a(\beta_k v_{00} - \beta_k v_{0L})\}$$

$$+ x\psi(r\beta_r + \mu_0 + b_0 - \lambda_p - a v_{0L} + a v_{LL}) + \psi\big(B + Aa(v_{00} - v_{0L})\big)\Big]dt + x_0\psi_0$$

which leads us to the conditions

$$\boxed{\begin{aligned} &\beta_k f_p - \delta - \beta_k v_{0L} - r\beta_r + a\beta_k(v_{00} - v_{0L}) = 0 && (15)\\ &r\beta_r + \mu_0 + b_0 - \lambda_p + a(v_{LL} - v_{0L}) = 0 && (16)\\ &u_c(c^*, k_g^*) = \beta_c^{-1}\psi && (17)\end{aligned}}$$

In principle, this allows us to build solutions. Once we've chosen β_c (perhaps constant), from (17) we deduce ψ, which fixes a and b, both functions of k^*. Then (16) tells us what $r\beta_r$ is going to be, (15) determines β_k, and from (14) we'd deduce β_W in terms of Γ.



$x_0 + (1-\delta)x_1 = 0$
$x_0 + (1+\delta)x_1 = 0$

The Cvitanić–Karatzas example (19/3/02)

1) Consider a single asset with transaction costs, as studied by Davis & Norman, Cvitanić–Karatzas with dynamics

$$dX_0 = rX_0\,dt + (1-\varepsilon)\,dM - (1+\delta)\,dL$$

$$dX_1 = X_1(\sigma dW + \alpha dt) - dM + dL$$

where M, L are increasing and adapted, and $r, \sigma, \alpha, \sigma^{-1}$ are assumed bounded in the CK paper, and for what follows.

The solvency cone C for this problem is defined by

$$C = \{x : x\cdot\eta^0 \geq 0,\ x\cdot\eta^1 \geq 0\} = \{a_0\xi^0 + a_1\xi^1 : a_0, a_1 \geq 0\}$$

where

$$\xi^0 \equiv (\varepsilon+\delta)^{-1}\begin{pmatrix} -1+\varepsilon \\ 1 \end{pmatrix}, \qquad \xi^1 \equiv (\varepsilon+\delta)^{-1}\begin{pmatrix} 1+\delta \\ -1 \end{pmatrix} \qquad (\xi^i\cdot\eta^j = \delta_{ij})$$

$$\eta^0 \equiv \begin{pmatrix} 1 \\ 1+\delta \end{pmatrix}, \qquad \eta^1 \equiv \begin{pmatrix} 1 \\ 1-\varepsilon \end{pmatrix}$$

with dual cone

$$C^* \equiv \{\lambda_0\eta^0 + \lambda_1\eta^1 : \lambda_0, \lambda_1 \geq 0\} = \{y : y\cdot\xi^0 \geq 0,\ y\cdot\xi^1 \geq 0\}.$$

2) In order to approach the super-replication problem, let's consider the utility-maximisation problem

$$\sup_{X\in\mathcal{X}(x)} E\,U(X(T) - f) \qquad (x\in C \text{ fixed})$$

where the contingent claim f is C-valued, $f \equiv \varphi_0\xi^0 + \varphi_1\xi^1$, and supposed to be L^2:

$$E(\varphi_0^2 + \varphi_1^2) < \infty.$$

We shall suppose a very specific form for U:

$$\boxed{U(x) = \psi_n(\eta^0\cdot x) + \psi_n(\eta^1\cdot x)}$$

so that

$$\boxed{V(y) \equiv \tilde\psi_n(\xi^0\cdot y) + \tilde\psi_n(\xi^1\cdot y)}$$

where $\psi_n(x) \equiv \frac{1}{n}\psi(nx)$, $\tilde\psi_n(y) \equiv \sup_x\{\psi_n(x) - x\cdot y\}$ and the concave function ψ is defined by

$$\boxed{\psi(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(x-1)^2, & x \leq 0 \\ 1 - e^{-x}, & x \geq 0 \end{cases}}$$

p^* is the risk-neutral probability, of course.

In this case, we get explicitly

$$\tilde{\psi}(\lambda) = \begin{cases} \frac{1}{2}(\lambda-1)^2, & \lambda \geq 1 \\ \lambda \log \lambda - \lambda + 1, & 0 \leq \lambda \leq 1 \\ +\infty, & \lambda < 0 \end{cases}$$

and $\tilde{\psi}_n(\lambda) = \frac{1}{n}\tilde{\psi}(\lambda)$.

3) <u>The primal solution</u>. The aim here is to show that the sup is in fact attained. Notice that we can re-express the dynamics as

$$d\tilde{X}_0 = \beta(1-\varepsilon)\,dM - \beta(1+\delta)\,dL$$
$$d\tilde{X}_1 = \tilde{X}_1\,\sigma\,dW^* - \beta\,dM + \beta\,dL$$

where $\beta_t \equiv \exp(-\int_0^t r_s\,ds)$ is the discount factor, $\tilde{X}_i = \beta X_i$, $dW^* = dW + \sigma^{-1}(\alpha - r)\,dt$. Thus

$$\begin{cases} \eta^0 \cdot \tilde{X} = P^* - \text{loc. mg} - \int \beta(\varepsilon+\delta)\,dM \\ \eta^1 \cdot \tilde{X} = P^* - \text{loc mg} - \int \beta(\varepsilon+\delta)\,dL \end{cases}$$

and so

$$\eta^0 \cdot X(0) \geq E^* \int_0^T \beta(\varepsilon+\delta)\,dM \geq \text{const}\cdot E^* M_T$$
$$\eta^1 \cdot X(0) \geq E^* \int_0^T \beta(\varepsilon+\delta)\,dL \geq \text{const}\; E^* L_T$$

Since all wealth processes X stay within C and we can thus use Fatou's Lemma. From this, if we have processes $X^{(k)} \in \mathcal{X}(x)$ – with corresponding buy/sell processes $L^{(k)}, M^{(k)}$ – such that

$$E\,U(X^{(k)}(T) - f) \geq \sup E\,U(X(T) - f) - 1/k$$

we can by taking convex combinations and passing to a fast subsequence assume that a.s. $M_q^{(k)} \to M_q$, $L_q^{(k)} \to L_q$ for all rational $q \in [0,T]$. The $L^1(P^*)$ bound on $M_T^{(k)}$ and $L_T^{(k)}$ goes through to the limit. (This is in effect the Kabanov-Last result)

Do the corresponding $\tilde{X}^{(k)}$ processes converge? In fact, all we need is $\tilde{X}^{(k)}(T)$ convergent, and it's easy to get this:

$$\tilde{X}_0^{(k)}(T) = x_0 + \int_0^T \beta_s\{(1-\varepsilon)\,dM_s^{(k)} - (1+\delta)\,dL_s^{(k)}\}$$

$$S_T^{-1} X_1^{(k)}(T) = x_1 + \int_0^T S_u^{-1}(dL_u^{(k)} - dM_u^{(k)})$$

and so since β and S are both continuous there is a.s. convergence. What happens to the expectation in the limit? Since

$$-U(X(T)-f) \leq -U(-f) \leq A + B(\varphi_0^2 + \varphi_1^2) \in L^1$$

for some constants A, B, the negative parts of $U(X^{(k)}(T) - f)$ are UI, so Fatou goes the correct way: expectation doesn't decrease in the limit, and so the limit point <u>does</u> attain the sup.

$$\mathcal{Y} = \bigcup_{y \in C^*} \mathcal{Y}(y)$$

4) <u>The dual problem</u> Let's introduce dual multiplier process Y_0, Y_1

$$dY_i = Y_i(a_i\,dW + b_i\,dt)$$

and take the Lagrangian form of the problem:

$$\sup E\Big[U(X(T)-f) + \int_0^T X_0Y_0(r+b_0)\,dt + \int_0^T X_1Y_1(\alpha+b_1+a_1\sigma)\,dt - X(T)\cdot Y(T) + X(0)\cdot Y(0)$$
$$+ \int_0^T \{(1-\varepsilon)Y_0 - Y_1\}\,dM + \int_0^T \{Y_1-(1+\delta)Y_0\}\,dL\Big]$$

$$\le E\big[V(Y(T)) - Y(T)\cdot f + X(0)\cdot Y(0)\big]$$

if we have the dual-feasibility conditions $r+b_0=0$, $\alpha+b_1+a_1\sigma=0$, $(1-\varepsilon)\le Y_1/Y_0 \le 1+\delta$.
So the space of dual processes is

$$\mathcal{Y}(y) = \{Y : Y(t)\in C^*\ \forall t,\ Y_0(t)B_t \text{ and } Y_1(t)S_t \text{ are martingales}, Y(0)=y\}.$$

The space is non-empty: $Y_i(t) = y_i\,\zeta(t)$. Each $\mathcal{Y}(y)$ is convex. Because of the boundedness assumptions, $\zeta_T = \bigcap_{p>1} L^p$, and so the dual value

$$\inf_y \inf_{Y\in\mathcal{Y}(y)} E\big[V(Y(T)) - Y(T)\cdot f + X(0)\cdot Y(0)\big]$$

is $<\infty$. Noticing that

$$V(Y) - Y\cdot f = \tilde\psi(\xi^0\cdot Y) + \tilde\psi(\xi^1\cdot Y) - \varphi_0(\xi^0\cdot Y) - \varphi_1(\xi^1\cdot Y)$$

$$\ge -(\varphi_0+\varphi_1) - \tfrac12(\varphi_0^2+\varphi_1^2) \qquad (*)$$

by elementary calculations, we have that the dual value is $>-\infty$. Moreover, the negative parts of $\{V(Y(T)) - Y(T)\cdot f : Y\in\mathcal{Y}\}$ are UI.

Now select a sequence $Y^{(k)}\in\mathcal{Y}$ approximating the infimum. For now, we'll

ASSUME $X(0)\in C^0$

so that the $Y^{(k)}(0)$ remain bounded. By taking convex combinations, we can suppose that the $Y^{(k)}(T)$ converge a.s., and the $Y^{(k)}(0)$ converge. Now for $z\ge0$, $a\ge0$, we have the elementary inequality

$$\tilde\psi(z) - az \ge -a\,I_{[0,1]}(z) + \left\{\tfrac14 z^2 + \tfrac12 - \tfrac34(1+a)^2\right\} I_{(1,\infty)}(z).$$

Applying this with $a=\varphi_i$, $z=\xi^i\cdot Y^{(k)}(T)$, we see that the sequence $(Y^{(k)}(T))_{k=1}^\infty$ is bounded in L^2, and hence the $Y^{(k)}(T)$ converge not only a.s., but also in any L^p, $1\le p<2$. Hence we get good convergence of the limit variables of the martingales $Z_0^{(k)} \equiv Y_0^{(k)}B$, $Z_1^{(k)} \equiv Y_1^{(k)}S$. Indeed, the $Z_0^{(k)}(T)$ converge in $L^{3/2}$, and since $S(T)\in\bigcap_{p>1}L^p$, $Z_1^{(k)}(T)$ also converge in $L^{3/2}$. We deduce that the $Y^{(k)}$ processes converge to a limit process $Y^* \in \mathcal{Y}$. Once again, the Fatou inequality goes the right way, and so the infimum in the dual problem is attained.

The big question of course is:

"The values of primal and dual problems are attained: are they equal?"

5) Properties (XY) are always elusive, but one thing which would help here is if we could prove that

$$\mathcal{Y}(y_1+y_2) = \mathcal{Y}(y_1) + \mathcal{Y}(y_2) \qquad \forall y_1, y_2 \in C^*$$

then property (XY2) is immediate. In view of the property $\mathcal{Y}(\lambda y) = \lambda \mathcal{Y}(y) \quad \forall \lambda \geq 0$, it will be enough to show that

$$\mathcal{Y}(\pi_1 y_1 + \pi_2 y_2) = \pi_1 \mathcal{Y}(y_1) + \pi_2 \mathcal{Y}(y_2)$$

where $0 \leq \pi_1 \leq 1$, $\pi_2 = 1-\pi_1$; it's obvious that $\supseteq$ holds, but how about the other inclusion? To show this, we need to show that if $Y \in \mathcal{Y}(y)$, and $y_1, y_2 \in C^*$ are such that $\pi_1 y_1 + \pi_2 y_2 = y$, then there exist $Y_i \in \mathcal{Y}(y_i)$ such that $Y = \pi_1 Y_1 + \pi_2 Y_2$. Write $Y_i(t) = \begin{pmatrix} Y_{i0}(t) \\ Y_{i1}(t) \end{pmatrix}$ etc. The defining martingale property of $\mathcal{Y}(y)$ shows that we must have for some a_0, a_1

$$dY_0 = Y_0\{-r\,dt + a_0\,dW\}$$

$$dY_1 = Y_1\{-\alpha\,dt + a_1(dW - \sigma dt)\} \equiv Y_1\{-\alpha\,dt + a_1\,d\tilde{W}\} \quad \text{say}$$

Similar expressions hold for Y_1 and Y_2, and the convex combination condition tells us that

$$\left.\begin{aligned} \pi_1 Y_{10} a_{10} + \pi_2 Y_{20} a_{20} &= a_0 Y_0 \\ \pi_1 Y_{11} a_{11} + \pi_2 Y_{21} a_{21} &= a_1 Y_1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \pi_1 Y_{10}(a_{10} - a_{20}) &= (a_0 - a_{20}) Y_0 \\ \pi_1 Y_{11}(a_{11} - a_{21}) &= (a_1 - a_{21}) Y_1 \end{aligned}$$

If we define $\varphi_0 \equiv (a_0 - a_{20})/(a_{10} - a_{20})$, $\varphi_1 \equiv (a_1 - a_{21})/(a_{11} - a_{21})$, then

$$\pi_1 Y_{10} = \varphi_0 Y_0 \quad , \quad \pi_1 Y_{11} = \varphi_1 Y_1 \qquad (\text{hence } \pi_2 Y_{20} = (1-\varphi_0) Y_0, \; \pi_2 Y_{21} = (1-\varphi_1) Y_1)$$

from which a few calculations lead us to

$$d\varphi_0 = \varphi_0 (a_{10} - a_0)\{dW - a_0\,dt\} \quad , \quad d\varphi_1 = \varphi_1 (a_{11} - a_1)\{d\tilde{W} - a_1\,dt\}$$

with $d\tilde{W} \equiv dW - \sigma dt$, and symmetrically

$$d(1-\varphi_0) = (1-\varphi_0)(a_{20} - a_0)\{dW - a_0\,dt\} \;, \quad d(1-\varphi_1) = (1-\varphi_1)(a_{21} - a_1)\{d\tilde{W} - a_1\,dt\}$$

So we can choose one of $\{a_{10}, a_{20}\}$, one of $\{a_{11}, a_{21}\}$ and the rest is determined. We require

$$d\log\left(Y_{i1}/Y_{i0}\right) = (a_{i1} - a_{i0})\,dW + \{r - \alpha - a_{i1}\sigma - a_{i0}(a_{i1} - a_{i0}) - \tfrac{1}{2}(a_{i1} - a_{i0})^2\}\,dt$$

to be the differential of a process bounded in $[\log(1-\varepsilon), \log(1+\delta)]$, for $i = 1, 2$.

Any hope of doing this?

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k\geq 0} \left(\frac{x}{2}\right)^{2k} \Big/ k!\,\Gamma(k+\nu+1)$$

If Z is a $BESQ^d$ then

$$E^z\left(e^{-\lambda Z_t}\right) = (1+2\lambda t)^{-d/2} \exp\{-\lambda z/(1+2\lambda t)\}$$

As $E\, Z_t^{-R} = \int_0^\infty \lambda^{R-1} e^{-\lambda Z_t} \frac{d\lambda}{\Gamma(R)} = \frac{1}{\Gamma(R)} \int_0^\infty \frac{\lambda^{R-1}}{(1+2\lambda t)^{d/2}} e^{-\lambda z/(1+2\lambda t)}\, d\lambda$

As for finiteness we need $\boxed{d/2 > R}$

This is a hypergeometric function

$${}_1F_1(a,b,z) = \sum_{n\geq 0} \frac{\Gamma(a+n)}{\Gamma(a)} \cdot \frac{\Gamma(b)}{\Gamma(b+n)} \frac{z^n}{n!}$$

Option pricing in an almost complete market (10/4/02)

1) Consider the following set-up. There are J assets, asset j generating dividend process δ_j, where

$$d\delta_j = \sigma\sqrt{\delta_j}\, dW^j + (\alpha_j - \beta\delta_j)\,dt$$

and total production $\Delta \equiv \sum_j \delta_j$ therefore satisfies

$$\boxed{d\Delta = \sigma\sqrt{\Delta}\, d\tilde{W} + (\alpha - \beta\Delta)\,dt} \qquad \left(\begin{array}{l} d\tilde{W} = \sum \sqrt{\delta_j}\, dW^j/\sqrt{\Delta} \\ \alpha = \sum \alpha_j \end{array}\right)$$

Suppose we have a single representative agent maximising

$$E\int_0^\infty e^{-\rho t}\, U(c_t)\,dt$$

Where $U'(x) = x^{-R}$, then usual story tells us that $e^{-\rho t}U'(\Delta) = \lambda \zeta_t$, so we may take the State-price density process to be

$$\boxed{\zeta_t = e^{-\rho t}\Delta_t^{-R}}$$

2) What would bond prices be here? What is the transition density of Δ? How does the spot-rate process evolve?

As for the last,

$$\frac{d\zeta}{\zeta} = -\frac{R\sigma}{\sqrt{\Delta}}\, d\tilde{W} - \left\{\rho - \beta R + \frac{R}{\Delta}\left(\alpha - \tfrac{1}{2}\sigma^2(R+1)\right)\right\}dt$$

So looks like we'll demand $\alpha \geq \frac{1}{2}\sigma^2(R+1)$, and then the spot rate process will be

$$r = \rho - \beta R + \frac{R}{\Delta}\left(\alpha - \tfrac{1}{2}\sigma^2(R+1)\right)$$

where $d\Delta = \sigma\sqrt{\Delta}\, dW^* + (\alpha - R\sigma^2 - \beta\Delta)\,dt$ in risk-neutral terms.

Routine transformations show that if X solves

$$dX = \sigma\sqrt{X}\, dW + (\alpha - \beta X)\,dt$$

then $(X_t) \overset{\mathcal{D}}{=} \left(e^{-\beta t} Z(\lambda(e^{\beta t}-1))\right)$ where Z is a BESQ$(4\alpha/\sigma^2)$, $\lambda = \sigma^2/4\beta$. If we set $\delta = 4\alpha/\sigma^2$, $\nu = (\delta/2)-1$, then

$$\boxed{P(X_t \in dy \mid X_0 = x)/dy = e^{\beta t}\, q^\delta_s(x, y e^{\beta t})}, \quad s \equiv \frac{\sigma^2}{4\beta}(e^{\beta t}-1)$$

and q^δ is the transition density of BESQ$^\delta$;

$$q^\delta_t(x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} e^{-(x+y)/2t}\, I_\nu\!\left(\frac{\sqrt{xy}}{t}\right).$$

A few calculations give us

$$\boxed{E(X_t^{-R} \mid X_0 = x) = e^{R\beta t - x/2s}\sum_{k\geq 0}\frac{\Gamma(k+\nu+1-R)}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2s}\right)^k (2s)^{-R}} \qquad s = \frac{\sigma^2(e^{\beta t}-1)}{4\beta}$$

From this we can find bond prices.

3) How about shares, options on shares? It's clear that the price of a share on asset j must be a function only of Δ_t, and $\delta_j(t)$, so we have $S_j(t) = f(\Delta_t, \delta_j(t))$, where

$$\zeta_t f(\Delta_t, \delta_j(t)) + \int_0^t \zeta_s \delta_j(s)\,ds \quad \text{is a martingale.}$$

From the explicit form of ζ, we shall conclude that f must solve

$$\mathcal{G}f + \delta_j = 0$$

where

$$\boxed{\begin{aligned}\mathcal{G}f \equiv\ & \tfrac{1}{2}\sigma^2\Delta\left\{f_{11} - \frac{2R}{\Delta}f_1 + \frac{R(R+1)}{\Delta^2}f\right\} + \sigma^2\delta_j\left\{f_{12} - \frac{R}{\Delta}f_2\right\} + \tfrac{1}{2}\sigma^2\delta f_{22} \\ & + (\alpha - \beta\Delta)\left(f_1 - \frac{R}{\Delta}f\right) + (\alpha_j - \beta\delta_j)f_2\end{aligned}}$$

As for an option, this has to be worth $\varphi(T-t, \Delta_t, \delta_j(t))$ at time t, if T is the expiry. So the PDE here is an IVP,

$$\mathcal{G}\varphi - \frac{\partial\varphi}{\partial t} = 0 \quad , \quad \varphi(0, \Delta, \delta_j) = (S_j(\Delta, \delta_j) - K)^+$$

Finite difference schemes for 2 dimensions. Want to do a 2nd order PDE in a rectangular domain, and in particular want 9-point schemes for the second derivative $\frac{\partial^2 f}{\partial x \partial y}$ (others are quite easy, as they reduce to one-dimensional problem. In the middle of the grid, can use any combination of

$$\begin{array}{c|c} x \backslash y & \begin{matrix} -h & \quad 0 \quad & h \end{matrix} \\ \begin{matrix} -h \\ 0 \\ h \end{matrix} & \begin{bmatrix} 0 & \frac{1}{2h^2} & -\frac{1}{2h^2} \\ \frac{1}{2h^2} & -\frac{1}{h^2} & \frac{1}{2h^2} \\ -\frac{1}{2h^2} & \frac{1}{2h^2} & 0 \end{bmatrix} \end{array} \quad \text{and} \quad \begin{array}{c|c} x \backslash y & \begin{matrix} -h & \quad 0 \quad & h \end{matrix} \\ \begin{matrix} -h \\ 0 \\ h \end{matrix} & \begin{bmatrix} \frac{1}{4h^2} & 0 & -\frac{1}{4h^2} \\ 0 & 0 & 0 \\ -\frac{1}{4h^2} & 0 & \frac{1}{4h^2} \end{bmatrix} \end{array}$$

to get $\frac{\partial^2 f}{\partial x \partial y}$ at the circled point of the grid

At the edge $x = 0$, we can use (for y not at an edge)

$$\begin{array}{c|c} x \backslash y & \begin{matrix} -h & \quad 0 \quad & h \end{matrix} \\ \begin{matrix} 0 \\ h \\ 2h \end{matrix} & \begin{bmatrix} \frac{3}{4h^2} & 0 & -\frac{3}{4h^2} \\ -\frac{1}{h^2} & 0 & \frac{1}{h^2} \\ \frac{1}{4h^2} & 0 & -\frac{1}{4h^2} \end{bmatrix} \end{array} \quad \text{and} \quad \begin{bmatrix} \frac{1}{h^2} & -\frac{1}{2h^2} & -\frac{1}{2h^2} \\ -\frac{3}{2h^2} & \frac{1}{h^2} & \frac{1}{2h^2} \\ \frac{1}{2h^2} & -\frac{1}{2h^2} & 0 \end{bmatrix}$$

At a corner $x=0, y=0$ we can use any combination of

$$\begin{array}{c|c} x \backslash y & \begin{matrix} 0 & \quad h \quad & 2h \end{matrix} \\ \begin{matrix} 0 \\ h \\ 2h \end{matrix} & \begin{bmatrix} 0 & \frac{3}{2h^2} & -\frac{3}{2h^2} \\ \frac{3}{2h^2} & -\frac{5}{h^2} & \frac{7}{2h^2} \\ -\frac{3}{2h^2} & \frac{7}{2h^2} & -\frac{2}{h^2} \end{bmatrix} \end{array} \quad \text{and} \quad \begin{array}{c|c} x \backslash y & \begin{matrix} 0 & \quad h \quad & 2h \end{matrix} \\ \begin{matrix} 0 \\ h \\ 2h \end{matrix} & \begin{bmatrix} \frac{2}{h^2} & -\frac{5}{2h^2} & \frac{1}{2h^2} \\ -\frac{5}{2h^2} & \frac{3}{h^2} & -\frac{1}{2h^2} \\ \frac{1}{2h^2} & -\frac{1}{2h^2} & 0 \end{bmatrix} \end{array}$$

$\times 3$ $\quad + \quad$ $\times 5$

$$\begin{pmatrix} 10 & -16 & -4 \\ -16 & 0 & 16 \\ -4 & 16 & -6 \end{pmatrix}$$

Interesting questions.

1) We have a dual characterisation of the value of an American option, but is there something analogous for the optimal stopping of a controlled process?

2) J. Scheinkman & Thaleia Z. study the following problem. You have a total of A American options which you exercise between 0 and T so as to $\max E U(x_T)$, where x_0 is given, $x_T = \int_0^T \varphi_s \, dm_s + x_0$ where φ_s is payoff for exercising at time s. (Assume wlog $r \equiv 0$)

The dual form is easily shown to be

$$\min_{y, \eta} E\left[\tilde{U}(y_T) + \eta_T A + x_0 y_0 \right]$$

where $dy/y = \alpha dW - \rho dt$ for some $\rho \geq 0$, some α, and η is a positive martingale, and

$$y_t \varphi_t \leq \eta_t \qquad \forall \quad 0 \leq t \leq T.$$

Any ideas? Notice that if $U(x) = x$ $(x \geq 0)$; $-\infty$ $(x<0)$, then we've got standard American option pricing prob.

3) There is apparently a strand of the economics literature where you consider an optimisation problem of the form $\max \int_0^\infty e^{-\rho t} \phi(x_t, \dot{x}_t)\, dt$, where φ is concave, increasing in the first argument, decreasing in the second. More restrictively, we might ask that $\varphi(x, \dot{x}) = U(f(x) - \dot{x}) \equiv U(c)$ if $\dot{x} = f(x) - c$, the Ramsey dynamics.

The question then is, if we solve and find $\dot{x} = h(x)$, what h can arise this way?

4) Phelim Boyle proposes an American-style Asian option, where you get $\left(\int_{\tau-1}^{\tau} S_u \, du - K\right)^+$ upon exercising at time τ. Such an option could be used for example to cut PDE guys down to size.